The Series RLC circuit

Second order circuits contain two energy storage elements that cannot be replaced by a single equivalent element.

Then, we can write the circuit to get

This is known as a series RLC circuit.

On the left hand side use KVL to get

On the right hand side use KVL to get

The i-v characteristics of the inductor and capacitor give

There are four equations in four unknowns \(i, v, v_c, v_L\)

To get a single equation in \(v_c(t)\)

Substituting

\[ v_c(t) = v_T(t) + R_T C \frac{d^2 v_c}{dt^2} \]
This is a second-order linear differential equation with constant coefficients.

The initial conditions are

\[ V_c(0) = V_0 \]

\[ \frac{dV_c(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C} \]

**Zero-input response**

With \( V_I = 0 \) the differential equation for \( V_c \) becomes

\[ LC \frac{d^2V_c}{dt^2} + R_T C \frac{dV_c}{dt} + V_c = 0 \] (1)

We could also solve the circuit in terms of the inductor current \( i \).

\[ V_L(t) + V_c(t) - V_I(t) + i(t) R_T = 0 \]

\[ V_L = L \frac{di}{dt} - \frac{1}{C} \int_0^t i(x) \, dx + V_c(0) \]

\[ L \frac{di}{dt} + \frac{1}{C} \int_0^t i(x) \, dx + V_c(0) + i R_T = 0 \]

differentiate once more to get

\[ LC \frac{d^2i(t)}{dt^2} + R_T C \frac{di(t)}{dt} + i(t) = 0 \] (2)

Equations (1) and (2) are identical in form so we will analyze just (1)
Example 7-14

A series RLC circuit has $C = 0.25 \mu F$ and $L = 1H$. Find the roots of the characteristic equation for $R_T = 8.5k, 4k$ and $1k\Omega$.

The characteristic equation for a series RLC is

$$LCs^2 + R_TCs + 1 = 0$$

**For $R_T = 8.5k$**

$$LC = (1)(0.25 \times 10^{-6}) = 0.25 \times 10^{-6}$$

$$R_TC = (8.5 \times 10^3)(0.25 \times 10^{-6}) = 2.125 \times 10^{-3}$$

$$S_1, S_2 = \frac{-2.125 \times 10^{-3} \pm \sqrt{(2.125 \times 10^{-3})^2 - 4(0.25 \times 10^{-6})(1)}}{2(0.25 \times 10^{-6})}$$

$$S_1, S_2 = \frac{-2.125 \times 10^{-3} \pm 1.875 \times 10^{-3}}{0.5 \times 10^{-6}} = -4250 \pm 3750$$

$$S_1 = -500 \quad S_2 = -8000$$

**For $R_T = 4k$**

$$R_TC = (4 \times 10^3)(0.25 \times 10^{-6}) = 10^{-3}$$

$$S_1, S_2 = \frac{-10^{-3} \pm \sqrt{(10^{-3})^2 - 4(0.25 \times 10^{-6})(1)}}{2(0.25 \times 10^{-6})} = \frac{-10^{-3} \pm \sqrt{10^{-6} - 10^{-6}}}{2(0.25 \times 10^{-6})}$$

$$S_1, S_2 = -2000$$

**For $R_T = 1k$**

$$R_TC = (1 \times 10^3)(0.25 \times 10^{-6}) = 25 \times 10^{-3}$$

$$S_1, S_2 = \frac{-25 \times 10^{-3} \pm \sqrt{(25 \times 10^{-3})^2 - 4(0.25 \times 10^{-6})(1)}}{2(0.25 \times 10^{-6})}$$

$$S_1, S_2 = \frac{-25 \times 10^{-3} \pm \sqrt{6.25 \times 10^{-8} - 1 \times 10^{-6}}}{0.5 \times 10^{-6}}$$

$$S_1, S_2 = \frac{-25 \times 10^{-3} \pm \sqrt{-9.375 \times 10^{-7}}}{0.5 \times 10^{-6}} = -25 \times 10^{-3} \pm j9.682 \times 10^{-4}$$

$$S_1, S_2 = -500 \pm j1936.5$$
Based upon previous experience we try a solution of the form $u_c = ke^{st}$ to get the characteristic equation

$$LC \cdot ke^{st} + R_T C \cdot kse^{st} + kse^{st} = 0$$

$$ke^{st}(LCs^2 + R_T Cs + 1) = 0$$

This equation has two roots, using the quadratic formula

$$s_1, s_2 = \frac{-R_T C \pm \sqrt{(R_T C)^2 - 4LC}}{2LC}$$

This has three cases depending upon the square root.

CASE A: $(R_T C)^2 - 4LC > 0$  two real, unequal roots

$$s_1 = -x_1, \quad s_2 = -x_2$$

CASE B: $(R_T C)^2 - 4LC = 0$  two equal, real root

$$s_1 = s_2 = -x$$

CASE C: $(R_T C)^2 - 4LC < 0$  two complex conjugate roots

$$s_1 = -x - j\beta \quad \text{and} \quad s_2 = -x + j\beta$$
CASE A: \( s_1 = -\alpha_1 \), \( s_2 = -\alpha_2 \)

\[
\nu_c(t) = \frac{-\alpha_2 V_0 - I_o / C}{-\alpha_2 + \alpha_1} e^{-\alpha_1 t} + \frac{\alpha_1 V_0 + I_o / C}{-\alpha_2 + \alpha_1} e^{-\alpha_2 t}
\]

\[
\nu_c(t) = \frac{\alpha_2 V_0 + I_o / C}{\alpha_2 - \alpha_1} e^{-\alpha_1 t} + \frac{-\alpha_1 V_0 - I_o / C}{\alpha_2 - \alpha_1} e^{-\alpha_2 t}
\]

This is a sum of two exponential functions.

See Example 5-14.

CASE B: \( s_1 = s_2 = -\alpha \)

The solution becomes

\[
\nu_c(t) = \frac{-\alpha V_0 - I_o / C}{-\alpha + \alpha} e^{-\alpha t} + \frac{-\alpha V_0 + I_o / C}{-\alpha + \alpha} e^{-\alpha t}
\]

The denominators go to zero but the numerator also goes to zero.

Let \( s_1 = -\alpha \) and \( s_2 = -\alpha + \alpha \) and take limit as \( x \to 0 \)

\[
\nu_c(t) = \frac{(-\alpha + x) V_0 - I_o / C}{-\alpha + x + \alpha} e^{-\alpha t} + \frac{\alpha V_0 + I_o / C}{-\alpha + x + \alpha} e^{-\alpha t}
\]

\[
\nu_c(t) = e^{-\alpha t} \left[ \frac{-\alpha V_0 + x V_0 - I_o / C + \alpha V_0 e^{\alpha t} + I_o / C e^{\alpha t}}{\alpha} \right]
\]

\[
\nu_c(t) = e^{-\alpha t} \left[ V_0 - \frac{\alpha V_0 - I_o / C}{x} e^{\alpha t} + \frac{\alpha V_0 + I_o / C}{x} e^{\alpha t} \right]
\]

\[
\nu_c(t) = e^{-\alpha t} \left[ V_0 - (\alpha V_0 + I_o / C) \frac{1 - e^{\alpha t}}{x} \right]
\]
Zero-input response

There are two roots to the characteristic equation which correspond to two solutions.

The general solution for zero-input is the sum of these solutions

\[ v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \]

The initial conditions are

\[ v_c(0) = V_0 \]

\[ \frac{dv_c(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C} \]

For $t = 0$

\[ v_c(0) = K_1 e^{s_1} + K_2 e^{s_2} = K_1 + K_2 = V_0 \]

To use the second initial condition we need to differentiate our expression for $v_c(t)$

\[ \frac{dv_c(t)}{dt} = K_1 s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t} \]

\[ \frac{dv_c(0)}{dt} = K_1 s_1 + K_2 s_2 = \frac{I_0}{C} \]

\[ s_1 K_1 + s_2 K_2 = \frac{I_0}{C} \]

\[ K_1 + K_2 = V_0 \]

\[ s_1 K_1 + s_2 K_2 = \frac{I_0}{C} \]

\[ s_1 K_1 + s_1 K_2 = s_1 V_0 \]

\[ (s_2 - s_1) K_2 = \frac{I_0}{C} - s_1 V_0 \]

\[ K_2 = \frac{-s_1 V_0 + \frac{I_0}{C}}{s_2 - s_1} \]

\[ K_1 = V_0 - K_2 = V_0 - \frac{-s_1 V_0 + \frac{I_0}{C}}{s_2 - s_1} = \frac{s_2 V_0 - \frac{I_0}{C}}{s_2 - s_1} \]

This response is different depending upon cases A, B, or C.
The indeterminacy in this equation comes from the second term. We can use L'Hopital's rule to evaluate it

\[
\lim_{{x \to 0}} \frac{{1 - e^{xt}}}{{x}} = \lim_{{x \to 0}} \left( -\frac{te^{xt}}{1} \right) = -t
\]

take derivatives with respect to \( x \)

Substituting this result gives

\[
\nu_c(t) = e^{-\alpha t} \left[ v_0 - (\alpha V_0 + \frac{I_0}{C})(-t) \right]
\]

\[
\nu_c(t) = V_0 e^{-\alpha t} + (\alpha V_0 + \frac{I_0}{C}) t e^{-\alpha t}, \quad t \geq 0
\]

This solution is an exponential and a damped ramp.

CASE C is perhaps the most interesting.

\( s_1 = -\alpha - j\beta \), \( s_2 = -\alpha + j\beta \)

Substituting gives

\[
\nu_c(t) = \frac{(-\alpha + j\beta)V_0 - \frac{I_0}{C}}{(-\alpha + j\beta) - (-\alpha - j\beta)} e^{-\alpha t} e^{-j\beta t} + \frac{(-\alpha - j\beta)V_0 + \frac{I_0}{C}}{(-\alpha + j\beta) - (-\alpha - j\beta)} e^{-\alpha t} e^{j\beta t}
\]

\[
\nu_c(t) = \frac{-\alpha V_0 + j\beta V_0 - \frac{I_0}{C}}{j^2\beta} e^{-\alpha t} e^{-j\beta t} + \frac{\alpha V_0 + j\beta V_0 + \frac{I_0}{C}}{j^2\beta} e^{-\alpha t} e^{j\beta t}
\]

\[
\nu_c(t) = \frac{j\beta V_0 e^{-\alpha t} e^{-j\beta t}}{j^2\beta} + \frac{-\alpha V_0 e^{-j\beta t} - \frac{I_0}{C} e^{j\beta t} + \alpha V_0 e^{j\beta t} + \frac{I_0}{C} e^{-j\beta t}}{j^2\beta}
\]

\[
\nu_c(t) = v_0 e^{-\alpha t} \left[ \frac{e^{-j\beta t} + e^{j\beta t}}{2} \right] + \frac{\alpha V_0 + \frac{I_0}{C}}{\beta} e^{-\alpha t} \left[ \frac{e^{j\beta t} - e^{-j\beta t}}{j^2} \right]
\]
We can use Euler's identity to re-write the expressions in the square brackets.

\[ e^{j\theta} = \cos \theta + j \sin \theta \]
\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

\[
\begin{align*}
\therefore \cos \theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\
\sin \theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j}
\end{align*}
\]

Adding and subtracting the above equations gives these useful expressions for the sine and cosine.

We can use these expressions to simplify our solution

\[
V_c(t) = V_0 e^{-\alpha t} \cos \beta t + \frac{\alpha V_0 + I_0 \omega}{\beta} e^{-\alpha t} \sin \beta t
\]

These are two damped sinusoids. The real part of the roots describes the exponential decay, and the imaginary part of the roots defines the frequency of the sinusoidal oscillations.

- \( \alpha \) has units of complex frequency
- \( \beta \) has units of radian frequency
- \( \omega \) has units of radian frequency
Example 7-15

The circuit shown below has \( C = 0.25 \mu F \) and \( L = 1 H \). The switch has been open for a long time and is closed at \( t = 0 \). Find the capacitor voltage for \( t \geq 0 \) for (a) \( R_T = 8.5 k \), (b) \( R = 4 k \) and (c) \( R = 1 k \). The initial conditions are \( I_0 = 0 \) and \( V_0 = 15 \).

![Circuit Diagram]

**Solution:**

The roots of the characteristic equation were found in Example 7-14.

(a) \( R_T = 8.5 k \)  
\( s_1 = -500 \)  
\( s_2 = -8000 \)

\[
V_c(t) = K_1 e^{-500t} + K_2 e^{-8000t}
\]

\( v_c(0) = K_1 + K_2 = V_0 = 15 \)

\[
\frac{dV_c(t)}{dt} = \frac{I_0}{C} = K_1(-500)e^{-500t} + K_2(-8000)e^{-8000t}
\]

\[
\frac{dV_c(0)}{dt} = 0 = -500K_1 - 8000K_2
\]

\[
K_1 + K_2 = 15
\]

\[
K_1 + 16K_2 = 0
\]

\[
K_2 = 1 \quad K_1 = 16
\]

\[
V_c(t) = 16e^{-500t} - e^{-8000t} \quad t \geq 0
\]
(b) \( R_T = 4 \, k \Omega \quad s_1 = s_2 = -2000 \).

The zero-input solution is an exponential plus an exponentially damped ramp

\[ v_{\xi}(t) = K_1 e^{-2000t} + K_2 t e^{-2000t} \]

We now solve for the initial conditions in the same manner as (a).

\[ v_{\xi}(0) = K_1 = V_0 = 15 \]

\[ \frac{dv_{\xi}(t)}{dt} = K_1 (-2000) e^{-2000t} + K_2 (-2000) t e^{-2000t} + K_2 e^{-2000t} = \frac{I_0}{C} \]

\[ \frac{dv_{\xi}(0)}{dt} = -2000 K_1 e^0 + 0 + K_2 e^0 = \frac{0}{C} \]

\[ -2000 K_1 + K_2 = 0 \quad \Rightarrow \quad K_2 = 2000 \, K_1 = 30000 \]

\[ v_{\xi}(t) = 15e^{-2000t} + 30000te^{-2000t}, \quad t \geq 0 \]

(c) \( R_T = 1 \, k \Omega \quad s_1 = -500 - j \, 1936.5 \quad s_2 = -500 + j \, 1936.5 \)

\[ v_{\xi}(t) = K_1 e^{-500t} \cos(1936.5t) + K_2 e^{-500t} \sin(1936.5t) \]

\[ v_{\xi}(0) = K_1 e^0 + K_2 e^0 \cdot 0 = 15 \quad \therefore \quad K_1 = 15 \]

\[ \frac{dv_{\xi}(t)}{dt} = K_1 (-500) e^{-500t} \cos(1936.5t) + K_1 e^{-500t} (-1936.5) \sin(1936.5t) \]

\[ + K_2 (-500) e^{-500t} \sin(1936.5t) + K_2 e^{-500t} (1936.5) \cos(1936.5t) \]

\[ \frac{dv_{\xi}(0)}{dt} = K_1 (-500) e^0 \cdot 1 + K_1 e^0 (-1936.5)(0) + K_2 (-500) e^0 (0) + K_2 e^0 (1936.5)(1) = 0 \]

\[ -500 K_1 + 1936.5 K_2 = 0 \]

\[ K_2 = \frac{500}{1936.5}, \quad K_1 = \frac{500}{1936.5} (15) = 3.873 \]

\[ v_{\xi}(t) = 15 e^{-500t} \cos(1936.5t) + 3.873 e^{-500t} \sin(1936.5t), \quad t \geq 0 \]
The plots of these are very interesting.

All three cases start at $V_c(0) = 15$ and all damp out but in different ways.
Example 7-16

In a series RLC circuit the zero-input voltage across the 1\mu F capacitor is
\[ v_c(t) = 10e^{-1000t}\sin 2000t \quad V \geq 0 \]

(a) Find the circuit characteristic equation.

The circuit is underdamped because it has a sine
By inspection \( \alpha = 1000, \beta = 2000 \)
the solutions are then
\[
(s + \alpha + j\beta)(s + \alpha - j\beta) = (s + 1000 + j2000)(s + 1000 - j2000)
= s^2 + (1000 + j2000 + 1000 - j2000)s + (1000^2 + 2000^2)
= s^2 + 2000s + 5 \times 10^6 = 0
\]

(b) Find the R and L.

The characteristic equation can also be written as
\[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \]
Since \( C = 1\mu F \) we can solve for L and R.
\[
\frac{1}{LC} = 5 \times 10^6 \quad \Rightarrow \quad L = \frac{1}{(5 \times 10^6)(1 \times 10^{-6})} = 0.2 \ \text{H}
\]
\[
\frac{R}{L} = 2000 \quad \Rightarrow \quad R = 2000L = 2000(0.2) = 400 \ \Omega
\]

(c) Find \( i_L(t) \) for \( t \geq 0 \)

In a series RLC circuit \( i_L(t) = i_c(t) \) and we know \( i_c(t) = C \frac{dv_c(t)}{dt} \)
\[
i_L(t) = (1 \times 10^{-6}) \left[ (10)(-1000) e^{-1000t}\sin 2000t + 10 e^{-1000t}\cos 2000t \right] 
\]
\[
i_L(t) = 0.01 e^{-1000t}\sin 2000t + 0.02 e^{-1000t}\cos 2000t
\]

(d) Find the initial values of the state variables.
By inspection \( v_c(0) = 0 \)
\( i_L(0) = 0.02 \ \text{amps} \).