Example 7-2

Find the response of the state variable of the RL circuit in the figure below using \( L_1 = 10 \text{mH}, L_2 = 30 \text{mH}, R_1 = 2k \Omega, \) \( R_2 = 6k \Omega, \) and \( i_0 = 100mA. \)

The inductors are in series and can be replaced by an equivalent \( L_{eq}. \)

\[
L_{eq} = L_1 + L_2 = 10\text{mH} + 30\text{mH} = 40\text{mH}
\]

The resistors are in parallel and can be replaced by

\[
R_{eq} = \frac{R_1 R_2}{R_1 + R_2} = \frac{(2k)(6k)}{2k + 6k} = 1.5k
\]

\( i(t) \) remains the same since \( L_1 \) & \( L_2 \) were in series

Using KVL

\[
+ i(t)R_{eq} + v(t) = 0
\]

The inductor constraint is

\[
v(t) = L_{eq} \frac{di(t)}{dt}
\]

\[
i(t)R_{eq} + L_{eq} \frac{di(t)}{dt} = 0
\]

\[
\frac{L_{eq}}{R_{eq}} \frac{di(t)}{dt} + i(t) = 0
\]

Assume \( i(t) = Ke^{st}, \) then

\[
\frac{L_{eq}}{R_{eq}} Ks e^{st} + Ke^{st} = 0
\]

\[
\frac{L_{eq}}{R_{eq}} s + 1 = 0 \quad \Rightarrow \quad s = -\frac{R_{eq}}{L_{eq}} = -\frac{1500}{40 \times 10^{-3}}
\]

\( s = -37500 \)

\[
i(t) = Ke^{st} = Ke
\]

\( i(t=0) = Ke^0 = 100mA \)

\[
i(t) = 0.1e^{-\frac{t}{26.7\text{mS}}} \text{A.}
\]
Consider a circuit which is difficult to Thevenize. Derive equations in terms of a more convenient variable.

Using KCL at node A
\[ \sum i = 0 \]
\[ + i_1(t) - i_N(t) + i_c(t) + i_2(t) = 0 \]

\[
\frac{v_s - 0}{R_1} + 0 + c \frac{d}{dt} (v_o - 0) + \frac{v_o - 0}{R_2} = 0
\]

\[
c \frac{dv_o}{dt} + \frac{v_o}{R_2} = - \frac{v_s}{R_1}
\]

\[
R_2c \frac{dv_o}{dt} + v_o = - \frac{R_2}{R_1} v_s(t)
\]

\[\text{differential equation in } v_o\]

Rather that the capacitor voltage \(v_c\)
7.2 First Order Circuit Step Response

Consider the following circuit:

![Circuit Diagram]

We solved this circuit previously for $v_T(t) = 0, \quad t > 0$

Consider the case where $v_T(t) = V_A u(t)$

The circuit differential equation is

$$R_TC \frac{dv}{dt} + v = V_A u(t)$$

or

$$R_TC \frac{dv}{dt} + v = V_A \quad \text{for} \quad t > 0$$

While there are many methods to solve this equation, we will use superposition.

$$v(t) = v_{N}(t) + v_{F}(t)$$

- **natural response**
  - when input is set to zero.
- **forced response**
  - to the input step function

**Natural response**:

$$R_TC \frac{dv_{N}}{dt} + v_{N} = 0$$

Solution is $v_{N}(t) = KE \left( -\frac{t}{R_TC} \right) \quad t > 0$

which we have seen previously.
Forced response

\[ R_C \frac{dV_F(t)}{dt} + V_F(t) = V_A \quad t > 0 \]

A solution is \( V_F(t) = V_A \quad t > 0 \)

since \( \frac{dV_F}{dt} = 0 \) for \( t > 0 \)

The total response is the sum of the natural and forced response,

\[ V(t) = V_N(t) + V_F(t) \]

\[ V(t) = k e^{-\frac{t}{RC}} + V_A \]

This is the general solution and is plotted below.

From this plot we can easily see that

\[ \lim_{t \to 0^+} V(t) = V_0 \quad \text{initial value} \]

\[ \lim_{t \to \infty} V(t) = V_A \quad \text{final value} \]
For a RL circuit with a $I_A u(t)$ input, \[ i_N(t) \]

The differential equation is \[ \frac{L}{R_N} \frac{d i(t)}{dt} + i(t) = I_A \text{ for } t > 0 \]

The natural response \[ \frac{L}{R_N} \frac{d i_N(t)}{dt} + i_N(t) = 0 \]

has the solution \[ i_N(t) = ke^{-\frac{R_N t}{L}} = ke^{-\frac{t}{LR_N}} \text{ for } t > 0 \]

The forced response is from \[ \frac{L}{R_N} \frac{d i_F(t)}{dt} + i_F(t) = I_A \text{ for } t > 0 \]

which has solution \[ i_F(t) = I_A \text{ for } t > 0 \]

The general solution is then \[ i(t) = i_N(t) + i_F(t) = ke^{-\frac{t}{LR_N}} + I_A \text{ for } t > 0 \]

A step function drives the state variable from an initial value (determined by what happened for $t < 0$) to a final value (determined by the magnitude of the step at $t = 0$).
Example 7-4  Find the response of the given RC circuit.

\[ V_A(u(t)) \]

\[ V_A = 100 \text{ V} \]
\[ a(t) = V_{o1} = 5 \text{ V} \]
\[ V_{o2} = 10 \text{ V} \]
\[ C_1 = 0.1 \mu F \]
\[ C_2 = 0.5 \mu F \]
\[ R_1 = 30k \]
\[ R_2 = 10k \]

The two capacitors can be replaced by a single equivalent capacitor

\[ C_{EQ} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2} \]

\[ C_{EQ} = \frac{(0.1)(0.5)}{0.1 + 0.5} = 0.0833 \mu F \]

The initial voltage on \( C_{EQ} \) is the sum of the initial voltages on \( C_1 \) and \( C_2 \)

\[ V_0 = V_{o1} + V_{o2} = 5 + 10 = 15 \text{ volts} \]

We next find the Thevenin equivalent seen by \( C_{EQ} \).

\[ V_T = \frac{R_2}{R_1 + R_2} V_A u(t) = \frac{10k}{30k + 10k} \cdot 100 u(t) \]

\[ V_T(t) = 25 u(t) \]

Replacing the voltage source by a short, we see that

\[ R_T = R_1 || R_2 = \frac{(30k)(10k)}{80k + 10k} = 7.5k \]

\[ T_c = R_T C_{EQ} = (7.5 \times 10^3)(0.0833 \times 10^{-6}) = 0.625 \text{ ms} \]

The circuit equation is

\[ -V_T(t) + i(t) R_T + V(t) = 0 \]

but \( i(t) = \frac{C_{EQ}}{R_T} \frac{dV(t)}{dt} \)

\[ -V_T(t) + RC_{EQ} \frac{dV}{dt} + V = 0 \]

\[ R_T C_{EQ} \frac{dV}{dt} + V = V_T(t) \]