Chapter 7 - First- and Second- Order Circuits

7-1 RC and RL Circuits

- Linear Circuit
- Use device and connection equations to write differential equation describing the circuit
- Differential Equation
- Solve differential equation
- Classical Techniques
- Use classical techniques for solving D.E.
- (Other techniques include phasors & Laplace Transform)

RC and RL Equations

\[ -V_T(t) + i(t)R_T + V(t) = 0 \]

From the previous chapter the capacitor is described by

\[ i(t) = C \frac{dV_T(t)}{dt} \]

Substituting

\[ -V_T(t) + C \frac{dV_T(t)}{dt} R_T + V(t) = 0 \]

Re-arranging

First order linear differential equation with constant coefficients

\[ R_T C \frac{dV(t)}{dt} + V(t) = V_T(t) \]

\( V_T(t) \) is the input and \( V(t) \) is the response

\( V(t) \) is called the state variable and determines the amount of energy stored in the RC circuit.
We can do the same with an inductor.

Using KCL at the node gives
\[ \sum_i i = 0 \]
\[ + i_N(t) = v(t) - i(t) = 0 \]

The element constraint is
\[ v(t) = L \frac{di(t)}{dt} \]

Substituting
\[ i_N(t) = \frac{1}{R_N} \frac{di(t)}{dt} - i(t) = 0 \]

Re-arranging
\[ \frac{L}{R_N} \frac{di(t)}{dt} + i(t) = i_N(t) \]

- First-order linear differential equation with constant coefficients
- \( i_N(t) \) is the forcing function
- \( i(t) \) is the state variable as it defines the amount of energy stored in the RL circuit.

Any circuit containing a single capacitor or inductor and resistors is a first-order circuit described by a first-order differential equation.
Zero-input response of First-order circuits

The response $v(t)$ for an RC circuit depends upon

1. The input $v_i(t)$
2. The circuit values $R_T$ and $C$
3. The initial condition, i.e., $v(t=0)$
   
   This can use a response even when $v_i(t)=0$.

Consider the zero-input response when $v_i(t)=0$. for $t>0$

$$R_T C \frac{dv(t)}{dt} + v(t)=0$$

This is a homogeneous differential equation with a solution of the form

$$v(t) = Ke^st$$

Substituting gives

$$R_T C (Kse^{st}) + Ke^{st} = 0$$

$$Ke^{st} (R_T Cs + 1) = 0$$

This can only be zero if

$$R_T Cs + 1 = 0$$

This is called the characteristic equation of the differential equation.

Solving gives $s = \frac{-1}{R_T C}$

The solution $v(t)$ is then

$$v(t) = Ke^{-\frac{t}{R_T C}} \quad t\geq 0$$

The constant $K$ comes from the initial condition $v(t=0) = V_0$

$$v(t=0) = Ke^0 = K = V_0$$

The final zero-input response is then

$$v(t) = V_0 e^{-\frac{t}{R_T C}} \quad t>0$$
We can do the same for the RL circuit.

\[
\frac{L}{R_N} \frac{di(t)}{dt} + i(t) = 0 \quad \text{where we set } i_n(t) = 0, \; t > 0
\]

This is also a homogeneous linear differential equation with a solution of the form

\[i(t) = Ke^{st}\]

Substituting gives

\[
\frac{L}{R_N} Ke^{st} + Ke^{st} = 0
\]

\[Ke^{st}\left(\frac{L}{R_N} s + 1\right) = 0\]

This requires the characteristic equation

\[
\frac{L}{R_N} s + 1 = 0
\]

Solving gives

\[s = -\frac{L}{R_N} = -\frac{R_N}{L}\]

The solution is

\[i(t) = Ke^{-\frac{R_N}{L}t}, \quad t > 0\]

The constant \(K\) comes from the initial condition \(i(t=0) = I_0\)

\[i(t=0) = Ke^0 = K = I_0\]

The final zero-input response is then

\[i(t) = I_0 e^{-\frac{R_N}{L}t}, \quad t > 0\]
Example 7-1

The switch in the circuit shown below is closed at \( t=0 \), connecting a capacitor with an initial voltage of 30V to the resistances shown.

Find the responses \( v_c(t) \), \( i(t) \), \( i_1(t) \), and \( i_2(t) \) for \( t > 0 \).

\[ C = 0.5 \text{ pF} \]

\[
\begin{align*}
\begin{array}{c}
\text{\( v_c(t) \)} \\
\text{\( i(t) \)} \\
\text{\( i_1(t) \)} \\
\text{\( i_2(t) \)}
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
10k \\
20k \\
20k
\end{array}
\]

\[ \text{Re} \]

\[ t=0 \]

30volts

We first determine the equivalent resistance \( \text{Re} \) as seen by the capacitor,

\[
\text{Re} = 10k + 20k || 20k = 10k + 10k = 20k
\]

Doing KVL gives

\[
-v_c(t) - i(t) \text{Re} = 0
\]

The element constraint for \( C \) is \( i_c = C \frac{dv_c}{dt} \)

Since we chose \( i_c(t) \) to be in the same direction as \( i(t) \), \( i_c(t) = i(t) \)

Substituting,

\[
-v_c(t) - C \frac{dv_c(t)}{dt} \text{Re} = 0
\]

\[
\text{Re} C \frac{dv_c(t)}{dt} + v_c(t) = 0
\]

The solution is \( v_c(t) = Ke^{st} \). Substituting gives

\[
\text{Re} C Ke^{st} + Ke^{st} = 0
\]

\[
\text{Re} Cs + 1 = 0
\]

\[
S = -\frac{1}{\text{Re} C} = -\frac{1}{(20 \times 10^3)(0.5 \times 10^{-6})} = -100
\]

Using the initial condition \( v_c(t=0) = 30 \text{ volts} \)

\[
v_c(t=0) = Ke^0 = 30 \text{ volts} \implies K = 30
\]

The solution is \( v_c(t) = 30e^{-100t}, \ t \geq 0 \)
We can now calculate the other circuit values using $v_c(t)$

$$i_c(t) = \frac{d}{dt} \frac{d v_c}{d t} = (0.5 \times 10^{-6}) \frac{d}{dt} \left( 30e^{-100t} \right) = (0.5 \times 10^{-6})(30)(-100)e^{-100t} = -1.5 \times 10^{-3} e^{-100t}, \quad t > 0$$

$i_1$ and $i_2$ are given by a current divider:

$$i_1 = \frac{20k}{20k + 20k} i(t) = \frac{1}{2} i(t) = -0.75 \times 10^{-3} e^{-100t}, \quad t > 0$$

$$i_2 = \frac{20k}{20k + 20k} i(t) = \frac{1}{2} i(t) = -0.75 \times 10^{-3} e^{-100t}, \quad t > 0$$