7-1 RC and RL Circuits

- **Linear Circuit**
  - use device and connection equations to write differential equation describing the circuit
  - solve differential equation
  - use classical techniques for solving D.E.
    (Other techniques include phasors & Laplace transform)

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- **RC and RL Equations**
  - Rest of the circuit (resistor sources)
  - \[ i(t) + v(t) + v_T(t) = 0 \]

- **Voltage and Current Equation**
  - From the previous chapter, the capacitor is described by \[ i(t) = C \frac{dv(t)}{dt} \]

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- **Substituting**
  - \[ -v_T(t) + C \frac{dv(t)}{dt} R_T + v(t) = 0 \]

- **Re-arranging**
  - First order linear differential equation with constant coefficients
  - \[ R_T C \frac{dv(t)}{dt} + v(t) = v_T(t) \]

- **v(t)** is called the state variable and determines the amount of energy stored in the RC circuit.
We can do the same with an inductor.

Using KCL at the node gives
\[ \sum_i i = 0 \]
\[ + i_N(t) - \frac{v(t)}{R_N} - i(t) = 0 \]

The element constraint is
\[ v(t) = L \frac{di(t)}{dt} \]

Substituting
\[ i_N(t) - \frac{L}{R_N} \frac{di(t)}{dt} - i(t) = 0 \]

Re-arranging
\[ \frac{L}{R_N} \frac{di(t)}{dt} + i(t) = i_N(t) \]

- first-order linear differential equation with constant coefficients
- \( i_N(t) \) is the forcing function
- \( i(t) \) is the state variable as it defines the amount of energy stored in the RL circuit

Any circuit containing a single capacitor or inductor and resistors is a first-order circuit described by a first-order differential equation.
7.2 First Order Circuit Step Response

Consider the following circuit:

\[ V_T(t) \quad \square \quad i(t) \]

\[ R \quad + \quad i(t) \]

\[ C \quad - \quad V_T(t) \quad \square \quad V(t) \]

We solved this circuit previously for \( V_T(t) = 0 \), \( t > 0 \)

Consider the case where \( V_T(t) = V_A u(t) \)

The circuit differential equation is

\[ R \cdot C \frac{dV}{dt} + V = V_A u(t) \]

or

\[ R \cdot C \frac{dV}{dt} + V = V_A \quad \text{for} \quad t > 0 \]

While there are many methods to solve this equation, we will use superposition.

\[ V(t) = V_N(t) + V_F(t) \]

\[ \quad \text{natural response} \quad \quad \quad \quad \quad \text{forced response} \]

\[ \text{when input is set to zero.} \quad \text{to the input step function} \]

**Natural response:**

\[ R \cdot C \frac{dV_N}{dt} + V_N = 0 \]

Solution is

\[ V_N(t) = K e^{-\frac{t}{R \cdot C}} \quad \text{for} \quad t > 0 \]

which we have seen previously.
7-4 First-order Circuit Sinusoidal Response

In previous sections we determined that the solution of a linear first-order differential equation consisted of a forced and natural response. We determined the response to a step-input. In this section we consider the response to a causal sinusoid, \( V_A \cos \omega t u(t) \).

The complete differential equation for a first order RC circuit is

\[
R_C \frac{dV(t)}{dt} + V(t) = V_A \cos \omega t u(t)
\]

The natural response does not change and is given by

\[
V_N(t) = Ke^{-\frac{t}{R_C}} \quad t > 0
\]

The forced response is a solution of

\[
R_C \frac{dV_F(t)}{dt} + V_F(t) = V_A \cos \omega t \quad t > 0
\]

The only solution of this equation is another sinusoid.

Consider \( V_F(t) = a \cos \omega t + b \sin \omega t \)

This is called the method of undetermined coefficients.

\[
R_C \frac{d}{dt}(a \cos \omega t + b \sin \omega t) + (a \cos \omega t + b \sin \omega t) = V_A \cos \omega t
\]

\[
R_C (-aw \sin \omega t + bw \cos \omega t) + (a \cos \omega t + b \sin \omega t) = V_A \cos \omega t
\]

\[
[-awR_C + b] \sin \omega t + [bwR_C + a - V_A] \cos \omega t = 0
\]

This is only true when the sine and cosine coefficients are identically zero.

\[
a + (R_C \omega) b = V_A
\]

\[
(-wR_C)a + b = 0
\]
\[ a + \omega R_T C b = V_A \]
\[ -(\omega R_T C)^2 a + (\omega R_T C) b = 0 \]

Subtracting \( 1 + (\omega R_T C)^2 \) a = \( V_A \)
\[ a = \frac{V_A}{1 + (\omega R_T C)^2} \]

Substituting \( b = (\omega R_T C) a = \frac{\omega R_T C V_A}{1 + (\omega R_T C)^2} \)

The forced response is then
\[ v_e(t) = \frac{V_A}{1 + (\omega R_T C)^2} \left[ \cos \omega t + \omega R_T C \sin \omega t \right] \quad t \geq 0 \]

The total solution is then
\[ v(t) = K e^{-\frac{t}{R_T C}} + \frac{V_A}{1 + (\omega R_T C)^2} \left[ \cos \omega t + \omega R_T C \sin \omega t \right] \quad t \geq 0 \]

The initial condition \( v_0(t=0) = V_0 \) requires
\[ v(0) = K e^{0} + \frac{V_A}{1 + (\omega R_T C)^2} \left[ 1 + 0 \right] = V_0 \]
\[ \therefore K = V_0 - \frac{V_A}{1 + (\omega R_T C)^2} \]

The total solution is then
\[ v(t) = \left[ V_0 - \frac{V_A}{1 + (\omega R_T C)^2} \right] e^{-\frac{t}{R_T C}} + \frac{V_A}{1 + (\omega R_T C)^2} \left[ \cos \omega t + \omega R_T C \sin \omega t \right] \quad t \geq 0 \]

\underbrace{\text{natural response}}_{\text{forced response}}
The more commonly used form of the solution requires converting the second term to magnitude and phase format.

$$u(t) = \left[ V_0 - \frac{V_A}{1 + (\omega R T C)^2} \right] e^{-\frac{t}{R T C}} + \frac{V_A}{\sqrt{1 + (\omega R T C)^2}} \cos(\omega t + \theta) \quad t \geq 0$$

where we used

$$\cos \omega t + \omega R T C \sin \omega t = \sqrt{1 + (\omega R T C)^2} \cos(\omega t + \theta)$$

$$\theta = \tan^{-1}\left(\frac{-\omega R T C}{1}\right) = \tan^{-1}(-\omega R T C)$$

Observations:

1. The forced sinusoidal response lasts whereas the natural response decays to zero.
2. The forced sinusoidal response is of the same frequency ($\omega$) as the input but with a different magnitude and phase.
3. The forced response is proportional to $V_A$.

* The forced response is called
  - the sinusoidal steady-state response
  - the ac steady-state response
  - the ac response

Technically we have found the solution to the step-function

$$V_A [\cos \omega t] u(t)$$

If $\omega = 0$ this reduces to the previous solution for

$$V_A u(t)$$
Example 7-12

The switch in the figure below has been open for a long time and is closed at $t=0$. Find the voltage $V(t)$ for $t>0$ when $V_s(t) = 20\sin 1000t \ u(t)$

We have to derive the circuit differential equation by Thévenizing the circuit.

Shunting $V_s(t)$, $R_T = 4k \parallel 4k = 2k\Omega$

$V_T(t) = \frac{4k}{4k+4k}V_s(t) = 10\sin 1000t \ u(t)$

$V_T(t) = 10\sin 1000t$

$R_T C = (2 \times 10^3)(1 \times 10^{-6}) = 2 \times 10^{-3}$

$-V_T(t) + i(t)R_T + V_T(t) = 0$

$i(t) = C \frac{dV_T}{dt}$

$R_T C \frac{dV_T}{dt} + V_T(t) = V_T(t)$

$(2 \times 10^{-3}) \frac{dV_T(t)}{dt} + V_T(t) = 10 \sin 1000t$

Now we can compute the natural and forced responses.

$V_N(t) = Ke^{-\frac{t}{R_T C}} \quad t>0$

Using undetermined Fourier coefficients we write

$V_f(t) = a \cos 1000t + b \sin 1000t$
We substitute $V_F(t)$ into the differential equation

$$
(2 \times 10^{-3}) \frac{d}{dt}(a \cos 1000t + b \sin 1000t) + (a \cos 1000t + b \sin 1000t)
= 10 \sin 1000t
$$

$$
\frac{1}{500} \left[ -a \cdot 1000 \sin 1000t + b \cdot 1000 \cos 1000t \right] + \left[ a \cos 1000t + b \sin 1000t \right] = 10 \sin 1000t
$$

$$
\left[ -a + b - 10 \right] \sin wt + \left[ b + a \right] \cos wt = 0
$$

$$
a + 2b = 0
\quad -2a + b = 10
\quad 2a + 4b = 0
\quad -2a + b = 10
$$

$$
5b = 10 \Rightarrow b = 2
\quad a = -2b = -2(2) = -4
$$

$$
\therefore V(t) = V_N(t) + V_F(t)
$$

$$
V(t) = Ke^{-0.002t} - 4 \cos 1000t + 2 \sin 1000t
$$

$K$ is found from the initial condition that $V(0) = 0$

since the switch was open for a long time the capacitor is uncharged.

$$
\therefore V(0) = 0 = Ke^{0} - 4 + 0 \Rightarrow K = 4
$$

The complete response is

$$
V(t) = 4e^{-0.002t} - 4 \cos 1000t + 2 \sin 1000t
$$

In magnitude and phase

$$
V(t) = 4e^{-0.002t} + \sqrt{(4)^2 + (2)^2} \cos (1000t + \tan^{-1}(\frac{2}{4}))
$$

$$
V(t) = 4e^{-0.002t} + 4.472 \cos (1000t + 26.5^\circ)
$$
Example 7-13

Find the sinusoidal steady-state response of the output voltage \( v_o(t) \) in the circuit shown below, when the input current is

\[
i_s(t) = [I_A \cos \omega t] u(t)
\]

\[
\text{The switch guarantees that } i(t<0) = 0
\]

For \( t>0 \) using KCL at \( \sum i = 0 \)

\[
+i_s(t) - \frac{v_o(t)}{R} - i(t) = 0
\]

Using the inductor constraint \( v_o(t) = L \frac{di}{dt} \)

\[
-\frac{L}{R} \frac{di}{dt} i(t) + i_s(t) = 0
\]

\[
\frac{L}{R} \frac{di}{dt} + i(t) = i_s(t) = I_A \cos \omega t \quad t \geq 0
\]

We are only asked to find the forced component.

Using the method of undetermined coefficients we write

\[
i_F(t) = a \cos \omega t + b \sin \omega t
\]

Substituting this into the differential equation gives

\[
-\frac{L}{R} \frac{da}{dt} \cos \omega t + \frac{L}{R} \frac{db}{dt} \sin \omega t + [a \cos \omega t + b \sin \omega t] = I_A \cos \omega t
\]

\[
-\frac{L}{R} \omega a \sin \omega t + \frac{L}{R} \omega b \cos \omega t + a \cos \omega t + b \sin \omega t = I_A \cos \omega t
\]

Collecting like terms

\[
[-\frac{L}{R} \omega a + b] \sin \omega t + \left[\frac{L}{R} \omega b + a - I_A\right] \cos \omega t = 0
\]
we require
\[-\frac{L}{R} \omega a + b = 0\]
\[a + \frac{L}{R} \omega b = I_A\]
\[\frac{L}{R} \omega a + (\frac{L}{R} \omega)^2 b = (\frac{L}{R} \omega) I_A\]
\[\left[1 + (\frac{L}{R} \omega)^2\right] b = (\frac{L}{R} \omega) I_A\]
\[b = \frac{(\frac{L}{R} \omega) I_A}{1 + (\frac{L}{R} \omega)^2}\]
\[a = \frac{b}{(\frac{L}{R} \omega)} = \frac{I_A}{1 + (\frac{L}{R} \omega)^2}\]

The forced component is then
\[i_F(t) = \frac{I_A}{1 + (\frac{L}{R} \omega)^2} \cos \omega t + \frac{(\frac{L}{R} \omega) I_A}{1 + (\frac{L}{R} \omega)^2} \sin \omega t\]

The output voltage \(v_o(t) = L \frac{di_f(t)}{dt}\)
\[v_o(t) = L \frac{d}{dt} \left[\frac{I_A}{1 + (\frac{L}{R} \omega)^2} \cos \omega t + \frac{(\frac{L}{R} \omega) I_A}{1 + (\frac{L}{R} \omega)^2} \sin \omega t\right]\]
\[v_o(t) = -L \frac{I_A}{1 + (\frac{L}{R} \omega)^2} \sin \omega t + \frac{L^2 \omega^2 I_A}{R} \cos \omega t\]
\[v_o(t) = \frac{I_A \omega L}{1 + (\frac{L}{R} \omega)^2} \left[-\sin \omega t + \omega \frac{L}{R} \cos \omega t\right]\]
\[v_o(t) = \frac{I_A \omega L}{1 + (\frac{L}{R} \omega)^2} \sqrt{1 + (\frac{L}{R} \omega)^2} \cos \left(\omega t + \tan^{-1}\left(\frac{\frac{L}{R}}{1}\right)\right)\]
\[ V_o(t) = \frac{I_A \omega L}{\sqrt{1 + \left(\frac{1}{R} \omega \right)^2}} \cos \left(\omega t + \tan^{-1}\left(\frac{1}{\omega L R}\right)\right) \]

Amplitude changes with the frequency of the sinusoidal input.

At dc (\( \omega = 0 \)) the amplitude goes to zero since the inductor acts like a short.

As \( \omega \to \infty \) the amplitude goes to \( \frac{I_A \omega L}{\frac{L}{R}} = I_AR \)

which makes sense since the inductor acts like an open forcing all current through the resistor.

This is a frequency dependent response.
Chapter 8 - Sinusoidal steady-state response

The forced sinusoidal remaining after the natural component disappears is called the sinusoidal steady-state response.

8-1 Sinusoids and Phasors

The fundamental relationship between sinewaves and complex numbers comes from Euler's identity

$$e^{j\theta} = \cos \theta + jsin \theta$$

Define

$$\cos \theta = \Re \{ e^{j\theta} \}$$

$$\sin \theta = \Im \{ e^{j\theta} \}$$

Expanding upon the general sinusoid

$$v(t) = V_A \cos (\omega t + \phi)$$

$$v(t) = V_A \Re \{ e^{j(\omega t + \phi)} \} = \Re \{ V_Ae^{j\phi}e^{j\omega t} \}$$

This is defined to be the phasor representation of the sinusoid $v(t)$.

$$\frac{v}{V} \triangleq V_A e^{j\phi} = V_A \cos \phi + jV_A \sin \phi$$

The phasor $\frac{v}{V}$ is a complex number.

1. Phasors will be written with an underline (\(_v \)) to distinguish them from signal waveforms such as $v(t)$.

2. A phasor is determined by amplitude and phase angle and does not contain any information about the frequency.