Chapter 6  Capacitance and Inductance

For small d the electric field is given by

\[ \vec{E}(t) = \frac{q(t)}{\varepsilon A} \]

but \[ \vec{E}(t) = \frac{\nu_C(t)}{d} \]

Solving for \( q \)

\[ q(t) = EA \vec{E}(t) = \frac{EA}{d} \nu_C(t) \]

This is a constant dependent upon the physical construction called the capacitance \( C \)

\[ q(t) = C \nu_C(t) \]

Differentiating

\[ i_c(t) = \frac{dq(t)}{dt} = C \frac{d\nu_C}{dt} + \frac{\nu_C(t)}{C} \]

This the the i-v characteristic of the capacitor.

Consider what this means

1. When \( \nu_C = \text{constant (dc)} \) the current is zero.
2. Discontinuous voltage like \( S(t) \) would require and infinite current, because of this voltage across a capacitor must be continuous.
Integrate \( i_c(t) = C \frac{dV_c}{dt} \)

\[
\frac{1}{C} \int_{t_0}^{t} i_c(x) \, dx = \int_{t_0}^{t} \frac{dV_c}{dt} = V_c(t) - V_c(t_0)
\]

\[
\therefore V_c(t) = V_c(t_0) + \frac{1}{C} \int_{t_0}^{t} i_c(x) \, dx
\]

In most of our problems we will
1. define \( t_0 = 0 \)
2. recognize \( V_c(t_0) \) as the initial voltage on the capacitor.

\[
\begin{array}{c}
\hline
\text{Consider the power} \\
\quad P_c(t) = i_c(t) V_c(t) \\
\quad = \left[ C \frac{dV_c}{dt} \right] V_c(t) \\
\quad P_c(t) = \frac{d}{dt} \left[ \frac{1}{2} C V_c^2(t) \right]
\end{array}
\]

This is \textbf{VERY} interesting.

\[ P_c(t) > 0 \text{ capacitor absorbing power} \]

\[ P_c(t) < 0 \text{ capacitor releasing stored power} \]

Since \( P_c(t) = \frac{d}{dt} [\omega_c(t)] \) \( \uparrow \) energy in the capacitor

\[ \omega_c(t) = \frac{1}{2} C V_c^2(t) + \text{constant} \]

\[ \text{energy in capacitor} \quad \omega_c(t) = \frac{1}{2} C V_c^2(t) + \text{constant} \]

\[ \text{since} \ \xi = 0 \text{ when } V_c = 0 \]
Example 6-1
If the capacitor voltage is as shown across a $\frac{1}{2}$ µF capacitor. What is the corresponding current?

![Graph showing capacitor voltage over time]

Solution: Since $i_c = C \frac{dv_c}{dt}$

\[
\left( \frac{1}{2} \times 10^{-6} \right) \left( \frac{10 - 5}{2 - 3 \times 10^{-3}} \right) + 2.5 \times 10^{-3}
\]

\[i_c = 0 \text{ since } \frac{dv_c}{dt} = 0\]

\[i_c = -2.5 \times 10^{-3} \text{ A} = -2.5 \text{ mA}\]

Example 6-2
If \[i_c(t) = I_o e^{-\frac{t}{\tau_c}} u(t)\] find \[v_c(t)\] if \[v_c(t=0) = 0\].

\[i_c = C \frac{dv_c}{dt}\]

\[\frac{1}{C} \int i_c dt = d v_c\]

\[v_c(t) = v_c(t=0) + \frac{1}{C} \int_0^t I_o e^{-\frac{x}{\tau_c}} dx\]

\[v_c(t) = 0 + \frac{I_o}{C} \left( e^{-\frac{t}{\tau_c}} \right) \left. \right|_0^t = \frac{I_o \tau_c}{C} \left( 1 - e^{-\frac{t}{\tau_c}} \right)\]

Thus is continuous
Example 6-3

The voltage across a capacitor is given below. Find the capacitor's power and energy.

Solution: We found the corresponding current in Example 6-1.

The power $P_c(t)$ is the product of these waveforms.

The energy $W_c = \frac{1}{2} C V^2$

$W_c(t=2) = \frac{1}{2}(\frac{1}{2} \times 10^{-6})(10) = 25 \times 10^{-6} J$

$W_c(t=3) = \frac{1}{2}(\frac{1}{2} \times 10^{-6})(5) = 6.25 \times 10^{-6} J$
Example 6-4

The current through a capacitor is given by:

\[ i_c(t) = I_0 e^{-\frac{t}{RC}} u(t) \]

Find the capacitor’s energy and power.

The voltage was found in Example 6-2 to be:

\[ v_c(t) = \frac{I_0 T_c}{C} \left[ 1 - e^{-\frac{t}{RC}} \right] \]

The power is given by:

\[ P_c(t) = i_c(t) v_c(t) \]

\[ P_c(t) = \left[ I_0 e^{-\frac{t}{RC}} \right] \left[ \frac{I_0 T_c}{C} \left( 1 - e^{-\frac{t}{RC}} \right) \right] \]

\[ P_c(t) = \frac{I_0^2 T_c}{C} \left[ e^{-\frac{t}{RC}} - e^{-\frac{2t}{RC}} \right] \]

The energy is most easily calculated as:

\[ W_c(t) = \frac{1}{2} C v_c(t)^2 \]

\[ W_c(t) = \frac{1}{2} C \left[ \frac{I_0 T_c}{C} \left( 1 - e^{-\frac{t}{RC}} \right) \right]^2 \]
6-2 The Inductor

From magneto statics

\[ \phi(t) = k_1 N i_L(t) \]

\[ \text{number of turns} \]

\[ \text{constant of proportionality} \]

\[ \text{magnetic flux (webers)} \]

\[ \lambda(t) = N \phi(t) \]

\[ \text{magnetic flux} \]

\[ \text{number of turns} \]

\[ \text{flux linkage, flux/unit area} \]

Substituting

\[ \lambda(t) = N \phi(t) = N k_1 N i_L(t) = \left[ N^2 k_1 \right] i_L(t) \]

\[ \text{this is the inductance} \ L \]

\[ \lambda(t) = L i_L(t) \]

Differentiate this to get the i-v relationship

\[ \frac{d\lambda(t)}{dt} = L \frac{di_L(t)}{dt} \]

\[ \text{by Faraday's law} \ V_L(t) = \frac{d\lambda(t)}{dt} \]

\[ V_L(t) = L \frac{di_L(t)}{dt} \]

Observations

1. If \( i_L(t) \) is constant, \( V_L = 0 \) and the inductor looks like a short.

2. A discontinuity \( u(t) \) in \( s(t) \) in \( i_L \) would create an infinite voltage so \( i_L(t) \) must be continuous.
We often want \( i_L \) in terms of voltage

\[
\dot{V}_L(t) = L \frac{di_L(t)}{dt}
\]

\[
di_L(t) = \frac{1}{L} \dot{V}_L(t) dt
\]

Assuming \( i_L(t_0) \) is known we can integrate this to get

\[
\int_{i_L(t_0)}^{i_L(t)} di_L = \frac{1}{L} \int_{t_0}^{t} \dot{V}_L(x) dx
\]

\[
i_L(t) - i_L(t_0) = \frac{1}{L} \int_{t_0}^{t} \dot{V}_L(x) dx
\]

\[
i_L(t) = i_L(t_0) + \frac{1}{L} \int_{t_0}^{t} \dot{V}_L(x) dx
\]

**Power & Energy**

\[
P_L(t) = i_L(t) \dot{V}_L(t)
\]

\[
P_L(t) = i_L(t) L \frac{di_L(t)}{dt}
\]

\[
P_L(t) = \frac{d}{dt} \left[ \frac{1}{2} L i_L^2(t) \right]
\]

recognize this as the energy \( \omega_L \)

stored in the inductor

\[
\omega_L(t) = \frac{1}{2} L i_L^2(t) + \text{constant}^{t_0}
\]

since it is the energy stored when \( i_L = 0 \).
Example 6-6

The current through a 2-mH inductor is \( i_L(t) = 4 \sin 1000t + \sin 3000t \).

Find the corresponding \( v_L(t) \).

\[
    v_L(t) = L \frac{di_L}{dt} = L \frac{d}{dt} (4 \sin 1000t + \sin 3000t)
\]

\[
    v_L(t) = (0.002) \left[ 4 (1000) \cos 1000t + 3000 \cos 3000t \right]
\]

\[
    v_L(t) = 8 \cos 1000t + 6 \cos 3000t.
\]

Example 6-7

The following figures show the current and voltage across an unknown energy storage element.

(a) What is the element and its numerical value?

This is an inductor since \( v = 0 \) when \( i = \) constant.

Since \( v = L \frac{di}{dt} \) for \( 0 < t < 1 \)

\[
    0.1 = L \left( \frac{10 - 0}{1 - 0} \right) = 10L
\]

\[
    L = 0.1 \, \text{H} = 10 \, \text{mH}
\]

(b) What is the energy stored at \( t = 1 \) sec?

\[
    w_L = \frac{1}{2} L i^2 = \frac{1}{2} (0.1)(10)^2 = \frac{1}{2} J.
\]