PRINCIPLES AND APPLICATIONS OF
Electromagnetic Fields

ROBERT PLONSEY

ROBERT E. COLLIN

Professors of Electrical Engineering
Case Institute of Technology

McGRAW-HILL BOOK COMPANY
New York    Toronto    London
1961
CHAPTER 3

ELECTROSTATIC FIELDS IN MATERIAL BODIES,
ENERGY, AND FORCES

In the previous chapter the properties of conducting materials in electrostatic fields were discussed. We turn now to a consideration of the properties of insulators in an electric field. In order to accomplish this aim, a short discussion of the microscopic (atomic) properties of insulators will be required. For this purpose simplified physical models of the atom will be used. Despite this simplicity it will be possible to satisfactorily predict the macroscopic behavior of dielectrics.

After discussing the basic properties of dielectrics, a study is made of capacitance and capacitors. This is followed by an evaluation of the work required to assemble a charge configuration. The work done is next related to the energy stored in the field. The latter part of the chapter introduces the principle of virtual work to evaluate the force acting on a body in an electrostatic field. The virtual-work principle of evaluating forces is a very powerful technique which greatly simplifies the solution of some otherwise difficult problems.

3.1. Polarizability

Electronic Polarizability

Let us begin by considering a monatomic gas in an electric field. This choice takes advantage of the fact that the spacing between the molecules of a gas is very much greater than the size of the molecule, so that, as we shall confirm, interaction between molecules can be neglected. This means that the effect of the field on any molecule is substantially the same as if it were the only particle present. A further geometric simplification arises in the choice of a single atom molecule.

Figure 3.1a illustrates a simplified model of the atom consisting of a positively charged nucleus surrounded by a spherically symmetric cloud of electrons. Since the nucleus has a diameter of the order of $10^{-15}$ meter while that of the electron cloud is of the order of $10^{-9}$ meter, the nucleus is essentially a point source. If, now, an external $E$ field is applied, then a relative displacement of the nucleus from the center will occur. The
relative displacement of the nucleus $z$ can be computed by equating the force exerted by the external field to the coulomb restoring force.

Let $R$ be the radius of the electron cloud, as in Fig. 3.1. The electron cloud is considered to be equivalent to a uniform sphere of charge with a density $-3q/4\pi R^3$. The field due to this charge at a distance $z$ from the center may be found by applying Gauss' law to the region bounded by a sphere of radius $z$, as in Fig. 3.1b. We have

$$\epsilon_0 4\pi z^2 E_r = -\frac{4\pi z^3}{3} \frac{3q}{4\pi R^3}$$

or

$$E_r = -\frac{qz}{4\pi R^3 \epsilon_0}$$

Since the charge on the nucleus is $q$, the coulomb restoring force $F$ acting on the nucleus when displaced an amount $z$ is

$$F = E_r q = -\frac{zq^2}{4\pi R^3 \epsilon_0}$$

(3.1)

Note that the restoring force, given by (3.1), is proportional to the displacement. This consequence is confirmed in atomic physics for small displacements $z$. Equating the external force $qE$ to $-F$ gives

$$qz = p = 4\pi \epsilon_0 R^3 E$$

(3.2a)

or in vector form,

$$p = 4\pi \epsilon_0 R^3 E$$

(3.2b)

The dipole moment per atom, $p$, is proportional to $E$, according to (3.2). If $N$ is the number of atoms per cubic meter, then $P = Np$, where $P$ is the dipole moment per unit volume. In view of the proportionality expressed in (3.2), we may relate $p$ to $E$ as $p = \alpha E$, where $\alpha$ is called the
polarizability. When the polarization arises in the way just described, \( a \) is designated the electronic polarizability \( a_e \). In terms of the above parameters,

\[
a_e = 4\pi R^2 \epsilon_0
\]

The quantity \( a_e/\epsilon_0 \) is seen, from (3.3), to be of the order of magnitude of the atomic volume. In Table 3.1 the ratio \( a_e/4\pi R^2 \epsilon_0 \) is given for several of the inert gases. Despite the crude model used, this ratio comes out in the order of unity.

**Table 3.1. Polarizabilities and Radii for Some Common Elements**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Gas</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_e \times 10^{40} ) cu m (measured)</td>
<td>He</td>
<td>Ne</td>
<td>A</td>
<td>Kr</td>
<td>Xe</td>
</tr>
<tr>
<td>( R ) (Ang)</td>
<td>0.95</td>
<td>1.15</td>
<td>1.4</td>
<td>1.6</td>
<td>1.75</td>
</tr>
<tr>
<td>( a_e/4\pi R^2 \epsilon_0 )</td>
<td>0.28</td>
<td>0.29</td>
<td>0.66</td>
<td>0.67</td>
<td>0.83</td>
</tr>
</tbody>
</table>

In summary, we note that the effect of an applied electric field on the molecules of matter may be to create electrostatic dipoles. These in turn will set up a secondary (induced) field so that the net field in the presence of matter is modified from its free-space value. Before proceeding to a consideration of the fields, we mention several other polarization-producing mechanisms.

**Ionic Polarizability**

In a molecule characterized essentially by ionic bonds, we can think of that molecule as composed of positively and negatively charged ions. It is the coulomb forces between these ions which mainly account for the binding force. The application of an electric field to any such molecule will tend to displace the positive ions relative to the negative ones. This process will induce a dipole moment in the molecule. Note that it is quite distinct from electronic polarizability, where the displacement between the nucleus and electron cloud accounts for the polarization. In a polyatomic gas one can expect both processes to occur. We shall designate ionic polarizability by \( a_i \).

**Orientational Polarizability**

In certain polyatomic molecules where the atomic bond is at least partially ionic, the individual atoms tend to be either positively or negatively charged. A two-atom molecule will thus have a permanent dipole moment, the magnitude of which depends on the time-average transfer of charge between atoms and the internuclear distance. For polyatomic
molecules several bonds may have a permanent dipole moment; the dipole moment of the entire molecule is the vector sum of the component moments. For certain kinds of symmetry the latter may come out zero.

The application of an electric field to molecules with a permanent dipole moment would ordinarily cause all molecules to align themselves with the applied field. This orienting tendency is opposed by the random thermal agitation of the molecule, and in solids and some liquids by mutual interactions of the molecules. Assuming that the molecules are free to move, we can calculate their effective orientational or dipolar polarizability.

Consider a system of $N$ molecules per unit volume, each possessing a permanent dipole moment of $p$. In the absence of an electric field, their orientation is completely random, so that there is no net time-average dipole moment. Now let an electric field be applied in the $z$ direction.

We represent in Fig. 3.2 a typical molecule, with moment $p = ql$, making an angle $\theta$ with the $z$ direction.

For the potential energy of the dipole, let the zero reference be taken to be when $p \cdot E = 0$, that is, $\theta = 90^\circ$. If the dipole is allowed to rotate into alignment with the field, then the decrease in potential energy of $+q$, from the position shown in Fig. 3.2, would be

$$W_+ = \int_{\pi/2}^\theta \frac{qIE}{2} \sin \theta \, d\theta = -\frac{qIE \cos \theta}{2} \tag{3.4a}$$

and similarly for the negative charge,

$$W_- = -\frac{qIE \cos \theta}{2} \tag{3.4b}$$

since the torque acting on each charge of the dipole is $T = (qIE/2) \sin \theta$. Consequently, the potential energy of the dipole in any position can be written as

$$W = -p \cdot E = -pE \cos \theta \tag{3.5}$$

The contribution to the total dipole moment in the $z$ direction from the afore-mentioned molecule is $p \cdot a_z = p \cos \theta$. To find the total moment it is necessary to know the relative number of molecules making different angles $\theta$ with $z$. With an applied field, the distribution is no longer uniform, but it can be found from the Boltzmann distribution law. The latter states that the probability that the direction of $p$ lies between $\theta$
and $\theta + d\theta$ is proportional to $e^{-\frac{W}{kT}} d\omega$, where $d\omega$ is an element of solid angle corresponding to $d\theta$, $k$ is Boltzmann’s constant and equals $1.38 \times 10^{-23}$ joule per degree Kelvin, while $T$ is absolute temperature. From Fig. 3.3 we note that

$$d\omega = 2\pi \sin \theta \, d\theta$$

The number of dipoles, per unit volume, whose moment lies between $\theta$ and $\theta + d\theta$, is now given by

$$dN = A e^{(pE \cos \theta)/kT} \sin \theta \, d\theta \quad (3.6)$$

where $A$ is a constant of proportionality. The latter can be determined from the requirement that the total number of molecules per unit volume be given by the integral of (3.6) from $\theta = 0$ to $\pi$; that is,

$$A = \frac{N}{\int_0^\pi e^{(pE \cos \theta)/kT} \sin \theta \, d\theta} \quad (3.7)$$

The partial contribution to the polarization from dipoles lying between $\theta$ and $\theta + d\theta$ is

$$dP = dN \, p \cos \theta$$

The total polarization $P$ is given by

$$P = Ap \int_0^\pi e^{(pE \cos \theta)/kT} \sin \theta \cos \theta \, d\theta$$

$$= pN \frac{\int_0^\pi e^{(pE \cos \theta)/kT} \sin \theta \cos \theta \, d\theta}{\int_0^\pi e^{(pE \cos \theta)/kT} \sin \theta \, d\theta} \quad (3.8)$$

Let $x = \cos \theta$, $a = pE/kT$; then

$$P = pN \frac{\int_{-1}^1 e^{ax} \, dx}{\int_{-1}^1 e^{ax} \, dx} = Np \left( \coth a - \frac{1}{a} \right) = NpL(a) \quad (3.9)$$

where integration by parts was used to evaluate the numerator. The function $L(a)$ was first introduced in connection with a similar study of magnetic dipoles by Langevin (1905), and it is called the Langevin function. A plot of $L(a)$ is shown in Fig. 3.4. The curve depicts the saturation property of the orientational polarization. At room temperatures laboratory fields are too weak to approach saturation and $pE \ll kT$ can
be assumed. As a consequence,

\[ L(a) \approx \frac{a}{3} = \frac{pE}{3kT} \]

and the polarization per unit volume is

\[ P = \frac{Np^2E}{3kT} \tag{3.10} \]

The orientational polarizability per molecule, \( \alpha_0 \), is thus given by

\[ \alpha_0 = \frac{p^2}{3kT} \tag{3.11} \]

Typical values of \( p \) are around \( 10^{-30} \) coulomb-meter, so that at room temperatures \( \alpha_0 \) comes out around \( 10^{-40} \) cubic meter. This is the same order of magnitude as the electronic polarizability. Note, however, that while both electronic and ionic polarizability depend only on atomic configurations, and hence are essentially independent of temperature, the orientational polarizability is inversely proportional to temperature. This corresponds to the observation that at elevated temperatures it becomes more difficult to align the dipoles against the thermal motion.

The total polarization of a polyatomic gas may arise as a result of electronic, ionic, and orientational polarizability. Therefore, in general, we have

\[ \mathbf{P} = N \left( \alpha_e + \alpha_i + \frac{p^2}{3kT} \right) \mathbf{E} \tag{3.12} \]

where the vector notation, which is implicit in the previous work, has been restored.

If the foregoing analysis is applied to solid dielectrics, then the field \( \mathbf{E} \) in (3.12) cannot be taken as that which exists prior to the introduction of the dielectric. This is because adjacent dipoles become sufficiently influential in modifying the field at a point within the dielectric. The field \( \mathbf{E} \) in (3.12) must be interpreted as being the molecular field in the dielectric. In a later section we shall evaluate the first-order interaction, and hence relate the molecular field to the external field.

We turn now to a consideration of the macroscopic effects of the polarizability of dielectric materials. It will be useful to lump together
the various contributions to the polarization in the following expression:

\[ P = \varepsilon_0 \chi_e E \quad (3.13) \]

where \( \chi_e \) is a dimensionless quantity called the electric susceptibility, and \( E \) is the total field; that is, \( \chi_e \) is assumed to absorb the factor relating the molecular field to the total field. Equation (3.13) assumes that the polarization is proportional to \( E \), a relationship usually confirmed in practice. Note that, in part, it is based on an assumption of being far from saturation for dipolar molecules. Equation (3.13) fails to describe the polarization of certain substances which exhibit spontaneous polarization. The latter class are known as ferroelectrics. A description of their properties can be found in the references on solid-state physics.

### 3.2. Electric Flux Density \( D \)

In Chap. 2 we learned how to calculate the electric field from a given distribution of charge. This can be accomplished by first determining the scalar potential field that is set up. The electric field is then derived from the negative gradient of the scalar potential. The pertinent formulas follow (the reader is reminded that they apply under free-space conditions):

\[ \Phi = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho \, dV'}{R} \quad (3.14) \]

\[ E = -\nabla \Phi \quad (3.15) \]

where \( R = |r - r'| \).

Let us consider what happens if a dielectric is introduced into the electric field set up by an arbitrary charge distribution, as described above. As a consequence of its polarizability, the volume occupied by the dielectric now contains a dipole moment distribution \( P \). This distribution constitutes a secondary source for the electric field. Thus, in addition to the original charges that set up the field, the dipole moment \( P \) must also be included as a source for the complete field, i.e., the field in the presence of the dielectric.

The scalar potential due to \( P \) has already been determined and is given by (2.97). The total potential, by superposition, must then be

\[ \Phi = \frac{1}{4\pi\varepsilon_0} \left( \int_V \frac{\rho - \nabla' \cdot P}{R} \, dV' + \oint_S \frac{P \cdot n}{R} \, dS' \right) \]

\[ = \frac{1}{4\pi\varepsilon_0} \left( \int_V \frac{\rho + \rho_p}{R} \, dV' + \oint_S \frac{\rho_{sp}}{R} \, dS' \right) \quad (3.16) \]

In the last expression, \( \rho_p = -\nabla' \cdot P \) is the equivalent volume polarization charge, \( \rho_{sp} \) the surface polarization charge, while \( \rho \) is called the "true" charge. Note that \( \rho \) in (3.16) is, in general, not exactly the same as in
(3.14). This comes about because $\rho_p$ may react back on the original
distribution of charge, thus affecting that distribution. In many cases,
however, such effects will be small.

The polarization charge does not only arise as a physical interpretation
of the mathematical equivalence expressed in (2.97). If one could actu-
ally measure the excess charge in a small volume within a dielectric, one
would confirm that $\rho_p = -\nabla \cdot P$. [See remarks following (2.97).] Nevertheless, a distinction between true and polarization charge is made,
and this difference is related to the origin of the volume charge density.
True charge is essentially accessible for measurement; it is a free charge.
Polarization charge would also be accessible if one could make measure-
ments on an atomic scale within dielectric materials. However, we ordi-
narily consider the polarization charge as arising from constituent
dipoles so that the "bound" nature of the charges (hence "inaccessibil-
ity" of individual charges) is evident. We consequently distinguish the
polarization charge as a separate entity.

As a consequence of (3.16), the sources of electric field must be general-
ized to include the polarization charge. In the dielectric we must con-
sider $\rho_p$ as fully equivalent to the true charge $\rho$. Thus the divergence of
$E$ must be related to the sum of $\rho + \rho_p$ as follows:

$$\nabla \cdot E = \frac{\rho + \rho_p}{\varepsilon_0}$$

(3.17)

This result is readily verified by taking the Laplacian of (3.16) and using
the singularity property of $\nabla^2(1/R)$. Since (3.17) relates the divergence of $E$ to the charge density at a point, the surface charge term does not enter as long as we are in the interior of the dielectric. Later on we
shall show that the surface polarization charge is readily taken into
account in practice by imposing a discontinuity condition on the normal
component of $E$ at the surface of the dielectric body. If we express $\rho_p$ in
terms of $P$, we have

$$\nabla \cdot (\varepsilon_0 E + P) = \rho$$

(3.18)

In practice, it is inconvenient to take explicit account of the polarization
$P$. We can avoid this by introducing a new field vector $D$ defined as

$$D = \varepsilon_0 E + P$$

(3.19)

Equation (3.18) now becomes

$$\nabla \cdot D = \rho$$

(3.20)

The vector $D$ has the dimensions of coulombs per square meter and is
called the electric displacement, or electric-flux-density vector. The
source for the vector $D$ is the true charge density $\rho$. 
With materials for which (3.13) holds we have

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} \]  

(3.21)

The parameter \( \varepsilon = \varepsilon_0 (1 + \chi_e) \) is called the permittivity of the dielectric. The permittivity relative to that of free space is \( \varepsilon / \varepsilon_0 = 1 + \chi_e \) and is called the relative dielectric constant and will be designated by the symbol \( \kappa \). If \( \varepsilon \) is known, we can solve for the polarization \( \mathbf{P} \) since

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} = (\varepsilon - \varepsilon_0) \mathbf{E} \]  

(3.22)

Values of the relative dielectric constant of several typical materials are given in Table 3.2.

**Table 3.2. Relative Dielectric Constant of Several Materials**

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative dielectric constant</th>
<th>Dielectric strength, kv/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>1.00</td>
<td>3,000</td>
</tr>
<tr>
<td>Oil</td>
<td>2.3</td>
<td>15,000</td>
</tr>
<tr>
<td>Paper</td>
<td>1.5–4</td>
<td>15,000</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.7</td>
<td>20,000</td>
</tr>
<tr>
<td>Glass</td>
<td>6.0</td>
<td>30,000</td>
</tr>
<tr>
<td>Paraffin</td>
<td>2.1</td>
<td>30,000</td>
</tr>
<tr>
<td>Quartz</td>
<td>5.0</td>
<td>30,000</td>
</tr>
<tr>
<td>Mica</td>
<td>6.0</td>
<td>200,000</td>
</tr>
</tbody>
</table>

Sometimes (3.21) is taken as a definition of \( \mathbf{D} \). It is clear that such a relationship holds only for a class of dielectrics under certain conditions. It depends primarily on the linear relation between the polarization and the electric field as expressed in (3.13). In addition to this, it also requires that the material be isotropic, that is, \( \chi_e \) should be independent of the direction of \( \mathbf{E} \). When this is not the case, the relation between \( \mathbf{P} \) and \( \mathbf{E} \), and hence between \( \mathbf{D} \) and \( \mathbf{E} \), becomes a matrix, i.e., a tensor, relation. For example, if the molecule is not symmetrical, its dipole moment will in general not be collinear with the field \( \mathbf{E} \) and each component of \( \mathbf{P} \) will be related to each component of \( \mathbf{E} \), so that

\[
\begin{bmatrix}
P_z \\
P_y \\
P_z
\end{bmatrix} = \varepsilon_0
\begin{bmatrix}
\alpha_{zz} & \alpha_{zy} & \alpha_{zx} \\
\alpha_{zy} & \alpha_{yy} & \alpha_{zy} \\
\alpha_{zx} & \alpha_{zy} & \alpha_{zz}
\end{bmatrix}
\begin{bmatrix}
E_z \\
E_y \\
E_z
\end{bmatrix}
\]

It is clear that in this case \( \chi_e \) and \( \varepsilon \) are matrix (tensors of rank 2) quantities. Materials characterized by a tensor permittivity are called anisotropic materials. For our purposes in the remainder of this book we shall assume that \( \varepsilon \) is a scalar constant. Most materials fall into this class.
If (3.20) is integrated throughout a volume $V$ and Gauss' law is used, then an integral relationship between $\mathbf{D}$ and the total true charge results; i.e.,

$$\oint_{\partial V} \mathbf{D} \cdot d\mathbf{S} = \int_{V} \rho \, dV = Q$$

(3.23)

where $Q$ is the total charge within $V$. Utilizing the flux concept, we see that the number of flux lines originating within $V$ is proportional to the quantity of charge $Q$.

3.3. Boundary Conditions

In solving problems in electrostatics it is necessary to relate the inaccessible (polarization) charges to the accessible (true) charges, or to the fields produced by the latter. Such relationships which link the inaccessible charge sources to the external fields which produce them are called the constitutive equations. An example of such an equation is (3.13), although sometimes the nomenclature is applied to (3.21) as well. Such equations depend on the properties of the material to which they apply. As was noted earlier, (3.21) is restricted to linear, isotropic materials. However, the material need not be homogeneous; that is, $\varepsilon$ may be a function of position.

One very common case of nonhomogeneity occurs when the dielectric constant varies discontinuously as between two different homogeneous media. The way in which $\mathbf{D}$ and $\mathbf{E}$ behave in crossing the boundary between two dielectrics is of much interest and will be discussed now.

Figure 3.5 illustrates a very small element of the interface between dielectrics 1 and 2 whose permittivities are $\varepsilon_1$ and $\varepsilon_2$, respectively. Since the element of surface is of differential extent, it may be considered to be plane. A coin-shaped surface is placed with its broad face parallel to the
interface and so that one surface is in region 1 and the other in region 2. The area of the broad face is $\Delta S$, and the thickness is $h$. Let us apply Gauss' flux theorem to the volume of the coin. If we make use of (3.23), then

$$ (n \cdot D_1) \Delta S - (n \cdot D_2) \Delta S = \rho_s \Delta S $$  

(3.24)

where $\rho_s$ is the true surface charge density on the interface. Equation (3.24) does not include the outflow of flux of $D$ through the sides, because this flow can be made negligible by letting $h \to 0$, while at the same time the terms in (3.24) remain unaffected.

For the simple case of a surface charge $\rho_s$ in a free-space medium, we can set $D_1 = \epsilon_0 E_1$ and $D_2 = \epsilon_0 E_2$ in (3.24), with the result that

$$ n \cdot (E_1 - E_2) = \frac{\rho_s}{\epsilon_0} $$  

(3.25a)

In other words, the normal component of electric field is discontinuous through a charged surface, the magnitude of the discontinuity being given by (3.25a). For a dielectric interface $\rho_s$ is ordinarily equal to zero unless a surface charge is actually placed at the interface. Taking $\rho_s = 0$, (3.24) becomes

$$ n \cdot D_1 = n \cdot D_2 $$  

(3.25b)

That is, the normal component of $D$ is continuous across a dielectric boundary. The normal component of $E$, on the other hand, is discontinuous. This is clear if (3.21) is substituted into (3.25b) to give

$$ \frac{n \cdot E_1}{n \cdot E_2} = \frac{\epsilon_2}{\epsilon_1} $$  

(3.26a)

Using a somewhat simpler notation we have

$$ \epsilon_1 E_{1n} = \epsilon_2 E_{2n} $$  

(3.26b)

The discontinuity in $n \cdot E$ is readily explained physically. The field $E$ arises from the total effective charge consisting of the true charge $\rho$, the volume polarization charge $\rho_P = -\nabla \cdot P$, and the surface polarization charge $\rho_{sp}$. At the surface of a dielectric the normal component of $E$ is discontinuous by an amount equal to $\rho_{sp}/\epsilon_0$, just as it would be if we considered a surface layer of true charge equal to $\rho_{sp}$. Equation (3.26b) is readily shown to verify this result. From (3.22) the normal component of $P$ at the surface is seen to be given by

$$ P_{1n} = (\epsilon_1 - \epsilon_0) E_{1n} \quad \text{medium 1} $$

$$ P_{2n} = (\epsilon_2 - \epsilon_0) E_{2n} \quad \text{medium 2} $$

The surface polarization charge is given by $P_{2n} - P_{1n}$ since this repre-
sents the amount of charge on the positive ends of the dipoles in medium 2 that is not canceled by the opposite charge on the negative ends of the dipoles in medium 1. From the above relations we see that

\[ P_{2n} - P_{1n} = \rho_{sp} = \varepsilon_0(E_{1n} - E_{2n}) + \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} \]  

(3.27)

We can now see that a discontinuity of \( E_n \) by the amount \( \rho_{sp}/\varepsilon_0 \) corresponds to the requirement that \( \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \). In other words, satisfaction of (3.26) is consistent with the necessity that \( E_n \) be discontinuous by an amount \( \rho_{sp}/\varepsilon_0 \).

---

Fig. 3.6. Illustration for derivation of boundary conditions on \( E_\parallel \).

---

In practice, we usually find a suitable solution for \( \mathbf{E} \) and \( \mathbf{D} \) in the two dielectric regions. We then adjust the magnitudes of these solutions so that (3.26) holds at the boundary. By this means we avoid the necessity of taking the surface polarization charge into account explicitly.

Boundary conditions on the tangential components of the field can be found in the following manner. Figure 3.6 is a cross section normal to the interface separating two media of different permittivity. Considering the small rectangular path of length \( \Delta l \) and width \( \Delta \omega \), where opposite sides of the long dimension lie in the separate media, we have

\[ \int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} + \int_d^a \mathbf{E} \cdot d\mathbf{l} = 0 \]  

(3.28)

since the line integral of \( \mathbf{E} \) around any closed path is zero. Without affecting the remaining two integrals,

\[ \int_b^c \mathbf{E} \cdot d\mathbf{l} = \int_d^a \mathbf{E} \cdot d\mathbf{l} = 0 \]

by letting \( \Delta \omega \to 0 \), while keeping \( \Delta l \) fixed. If we symbolize the tangential component of \( \mathbf{E} \) in region 1 by \( E_{1\parallel} \) and that in region 2 by \( E_{2\parallel} \), then, with the choice of directions given in Fig. 3.6,

\[ E_{1\parallel} \Delta l - E_{2\parallel} \Delta l = 0 \]

or

\[ E_{1\parallel} = E_{2\parallel} \]  

(3.29a)
This may be written in vector form as

\[ \mathbf{n} \times \mathbf{E}_1 = \mathbf{n} \times \mathbf{E}_2 \quad (3.29b) \]

For the tangential components of \( \mathbf{D} \) we must now have

\[ \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2} \quad (3.30) \]

Thus the tangential component of electric field is continuous across a boundary between two dielectrics, while tangential \( \mathbf{D} \) is discontinuous. Note that this result would not be affected by the presence of a surface charge layer at the interface.

The total change in electric field in crossing an interface may be found from the above equations. The net result is analogous to the refraction of a light ray in passing from one medium to another. Thus in terms of the geometry of Fig. 3.7 and using (3.26) and (3.29), we have

\[
E_1 \sin \theta_1 = E_2 \sin \theta_2 \tag{3.31a}
\]

\[
\varepsilon_1 E_1 \cos \theta_1 = \varepsilon_2 E_2 \cos \theta_2 \tag{3.31b}
\]

Dividing (3.31a) by (3.31b) yields an electrostatic "Snell's law," relating the angle of incidence to the angle of refraction in terms of the properties of the media; i.e.,

\[
\frac{\tan \theta_1}{\tan \theta_2} = \frac{\varepsilon_1}{\varepsilon_2} \tag{3.32}
\]

Let us consider a problem that illustrates some of the concepts just considered. Figure 3.8 depicts two very large (essentially infinite) parallel conducting planes which are maintained at a potential difference \( V \) by means of a battery. Because of the uniformity (the very large size is chosen so that fringing of the field at the edges can be neglected), the
electric field must be given by
\[ E = \frac{V}{d} \] (3.33)

The uniformity of the electric field is represented in the flux plot in Fig. 3.8.†

Suppose we now insert a uniform slab of dielectric, of relative permittivity \( \kappa \), between the plates, as shown in Fig. 3.9. Let the thickness of the slab be slightly less than \( d \), leaving a small air gap. Since \( V \) is unchanged, \( E \) within the dielectric is still given by \( V/d \); that is, \( \int_0^d E \cdot dl \) must equal \( V \) as before. (We are neglecting the small error that the air gap introduces in this argument.)

In passing from the dielectric into the air gap, the electric field increases by virtue of (3.26). Thus, letting \( E_a \) be the field in the air gap and \( E_d \) that in the dielectric and noting that both are normal to the interface and the conducting plates, we have
\[ E_a = \kappa E_d \] (3.34)

where, of course, \( \kappa > 1 \). In Fig. 3.9, \( \kappa = 9 \), since the field strength is proportional to the number of lines per unit area.

† Fringing of the field at the sides of the plates may be avoided by increasing the size of the bottom plate and surrounding the upper plate by a guard ring, insulated from the upper plate and kept at the same potential \(-V\) as in Fig. 3.8.
The discontinuity in $\mathbf{E}$ is explained by the polarization charge which terminates the flux lines. The polarization of the dielectric is represented in Fig. 3.9, and it should be clear that an equivalent surface charge layer is available at the dielectric surface. If the charge density on the conductor is $\rho_s$, then $E_a = \rho_s/\varepsilon_0$. Now since $\kappa = 9$, $\chi_s = \kappa - 1 = 8$, and $P = \varepsilon_0 \chi_s E_d = 8\rho_s/9$. In view of (3.27),

$$\rho_{sp} = \mathbf{n} \cdot \mathbf{P} = \begin{cases} -\frac{8}{9}\rho_s & \text{lower interface} \\ \frac{9}{5}\rho_s & \text{upper interface} \end{cases}$$

where $\mathbf{n}$ is the outward normal to the dielectric surface. From Fig. 3.9 we see that this is precisely the charge density needed to terminate the lines of flux of $E_a$ at the surface of the dielectric.

### 3.4. Dielectric Strength

As we have noted, the process of induced polarization involves the displacement, from an equilibrium position, of the nucleus relative to the surrounding electron cloud. In our discussion of this phenomenon we tacitly assumed that an elastic process was involved. It might be anticipated, though, that if the field strength is increased sufficiently, new phenomena would be involved, including the possibility of a permanent change in the dielectric. This does, indeed, happen. We say that dielectric breakdown occurs, and the field strength at which this takes place is called the dielectric strength of the material. Table 3.2 lists the breakdown field strength for several common materials.

In general, breakdown begins as a result of the movement of electrons within the insulator under the influence of the applied field. The source of the electrons comes from impurities in the insulator, or crystal-lattice defects, or by field emission due to the applied field itself. If the energy gained by the electron from the accelerating field is greater than that lost in collision with the lattice structure, the electron will accumulate sufficient energy to create a hole-electron pair. The latter will then also be accelerated, so that a continuous discharge through the material can result.

If the solid dielectric contains gas bubbles or layers of gas, breakdown may first occur in that region because the breakdown field strength in a gas is lower than in the solid dielectric, while at the same time the field strength in the gas will be larger than in the dielectric [e.g., see (3.34)]. As a consequence of the gas discharge so initiated, the solid dielectric will be subjected to ionic bombardment, leading, finally, to complete breakdown.

It is not the intention to provide here a detailed physical description and analysis of dielectric breakdown. What is important is that it serves as a motivation in the solution of potential problems where exces-
sive field strengths that can cause breakdown must be avoided. In problems involving composite dielectrics one is faced with a need to reconcile the different dielectric constants and dielectric strengths. The following is a simple example of this.

**Example 3.1. Dielectric Breakdown.** For the parallel conducting planes as in Fig. 3.10, but with an air dielectric, the field is approximately uniform and of magnitude \( V/d \). Consequently, for \( d = 0.01 \text{ meter} \), the maximum potential is 30 kilovolts, as this results in a field of 3,000 kilovolts per meter, which is the breakdown stress for air.

If the region between the plates were filled with glass, then a voltage of 150 kilovolts would be permitted, since the dielectric strength of glass is approximately five times greater than that of air. Suppose, however, that the glass is only half the thickness of the air gap. Then since \( \kappa = 6 \) for glass, the field in the air is six times that in the glass, so that \( 6/\kappa V \) exists across the air gap. In this case

\[
\frac{6/\kappa V_{\max}}{0.005} = 3,000 \text{ kv/m}
\]

and

\[
V_{\max} = 17.5 \text{ kv}
\]

Thus breakdown in the air will occur at a lower voltage than it would in the absence of the glass. Once breakdown occurred, the entire voltage would appear across the glass, but since the field strength would be only \( 17.5/0.005 = 35,000 \) volts per meter, the glass would not break down.

**3.5. Capacitance**

Consider two perfectly conducting bodies of arbitrary shape, as illustrated in Fig. 3.11, and let a quantity of charge be transferred from one to the other, such as would be accomplished by connecting a battery between the two. The charge on one is \( Q \) and on the other \(-Q\). According to the uniqueness theorem, the difference in potential between the bodies
is uniquely determined by the charge $Q$ and the geometry of the conduct- 
ing bodies. In view of the linear dependence of potential on charge, as 
revealed for example in (2.32), the difference of potential between the 
bodies, $V$, can be expressed as

$$V = \frac{1}{C} Q$$  \hspace{1cm} (3.35)

The parameter $C$ depends only on the geometry and for a given geometry 
is a constant. It is called the capacitance and is measured in units of 
farads or coulombs per volt.

![Fig. 3.12. A parallel-plate capacitor with guard ring.](image)

The parallel-plate capacitor shown in Fig. 3.12 presents a simple 
geometry for illustrating the calculation of capacitance. If the smallest 
linear dimension of the plates is large compared with the spacing, then 
the field may be assumed to be uniform and the fringing at the edges 
eglected. Alternatively, a guard ring may be used as suggested in 
Fig. 3.12. In this case the relationship expressed by (3.33) applies, that 
is, $E = V/d$, where $V$ is an assumed potential difference between the 
plates. For an air dielectric, $D = \varepsilon_0 E$, and $\rho_s = D$; consequently, the 
total charge $Q$ on each plate, of area $A$, is

$$Q = \rho_s A = \frac{\varepsilon_0 V A}{d}$$  \hspace{1cm} (3.36)

and

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$$  \hspace{1cm} (3.37)

**Example 3.2. Capacitance between Concentric Spheres.** It is 
required to find the capacitance between the two conducting concentric 
spherical shells illustrated in Fig. 3.13. If we assume a charge $+Q$ on 
the inner sphere and $-Q$ on the outer, then the electric field between the 
two shells is

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$$  \hspace{1cm} (3.38)
This result can be verified by applying Gauss' flux theorem to the concentric spherical surface of radius $r$, where $a < r < b$. The potential difference between the spheres, $V$, is given by

$$V = \int_a^b E_r \, dr = \frac{Q}{4\pi \varepsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right)$$  \hspace{1cm} (3.39)

Consequently, the capacitance between the two spheres is

$$C = \frac{Q}{V} = 4\pi \varepsilon_0 \frac{ab}{b - a}$$  \hspace{1cm} (3.40)

If $b \rightarrow \infty$, the capacitance of an isolated sphere results; that is, we have a uniformly charged sphere with lines of flux extending radially outward to terminate at "infinity." Such a condition is essentially obtained in the case of a small charged sphere in a relatively large laboratory room, where the flux tends to terminate on the distant walls. From (3.40) the capacitance of the isolated sphere is seen to be

$$C = 4\pi \varepsilon_0 a$$  \hspace{1cm} (3.41)

**Multicapacitor System**

The concept of capacitance can be extended to a region containing more than two conducting bodies. In Fig. 3.14 we show $N$ arbitrary conducting bodies and the earth. The latter may or may not figure in a practical problem. We can simply consider it as just another conducting body, chosen as the reference, in the following analysis.

On the basis of the uniqueness theorem in Sec. 2.9, specification of the charge on each body plus specification that the earth be a zero reference potential uniquely determines the potential everywhere. Furthermore,
because of the linear dependence of potential on charge, the following equations result:

\[
\begin{align*}
\Phi_1 &= p_{11}q_1 + p_{12}q_2 + \cdots + p_{1N}q_N \\
\Phi_2 &= p_{21}q_1 + p_{22}q_2 + \cdots + p_{2N}q_N \\
&\quad \quad \vdots \\
\Phi_N &= p_{N1}q_1 + p_{N2}q_2 + \cdots + p_{NN}q_N
\end{align*}
\]  

(3.42)

In the above set of equations, the \( p_{ij} \)'s are constants, called the coefficients of potential, and they depend only on the geometry. We note that the potential on each body is properly a function of the total charge on each and that the relationship is linear.

The \( N \) equations can be solved to give the charges as functions of the potentials. The result will be in the form

\[
\begin{align*}
q_1 &= c_{11} \Phi_1 + c_{12} \Phi_2 + \cdots + c_{1N} \Phi_N \\
q_2 &= c_{21} \Phi_1 + c_{22} \Phi_2 + \cdots + c_{2N} \Phi_N \\
&\quad \vdots \\
q_N &= c_{N1} \Phi_1 + c_{N2} \Phi_2 + \cdots + c_{NN} \Phi_N
\end{align*}
\]  

(3.43)

Since the \( c_{ij} \)'s depend only on the \( p_{ij} \)'s of (3.42), they also depend only on the geometry. The terms \( c_{11}, c_{22}, \ldots, c_{NN} \) are called coefficients of capacitance, while the \( c_{12}, c_{13}, \ldots \) are coefficients of induction. The coefficient \( c_{ii} \) can be obtained by evaluating the ratio \( q_i/\Phi_i \) of the \( i \)th body, with all others grounded. Since a positive charge on \( i \) produces a positive potential on \( i \), the \( c_{ii} \)'s are all positive. The \( c_{ij} \) can be measured by grounding all but the \( i \)th body and evaluating the ratio \( q_j/\Phi_i \). Note that if \( q_i \) is positive, then \( \Phi_i \) is positive, but the charge induced on \( j \) (\( q_j \) for \( j \neq i \)) will be negative. Accordingly, \( c_{ji} \) (\( i \neq j \)) is negative.

The coefficients of induction must satisfy a condition of reciprocity; that is, \( c_{ij} = c_{ji} \). We can show this in the following way. Let all bodies be grounded but the first and second, which we take to be initially uncharged. If we begin charging body 1 to a final value of \( q_1 \) by adding very small increments of charge, then at some intermediate point the accumulated charge is \( q \) and the potential [from (3.42)] is \( \Phi_1 = p_{11}q \). The energy required to add a charge \( dq \) is then, by definition of potential,

\[ dW = p_{11}q \, dq \]

Accordingly, the total energy required to place a charge of \( q_1 \) on 1 is

\[ W = \int_0^{q_1} p_{11}q \, dq = \frac{p_{11}q_1^2}{2} \]

(3.44)

If we proceed now to charge body 2 until \( q_2 \) is accumulated, then, corresponding to the presence of a charge \( 0 < q < q_2 \), the potential is \( \Phi_2 = p_{21}q_1 + p_{22}q \) and the work done in adding an increment of charge \( dq \)
to body 2 is
\[ dW = (p_{21}q_1 + p_{22}q) \, dq \]
Altogether, the following amount of work is required to charge body 2 with \( q_2 \) coulombs:
\[ W = \int_0^q (p_{21}q_1 + p_{22}q) \, dq = p_{21}q_1q_2 + \frac{p_{22}q_2^2}{2} \tag{3.45} \]
The total energy required to put \( q_1 \) on 1 and \( q_2 \) on 2 is
\[ W_T = p_{21}q_1q_2 + \frac{p_{11}q_1^2}{2} + \frac{p_{22}q_2^2}{2} \tag{3.46} \]
If, however, we had first charged body 2 with \( q_2 \) and then body 1 with \( q_1 \), the total energy required is expressed by [we have only to interchange the subscripts 1 and 2 in (3.46)]
\[ W_T = p_{12}q_2q_1 + \frac{p_{22}q_2^2}{2} + \frac{p_{11}q_1^2}{2} \tag{3.47} \]
Since the final result is the same, the energies expressed by (3.46) and (3.47) are equal; consequently,
\[ p_{12} = p_{21} \tag{3.48} \]
The above proof can be repeated between any two bodies, say the \( i \)th and \( j \)th, so that
\[ p_{ij} = p_{ji} \tag{3.49} \]
From purely algebraic considerations it then follows that
\[ c_{ij} = c_{ji} \tag{3.50} \]
Equation (3.43) can now be rewritten in a way that is more informative. We let \( C_{ij} = -c_{ij} \), noting that \( C_{ij} \) will be positive \((i \neq j)\), and let
\[ C_{ii} = -c_{11} + c_{12} + c_{13} + \cdots + c_{iN} \tag{3.51} \]
Then by adding and subtracting additional terms that will be clear upon examination, we can write in place of (3.43) the following:
\[
q_1 = C_{11}\Phi_1 + C_{12}(\Phi_1 - \Phi_2) + C_{13}(\Phi_1 - \Phi_3) + \cdots + C_{1N}(\Phi_1 - \Phi_N) \\
q_2 = C_{21}(\Phi_2 - \Phi_1) + C_{22}\Phi_2 + C_{23}(\Phi_2 - \Phi_3) + \cdots + C_{2N}(\Phi_2 - \Phi_N) \\
\vdots \\
q_N = C_{N1}(\Phi_N - \Phi_1) + C_{N2}(\Phi_N - \Phi_2) + C_{N3}(\Phi_N - \Phi_3) + \cdots + C_{NN}\Phi_N \tag{3.52} \\
\]
Of the total charge \( q_1 \) on body 1, a portion \( C_{12}(\Phi_1 - \Phi_2) \) depends on the difference of potential with body 2. Correspondingly, on body 2, we note that since \( C_{12} = C_{21} \), an equal but opposite charge is bound. But
of lines \( N_{12} \) terminates on an amount of charge \( -q_2 = -Q_1 N_{12} / N_1 \) on body 2, while \( N_{11} \) corresponds to a total charge

\[
-q_1 = \frac{-Q_1 N_{11}}{N_1} = -(Q_1 - q_2)
\]

residing on the ground plane. If the total charge on body 2 is \( Q_2 \), then since \( N_{12} \) lines of flux from body 1 terminate in a charge \( -q_2 \) on body 2, there must be \( N(Q_2 + q_2) = N_{22} \) lines of flux leaving body 2 and terminating on the ground plane. If body 1 is at a potential \( \Phi_1 \) and body 2 is at a potential \( \Phi_2 \), then the capacitances \( C_{ij} \) are given by

\[
C_{11} = \frac{N_{11}}{N \Phi_1} \quad C_{12} = \frac{N_{12}}{N(\Phi_1 - \Phi_2)} \quad C_{22} = \frac{N_{22}}{N \Phi_2}
\]

and are just proportional to the fraction of the number of lines of flux leaving the body and terminating on adjacent bodies, divided by the potential difference. These results follow from (3.52) as follows. We have

\[
Q_1 = C_{11} \Phi_1 + C_{12} (\Phi_1 - \Phi_2) = q_1 + q_2
\]

and thus \( q_1 = C_{11} \Phi_1 = N_{11} / N \), \( q_2 = C_{12} (\Phi_1 - \Phi_2) = N_{12} / N \). Hence, \( C_{11} = N_{11} / N \Phi_1 \), etc., as stated above.

**Electrostatic Shielding**

The electric field within a closed conducting surface must be zero if no charge is placed within. For suppose that a field were present; then just inside the conducting surface, which is an equipotential, we could define another equipotential surface but at a different potential. Then over this surface \( E \) is everywhere outward (or inward). However, by Gauss’ law, this requires the presence of an interior charge. But this contradicts the original assumption.

This characteristic leads to the use of hollow shells for electrostatic screening. If a body is placed within, it is un influenced by external fields. If the internal body is charged, the interior electric field is independent of external effects. The potential difference to the shell is also independent of external sources, but the potential with respect to ground depends on the potential of the shell with respect to ground. This variability is eliminated by grounding the shell.

Some of these remarks are more rigorously formulated through the use of (3.52). For example, based on Fig. 3.17 we have the following:

\[
\begin{align*}
q_1 &= C_{11} \Phi_1 + C_{12} \Phi_1 + C_{13} (\Phi_1 - \Phi_2) \\
q_2 &= -C_{12} \Phi_1 - C_{23} \Phi_2 \\
q_3 &= C_{13} (\Phi_3 - \Phi_1) + C_{23} \Phi_3 + C_{33} \Phi_3
\end{align*}
\]

(3.53)

since \( \Phi_2 = 0 \) because body 2 is grounded. If we choose \( q_1 = 0 \), then the
field inside body 2 is zero since there is no charge contained within. Therefore the potential \( \Phi_1 \) must equal \( \Phi_2 \), that is, equal zero. From the first equation in the set (3.53), we see this requires that \( C_{12} = 0 \). Thus in place of (3.53), we have

\[
\begin{align*}
q_1 &= (C_{11} + C_{12})\Phi_1 \\
q_2 &= -C_{12}\Phi_1 - C_{23}\Phi_3 \\
q_3 &= (C_{23} + C_{33})\Phi_3
\end{align*}
\]

Since body 2 is grounded, we may consider it as part of the ground plane, and then \( C_{11} + C_{12} \) is the capacitance of body 1 with respect to ground while \( C_{23} + C_{33} \) is the capacitance of body 3 with respect to ground. Since the potentials of body 1 and 3 depend only on their own charges, there is no interaction between them. Consequently, body 2 behaves as an electrostatic shield.

3.6. Electrostatic Energy

The absolute potential due to a single charge \( q \) has been found to be

\[
\Phi = \frac{q}{4\pi\varepsilon_0 R}
\]

where \( R \) is the distance between the field point and the position of \( q \). Consequently, the energy required to bring a charge \( q' \) from infinity to a distance \( R \) from \( q \) is

\[
W_e = \frac{qq'}{4\pi\varepsilon_0 R}
\]

(3.55)

This energy is what is meant by the potential energy of the charges and is a function of their final position only.

Let us evaluate the energy required to assemble an arbitrary distribution of charges of arbitrary magnitudes, such as illustrated in Fig. 3.18. We can think of bringing each charge in from infinity in succession and evaluating the energy required. Thus no energy is required to introduce the first charge. The second charge brought in requires an energy as given by (3.55). For the third charge we must consider the net energy arising from interaction with the first and second charges, and so on.
The total energy to establish $N$ charges can be formulated in a relatively simple way. Consider that the $k$th charge is the last one to be brought in from infinity. Since all others are present, then by superposition the energy required would be

$$W_e)_k = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N'} \frac{q_k q_i}{R_{ik}}$$

(3.56)

where $R_{ik}$ is the distance from the $i$th to $k$th charge, and the primed summation symbolizes that $i = k$ is not included. The total energy required to assemble all $N$ charges would not be correctly given if (3.56) were summed from $k = 1$ to $k = N$ since that equation is valid only for the last charge brought in; that is, no matter what order the charges are assembled in, the partial energy associated with any two charges, say, $k$th and $j$th, equals $q_k q_j/4\pi \varepsilon_0 R_{jk}$, and this term occurs only once during the assembly process. But if (3.56) is summed over $k$, each interaction occurs twice; for example, a partial energy contribution due to charges $q_3$ and $q_5$ is included not only when $k = 3$, $i = 5$, but also when $k = 5$, $i = 3$. This means that a summation of (3.56) over $k$ yields exactly twice the desired value of energy. Then the energy $W_\Phi$ to assemble the charges is simply

$$W_\Phi = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{N'} \frac{q_k q_i}{4\pi \varepsilon_0 R_{ik}}$$

(3.57)

If $\Phi_k$ designates the potential at the $k$th charge due to all other charges, i.e.,

$$\Phi_k = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N'} \frac{q_i}{R_{ik}}$$

(3.58)

then (3.57) can be written as

$$W_\Phi = \frac{1}{2} \sum_{k=1}^{N} q_k \Phi_k$$

(3.59)

We may inquire where the energy associated with a charge distribution is stored. An analogous question can be raised in mechanics if we are given two masses that are attached to the opposite ends of a compressed spring. The seat of the stored energy may be considered to be in the masses. We say that initially they lie in a region of higher potential energy; after expansion of the spring, potential energy is converted into kinetic energy. But one could also consider that the stored energy reposed in the stressed state of the spring. The first viewpoint coincides
with the expression (3.57), where the stored energy is linked to the charges and their positions. This view is the counterpart of the action-at-a-distance concept, in that it ascribes physical reality to the charges and their spatial distribution alone. On the other hand, the field concept should be capable of expressing the stored energy without recourse to a description of the charges causing the field, if the concept is to be fully complete. The energy would then be described in terms of the "elastic" quality of the electric field, in similarity with the viewpoint that the spring stores the energy in the mechanical system considered. It turns out that the energy associated with a charge distribution can be expressed in terms of the fields alone, and we now proceed to show this.

For a continuous charge distribution, (3.59) can be generalized to read

$$W_e = \frac{1}{2} \int_V \rho \Phi \, dV$$

(3.60)

where \( \Phi \) is now a continuous function of position and is the potential distribution due to all charges. It is no longer necessary to exclude the contribution due to \( \rho \Delta V \) at the point where \( \Phi \) is evaluated, since in the limit \( \Delta V \to 0 \) the contribution is zero anyway. Since \( \nabla \cdot D = \rho \), we have

$$W_e = \frac{\varepsilon_0}{2} \int_V (\nabla \cdot E) \Phi \, dV$$

Now \( \nabla \cdot (\Phi E) = \Phi \nabla \cdot E + E \cdot \nabla \Phi \), and hence

$$W_e = \frac{\varepsilon_0}{2} \int_V \nabla \cdot (\Phi E) \, dV - \frac{\varepsilon_0}{2} \int_V \nabla \Phi \cdot E \, dV$$

Using the divergence (Gauss') theorem and noting that \( E = -\nabla \Phi \),

$$W_e = \frac{\varepsilon_0}{2} \oint_S \Phi E \cdot dS + \frac{\varepsilon_0}{2} \int_V E^2 \, dV$$

(3.61)

If the original volume extends indefinitely, as it should to include all possible charge, then the surface \( S \) in (3.61) extends to infinity. Considering \( S \) as the surface of a sphere of radius \( R \), where \( R \to \infty \), we note that \( \Phi \propto 1/R \), \( E \propto 1/R^2 \), so that even though \( S \propto R^2 \), the integral is of order \( 1/R \) and vanishes in the limit \( R \to \infty \). We are then left with

$$W_e = \frac{\varepsilon_0}{2} \int_V E^2 \, dV$$

(3.62)

We have consequently found a way of expressing the stored energy of a charge distribution in terms of the field alone.

The previous analysis, which applies to charges in free space, can be extended to regions involving dielectrics. In this derivation we shall again consider the true charges as continuously distributed. If an
amount of charge $\Delta \rho$ is introduced into an existing field $\Phi$, then the energy is increased by an amount $\Delta W_*$, where

$$\Delta W_* = \int_V \Phi \Delta \rho \, dV$$

This can also be written as

$$\Delta W_* = \int_V \Phi \Delta (\nabla \cdot D) \, dV = \int_V \Phi (\nabla \cdot \Delta D) \, dV$$  \hspace{1cm} (3.63)

Using the identity $\nabla \cdot \Phi \Delta D = \Phi \nabla \cdot \Delta D + \Delta D \cdot \nabla \Phi$ and the divergence theorem gives

$$\Delta W_* = \int_V [\nabla \cdot (\Phi \Delta D) - \Delta D \cdot \nabla \Phi] \, dV = \oint_S \Phi \Delta D \cdot dS - \int_V \Delta D \cdot \nabla \Phi \, dV$$

The surface integral term goes out for the same reasons as given in discussing (3.61); thus

$$\Delta W_* = \int_V \Delta D \cdot E \, dV$$

To obtain the total energy in the field, the above equation must be integrated. This involves the unknown functional relationship between $D$ and $E$. For the case where $D = \epsilon E$ and $\epsilon$ is a constant, we have

$$W_* = \int_0^D \Delta W_* = \int_0^E \int_V E \cdot \epsilon \Delta E \, dV = \int_V \int_0^E \frac{\epsilon E^2}{2} \, dV$$

or

$$W_* = \frac{\epsilon}{2} \int_V E^2 \, dV = \frac{1}{2} \int_V E \cdot D \, dV$$  \hspace{1cm} (3.64)

The above expression reduces to the earlier one if free-space conditions are supposed.

In (3.64), as in (3.62), we have shown that the potential energy of a charge distribution can be calculated from the field produced by the charges. In the expressions themselves, the quantity $U_* = E \cdot D/2$ has the dimensions of energy density. It is often referred to as if it were truly the energy density of the field. However, just as it was impossible to localize the energy as being associated with the charge or the field, it is impossible to relate the total energy of the field with any particular distribution of component energy. Thus $E \cdot D/2$ is an energy density only to the extent that its volume integral over all space correctly evaluates the total potential energy.

**Example 3.3. Energy Storage in a Capacitor.** The parallel-plate capacitor provides a simple example for the application of the results of this section. Conditions are as illustrated in Fig. 3.19, where the plates have an area $A$ and spacing $d$. For simplicity, one plate may be chosen to have a potential zero; the other is then at potential $V$. By an exten-
sion of (3.60) we may write

$$W_* = \frac{1}{2} \int_S \Phi \rho_* dS$$  \hspace{1cm} (3.65)

This is zero over the lower surface, while for the upper surface we have $W_* = \frac{1}{2} V Q$. Since $C = Q/V$, this may also be expressed as

$$W_* = \frac{1}{2} C V^2$$  \hspace{1cm} (3.66)

From the method of derivation it is clear that (3.66) is valid for any two-body capacitor.

We may also calculate this result from the fields. We note that

$$D = \frac{Q}{A}$$ and is uniform within the capacitor and zero outside, neglecting fringing. Hence,

$$W_* = \frac{1}{2} \int_V D \cdot E \, dV = \frac{1}{2\epsilon_0} \left( \frac{Q}{A} \right)^2 A d = \frac{Q^2 d}{2\epsilon_0 A}$$  \hspace{1cm} (3.67)

But $C = \epsilon_0 A / d$; hence $W_* = Q^2 / 2C = CV^2 / 2$ as before.

This same result can also be obtained in a more familiar way. The total charge $Q$ on the upper plate is considered as having been transferred from the lower plate in differential increments. At an intermediate stage the charge is $q$, and the potential is therefore $\Phi = q/C$. The energy required to add a charge $dq$ is simply $\Phi \, dq = q \, dq / C$. Then the work to build up to a charge $Q$ is

$$W_* = \frac{1}{C} \int_0^Q q \, dq = \frac{Q^2}{2C}$$

thus confirming (3.66) once again.

Energy in a Multicapacitor System

The method used to derive (3.59) is also valid for a system of $N$ conducting bodies as illustrated in Fig. 3.14. The energy stored in a system of $N$ charged conducting bodies can be determined from (3.65). We note
that over each separate conductor $\Phi$ is a constant, and hence the surface integral, if expressed as the sum of surface integrals over each body separately, becomes

$$W_* = \frac{1}{2} \int_S \rho_s \Phi \, dS = \frac{1}{2} \Phi_1 \int_{S_1} \rho_s \, dS + \frac{1}{2} \Phi_2 \int_{S_2} \rho_s \, dS + \cdots + \frac{1}{2} \Phi_j \int_{S_j} \rho_s \, dS + \cdots \quad (3.68)$$

But each integral is just the total charge on each conducting surface. Thus, if we let

$$Q_i = \int_{S_i} \rho_s \, dS \quad (3.69)$$

we have

$$W_* = \frac{1}{2} \sum_j \Phi_j Q_j \quad (3.70)$$

If (3.43) is used to express $Q_i$ in terms of the potentials $\Phi_j$, we can rewrite (3.70) as

$$W_* = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N c_{ij} \Phi_i \Phi_j \quad (3.71)$$

This expression gives the energy stored in a system of $N$ conducting bodies in terms of the potentials of the bodies.

For a two-body problem the capacitance $C$ between the two bodies may be defined in terms of the stored energy in the electric field surrounding the bodies. From (3.66) we have

$$C = \frac{2W_*}{V^2} \quad (3.72)$$

This alternative definition of capacitance $C$ is often easier to evaluate than some of the other formulations.

### 3.7. Electrostatic Forces

Figure 3.20 shows a parallel-plate capacitor in a free-space medium with a total charge $+Q$ on one plate and $-Q$ on the other. Because of the presence of charge of opposite sign, one would expect that a force of attraction exists between the plates. Such a force does, of course, exist; our present purpose is to calculate its magnitude. One of the simplest ways of doing this makes use of the principle of virtual work. As applied to this problem one of the capacitor
plates, say the upper one, is visualized as being displaced by an amount $\Delta x$ away from the lower plate ($x$ is the plate separation and is positive from lower to upper plate, as in Fig. 3.20). As a consequence, work must be performed, and the amount is simply $-F_z \Delta x$, where $F_z$ is the electrostatic force that must be overcome by the external agent. From conservation of energy, this mechanical work must reappear as energy elsewhere, and in this case the stored energy of the field is the only other energy term that could possibly be involved. It must have increased, and the amount of increase is readily established from (3.67) to be

$$\Delta W_e = \frac{Q^2}{2\varepsilon_0 A} \Delta x = \frac{\varepsilon_0}{2} E^2 A \Delta x$$

where $A$ is the area of the plates, and fringing of the field near the edges of the plates is neglected. Consequently,

$$-F_z \Delta x = \frac{\varepsilon_0}{2} E^2 A \Delta x \quad (3.73a)$$

or in vector form,

$$\frac{\mathbf{F}}{A} = -\frac{\varepsilon_0 E^2}{2} \mathbf{a}_z \quad (3.73b)$$

A generalization of this technique can be used to determine the force acting on any conducting body in the presence of an arbitrary distribution of charged conductors in free space. The $x$ component of electrostatic force $F_i \cdot a_z$ on the $i$th body can be found by giving the $i$th body a virtual displacement $\Delta x_i a_z$ and equating the work done, $-F_i \cdot a_z \Delta x_i$, to the net increase in the algebraic sum of all other energy changes that are produced. It is necessary that all energy terms that might be affected by the virtual displacement be included in the energy balance. In general, this includes a change in stored energy that results from the new position of the displaced body. Furthermore, if the potential between two or more bodies is held constant by the presence of a battery, then charge may be transferred by the battery in the virtual process, hence requiring the inclusion of a battery energy term also. If $\Delta W$ represents the work done by the batteries, then

$$-(F_i)_z \Delta x_i + \Delta W = \Delta W_e \quad (3.74)$$

or

$$\frac{(F_i)_z}{\Delta x_i} = \frac{\Delta W}{\Delta x_i} - \frac{\Delta W_e}{\Delta x_i} \quad (3.75)$$

The remaining force components are found by replacing $x$ by $y$ and $z$ in turn.

Certain constraints are evident in a system of conducting charged bodies during the processes of displacing one or several of them. Those charged bodies that are isolated are constrained by the requirement that
the total charge be kept constant in the process; their potential may vary, however. Those bodies linked together by batteries will be constrained to maintain a constant relative potential; in this case the total charge on each body may change. If one or several bodies are actually moved through a finite distance and it is desired to calculate the total work done on the body by setting up an energy balance, then obviously the true constraints of the system must be known and utilized. However, if it is only required that the force, for a particular configuration, be evaluated, then the virtual work process is conceptual only. Consequently, we may set up hypothetical constraints in the knowledge that so long as the resulting energy balance is correctly described, the correct value of force will be determined. The force that exists under a given set of conditions has a definite value, and this value cannot be affected by the constraints that are assumed under a purely hypothetical displacement of one of the bodies. However, if a constraint is assumed, then the correct consequences of that constraint must be included in the energy calculations. We consider now two cases, that of constant charge and that of constant potential.

**Constant Charge**

Since the constraint requires that each conducting body be isolated, the possibility that energy be supplied by a battery source does not occur. In the limit $\Delta x_i \to 0$, (3.75) may be written as

$$ (F_i)_{x} = - \left. \frac{\partial W_z}{\partial x_i} \right|_{\text{constant charge}} \tag{3.76} $$

where $x_i$ is the $x$ variable at the $i$th body. Equation (3.70) can be used to evaluate $W_z$. When this equation is substituted into (3.76) and the fact that $Q_j$ is constant is made use of, we find that

$$ (F_i)_{x} = -\frac{1}{2} \sum_j Q_j \frac{\partial \Phi_j}{\partial x_i} \tag{3.77} $$

**Constant Potential**

In order to maintain each body at a constant potential when one body suffers a virtual displacement, it may be necessary to transfer charge to any or all of the conducting bodies in the system. This is performed by a battery, and the work involved must be included in an energy balance. If $dQ_j$ is added to the $j$th body as a consequence of the virtual displacement, the energy supplied is $\Phi_j dQ_j$, and the total of such energy is

$$ dW = \sum_j \Phi_j dQ_j \tag{3.78} $$
Combining this equation with (3.70) and (3.75), we have in the limit

\[
(F_i)_x = \sum_j \Phi_j \frac{\partial Q_j}{\partial x_i} - \frac{1}{2} \sum_j \Phi_j \frac{\partial Q_j}{\partial x_i} = \frac{1}{2} \sum_j \Phi_j \frac{\partial Q_j}{\partial x_i} = \frac{\partial W_\varepsilon}{\partial x_i} \bigg|_{\text{constant potential}}
\]

(3.79)

Note that the second summation makes use of the constant-potential constraint. Equation (3.79) expresses the very interesting fact that in a virtual displacement under a constant-potential constraint the work done by the field is equal to the increase in stored energy. The energy balance is explained by the fact that the battery supplies an energy equal to the sum of the mechanical and field terms.

The above results are applicable to the evaluation of torques also. It is only necessary to interpret \( F_i \) as a torque and \( dx_i \) as a small rotation \( d\theta_i \).

**Example 3.4. Force on a Capacitor Plate.** The force between plates of a parallel-plate capacitor is given by (3.73). The method used to obtain this result involved the principle of virtual work together with a constraint of constant charge. Note that the result can be obtained immediately from (3.77). As pointed out above, the constraint can also be chosen as that of constant potential. Such a constraint can be interpreted physically as being due to the connection of a battery across the capacitor. Let us calculate the force under such assumed conditions. For our present purpose Fig. 3.19 correctly describes the problem. We assume that the upper plate is pushed away from the lower plate by an amount \( dx \), where \( x \) is the plate separation. Since \( W_\varepsilon = \frac{1}{2} CV^2 \),

\[
dW_\varepsilon = \frac{1}{2} V^2 dC = \frac{1}{2} V^2 d \left( \frac{\varepsilon_0 A}{x} \right) = -\frac{1}{2} V^2 \left( \frac{\varepsilon_0 A}{x^2} \right) \bigg|_{x=d} dx
\]

\[
= -\frac{1}{2} \frac{V^2 C}{d} \ dx
\]

Since \( V \) remains constant, \( Q \) must vary in view of the change in \( C \). Thus

\[
Q = CV
\]

\[
dQ = V \ dC = \frac{V}{d} \ C \ dx
\]

The change in \( Q \) is accomplished by the transfer of the charge \( dQ \) by the battery. The battery therefore does an amount of work \( V \ dQ \). This plus the work done by an external force acting against the field, \( -F_x \ dx \),
must equal the increase in stored energy. Consequently,

\[
\frac{-V^2C}{d} \, dx - F_z \, dx = - \frac{1}{2} \frac{V^2C}{d} \, dx
\]

or

\[
F_z = - \frac{1}{2} \frac{V^2C}{d} = - \frac{\varepsilon_0 AE^2}{2}
\]

The result confirms that given by (3.73). Note that it is obtained almost at once by using (3.79).

The above results are easily generalized to the case of an arbitrary two-body capacitor with capacitance \( C \). For an arbitrary virtual displacement \( dr \), the electrostatic force acting in the direction of \( dr \) is readily found if the rate of change \( \partial C/\partial r \) can be found. For a constant-charge constraint we write \( W_e = Q^2/2C \), and hence from (3.76),

\[
F_r = - \frac{\partial W_e}{\partial r} = - \frac{Q^2}{2} \frac{\partial C^{-1}}{\partial r} = \frac{Q^2}{2C^2} \frac{\partial C}{\partial r}
\]

For a constant-potential constraint we may write \( W_e = CV^2/2 \) in place of \( Q^2/2C \), and since \( V \) is constant, we have, from (3.79),

\[
F_r = \frac{\partial W_e}{\partial r} = \frac{V^2}{2} \frac{\partial C}{\partial r} = \frac{Q^2}{2C^2} \frac{\partial C}{\partial r}
\]

since \( V = Q/C \). The force \( F_r \) is, of course, the same in both cases, since for a given charge on the capacitor a unique field exists around the conductors, and hence the force acting on the conductors is also unique.

The existence of the force does not depend on the virtual displacement or constraints that are assumed. The virtual displacement is conceptual and, together with assumed constraints and an energy balance, permits the force exerted by the field to be evaluated. Any change in the assumed constraints cannot change the force acting; it only changes the details of the energy-balance equation.

**Forces in the Presence of Dielectrics**

In the previous paragraphs we showed how the force on a charged body in an electrostatic field could be found. The equations that were derived assumed that the conducting bodies lie in a free-space medium. The results obtained, however, would still apply in the presence of dielectrics; that is, the process whereby a virtual displacement is given a particular body, and the consequent mechanical work equated to the increase in field energy less the work performed by the batteries, still applies. However, the change in energy stored in the field must now include energy storage in the dielectrics.

If a dielectric body which lies in an electric field is given a virtual displacement, then the energy stored in the field may be expected to change
in view of its dependence on the geometry. As a consequence it follows that dielectric bodies in electrostatic fields will experience a net force. This force may be calculated from the principle of virtual work, just as in the case for conducting bodies. The following example illustrates this technique.

**Example 3.5. Force on a Dielectric Slab.** A parallel-plate capacitor of width $W$, separation $d$, and length $L$ is partially filled with a uniform dielectric slab, of permittivity $\epsilon$, as illustrated in Fig. 3.21. The force acting on the dielectric slab is desired.

With the geometry as given, the total stored energy is calculated to be

$$W_s = \frac{1}{2} \epsilon_0 \left( \frac{V}{d} \right)^2 (W - x)Ld + \frac{1}{2} \epsilon \left( \frac{V}{d} \right)^2 xLd + W_1$$

where $W_1$ is a correction term that takes account of the fringing of the field at the sides. Let us postulate a virtual displacement $\Delta x$ of the dielectric body in the positive $x$ direction. We shall arbitrarily consider this to occur under the condition of constant potential. The work done by the field, $F_x \Delta x$, equals the increase in stored energy. We have

$$F_x \Delta x = \Delta W_s = \frac{1}{2} \epsilon \left( \frac{V}{d} \right)^2 Ld \Delta x - \frac{1}{2} \epsilon_0 \left( \frac{V}{d} \right)^2 Ld \Delta x + \frac{\partial W_1}{\partial x} \Delta x$$

$$= \frac{1}{2} E^2 (\epsilon - \epsilon_0)Ld \Delta x \quad (3.82)$$

since the correction term $W_1$ does not change if the ends of the dielectric slab are not too close to the plate edges, i.e., if the fringing field remains constant. The direction of the force is such as to draw the dielectric slab farther into the air gap of the capacitor since $\epsilon > \epsilon_0$ always. The pressure $P_x$, in the $x$ direction, is

$$P_x = \frac{F_x}{Ld} = \frac{1}{2} (\epsilon - \epsilon_0) E^2 \quad (3.83)$$

and is the force per unit area exerted by the field on the plane end of the slab.
3.8. Electrostatic Forces (Maxwell Stress Tensor)

The total force acting on a volume distribution of charge \( \rho \), which lies in an external electric field \( \mathbf{E} \), is given by

\[
\mathbf{F} = \int \mathbf{F}_s \, dV
\]

(3.84)

where the volume density of force is

\[
\mathbf{F}_s = \rho \mathbf{E}
\]

(3.85)

As expressed by (3.84) and (3.85), the point of view that the force arises from an action at a distance is still manifest, even though the sources of the \( \mathbf{E} \) field do not enter the calculation explicitly. The formulation is actually hybrid since it involves both fields and charge. According to the Faraday-Maxwell field theory, it should be possible to evaluate forces completely in terms of the field alone; that is, if the field is the means whereby forces are transmitted, one should be able to determine these forces by considering only the field. One can think of the transmission of forces via the field in the same way as one views that transmitted by a stretched rubber band. Indeed, one may view the flux lines as being elastic, the force action being a consequence of their state of stress. This implies that if a charge distribution in an external field is surrounded by an arbitrary surface area, then the total force acting on the charge contained must, in a sense, cross this area. But then it should be possible to calculate the total force by integrating over the arbitrary surface a "stress function" that depends, at each point, only on the field at that point and the direction of the surface area. Put mathematically, it suggests the existence of a force \( \mathbf{T} \, dS \) which acts on an element of an arbitrary closed surface that contains charge and which satisfies the relation

\[
\int_V \mathbf{F}_s \, dV = \int_S \mathbf{T} \, dS
\]

(3.86)

The existence of a function \( \mathbf{T} \) was asserted by Maxwell and forms an important part of the field concept.

We proceed now to show that the surface force \( \mathbf{T} \) does exist and to derive an expression for it. This will require some vector manipulation so that the volume integral of (3.86) may be transformed into surface integrals. We begin by substituting for \( \mathbf{F}_s \) the value given in (3.85) and note that \( \nabla \cdot \mathbf{D} = \rho \); thus

\[
\int_V \mathbf{F}_s \, dV = \int_V \rho \mathbf{E} \, dV = \int_V \mathbf{E} \nabla \cdot \mathbf{D} \, dV
\]

If \( \mathbf{E} \) is now expanded into rectangular components, we obtain

\[
\int_V \mathbf{F}_s \, dV = a_x \int_V E_x \nabla \cdot \mathbf{D} \, dV + a_y \int_V E_y \nabla \cdot \mathbf{D} \, dV + a_z \int_V E_z \nabla \cdot \mathbf{D} \, dV
\]

Using vector identity (1.118), each term may be written as

\[
\int_V \mathbf{F}_s \, dV = a_x \int_V \nabla \cdot E_x \mathbf{D} \, dV + a_y \int_V \nabla \cdot E_y \mathbf{D} \, dV + a_z \int_V \nabla \cdot E_z \mathbf{D} \, dV
\]

\[
- a_x \int_V \mathbf{D} \cdot \nabla E_x \, dV - a_y \int_V \mathbf{D} \cdot \nabla E_y \, dV - a_z \int_V \mathbf{D} \cdot \nabla E_z \, dV
\]

(3.87)

The first three right-hand terms of (3.87) can be integrated by Gauss' theorem, and then summed, whereupon the sum becomes equal to \( \oint_{\mathbf{E} \cdot (n \cdot \mathbf{D})} \, dS \). Concerning the
remaining three terms, let us consider the coefficient of \( a_z \) in some detail. Expanding in rectangular coordinates we obtain

\[
a_z \int_V \mathbf{D} \cdot \nabla E_z \, dV = a_z \int_V \left( D_x \frac{\partial E_z}{\partial x} + D_y \frac{\partial E_z}{\partial y} + D_z \frac{\partial E_z}{\partial z} \right) dV
\]

But since \( \nabla \times \mathbf{E} = 0 \), it follows that \( \partial E_z/\partial y = \partial E_x/\partial z \) and \( \partial E_z/\partial z = \partial E_y/\partial x \). Accordingly,

\[
a_z \int_V \mathbf{D} \cdot \nabla E_z \, dV = a_z \int_V \left( D_x \frac{\partial E_z}{\partial x} + D_y \frac{\partial E_z}{\partial x} + D_z \frac{\partial E_z}{\partial z} \right) dV
\]

\[
= a_z \varepsilon_0 \int_V \frac{\partial}{\partial x} (E_x^2 + E_y^2 + E_z^2) \, dV
\]

(3.88)

A similar expression is obtained for the \( a_y \) coefficient and the \( a_z \) coefficient, except that \( \partial/\partial x \) is replaced by \( \partial/\partial y \) and \( \partial/\partial z \), respectively. Equation (3.87) may now be written as

\[
\int_V \mathbf{F}_s \, dV = \oint_S \mathbf{E} (\mathbf{n} \cdot \mathbf{D}) \, dS - \frac{\varepsilon_0}{2} \int_V \nabla (E_z^2) \, dV
\]

(3.89)

The volume integral on the right-hand side can be converted to a surface integral using vector identity (1.127) (also see Prob. 3.10). As a result,

\[
\int_V \mathbf{F}_s \, dV = \oint_S \varepsilon_0 \left[ (\mathbf{n} \cdot \mathbf{E}) \mathbf{E} - \frac{E_z^2}{2} \mathbf{n} \right] dS
\]

(3.90)

We can now identify (3.86) with (3.90), and therefore

\[
\mathbf{T} = \varepsilon_0 \left[ (\mathbf{n} \cdot \mathbf{E}) \mathbf{E} - \frac{E_z^2}{2} \mathbf{n} \right]
\]

(3.91)

Some idea of the vector relationships involved between \( \mathbf{T}, \mathbf{E}, \) and \( \mathbf{n} \) can be obtained from Fig. 3.22. An element of surface is illustrated, and this determines the direction of \( \mathbf{n} \), of course. The field \( \mathbf{E} \) is illustrated as making an arbitrary angle \( \theta \) with \( \mathbf{n} \). The surface force \( \mathbf{T} \) is described by (3.91). From (3.91) we see that \( \mathbf{T} \) has a component in the direction of \( \mathbf{n} \) which is either a compression or tension force, while that in the direction \( \mathbf{E} \) involves a shearing force, in general. Furthermore, \( \mathbf{T} \) must lie in the plane determined by \( \mathbf{E} \) and \( \mathbf{n} \). If the components of \( \mathbf{T} \) in the direction normal (\( T_n \)) and tangential (\( T_t \)) to the surface are computed, we find

\[
T_n = \varepsilon_0 \left( E_z^2 \cos^2 \theta - \frac{E_z^2}{2} \right) = \frac{\varepsilon_0 E_z^2}{2} \left( 2 \cos^2 \theta - 1 \right)
\]

\[
= \frac{\varepsilon_0 E_z^2}{2} \cos 2\theta \quad (3.92a)
\]

\[
T_t = \varepsilon_0 E_z^2 \cos \theta \sin \theta = \frac{\varepsilon_0 E_z^2}{2} \sin 2\theta \quad (3.92b)
\]

This clearly requires that \( \mathbf{T} \) have a magnitude \( \varepsilon_0 E_z^2/2 \) and make an angle \( 2\theta \) with \( \mathbf{n} \), as illustrated in Fig. 3.22.

To summarize, the force exerted on a charged region can be computed by integrating the surface force \( \mathbf{T} \) over any bounding surface. At each surface element, \( \mathbf{E} \) bisects the angle between \( \mathbf{T} \) and the surface normal. Where \( \mathbf{E} \) and \( \mathbf{n} \) are in the same direction, the force is pure tension. For \( \theta = 45^\circ \), the stress transmitted is pure shearing stress. As

![Fig. 3.22. Relation between \( \mathbf{n} \), \( \mathbf{E} \), and \( \mathbf{T} \) at a plane surface.](image)
\( \theta \) increases further, the shearing stress diminishes until at \( \theta = 90^\circ \), it is pure pressure. The magnitude of \( T \) remains equal to \( \epsilon_0 E^2/2 \) and is independent of the angle between \( E \) and \( n \).

It is also possible to write
\[
\int_S T \, dS = \int_S \tau \cdot dS
\]
(3.93)
in which case \( \tau \) has the properties of a tensor. Maxwell actually expressed his results in this form; consequently \( \tau \) is referred to as the Maxwell stress tensor.

The result expressed by (3.91) has been derived only for forces involving charges in free space. If dielectric bodies are also present, then the volumetric force is no longer given by the simple expression (3.85) at every point. It turns out, however, that if the surface of integration lies completely in free space, then (3.90) correctly evaluates the net force exerted on enclosed dielectric bodies in addition to that exerted on enclosed true charge. This result is not surprising since we know that the external behavior of a dielectric body can be interpreted in terms of an equivalent true charge distribution (polarization charge).

The full meaning of the surface force \( T \) may be more apparent if we consider some examples. First let us note how it could have been used to determine the force between the charged parallel plates illustrated in Fig. 3.20. To find the force on the upper plate, a rectangular parallelepiped may be envisioned surrounding that plate, as shown in Fig. 3.23. Since \( E \) lies only between the plates, a nonzero value of \( E \) appears only at the lower surface of the parallelepiped. In view of the uniformity of \( E \),
\[
\int_S T \, dS = -\frac{\epsilon_0 E^2 a_z}{2} \int_S dS = -\frac{\epsilon_0 E^2}{2} A a_z
\]
and the result of (3.73) is given at once.

In Fig. 3.24 we have two equal and opposite charges separated by a distance \( 2h \). The force exerted on the negative charge will be found by integrating \( T \, dS \) over an infinite plane surface (closed at infinity). The line joining \(-q\) and \(+q\) is bisected by the surface and is perpendicular to it. The geometry is then precisely that in Fig. 2.18; consequently, the electric field over the plane is given by (2.66). Since \( E \) is always normal to the surface and in the negative \( z \) direction, the force exerted on \(-q\)
is in the positive \( z \) direction. Its magnitude is given by

\[
F_{-q} = \frac{\varepsilon_0}{2} \left( \frac{q^2 h}{4 \pi \varepsilon_0} \right)^2 \int_0^\infty \int_0^{2\pi} \frac{r \, dr \, d\theta}{(h^2 + r^2)^{3/2}} \int_0^\infty \frac{1}{4(h^2 + r^2)^2} \, dr 
\]

\[
= \frac{q^2}{4\pi \varepsilon_0 (2h)^3} 
\]

(3.94)

We note that the final result correctly corresponds to Coulomb's law of force.

### 3.9. Molecular Fields

In Sec. 3.1 some of the details of the polarization of dielectric materials were discussed. Basically, we noted that under the influence of an applied electric field the molecules of a dielectric would become polarized. Further, the dipole polarization was noted to be, in general, proportional to the applied field for field strengths well below the saturation level.

In order to actually calculate the polarization of a given material, it is necessary to know the polarizability of the material and the field. The latter must be the field that actually exists at the molecule. For gases, this is to a good approximation the applied field, i.e., the field that exists in the absence of the gas. Actually, the field at a particular molecule is the superposition of the applied field and that due to its neighboring polarized molecules. But for a gas, the separation of the molecules is sufficiently great so that the effect of adjacent dipoles on one another can be neglected.

For a solid dielectric, dipole-dipole interaction must be considered. In general, this may be fairly difficult to express analytically. We shall consider in some detail a specific, and common, configuration to illustrate the principle. We shall assume that the dielectric consists of a cubical array of atoms (or molecules) with spacing \( d \) between each atom along the \( x, y, z \) axis, as in Fig. 3.25. The medium is assumed infinite in extent, and a field \( E_0 \) is applied in the \( z \) direction. Under the influence of the field \( E_0 \), each atom becomes polarized with a dipole moment \( p \). Since there are \( N = 1/d^3 \) atoms per unit volume, the dipole moment per unit volume is \( p/d^3 = P \). The field acting to polarize the atom at the origin (or any other atom) is the sum of the applied field \( E_0 \) plus an interaction field \( E_i \). The interaction field is the field produced by all the neighboring dipoles. The polarizing field \( E_p \) may be written as

\[
E_p = E_0 + E_i 
\]

(3.95)

and hence the induced dipole moment in each atom is

\[
p = \alpha \varepsilon_0 
\]

(3.96)

where \( \alpha \) is the polarizability of the atom. When the material behaves linearly, \( p, E_0, \) and \( E_p \) are all proportional to the applied field \( E_0 \), and hence we may take \( E_i \) propor-
tional to \( p \). Thus

\[
E_i = Cp
\]

(3.97)

where \( C \) is called the field interaction constant. Equation (3.96) now becomes

\[
p = \alpha_e (E_0 + E_i) = \alpha_e E_0 + \alpha_e Cp
\]

or

\[
p = \frac{\alpha_e E_0}{1 - \alpha_e C}
\]

(3.98)

During the course of the analysis it will be shown that the average total electric field \( E \) in the dielectric is equal to the applied field \( E_0 \). Hence the displacement flux density \( D \) is given by

\[
D = \varepsilon E + P = \varepsilon_0 E_0 + \frac{N\alpha_e E_0}{1 - \alpha_e C} = \varepsilon E
\]

(3.99)

where \( N = d^{-3} \) is the number of dipoles per unit volume. Solving for the permittivity \( \varepsilon \) gives

\[
\varepsilon = \varepsilon_0 + \frac{N\alpha_e}{1 - \alpha_e C}
\]

(3.100)

a result known as the Clausius-Mossotti equation. Assuming that \( \alpha_e \) is known and that the interaction constant \( C \) can be found, we can compute \( \varepsilon \) from (3.100). The interaction constant \( C \) is a function of the lattice geometry only.

**Evaluation of Interaction Constant**

The interaction constant \( C \) may be found by evaluating the field at the origin as produced by all the neighboring dipoles in the lattice illustrated in Fig. 3.25. From symmetry considerations it is readily seen that this field has a \( z \) component only. The scalar potential due to a dipole of moment \( p \) and located at \((x',y',z')\) is

\[
\Phi(x,y,z) = \frac{p}{4\pi\varepsilon_0} \frac{\cos \theta}{\sqrt{R^2}} = \frac{p(z - z')}{4\pi\varepsilon_0 R^2}
\]

since \( z - z' = R \cos \theta \), and \( R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 \). The \( z \) component of electric field is

\[
E_z = -\frac{\partial \Phi}{\partial z} = -\frac{p}{4\pi\varepsilon_0} \left[ \frac{1}{R^2} - \frac{3(z - z')^2}{R^4} \right]
\]

\[
= \frac{p}{4\pi\varepsilon_0} \left[ \frac{2(z - z')^2 - (x - x')^2 - (y - y')^2}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \right]
\]

(3.101)

Now the dipoles are located at \( x' = nd, y' = md, z' = sd \) \((n, m, s = 0, \pm 1, \pm 2, \ldots, \) but \( n = m = s = 0 \) excluded). Summing (3.101) over all the dipoles and placing \( x = y = z = 0 \), we obtain the following expression for the interaction field at the origin:

\[
E_i = \frac{p}{4\pi\varepsilon_0} \sum_{n = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} \sum_{s = -\infty}^{\infty} \frac{2(sd)^2 - (nd)^2 - (md)^2}{[(sd)^2 + (nd)^2 + (md)^2]^{3/2}}
\]

(3.102)

where the prime means omission of the term \( n = m = s = 0 \), which corresponds to the dipole at the origin.

The above series may be transformed into a rapidly converging series by use of the Poisson summation formula, and the sum may then be evaluated. For our purpose we shall approximate the sum by a triple integral. Let us introduce the variables
$x = n$, $y = m$, $z = s$ and the differentials $dx = dn$, $dy = dm$, $dz = ds$. The integration extends from $|x|$, $|y|$, $|z|$ equal to unity out to infinity. Since all quantities are squared, we may integrate over one octant only and multiply the result by 8. We now have

$$E_i = \frac{8p}{4\pi \varepsilon_0 d^3} \int_1^\infty \int_1^\infty \int_1^\infty \frac{2z^2 - x^2 - y^2}{(z^2 + y^2 + z^2)^{3/2}} \, dx \, dy \, dz$$

Since the integrand was originally obtained by differentiating with respect to $z$, we may integrate with respect to $z$ at once to get

$$E_i = \frac{2p}{\pi \varepsilon_0 d^3} \int_1^\infty \int_1^\infty \int_1^\infty \frac{-z}{(z^2 + y^2 + z^2)^{3/2}} \, dx \, dy \, dz$$

With the substitution $y = (x^2 + 1)^{1/2} \tan \theta$ and the limits $\sin^{-1} \left(2 + x^2\right)^{-1/2}$ to $\pi/2$ on $\theta$, the integration over $y$ is readily performed (the integrand becomes simply $\cos \theta \, d\theta$) to give

$$E_i = \frac{2p}{\pi \varepsilon_0 d^3} \int_1^\infty \left[ \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)(x^2 + 2)^{1/2}} \right] \, dx$$

The first term gives

$$\int_1^\infty \frac{dx}{1 + x^2} = \tan^{-1} x \bigg|_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

In the second term replace $x^2 + 1$ by $u^2$ to get

$$- \int_{\sqrt{2}}^{\infty} \frac{du}{u(u^4 - 1)^{1/2}} = -\frac{1}{2} \sec^{-1} u \bigg|_{\sqrt{2}}^\infty = -\frac{\pi}{4} + \frac{\pi}{6}$$

The final value for $E_i$ is thus

$$E_i = \frac{2p}{\pi \varepsilon_0 d^3} \left( \frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{4} \right) = \frac{p}{3 \varepsilon_0 d^3} \tag{3.103}$$

From (3.97) we find that the interaction constant $C$ is

$$C = \frac{E_i}{p} = \frac{1}{3 \varepsilon_0 d^3} = \frac{N}{3 \varepsilon_0} \tag{3.104}$$

If the series (3.102) is summed, it is found that the interaction constant $C$ is equal to $0.34/d^3 \varepsilon_0$, and hence we can conclude that the triple integral is a good approximation to the triple sum.

If we use the above value of $C$ in (3.100), we find that the permittivity is given by

$$\varepsilon = \varepsilon_0 + \frac{N \alpha_s}{1 - N \alpha_s / 3 \varepsilon_0}$$

or

$$\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2 \varepsilon_0} = \frac{N \alpha_s}{3 \varepsilon_0} \tag{3.105}$$

For a gas, $\varepsilon$ is only slightly greater than $\varepsilon_0$, so that (3.105) becomes

$$\frac{\varepsilon - \varepsilon_0}{3 \varepsilon_0} = \frac{N \alpha_s}{3 \varepsilon_0}$$

or

$$\varepsilon = \varepsilon_0 + N \alpha_s \tag{3.106}$$
which, upon comparison with (3.100), is seen to be equivalent to neglecting the interaction field. By combining (3.104) and (3.95) we find that the polarizing field is given by

$$E_p = E_0 + C_p = E_0 + \frac{P}{3\varepsilon_0}$$

(3.107)

As a final step we shall show that the average electric field in the dielectric is equal to the applied field $E_0$. To show this we must evaluate the average field produced by all the dipoles. The average is to be taken over the volume of a unit cell. We have already found a value for the interaction field, which is the field set up by all the dipoles except one. If we add to this the average field set up by the individual dipole, we shall obtain the average field produced by all the dipoles in the lattice. At this point it should be noted that replacing the triple sum in (3.102) by a triple integral is equivalent to determining an average value $E_i$ for the interaction field.

The $z$ component of the field produced by a single dipole, say the dipole at the origin, is given by (3.101) with $x', y'$, and $z'$ equal to zero. The average value of this field is $E_{1z}$, where

$$E_{1z} = \frac{8p}{4\pi\varepsilon_0 d^3} \int_0^{d/2} \int_0^{d/2} \int_0^{d/2} \frac{2x^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{3/2}} \, dx \, dy \, dz$$

(3.108)

In (3.108) the integration is taken over one octant of the unit cell, which has a volume of $d^3/8$. This is equivalent to averaging over the unit cell because of the symmetry of the integrand. The above may be integrated in the same manner as the integral for $E_i$ was performed. The reader may readily verify by carrying out the details that the end result is simply

$$E_{1z} = -\frac{p}{3\varepsilon_0 d^3}$$

(3.109)

This result is interesting in that it shows that the average field produced by a single dipole in its own unit cell is the negative of the average field produced by all the neighboring dipoles. Consequently, the average field produced by all the dipoles in the lattice is $(E_d)_{av}$ and vanishes since

$$(E_d)_{av} = E_i + E_{1z} = \frac{p}{3\varepsilon_0 d^3} - \frac{p}{3\varepsilon_0 d^3} = 0$$

This completes the proof that the average field in the dielectric is just equal to the applied field $E_0$.

The above analysis clearly shows that the effective field acting to polarize a given molecule is not equal to $E_0$, for this is the average field in the dielectric and includes the contribution from the molecule itself. The field acting to polarize the molecule is equal to $E_0$ minus the contribution to the total average field from the molecule itself. This gives $E_0 - (-p/3\varepsilon_0 d^3) = E_0 + p/3\varepsilon_0 d^3$ for the polarizing field, a result in agreement with (3.107).
Chapter 3

3.1. (a) If an arbitrarily shaped dielectric of volume $V$ is placed in an electric field, a dielectric polarization $\mathbf{P}$ results which is also equivalent to a charge density $-\nabla \cdot \mathbf{P}$ and a surface charge density $\mathbf{P} \cdot \mathbf{n}$. Since the dielectric is electrically neutral, the total induced charge must equal zero. Show this by making use of the divergence theorem.

(b) Consider a specific example where the body is a rectangular parallelepiped whose axis extends from $z = -l/2$ to $z = l/2$ and with a cross-sectional area $A$. Given that $\mathbf{P} = (Az^2 + B)a_z$, determine the volume and surface charge density and show explicitly that the total charge is zero.

3.2. Consider a parallel-plate capacitor with sides $a$, $b$, and spacing $d$. The capacitor is half-filled with dielectric (0 to $a/2$, 0 to $b$) of relative dielectric constant $\varepsilon$. A potential $V$ exists between the plates. Calculate the charge density on the plates and also the equivalent surface polarization charge on the dielectric surfaces. Neglect fringing effects.
3.3. A solid dielectric cylinder of length \( L \) and radius \( a \) is uniformly polarized with polarization \( \mathbf{P} \), where \( \mathbf{P} \) is directed axially. Determine the electric field along the cylinder axis both within and outside the cylinder.

3.4. For a field strength of \( 3 \times 10^6 \) volts per meter determine the relative displacement and nucleus and electron cloud for He and Ne. Compare with the radius of the atom.

3.5. An infinite dielectric slab of thickness \( t \) is placed in a uniform external field \( \mathbf{E}_0 \). The slab is inclined at an angle \( \theta_1 \) to the field \( \mathbf{E}_0 \). Find the angle \( \theta_1 \) such that the electric lines of flux in the slab make an angle \( \theta_2 = \pi/4 \) with the sides of the slab, i.e., so that \( \theta_2 = \pi/4 \). The dielectric constant is \( \kappa = 4 \). Find the density of surface polarization charge on the two faces of the slab.

3.6. (a) Consider two coaxial cylinders of inner radius \( a \) and outer radius \( b \), as illustrated. The space between cylinders is filled with a dielectric with permittivity \( \varepsilon = \kappa \varepsilon_0 \). Find the capacitance per meter length of cylinders.

(b) For a given difference of potential between the inner and outer conductors determine the magnitude of \( b/a \) that causes the greatest value of \( \mathbf{E} \) to be a minimum under the restriction that the radius \( b \) is fixed.

3.7. Consider two coaxial cylinders, the inner having a radius \( a \) and the outer a radius \( c \). The space \( a < r < b \), where \( b < c \), is filled with a dielectric with a dielectric constant \( \kappa \). Find the capacitance per unit length. Show that the capacitance \( C \) is equal to the series capacitance of \( C_1 \) and \( C_3 \), where \( C_1 \) is given in Prob. 3.6 and \( C_2 \) is the capacitance of an air-filled coaxial cylinder capacitor with inner radius \( b \) and outer radius \( c \).

3.8. Consider two coaxial cylinders with the intervening space \( 0 < \theta < \theta_1 \) filled with a dielectric with a dielectric constant \( \kappa \). Find the capacitance per unit length,
and show that this capacitance is equal to the parallel combination of the capacitances of the air-filled section and the dielectric-filled section.

**HINT:** Note that the field \( E \) is independent of \( \theta \) and depends only on the potential difference between the cylinders.

**3.9.** A coaxial cable consists of a copper inner conductor of 0.5 centimeter radius, a solid dielectric of permittivity 4.8 to a radius \( r_o \), and a layer of oil (\( \kappa = 2.4 \)) from \( r_o \) to a radius of 2.0 centimeters. The dielectric strength of the solid dielectric is \( E_d = 40 \times 10^4 \) kilovolts per meter, while that for the oil is \( E_o = 30 \times 10^4 \) kilovolts per meter. Find the value of \( r_o \) and the value of potential that will result in maximum stress in both media.

**3.10.** Show that

\[
\int_V \nabla E^2 \, dv = \oint_S E^2 \mathbf{n} \, dS
\]

**HINT:** Consider \( \int_V \nabla \cdot (\Psi \mathbf{A}) \, dv \), where \( \mathbf{A} \) is a constant vector and \( \Psi = E^2 \), and utilize the divergence theorem.

**3.11.** Prove that the following relationship exists between the polarization \( \mathbf{P} \) and equivalent volume and surface charge densities \( \rho_P \) and \( \rho_{sp} \) for an arbitrary volume \( V \):

\[
\int_V \mathbf{P} \, dv = \int_V \rho_P \, dv + \int_S \rho_{sp} \, dS
\]

In the above, \( r \) is the position vector from an arbitrary origin, and \( r = a_x x + a_y y + a_z z \).

**HINT:** Note that \( \int_V x \nabla \cdot \mathbf{P} \, dv = \int_V \nabla \cdot (x \mathbf{P}) \, dv - \int_V \mathbf{P} \cdot \nabla x \, dv \), and similarly for \( y \) and \( z \).

**3.12.** Consider an arbitrary distribution of point charges \( q_i \) \((i = 1, 2, \ldots, n)\), and let the potential at the \( i \)th charge due to all others be given by \( \Phi_i \); that is,

\[
\Phi_i = \sum_{j=1}^{j=n} \frac{q_j}{4\pi \varepsilon r_{ij}}
\]

where \( r_{ij} \) is the separation of the \( i \)th and \( j \)th charge. Consider another system of charges located at the same points but with magnitudes \( q'_j \) \((i = 1, 2, 3 \ldots, n)\) and corresponding to which the potential at the \( i \)th point is \( \Phi_i' \), where

\[
\Phi_i' = \sum_{j=1}^{j=n} \frac{q'_j}{4\pi \varepsilon r_{ij}}
\]

Show that

\[
\sum_{i=1}^{i=n} q_i \Phi_i = \sum_{i=1}^{i=N} q'_i \Phi_i
\]

This result is known as Green's reciprocity theorem.

**3.13.** (a) Given \( N \) conducting bodies where under one set of conditions the charge on the \( i \)th body is \( Q_i \) and for another it is \( Q'_i \). Corresponding to each charge system
the potential of the \( i \)th body is \( \Phi_i \) and \( \Phi_i' \), respectively. Show that

\[
\sum_{i=1}^{N} Q_i \Phi_i' = \sum_{i=1}^{N} Q_i' \Phi_i
\]

**Hint:** Subdivide each body into infinitesimal elements, and apply Green's reciprocity theorem developed in Prob. 3.12.

(b) Using the result in (a) and with \( Q_i' = 1, \ Q_i' = 0 \ (i \neq j) \), confirm the formulation of (3.42). What physical significance can be attached to the coefficients of potential?

3.14. A spherical charge distribution of radius \( R \) is uniform and has a charge density \( \rho_b \). Calculate the self-energy from (3.60) and from (3.64).

3.15. A dielectric spherical shell has an inner radius \( r_i \), an outer radius \( r_o \), and a permittivity \( \epsilon \). What net energy is required to move it from infinity to a point where it is concentric with a point charge \( Q \)?

3.16. A variable capacitor is constructed from two coaxial cylinders as illustrated. The inner cylinder is solid and free to slide in an insulating bushing. Find the magnitude and direction of the force acting to displace the center cylinder when a potential difference \( V \) is maintained between the cylinders. What is the magnitude of the force when \( b = 2a = 1 \) centimeter and \( V = 1,000 \) volts?

**Hint:** Find the rate of change of the energy stored in the capacitor when \( l \) is varied.

3.17. In Prob. 3.16 let the variable capacitor be charged so that the total charge on the inner conductor is \( Q \). The battery is now disconnected. What are the magnitude and direction of the force in this case?

3.18. The capacitance of an air capacitor changes linearly from 25 to 350 microfarads during a rotation from 0 to 180°. When set at \( \theta \) \((0 < \theta < \pi)\), what is the electrostatic torque if a voltage of 400 volts is applied between the plates?

3.19. A homogeneous field \( \mathbf{E} \) exists within an infinite dielectric medium of permittivity \( \epsilon \). Find \( \mathbf{E} \) and \( \mathbf{D} \) in a hollow cavity within the dielectric if the cavity shape is

(a) A long thin cylinder parallel to \( \mathbf{E} \).

(b) A thin flat plate whose broad dimension is perpendicular to \( \mathbf{E} \).

(c) A sphere.

3.20. The dielectric constant of hydrogen gas at 0°C and atmospheric pressure is 1.000264.

(a) Compute the polarizability of the hydrogen molecule.

(b) Assuming ideal-gas laws apply, compute the dielectric constant for a pressure of 15 atmospheres and -150°C. (Neglect interaction effects.)

3.21. A dielectric sphere of radius \( a \) is placed in a uniform electric field. Find the resultant field inside and outside of the sphere.
**PROBLEMS**

**Hint:** The scheme outlined for the conducting sphere given for Prob. 2.15 may be utilized here. Continuity of \( \Phi \) and normal \( \mathbf{D} \) across the boundary will serve to determine the coefficients.

**3.22.** Consider two conducting bodies with an initial charge \( Q_i \), difference of potential \( V \), and capacitance \( C_i \). If the two bodies are displaced so that the new capacitance is \( C_2 \), compute the work done on the bodies by the field if

(a) \( V \) is kept constant (i.e., consider a battery connected between the two bodies).

(b) \( Q_i \) is kept constant (bodies isolated, no battery).

In each case account for the change in energy stored in the field. Note that the above process is not a virtual one, but involves finite displacements and finite-energy interchange.

**3.23.** Consider \( N \) conducting bodies at potential \( \Phi_i \), with total charge \( Q_i \) on the \( i \)th body, \( i = 1, 2, \ldots, N \). Prove that the charge distributes itself in such a way that the energy stored in the electrostatic field is a minimum (Thomson's theorem).

**Hint:** Let \( \Phi \) be the potential for the correct charge distribution and let \( \Delta \Phi \) be the change in potential when the charge distribution is perturbed by a small amount on each body, and consider the integral

\[
W_* = \frac{\varepsilon}{2} \int_V [\nabla(\Phi + \Delta \Phi)]^2 \, dV
\]

Show that

\[
\int_V (\nabla \Phi) \cdot (\nabla \Delta \Phi) \, dV
\]

vanishes since each body is an equipotential surface and the total charge is kept constant. Thus \( W_* \) is a minimum if the positive term

\[
\int_V [\nabla(\Delta \Phi)]^2 \, dV
\]

is zero, i.e., if \( \Delta \Phi = 0 \).

**3.24.** A uniform line charge of strength \( q_l \) coulombs per meter is placed a distance \( h \) above the plane surface of a semi-infinite dielectric of permittivity \( \varepsilon \). Compute the field everywhere in space.

**Hint:** Confirm and utilize the following image technique. For field points in the free-space region the source is \( q_l \), plus a line charge (image) \( q_l' \) at the image point, both taken as lying in an infinite free-space medium. For points within the dielectric we have \( q_l \), plus \( q_l'' \) superposed, both taken as existing in an infinite medium of permittivity \( \varepsilon \). The relation between \( q_l \) and \( q_l'' \) can be found from the boundary conditions at the interface.