Ch. 5  Cylindrical Waveguides and Cavity Resonators

5.1  Rectangular Waveguides

We assume perfectly conducting waveguide walls which require $E_{\text{tan}} = 0$ and $H_{\text{norm}} = 0$

$$E_x, E_z = 0 \quad \text{at} \quad y = 0 \text{ and } y = b$$

$$H_y = 0 \quad \text{"}$$

$$E_y, E_z = 0 \quad \text{at} \quad x = 0 \text{ and } x = b$$

$$H_x = 0 \quad \text{"}$$

We also want the fields to vary in the $z$-direction as $e^{-\gamma z}$.

Aside from the boundary conditions this is no different than the parallel plate guide and must satisfy the curl equations

$$\nabla \times H = j\omega E$$

$$\nabla \times E = -j\omega \mu H$$

which we already developed for the parallel plate guide.

Using $\frac{\partial}{\partial z} \rightarrow -\gamma$ we have
\[ \frac{\partial H_y}{\partial y} = j\omega eE_x \]
\[ \frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial y} = -j\omega e E_y \]
\[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega e E_z \]
\[ \nabla \times H = j\omega e E \] [3.1]
\[ \frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial x} = -j\omega \mu H_x \]
\[ \frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial y} = j\omega \mu H_y \]
\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \mu H_z \]
\[ \nabla \times E = -j\omega \mu H \]

The wave equations for \(E_z\) and \(H_z\) reduce to
\[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu e E_z \] [5.2]
\[ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu e H_z \]

The transverse field components can be written in terms of
\(E_z\) and \(H_z\):
\[ H_x = -\frac{\gamma}{h^2} \frac{\partial H_z}{\partial x} + j\frac{\omega e}{h^2} \frac{\partial E_z}{\partial y} \]
\[ H_y = -\frac{\gamma}{h^2} \frac{\partial H_z}{\partial y} - j\frac{\omega e}{h^2} \frac{\partial E_z}{\partial x} \] [5.3]
\[ E_x = -\frac{\gamma}{h^2} \frac{\partial E_z}{\partial x} - j\frac{\omega e}{h^2} \frac{\partial H_z}{\partial y} \]
\[ E_y = -\frac{\gamma}{h^2} \frac{\partial E_z}{\partial y} + j\frac{\omega e}{h^2} \frac{\partial H_z}{\partial x} \]

where \( h^2 = \gamma^2 + \omega^2 \mu e \)
Just as for the parallel plate waveguide the field solutions can be classified as

- TE where \( E_z = 0 \)
- TM where \( H_z = 0 \)

For waveguides we write the wave equations using a transverse operator \( \nabla_{tr} \) which can be written as

\[
\nabla_{tr} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}
\]

and

\[
\nabla_{tr}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

The wave equations become

\[
\nabla_{tr}^2 E_z + (\gamma^2 + \omega^2 \mu_e) E_z = 0
\]

\[
\nabla_{tr}^2 H_z + (\gamma^2 + \omega^2 \mu_e) H_z
\]

For TM modes the component equations become

\[
E_x = -\frac{\gamma}{h^2} \frac{\partial E_z}{\partial x}
\]

\[
E_y = -\frac{\gamma}{h^2} \frac{\partial E_z}{\partial y}
\]

\[
E_{tr} = \hat{x} E_x + \hat{y} E_y = -\hat{x} \frac{\gamma}{h^2} \frac{\partial E_z}{\partial x} - \hat{y} \frac{\gamma}{h^2} \frac{\partial E_z}{\partial y}
\]

\[
E_{-tr} = -\frac{\gamma}{h^2} \nabla_{tr} E_z = -\frac{\gamma}{\gamma^2 + \omega^2 \mu_e} \nabla_{tr} E_z
\]
Similarly,

\[ H_x = j \frac{\omega \varepsilon}{\mu} \frac{\partial E_z}{\partial y} \]

\[ H_y = -j \frac{\omega \varepsilon}{\mu} \frac{\partial E_z}{\partial x} \]

\[ H_{tr} = \hat{x} H_x + \hat{y} H_y = j \frac{\omega \varepsilon}{\mu} \hat{x} \frac{\partial E_z}{\partial y} - j \frac{\omega \varepsilon}{\mu} \hat{y} \frac{\partial E_z}{\partial x} \]

\[ = \frac{j \omega \varepsilon}{\mu} \left( \hat{x} \left( \frac{\partial E_z}{\partial y} \right) - \hat{y} \left( -\frac{\partial E_z}{\partial x} \right) \right) \]

\[ = \frac{j \omega \varepsilon}{\mu} \left( \hat{x} \left( \frac{\partial^2 E_y}{\partial y^2} \right) - \hat{y} \left( -\frac{\partial^2 E_x}{\partial x \partial y} \right) \right) \]

\[ = - \frac{j \omega \varepsilon}{\mu} \frac{\partial^2}{\partial y^2} \left( \hat{x} E_y + \hat{y} E_x \right) \]

\[ = - \frac{j \omega \varepsilon}{\mu} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial y} E_x & \frac{\partial}{\partial y} E_y & 0 \\ 0 & 0 & 1 \end{vmatrix} \]

\[ H_{tr} = - \frac{j \omega \varepsilon}{\mu} \left( E_{tr} \times \hat{z} \right) = \frac{j \omega \varepsilon}{\mu} \left( \hat{z} \times E_{tr} \right) \]

We can do the component equations for TE waves the same way.

\[ H_{tr} = \lambda H_x + \mu H_y = - \frac{\mu}{\varepsilon^2 + \mu^2 \omega^2} \nabla_{tr} \times H_z \]

\[ E_{tr} = \lambda E_x + \mu E_y = \frac{j \mu \omega}{\varepsilon} \left( H_{tr} \times \hat{z} \right) \]

where the boundary condition is that \( \hat{n} \cdot H_{tr} = 0 \)

or \( \frac{\partial H_z}{\partial x} = 0 \), \( \frac{\partial H_z}{\partial y} = 0 \)
Transverse Magnetic (TM) modes

We use separation of variables similar to that which we used for the parallel plate waveguide

\[ E_z (x, y, z) = E_0^z (x, y) e^{-\sqrt{\gamma}z} \]

now a function of two variables.

Let \( E_0^z (x, y) = f(x) g(y) \).

The wave equation becomes

\[ \nabla^2_{tr} E_z + (\beta^2 + \omega^2 \mu \epsilon) E_z = 0 \]

\[ \nabla^2_{tr} f g + (\beta^2 + \omega^2 \mu \epsilon) f g = 0 \]

The \( e^{-\sqrt{\gamma}z} \) drops out.

\[ \nabla^2_{tr} f g = g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} \]

\[ \frac{g}{f} \frac{\partial^2 f}{\partial x^2} + \frac{f}{g} \frac{\partial^2 g}{\partial y^2} + \beta^2 f g = 0 \]

where \( \beta^2 = \beta^2 + \omega^2 \mu \epsilon \)

divide by \( g \)

\[ \frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} + \beta^2 = 0 \]

Re-arranging

\[ \frac{1}{f} \frac{d^2 f}{dx^2} + \beta^2 = \frac{1}{g} \frac{d^2 g}{dy^2} \]

Each side must be equal to a constant, call it \( A^2 \), which is determined by the boundary conditions

\[ \frac{1}{f} \frac{d^2 f}{dx^2} + \beta^2 = A^2 \] and \[ \frac{1}{g} \frac{d^2 g}{dy^2} = -A^2 \]
The two equations have similar solutions

\[ f(x) = C_1 \cos (Bx) + C_2 \sin (Bx) \]
\[ \text{where } B = \sqrt{B^2 - A^2} \]

and \[ g(y) = C_3 \cos (Ay) + C_4 \sin (Ay) \]

The complete product solution is \[ E^0_z(x,y) = f(x)g(y) \]

\[ E^0_z(x,y) = C_1 C_3 \cos (Bx) \cos (Ay) + C_1 C_4 \cos (Bx) \sin (Ay) + C_2 C_3 \sin (Bx) \cos (Ay) + C_2 C_4 \sin (Bx) \sin (Ay) \]

The B.C.'s are that \( E^0_z = 0 \) at \( x = 0, x = a \), \( y = 0 \) and \( y = b \)

\[ E^0_z(0,y) = C_1 C_3 \cos (Bx) \cos (Ay) + C_1 C_4 \cos (Bx) \sin (Ay) + C_2 C_3 \sin (Bx) \cos (Ay) + C_2 C_4 \sin (Bx) \sin (Ay) \]

\[ E^0_z(0,y) = C_1 C_3 \cos (Ay) + C_1 C_4 \sin (Ay) \]

For \( E^0_z(0,y) = 0 \) we require \( C_1 = 0 \)

Requiring \( C_3 = C_4 = 0 \) will result in a trivial solution

\[ E^0_z(x,y) = C_2 C_3 \sin (Bx) \cos (Ay) + C_2 C_4 \sin (Bx) \sin (Ay) \]

Now requiring \( E^0_z(x,0) = 0 \)

\[ E^0_z(x,0) = C_2 C_3 \sin (Bx) \cos (Ay) + C_2 C_4 \sin (Bx) \sin (Ay) \]

\[ E^0_z(x,0) = C_2 C_3 \sin (Bx) \]

This requires that either \( C_2 \) or \( C_3 \) equals zero.

We pick \( C_3 = 0 \) since picking \( C_2 = 0 \) would be a trivial solution.
This gives the interim solution
\[ E_z^0 (x, y) = \frac{C_2 C_4}{\text{call this } C} \sin (Bx) \sin (Ay) \]

For \( E_z^0 (a, y) = C \sin (Ba) \sin (Ay) = 0 \)
requires that \( B = \frac{m \pi}{a} \quad m = 1, 2, 3 \)

For \( E_z^0 (x, b) = C \sin (Bx) \sin (Ab) = 0 \)
requires that \( A = \frac{n \pi}{b} \quad n = 1, 2, 3 \).

The final expression is
\[ E_z^0 (x, y) = C \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) \]

The final expressions for the fields are
\[ E_z (x, y, z) = C \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-j \beta_{mn} z} \quad \gamma = j \beta_{mn} \]
\[ E_z (x, y, z, t) = C \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) \cos (wt - \beta_{mn} z) \]

These are propagating waves since \( \gamma = j \beta_{mn} \). We also have the case \( \gamma = \alpha_{mn} \) for evanescent waves. In this case
\[ E_z (x, y, z) = C \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-\alpha_{mn} z} \]
\[ E_z (x, y, z, t) = C \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) \cos (wt) e^{-\alpha_{mn} z} \]
The other field components can be calculated using the component equations. For TM modes, \( H_y = 0 \) so

\[
H_x = \frac{j \omega \varepsilon}{\mu} \frac{\partial E_y}{\partial y}
\]

\[
H_y = -\frac{j \omega \varepsilon}{\mu} \frac{\partial E_x}{\partial x}
\]

\[
E_x = -\frac{\delta}{h^2} \frac{\partial E_y}{\partial x}
\]

and

\[
E_y = -\frac{\delta}{h^2} \frac{\partial E_x}{\partial y}
\]

so, for propagating rectangular TM\(_{mn}\) modes,

\[
E_z = C \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-j \beta_{mn} z}
\]

\[
E_x = \frac{-j \beta_{mn} C}{h^2} \frac{m \pi}{a} \cos \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-j \beta_{mn} z}
\]

since \( \delta = j \beta_{mn} \) derivative

Similarly,

\[
E_y = -\frac{j \beta_{mn} C}{h^2} \frac{n \pi}{b} \sin \left( \frac{m \pi}{a} x \right) \cos \left( \frac{n \pi}{b} y \right) e^{-j \beta_{mn} z}
\]

\[
H_x = \frac{j \omega \varepsilon}{\mu} \frac{C}{a} \sin \left( \frac{m \pi}{a} x \right) \cos \left( \frac{n \pi}{b} y \right) e^{-j \beta_{mn} z}
\]

\[
H_y = -\frac{j \omega \varepsilon}{\mu} \frac{C}{a} \cos \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-j \beta_{mn} z}
\]

where \( m = 1, 2, 3, \ldots \) and \( n = 1, 2, 3, \ldots \)
to find \( \overline{\gamma} = \sqrt{-h^2 - \omega^2 \mu e} \) we note that

\[
\frac{1}{f} \frac{d^2 f}{dx^2} + f^2 = A^2
\]

where \( f(x) = C_2 \sin(Bx) \)  

\[ A^2 = \rho^2 \left( \frac{d^2 f}{dx^2} \right) + C_2 (-B^2) \sin(Bx) \]

\[ \rho^2 = \rho^2 + B^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \]

Then knowing \( \rho^2 \)

\[ \overline{\gamma} = \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 - \omega^2 \mu e} \]

We see that \( \overline{\gamma} \) corresponds to a propagating wave only when \( \overline{\gamma} \) is imaginary, i.e., \( \omega > \omega_{cmn} \)

\[
\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 - \omega^2 \mu e = 0
\]

\[ \omega_{cmn}^2 = \frac{1}{\mu e} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] \]

\[ \omega_{cmn} = \frac{1}{\mu e} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]

for \( \omega > \omega_{cmn} \)

\[
\overline{\gamma} = j \overline{\beta}_{mn}
\]

\[ j \overline{\beta}_{mn} = j \sqrt{\omega^2 \mu e - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

\[ \overline{\beta}_{mn} = \sqrt{\omega^2 \mu e - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

\[ \overline{\beta}_{mn} = \beta \sqrt{1 - \left( \frac{f_{cmn}}{\rho} \right)^2} \]

where \( \beta = \omega \sqrt{\mu e} \) and \( f_{cmn} = \frac{\omega_{cmn}}{2\pi} \)
You can get corresponding expression for \( \lambda_{c_{mn}} \)

\[
\lambda_{c_{mn}} = \frac{\nu_p}{f_{c_{mn}}} = \frac{1}{\nu_p e \omega_{c_{mn}} \frac{2\pi}{f}}
\]

\[
\lambda_{mn} = \frac{2\pi}{\nu_p e} \left( \frac{1}{\sqrt{\frac{1}{(m\pi)^2} + \left(\frac{n\pi}{b}\right)^2}} \right) = \frac{2}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}
\]

For propagating waves

\[
\overline{V}_{pmn} = \frac{\omega}{\beta_{mn}} = \frac{\omega}{\beta \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{1}{\nu_p e \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}
\]

\[
\overline{\lambda}_{mn} = \frac{2\pi}{\beta_{mn}} = \frac{2\pi}{\sqrt{\omega^2 e - \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2}} = \frac{\beta \lambda}{\beta \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}
\]

\[
\overline{\lambda}_{mn} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}
\]

We can also define a wave impedance

\[
Z_{T_{mn}} = \frac{E_x}{H_y} = \frac{E_x^0 e^{-j\beta z}}{H_y e^{-j\beta z}} = \frac{E_x^0}{H_y}
\]

\[
Z_{T_{mn}} = \left( -j \frac{\beta_{mn}}{\sqrt{\omega e}} \mathcal{C} \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) - j \frac{\omega e}{\sqrt{\omega e}} \mathcal{C} \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \right) = \frac{\beta_{mn}}{\sqrt{\omega e}}
\]

\[
Z_{T_{mn}} = \frac{\beta \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}{\sqrt{\omega e}} = \frac{\omega \sqrt{\nu_p e \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}}{\sqrt{\omega e}}
\]

\[
Z_{T_{mn}} = \frac{\nu_p e \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}{\sqrt{\omega e}}
\]
Transverse Electric (TE) modes \( E_z = 0 \)

\[
H_z = C \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) e^{-j\beta_{mn} z}
\]

from which we can derive

\[
H_x = \frac{j \beta_{mn}}{\mu} C \frac{m\pi}{a} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) e^{-j\beta_{mn} z}
\]

\[
H_y = \frac{j \beta_{mn}}{\mu} C \frac{n\pi}{b} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{-j\beta_{mn} z}
\]

\[
E_x = \frac{j \omega n}{\epsilon} C \frac{n\pi}{b} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{-j\beta_{mn} z}
\]

\[
E_y = -\frac{j \omega n}{\epsilon} C \frac{m\pi}{a} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) e^{-j\beta_{mn} z}
\]

The formulas for \( \omega_{mn} \), \( \beta_{mn} \), etc. are identical.

One different formula is that for impedance

\[
Z_{TE_{mn}} = \frac{\gamma}{\sqrt{1 - \left( \frac{\omega_{mn}}{\omega} \right)^2}} = -\frac{E_y}{H_x}
\]
A very important mode is the TE\textsubscript{10} mode (a>b)

\[ H_x = C \cos \left( \frac{\pi x}{a} \right) e^{-j\beta_{10} z} \]

\[ H_x = \frac{j \beta_{10} a C}{\pi} \sin \left( \frac{\pi x}{a} \right) e^{-j\beta_{10} z} \]

\[ E_y = -\frac{j \omega \mu a C}{\pi} \sin \left( \frac{\pi x}{a} \right) e^{-j\beta_{10} z} \]

\[ E_x = H_y = 0 \]

\[ \beta_{10} = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{\pi}{a} \right)^2} = \sqrt{\left( \frac{2\pi}{\lambda} \right)^2 - \left( \frac{\pi}{a} \right)^2} \]

\[ \lambda_{10} = \frac{2\pi}{\beta_{10}} = \frac{\lambda}{\sqrt{1 - \left( \frac{\lambda}{2a} \right)^2}} \]

\[ f_{c,10} = \frac{1}{2a\sqrt{\mu \varepsilon}} \]

If propagation at a specified frequency is not possible in the TE\textsubscript{10} mode it is NOT possible for any mode.
5.1.2

We have not talked about how to couple power into particular modes.

The practice is to use a probe (source) that will produce lines of E and H that are roughly parallel to the lines of E and H for that mode, and that produce the maximum electric field where the field would be maximum for that mode.

This will excite a TE_{10} mode. It couples well to E & H fields of mode.

To excite a TE_{20} mode, use two vertical antenna probes.

A TE_{11} mode needs parallel excitation of the E field at the wall.
To excite the TM$_{11}$ mode, we need circular H fields.

In practice, waveguide dimensions are chosen to allow only a single mode to propagate.

Square guides, where a = b are undesirable since modes differ only by rotation. In practice, pick a ≈ 2b to separate modes and maximize power transmission.

Want single mode guides:

- Different phase velocities would give different transverse modes & make it difficult to extract energy.
- \( \frac{\lambda}{2} < a < \lambda \) to ensure transmission of only TE$_{10}$ mode.
- Often pick \( a = 0.7\lambda \) since values near \( \lambda \) may allow next mode to propagate and values near \( \frac{\lambda}{2} \) have large variation of \( \nu_p \) and \( Z_{TE \text{ or TM}} \) with \( f \).
For the dominant TE$_{10}$ mode the largest electric field is along the center of the wide wall.

\[ E_y = -\frac{j\omega \mu_0 c}{\pi} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{10} z} \]

The peak value of the electric field is then

\[ |E_y|_{\text{max}} = E_0 = \frac{\omega \mu_0 c}{\pi} \]

To avoid dielectric breakdown, \( E_0 \leq E_{BR} \)

The time average power density for the TE$_{10}$ mode can be calculated from the Poynting vector,

\[
S_{AV} = \frac{1}{2} \text{Re}\left\{ \mathbf{E} \times \mathbf{H}^* \right\} = \frac{1}{2} \text{Re}\left\{ E_y \mathbf{H}_y^* \times \mathbf{H}_x^* \right\} \\
= \frac{1}{2} \text{Re}\left\{ -\frac{j\omega \mu_0 c}{\pi} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_{10} z} \right\} \\
= \frac{1}{2} \text{Re}\left\{ + \frac{B_{10} \omega c^2}{2} \left(\frac{a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right) \right\}
\]

\[ S_{AV} = \frac{B_{10} \omega c^2}{2} \left(\frac{a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right) \]

The total average power is gotten by integrating \( S_{AV} \) over the cross-section

\[ P_{AV} = \int_0^a \int_0^b S_{AV} \cdot z \, dx \, dy = \int_0^b \int_0^a \frac{B_{10} \omega c^2}{2} \left(\frac{a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right) dx \, dy \\
= \frac{B_{10} \omega c^2}{2} \left(\frac{a}{\pi}\right)^2 \frac{ab}{2} \]
We can then relate the power to the peak electric field strength.

Since \( E_0 = \frac{\omega Pa}{\pi} \) \( C = \frac{\pi E_0}{\omega Pa} \) where \( E_0 \) is the peak field

\[
P_{\text{peak}} = 2P_{\text{ave}} = 2\beta_0 \omega P \mu \frac{\pi^2 E_0^2}{\omega^2 \mu^2 a^2} \left( \frac{a}{b} \right)^2 \frac{ab}{2}
\]

\[
P_{\text{peak}} = \frac{\beta_0 E_0^2}{\omega \mu} \frac{ab}{2} = \frac{2\pi}{\lambda} \sqrt{1 - \left( \frac{\lambda}{2a} \right)^2} \frac{E_0^2}{\omega \mu} \frac{ab}{2}
\]

\[
= \frac{2\pi}{\lambda \left( 2\pi \mu \right)} \frac{E_0^2}{\omega \mu} \frac{ab}{2} \sqrt{1 - \left( \frac{\lambda}{2a} \right)^2}
\]

\[
P_{\text{peak}} = \frac{1}{\sqrt{\nu \epsilon}} \frac{E_0^2}{\omega} \frac{ab}{2} \sqrt{1 - \left( \frac{\lambda}{2a} \right)^2} = \frac{E_0^2}{\eta} \frac{ab}{2} \sqrt{1 - \left( \frac{\lambda}{2a} \right)^2}
\]

For maximum power maximize \( a \) & \( b \). However, you can’t increase \( b \) too much if you want to keep the TE0 mode as the only propagating mode.
Example 5-2

Design an air-filled C-band (4-8 GHz) rectangular waveguide such that the center frequency of this bound \( f = 6 \text{ GHz} \) is at least 25% higher than the cutoff frequency of the \( \text{TE}_{10} \) mode and at least 25% lower than the cutoff frequency of the next higher mode, so that the dominant mode of propagation is \( \text{TE}_{10} \).

For the \( \text{TE}_{10} \) mode \( \frac{f_{c_{10}}}{2a} = \frac{c}{2a} \)

The first criterion can be written as

\[
f = 6 \text{ GHz} \geq 1.25 \frac{f_{c_{10}}}{2a} = 1.25 \left( \frac{c}{2a} \right) \]

25% above the cutoff

\[
a \geq \frac{1.25 c}{2f} = 1.25 \left( \frac{3 \times 10^8}{2 \cdot 6 \times 10^9} \right) = 0.03125 \text{ m}.
\]

The next highest mode would be \( \text{TE}_{20} \) (See Fig 5.5)

\[
\omega_{c_{20}} = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{\left( \frac{2 \pi}{a} \right)^2 + (0)^2} = \frac{2 \pi}{\sqrt{\mu \epsilon}} \frac{c}{a}
\]

\[
2 \pi f_{c_{20}} = \frac{2 \pi c}{a}
\]

\[
f_{c_{20}} = \frac{c}{a}
\]

The second design criterion can be written as

\[
f = 6 \text{ GHz} \leq 0.75 f_{c_{20}} = 0.75 \frac{c}{a}
\]

\[
a \leq \frac{0.75 c}{f} = \frac{0.75 \left( 3 \times 10^8 \right)}{6 \times 10^9} = 0.0375 \text{ m}.
\]

\[
\therefore 3.125 \text{ cm} \leq a \leq 3.75 \text{ cm}
\]

For the \( \text{TE}_{01} \) mode \( f_{c_{01}} = \frac{c}{2b} \). For this mode

\[
f = 6 \text{ GHz} \leq 0.75 f_{c_{01}} = 0.75 \frac{c}{2b}
\]

\[
b \leq 0.75 \frac{c}{2f} = \frac{0.75 \left( 3 \times 10^8 \right)}{2 \left( 6 \times 10^9 \right)} = 0.025 \text{ m}
\]

\[
b \leq 2.5 \text{ cm}.
\]
5.1.3. Attenuation in Rectangular Waveguides

Attenuation occurs through three mechanisms:

1. Losses due to surface currents flowing in the waveguide walls

2. Dielectric losses due to a dielectric with $\varepsilon' \neq \varepsilon''$ or $\varepsilon_c = \varepsilon' - \varepsilon''$ between the walls

3. Evanescent wave attenuation when $f < f_c$

1. Conduction losses

Surface current densities are given by $J_5 = \hat{n} \times H$

Restricting ourselves to the TE$_{10}$ mode:

$J_5^0 (x=0, y) = \hat{x} \times \hat{z} \frac{\partial}{\partial y} H_z^0 (x=0, y) = -\frac{\partial}{\partial y} H_z^0 (0, y) = -\frac{\partial}{\partial y} C$

$J_5^0 (x=a, y) = -\hat{x} \times \hat{z} \frac{\partial}{\partial y} H_z^0 (x=a, y) = \frac{\partial}{\partial y} H_z^0 (a, y) = -\frac{\partial}{\partial y} C = J_5^0 (x=0, y)$

$J_5^0 (x, y=0) = \hat{y} \times H = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 0 \\ H_x^0 (x, 0) & 0 & H_z^0 (x, 0) \end{vmatrix} = \hat{x} H_z^0 (x, 0) - \hat{z} H_x^0 (x, 0)$

$\hat{J}_5 (x, y=0) = \hat{x} \cos \left( \frac{\pi x}{a} \right) - \hat{z} \frac{\partial}{\partial y} \frac{\alpha C}{\pi} \sin \left( \frac{\pi x}{a} \right) = -\hat{J}_5^0 (x, y=0)$

The attenuation constant associated with power loss was shown to be:

$\alpha_c = \frac{\text{Power loss/unit length}}{2 \times \frac{P_{\text{transmitted}}}{P_{\text{Av}}}} = \frac{P_{\text{loss}}}{2P_{\text{Av}}}$

We can calculate $P_{\text{Av}} (z)$ for the TE$_{10}$ mode.
\[ P_{nv}(z) = \int \int \frac{1}{2} Re \left\{ E \times H^* \right\} \cdot \hat{z} \, dx \, dy \]
\[ = \int \int -\frac{1}{2} E_y^0 (H_x^0)^* \, dx \, dy \]
\[ = \int \int -\frac{1}{2} \left( j \omega \mu_0 c \right) \sin \left( \frac{\pi x}{a} \right) e^{-j \beta_1_0 z} \left( j \beta_1_0 a c \right) \sin \left( \frac{\pi y}{a} \right) e^{j \beta_1_0 z} \, dx \, dy \]
\[ = \int \int \frac{1}{2} \left( \frac{\omega \mu_0 \beta_1_0 c^2}{\pi^2} \right) \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) \, dx \, dy \]
\[ = \frac{\omega \mu_0 \beta_1_0 c^2}{2 \pi^2} \int_0^a \sin^2 \left( \frac{\pi x}{a} \right) \, dx \]
\[ = \frac{\omega \mu_0 \beta_1_0 c^2}{2 \pi^2} \left( \frac{a^2}{2} \right) \cdot \frac{a}{2} \]
\[ P_{nv}(z) = \frac{\omega \mu_0 \beta_1_0 c^2}{\pi^2} \cdot ab \left( \frac{a}{2} \right)^2 \]

There are four walls so
\[ P_{loss} = 2 \left[ P_{loss1} \right]_{y=0} + 2 \left[ P_{loss2} \right]_{y=0} \]
\[ \left[ P_{loss2} \right]_{y=0} = \int_0^a \left[ J_x^0(y=0) \right]^2 R_s \, dx + \int_0^a \left[ J_z^0(y=0) \right]^2 R_s \, dx \]
\[ = \int_0^a \left[ \left( \cos \left( \frac{\pi x}{a} \right) \right)^2 + \left( \frac{\beta_1_0 a c}{\pi} \sin \left( \frac{\pi x}{a} \right) \right)^2 \right] R_s \, dx \]
\[ = \frac{R_s c^2}{2} \left[ \frac{a}{2} + \frac{\beta_1_0 a ^2}{\pi^2} \frac{a}{2} \right] \]
\[
\begin{align*}
    [P_{\text{loss}1}] &= \int_0^b \left( \frac{1}{2} |J_{sy}^0 (x=0)|^2 \right) R_s \, dy \\
    &= \int_0^b \frac{1}{2} c^2 R_s \, dy = \frac{b}{a} c^2 R_s \quad \text{check math}
\end{align*}
\]

\[
P_{\text{loss}} = 2 [P_{\text{loss}2}]_{y=0} + 2 [P_{\text{loss}1}]_{y=0}
\]

\[
P_{\text{loss}} = R_s c^2 \left[ a + \frac{\beta_{10}^2 a^3}{\pi^2} + b \right] = R_s c^2 \left[ b + a \left[ 1 + \frac{\beta_{10}^2 a^2}{\pi^2} \right] \right]
\]

\[
= R_s c^2 \left[ b + a \left[ 1 + \frac{\beta_{10}^2 a^2}{\pi^2} \right] \right]
\]

\[
= R_s c^2 \left[ b + \frac{a}{2} + \frac{4 f^2}{c^2} \left[ 1 - \left( \frac{c_{10}}{f} \right)^2 \right] a^3 \right]
\]

\[
= R_s c^2 \left[ b + \frac{a}{2} \left( \frac{f}{c_{10}} \right)^2 \right]
\]

\[
\alpha_{C\text{TE}10} = \frac{P_{\text{loss}}}{2 P_{\text{AV}}} = \frac{R_s c^2 \left[ b + \frac{a}{2} + \frac{4 f^2}{c^2} \left[ 1 - \left( \frac{c_{10}}{f} \right)^2 \right] a^3 \right]}{2 \omega \mu \beta_{10} c^2 a b \left( \frac{a}{2} \right)^2}
\]

\[
= \frac{R_s \left[ 1 + \frac{2 b \left( \frac{c_{10}}{f} \right)^2}{a} \right]}{2 \omega \mu \beta_{10} c^2 a b \left( \frac{a}{2} \right)^2}
\]

\[
= \frac{1}{\gamma b \sqrt{1 - \left( \frac{c_{10}}{f} \right)^2}}
\]
losses for $\text{TE}_{mn}$ modes are given by (except for $m$ or $n = 0$)

\[
\alpha_{c,\text{TE}_{mn}} = \frac{2 \pi}{\lambda \sqrt{1 - \left(\frac{f_{\text{cmn}}}{f}\right)^2}} \left\{ 1 + \frac{1}{a} \left(\frac{f_{\text{cmn}}}{f}\right)^2 + \left[ 1 - \left(\frac{f_{\text{cmn}}}{f}\right)^2 \right] \frac{b}{a} \left[ \frac{b}{a} m^2 + n^2 \right] \right\}
\]

losses for $\text{TM}_{mn}$ modes are given by

\[
\alpha_{c,\text{TM}_{mn}} = \frac{2 \pi}{b \gamma \sqrt{1 - \left(\frac{f_{\text{cmn}}}{f}\right)^2}} \frac{m^2 (\frac{b}{a})^3 + n^2}{m^2 (\frac{b}{a})^2 + n^2}
\]

Dielectric losses occur when $\varepsilon_c = \varepsilon'$ - $j\varepsilon''$

In this case $\bar{\gamma} = \gamma + j\beta_{mn}$

\[
\bar{\gamma} = \frac{\omega \sqrt{\mu \varepsilon_0} \varepsilon''}{2 \sqrt{1 - \left(\frac{\omega_{\text{cmn}}}{\omega}\right)^2}} + j \sqrt{\frac{\omega^2 - \omega_{\text{cmn}}^2}{\varepsilon_0}} \varepsilon''
\]

$\gamma$: Huygens constant

For $f < f_{\text{cmn}}$

\[
\bar{\gamma} = \alpha_{\text{mn}} = \frac{2 \pi}{\lambda} \sqrt{\left(\frac{f_{\text{cmn}}}{f}\right)^2 - 1} = \frac{2 \pi}{\lambda e_{\text{mn}}} \sqrt{1 - \left(\frac{f}{f_{\text{cmn}}}\right)^2}
\]
5.2 Cylindrical Waveguides with Circular Cross Section

Three different types of cylindrical waveguides

- **Metal tube waveguide**: hollow or filled with dielectric
- **Coaxial waveguide**: center metal conductor, metal shield or braided shield usually dielectric filled.
- **Dielectric waveguide**: \( n_d > n_{cl} \) this is a step index fiber guide

**Mathematical analysis** for TM modes

\[
\nabla^2_{tr} E_z + (\gamma^2 + \omega^2 \mu_e) E_z = 0 \quad \text{as before for } a
\]

solution of form

\[
E_z(r,\phi,z) = E_z^0(r,\phi)e^{-\gamma_z}
\]

What happens is that \( \nabla^2_{tr} \) becomes \( \nabla^2 \) for cylindrical coordinates

\[
\nabla^2_{tr} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}
\]

We will also use separation of variables, i.e. \( E_z^0(r,\phi) = f(r)g(\phi) \)
Substituting \( E^0_\phi (r, \phi) = f(r) g(\phi) \) into the wave equation with \( \nabla^2 \) for cylindrical coordinates

\[
g(\phi) \frac{\partial^2 f}{\partial r^2} + \frac{f(\phi)}{r} \frac{\partial f}{\partial r} + \frac{f(r)}{r^2} \frac{\partial^2 g}{\partial \phi^2} + \left( \frac{\gamma^2}{r^2} + \omega^2 \right) f(r) g(\phi) = 0
\]

dividing by \( fg \) and defining \( \hbar^2 = \gamma^2 + \omega^2 \mu e \)

\[
\frac{1}{f} \frac{\partial^2 f}{\partial r^2} + \frac{1}{fr} \frac{\partial f}{\partial r} + \frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} + \hbar^2 = 0
\]
multiply by \( r^2 \)

\[
\frac{r^2}{f} \frac{d^2 f}{dr^2} + \frac{r}{f} \frac{df}{dr} + \frac{1}{g} \frac{d^2 g}{d\phi^2} + \hbar^2 r^2 = 0
\]

collecting terms:

\[
\frac{r}{f} \left[ \frac{d^2 f}{dr^2} + \frac{df}{dr} \right] + \hbar^2 r^2 + \frac{1}{g} \frac{d^2 g}{d\phi^2} = 0
\]

only a function of \( r \)

only a function of \( \phi \)

Recognizing that

\[
r \frac{d^2 f}{dr^2} + \frac{df}{dr} = \frac{d}{dr} \left( r \frac{df}{dr} \right)
\]

we can write the wave equation as

\[
\frac{r}{f} \frac{d}{dr} \left( r \frac{df}{dr} \right) + \hbar^2 r^2 = -\frac{1}{g} \frac{d^2 g}{d\phi^2} = n^2 \quad \text{(an integer constant)}
\]

The integer constant requirement comes from the equation for \( g(\phi) \) and requiring that it be continuous (i.e. periodic) at \( 2\pi \).
Looking at the $\phi$ equation

$$-\frac{1}{\rho} \frac{d^2\rho}{d\phi^2} = n^2$$

$$\frac{d^2\rho}{d\phi^2} + \rho^2 \rho = 0$$

which has general solutions

$$\rho(\phi) = c_1 \cos(n\phi) + c_2 \sin(n\phi)$$

Looking at the $r$ equation

$$\rho r \frac{d}{dr} \left( r \frac{df}{dr} \right) + \frac{\rho^2}{r^2} = n^2$$

we can re-write it as

$$\frac{d^2f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left( \frac{n^2}{r^2} - \frac{n^2}{\rho^2} \right) f = 0$$

This is a famous equation called Bessel's equation and has the general solution

$$f(r) = c_3 J_n(\rho r) + c_4 Y_n(\rho r)$$

$J_n(\cdot)$ is the $n$-th order Bessel function of the first kind

$Y_n(\cdot)$ is the $n$-th order Bessel function of the second kind

The two Bessel functions are tabulated and their properties are well known.

$$Y_n(r) \to \infty \text{ as } r \to 0$$

For a cylindrical waveguide we want finite fields at all points (no change in the guide) so we reject all solutions of the form $Y_n(r)$.
These functions behave a lot like a $\frac{\sin x}{x}$. They start at 1 or 0, decrease as $u$ increases, BUT the period changes as $r$ changes.

$J_0(u)$ has zeros at
- $u = 2.405$
- $u = 5.520$
- $u = 8.654$
- $u = 11.792$

$J_1(u)$ has zeros at
- $u = 3.832$
- $u = 7.016$
- $u = 10.173$
- $u = 13.323$

$J_2(u)$ has zeros at
- $u = 5.136$
- $u = 8.417$
- $u = 11.620$
- $u = 14.796$

You will rarely need to use any higher order Bessel functions.
The general solution is

\[ E^0_2(r, \phi) = C_3 J_n(\lambda r) \left[ C_1 \cos(n\phi) + C_2 \sin(n\phi) \right] \]

Looking at the \( \phi \) terms we expect the solutions to be independent of where we pick the origin, i.e. the \( \phi = 0 \) point. So just pick \( \cos(n\phi) \) making the solution

\[ E^0_2(r, \phi) = C_n J_n(\lambda r) \cos(n\phi) \]

This is almost exactly like the product solutions we got for rectangular waveguides.

For the metal tube waveguides we want

\[ E_{\text{tan}} = 0 \text{ at the metal wall} \]

which requires

\[ J_n(\alpha a) = 0 \]

where \( \alpha a \) is a zero (root) of the \( n \)-th order Bessel function of the first kind.

The modes are then given by

\[ \phi_{TM_{nl}} = \frac{\ell a}{a} \left( \frac{\alpha}{\lambda} \right) \]

\[ n \] will be the number of circumferential variations
\[ \ell \] will be the number of radial variations

The propagation constant

\[ \beta_{TM_{nl}} = \left[ \omega \varepsilon - \left( \frac{\ell a}{\lambda} \right)^2 \right]^{1/2} \]

propagation occurs for \( f > f_{TM_{nl}} \)
This gives \( f_{\text{c, TM}_{n\ell}} = \frac{t_{\text{nl}}}{2\pi \alpha \sqrt{\mu e}} \)

The lowest cutoff frequency is \( t_{01} = 2.405 \)
which makes \( \text{TM}_{01} \) the TM mode with the lowest cutoff frequency.

The field components for circular \( \text{TM}_{n\ell} \) modes are

\[
E_z = c_n J_n \left( \frac{t_{n\ell} r}{a} \right) \cos(n\phi) e^{-jB_{n\ell} z}
\]

\[
E_r = -j a^2 \overline{B_{n\ell}} \frac{t_{n\ell}}{c_n J_n'} \left( \frac{t_{n\ell} r}{a} \right) \cos(n\phi) e^{-jB_{n\ell} z}
\]

\[
E_\phi = \frac{j a^2 \overline{B_{n\ell}}}{t_{n\ell}^2 r} c_n J_n \left( \frac{t_{n\ell} r}{a} \right) \sin(n\phi) e^{-jB_{n\ell} z}
\]

\[
H_r = -j \omega e \frac{a^2}{t_{n\ell}^2} c_n J_n \left( \frac{t_{n\ell} r}{a} \right) \sin(n\phi) e^{-jB_{n\ell} z}
\]

\[
H_\phi = -j \omega e a \frac{t_{n\ell}}{c_n J_n'} \left( \frac{t_{n\ell} r}{a} \right) \cos(n\phi) e^{-jB_{n\ell} z}
\]

\[
H_z = 0
\]

where \( J_n'(\xi) = \frac{dJ_n(\xi)}{d\xi} \)
The $\text{TE}_{mn}$ modes are calculated in a similar manner except that $E_z = 0$.

Let $H_2^0(r, \phi) = c_n J_n \left( \frac{\alpha r}{a} \right) \cos n\phi$

the boundary condition is $E_\text{on} = 0$

$\Rightarrow E_\phi^0(r, \phi) = 0 \text{ at } r = a$

or $\frac{\partial H_2^0}{\partial r} = 0 \text{ at } r = a$

The eigenvalues of the solution come from the zeros of $J_n'(u)$ which we call $S_{nl}$

$$\beta_{\text{TE}_{nl}} = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{S_{nl}}{\alpha} \right)^2}$$

$$f_{\text{TE}_{nl}} = \frac{S_{nl}}{2\pi a \sqrt{\mu \varepsilon}}$$

roots of $J_n'(u)$

$J_0'(u)$ has zeros at $u = 3.832$

$u = 7.016$

$u = 10.173$

$u = 13.324$

$J_1'(u)$ has zeros at $u = 1.841$

$u = 5.331$

$u = 8.536$

$u = 11.706$

$J_2'(u)$ has zeros at $u = 3.054$

$u = 6.706$

$u = 9.969$

$u = 13.170$

It is very important to notice that $S_{11} = 1.841$ is the lowest zero and gives the $\text{TE}_{11}$ mode the lowest cutoff frequency.
The complete $\text{TE}_{n\ell}$ solutions are

\[ H_z = c_n J_n \left( \frac{\sin \theta}{a} \right) \cos(n\phi) e^{-j\beta_{n\ell} z} \]

\[ H_r = -j a^2 \beta_{\text{TE}_{n\ell}} c_n J_n' \left( \frac{\sin \theta}{a} \right) \cos(n\phi) e^{-j\beta_{n\ell} z} \]

\[ H_\phi = \frac{j n a^2 \beta_{\text{TE}_{n\ell}}}{S_{n\ell}} c_n J_n \left( \frac{\sin \theta}{a} \right) \sin(n\phi) e^{-j\beta_{n\ell} z} \]

\[ E_r = \frac{j d^2 \omega \mu_n}{S_{n\ell}} c_n J_n \left( \frac{\sin \theta}{a} \right) \sin(n\phi) e^{-j\beta_{n\ell} z} \]

\[ E_\phi = -j a \omega \mu c_n J_n' \left( \frac{\sin \theta}{a} \right) \cos(n\phi) e^{-j\beta_{n\ell} z} \]

\[ E_z = 0 \]
Notice that $TE_{11}$ is the overall dominant mode.

If we choose the wavelength $\lambda$ to lie between

$$\lambda_{c_{TE_{11}}} = \frac{2\pi a}{1.841} = 3.41a$$

and

$$\lambda_{c_{TM_{01}}} = \frac{2\pi a}{2.405} = 2.61a$$

only the $TE_{11}$ mode will propagate.
Attenuation in circular waveguides

For \( f > f_{cTmnl} \)

\[ \alpha_{cTmnl} = \frac{R_s}{\alpha \eta} \left[ 1 - \left( \frac{f_{cTmnl}}{f} \right)^2 \right]^{-\frac{1}{2}} \]

For \( f > f_{cTEml} \)

\[ \alpha_{cTEml} = \frac{R_s}{\alpha \eta} \left[ 1 - \left( \frac{f_{cTEml}}{f} \right)^2 \right]^{-\frac{1}{2}} \left[ \left( \frac{f_{cTEml}}{f} \right)^2 + \frac{n^2}{S_{nl}^2 - n^2} \right] \]

\[ \alpha = 1 \text{cm.} \]

Attenuation is high near cutoff, drops to a minimum near two or three times the cutoff, and increases \( \sim \sqrt{f} \) for frequencies well above cutoff.
5.2.2. Coaxial lines

used at frequencies less than 5 GHz
braided outer conductor & inner conductor

tangential $E$ \{ must be zero on conductor surfaces

normal $H$ \} \\
\Rightarrow \quad E_\phi = 0 \quad \text{at } r = a, b \\
H_r = 0

non-zero $E_\phi$ & $H_r$ can only exist between the conductors
if they vary with $r$.

A TEM solution can only exist with

$E = \hat{r} E_r$ and $H = \hat{\phi} H_\phi$

to give energy transport in the $z$-direction.

Radial variation of $E_\phi$ cannot be present since it would create $H_z$ which cannot exist for TEM

$-j \omega \mu H_z = [\nabla \times E]_z = \frac{1}{r} \left[ \frac{\partial}{\partial r} (rE_\phi) - \frac{\partial E_r}{\partial \phi} \right]$

this term would be non-zero

The component equations give

$-\frac{\partial H_\phi}{\partial z} = j\omega \epsilon E_r$

$+ j\beta H_\phi^0 (r) = j\omega \epsilon E_r^0 (r)$

and

$\frac{1}{r} H_\phi + \frac{\partial H_\phi}{\partial r} = 0$

$\frac{1}{r} H_\phi^0 (r) + \frac{\partial H_\phi^0 (r)}{\partial r} = 0$

The solution of this equation is $H_\phi^0 (r) = \frac{H_0}{r}$
\[ H = \frac{\hat{\phi}}{r} \frac{H_0}{e^{-j\beta z}} \]

\[ j \beta \mathbf{H}_\theta (r) = j \omega e E_r (r) \]

\[ E_r^0 (r) = \frac{\omega}{\omega e} H_\theta^0 (r) = \frac{\omega}{\omega e} H_\theta (r) \]

\[ E_r^0 (r) = \sqrt{\frac{\mu_0}{\varepsilon}} H_\theta^0 (r) = \gamma H_\theta (r) \]

\[ E = \frac{\hat{\phi}}{r} \gamma \frac{H_0}{e^{-j\beta z}} \]

There is no cutoff frequency since \( \beta \) is that of a plane wave.

The line exhibits loss due to surface currents on the conductors.

\[ J_s (z) = \hat{n} \times \mathbf{H} = \begin{cases} \hat{z} H_\theta (a, z) & \text{inner conductor} \\ -\hat{z} H_\theta (b, z) & \text{outer conductor} \end{cases} \]

\[ |S_{AV}| = \frac{1}{2} \text{Re} \left\{ (E \times H^*) \cdot \hat{z} \right\} = \frac{1}{2} E_r H_\theta^* = \frac{1}{2} \gamma \frac{H_0^2}{r^2} \]

\[ P_{AV} = \int_{a}^{b} \int_{0}^{2\pi} |S_{AV}| r \, d\phi \, dr = \int_{a}^{b} 2\pi \frac{1}{2} \gamma \frac{H_0^2}{r^2} r \, dr \]

\[ P_{AV} = \int_{a}^{b} \pi \gamma H_0^2 \frac{dr}{r} = \pi \gamma H_0^2 \ln \left( \frac{b}{a} \right) \]

The power loss/unit length is calculated as

\[ P_{\text{Loss}} (z) = \frac{1}{2} \int_{S} |E_s |^2 R_s \, ds = \frac{R_o}{2} \int_{0}^{1} \int_{0}^{2\pi} |H_\theta (a, z)|^2 a \, d\phi \]

\[ + \frac{R_o}{2} \int_{0}^{1} \int_{0}^{2\pi} |H_\theta (b, z)|^2 b \, d\phi \]
\[ P_{\text{loss}}(z) = \frac{R_s}{2} \int_0^1 \int_0^{2\pi} \left| \frac{H_0}{a} \right|^2 a \, d\phi + \frac{R_s}{2} \int_0^1 \int_0^{2\pi} \left| \frac{H_0}{b} \right|^2 b \, d\phi \]

\[ = \frac{2\pi R_s}{2} \frac{H_0^2}{a^2} a + \frac{2\pi R_s}{2} \frac{H_0^2}{b^2} b \]

\[ = \pi R_s H_0^2 \left[ \frac{1}{a} + \frac{1}{b} \right] \]

The attenuation constant is then

\[ \alpha_{c, \text{TEM}} = \frac{P_{\text{loss}}(z)}{2 P_{AV}} = \frac{\pi R_s H_0^2 \left[ \frac{1}{a} + \frac{1}{b} \right]}{2 \pi \eta H_0^2 \ln(b/a)} = \frac{R_s}{2 \eta \ln(b/a)} \left[ \frac{1}{a} + \frac{1}{b} \right] \]

and since \( R_s = \sqrt{\frac{\omega \eta}{2}} \), \( \alpha_{c, \text{TEM}} \) increases with frequency.

The formula for dielectric losses is the same as that for a dielectric coaxial waveguide [4.27]

\[ \alpha_d = \frac{\omega \sqrt{\mu_0 \varepsilon_0} \varepsilon''}{2 \sqrt{1 - \left( \frac{\omega}{\omega_c} \right)^2}} \bigg|_{\omega_c=0} = \frac{\omega \varepsilon''}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\sqrt{\mu_0}}{\varepsilon''} = \frac{\omega \varepsilon''}{2} \eta \]

where we note that \( \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \).
Just as for the (two-conductor) parallel plate waveguide, the TEM mode corresponds to voltage and current on a two-conductor transmission line where the voltage and current are given by:

\[
V(z) = - \int_a^b E \cdot dl = \int_a^b E \cdot r \cdot \hat{r} \cdot dr = - \int_a^b \frac{H_0 e^{-j\beta z}}{r} \cdot \hat{r} \cdot dr = - \int_a^b \gamma \frac{H_0 e^{-j\beta z}}{r} \cdot dr
\]

\[
V(z) = - \gamma H_0 \ln \left( \frac{b}{a} \right) e^{-j\beta z} = -V^+ e^{-j\beta z}
\]

\[
I(z) = \int_C H \cdot dl = \int_0^{2\pi} \frac{2\pi}{a} H_0 e^{-j\beta z} \phi \cdot \hat{a} \cdot d\phi
\]

\[
I(z) = 2\pi H_0 e^{-j\beta z}
\]

but from the above equation for \(V(z)\)

\[
H_0 = \frac{V^+}{\gamma \ln \left( \frac{b}{a} \right)}
\]

\[
I(z) = 2\pi \frac{V^+}{\gamma \ln \left( \frac{b}{a} \right)} e^{-j\beta z} = \frac{V^+}{Z_0} e^{-j\beta z}
\]

where \(Z_0 = \frac{\gamma \ln \left( \frac{b}{a} \right)}{2\pi} \approx 60 \ln \left( \frac{b}{a} \right)\) since \(\gamma \approx 120\pi\)

The average power is \(P_{\text{av}}(z) = \frac{1}{2} \frac{|V^+|^2}{Z_0}\)
You can optimize the coaxial cable in several ways.

Since \( E \sim \frac{1}{r} \) the maximum field occurs at \( r = a \)

For a coaxial geometry,

\[
E = \frac{V}{r \ln \left( \frac{b}{a} \right)}
\]

or

\[
E_{\text{max}} = \frac{V_{\text{max}}}{a \ln \left( \frac{b}{a} \right)} = \frac{V_{\text{max}}}{b \ln \left( \frac{b}{a} \right)} = \frac{V_{\text{max}}}{b \ln \left( \frac{b}{a} \right)} = \frac{V_{\text{max}}}{b \ln \left( \frac{b}{a} \right)} = \frac{V_{\text{max}}}{b \ln \left( \frac{b}{a} \right)}
\]

where \( j = \frac{b}{a} \)

The power transmitted is

\[
P_{\text{AV}} = \frac{V_{\text{max}}^2}{2Z_0} = \left[ \frac{E_{\text{max}} b \ln \left( \frac{b}{a} \right)}{2 \ln \left( \frac{b}{a} \right)} \right] = \frac{E_{\text{max}}^2 b^2 \ln \left( \frac{b}{a} \right)}{2 \ln \left( \frac{b}{a} \right)} = \frac{E_{\text{max}}^2 b^2 \ln \left( \frac{b}{a} \right)}{2 \ln \left( \frac{b}{a} \right)} = \frac{E_{\text{max}}^2 b^2 \ln \left( \frac{b}{a} \right)}{2 \ln \left( \frac{b}{a} \right)} = \frac{E_{\text{max}}^2 b^2 \ln \left( \frac{b}{a} \right)}{2 \ln \left( \frac{b}{a} \right)}
\]

\[
P_{\text{AV}} = K \frac{b \ln \left( \frac{b}{a} \right)}{\frac{b}{a}}
\]

To maximize power transmission

\[
\frac{dP_{\text{AV}}}{d\ln \left( \frac{b}{a} \right)} = K \left[ \frac{1}{\frac{b}{a}} - \frac{2 \ln \left( \frac{b}{a} \right)}{\left( \frac{b}{a} \right)^2} \right] = 0
\]

\[1 - 2 \ln \left( \frac{b}{a} \right) = 0 \quad \ln \left( \frac{b}{a} \right) = \frac{1}{2} \quad \Rightarrow \quad \frac{b}{a} \approx 1.65 \text{ for max power}
\]

This corresponds to a impedance

\[
Z_0 \approx 60 \ln (1.65) = 30 \Omega
\]

Incidentally, maximum \( V(z) \) occurs with 60\( \Omega \) line (maximum voltage handling)

minimum conduction losses with 77\( \Omega \) line
TE & TM modes in a coaxial line

The TEM mode is dominant since it is the lowest cutoff frequency, but TE & TM modes exist for a coaxial waveguide.

The presence of the inner conductor does not allow us to eliminate \( Y_n (kr) \) as a possible solution since \( r > a \).

For TM waves we now have

\[
E^0_z (r, \phi) = [C_3 J_n (kr) + C_4 Y_n (kr)] \cos n \phi
\]

with the boundary conditions \( E^0_z (r, \phi) = 0 \) at \( r = a \) and \( r = b \).

For TE waves we now have

\[
H^0_e (r, \phi) = [C_3' J_n ' (kr) + C_4' Y_n (kr)] \cos n \phi
\]

with the boundary conditions \( \frac{\partial H^0_e}{\partial r} = 0 \) at \( r = a \) and \( r = b \).

These equations have solutions of the form

\[
C_3 J_n (ha) + C_4 Y_n (ha) = 0
\]

\[
C_3 J_n (hb) + C_4 Y_n (hb) = 0
\]

\[
\frac{C_3 J_n (ha)}{C_3 J_n (hb)} = \frac{-C_4 Y_n (ha)}{-C_4 Y_n (hb)}
\]

\[
J_n (ha) Y_n (hb) = J_n (hb) Y_n (ha)
\]

which is a transcendental equation which is best solved numerically.
There are a few of the lowest order TE & TM modes. The lowest non-zero cutoff frequency is for the TE$_{11}$ mode

$$f_{c_{11}} \approx \frac{1}{\pi (a+b) \sqrt{\mu \epsilon}}$$

$$h = \frac{2}{a+b}$$

In practice, choose the coax dimensions so only the TEM mode propagates for the frequencies of interest.
Dielectric circular waveguides

We use a cladding as opposed to simple air because
(1) a cladding layer minimizes any effects on the fields in the guide which might cause higher order fields to exist
(2) the use of a cladding $\varepsilon_{cl}$ gives the designer freedom to design a waveguide so that only one mode propagates

Neither $E_z$ or $H_z$ can be zero so
The fields must satisfy the wave equations in both dielectrics

\[
\nabla_{tr}^2 E_z + \omega^2 E_z = 0
\]

and

\[
\nabla_{tr}^2 H_z + \omega^2 H_z = 0
\]

where \( \omega^2 = \omega^2_d = \gamma^2 + \omega^2 \mu_d \varepsilon_d \) in the dielectric
\( \omega^2 = -\omega^2_{cl} = -\gamma^2 + \omega^2 \mu_c \varepsilon_c \) in the cladding

Assume separable solutions
\[
E_z (r, \phi, z) = E_z^0 (r, \phi) e^{-\gamma z}
\]
\[
H_z (r, \phi, z) = H_z^0 (r, \phi) e^{-\gamma z}
\]

\[
= f(r) g(\phi) e^{-\gamma z}
\]
Inside the dielectric core we want \( \bar{\gamma} = j\beta \)

and \( \omega^2 \mu_d \varepsilon_d - \rho_d > 0 \)

The Bessel functions \( J_n(\cdot) \) and \( Y_n(\cdot) \) are well defined for this case and give solutions of the form

\[
\begin{align*}
E_z^0(r, \phi) = & \left[ C_d J_n(\rho_d r) + C'_d Y_n(\rho_d r) \right] \left[ A_d \cos \phi + B_d \sin \phi \right] \\
H_z^0(r, \phi) = & \left[ A_d J_n(\rho_d r) + B_d Y_n(\rho_d r) \right] \left[ A_d \cos \phi + B_d \sin \phi \right]
\end{align*}
\]

for \( r \leq a \).

However in the cladding we want solutions that decay with \( r \)

For the dielectric core \( \rho_c^2 = \rho_d^2 = \bar{\gamma}^2 + \omega^2 \mu_d \varepsilon_d \) and \( \delta^2 = \rho_d^2 - \omega^2 \mu_d \varepsilon_d < 0 \)

and we had solutions of the form

\[
C_d J_n(\rho_d r) + C'_d Y_n(\rho_d r)
\]

For the dielectric slab, we required solutions that decayed exponentially, i.e. \( \bar{\gamma} = \gamma > 0 \), outside the slab.

With Bessel functions we have a similar situation in the cladding. Bessel's equation in the core was

\[
\frac{d^2f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left( \frac{\rho_c^2 - \frac{n^2}{r^2}}{r^2} \right)f = 0 + \left( \frac{\rho_d^2 - \frac{n^2}{r^2}}{r^2} \right)f
\]

and gave the solutions \( J_n(\rho_d r) \) and \( Y_n(\rho_d r) \)

However, if we allow \( \rho_c^2 < 0 \) we get a slightly different equation

\[
\frac{d^2f}{dr^2} + \frac{1}{f} \frac{df}{dr} - \left( \frac{\rho_c^2 + \frac{n^2}{r^2}}{r^2} \right)f = 0
\]

For \( n=0 \) this equation has solutions of the form

\[
\begin{align*}
I_0(r) &= j^0 J_0(jr) \\
K_0(r) &= \frac{\pi}{2} j^0 \mathcal{H}^{(1)}_0(jr)
\end{align*}
\]
where \( H_n^{(1)}(jr) = J_n(jr) + j Y_n(jr) \)

This is a Homkel function

\( I_n \) and \( K_n \) are called modified Bessel functions and give better solutions in the cladding as they decay much faster than \( J_n(jr) \) which only decays as \( r^{-\frac{1}{2}} \)

See: Ramo, Whinnery, Van Duzer
Fields & Waves in Communications Electronics 2/e
7.14 Bessel Functions

The modified Bessel function \( K_n \) decay faster than \( r^{-1} \)
However, \( I_n \) increases and must be eliminated as a possible solution.
The complete solution in the cladding is

\[
\begin{align*}
E_z^0 (r, \phi) &= [C_{ce} K_n(P_{cr} r) + C'_{ce} I_n(P_{cr} r)][A_{ce} \cos n\phi + B_{ce} \sin n\phi] \\
H_z^0 (r, \phi) &= \{\} 
\end{align*}
\]

However we set \( C'_{ce} = 0 \)

The complete solution for propagating waves is then

\[
\begin{align*}
E_z (r, \phi, z) &= \begin{cases} 
C_d J_n(P_d r)[A_d \cos(n\phi) + B_d \sin(n\phi)]e^{-j\beta_z} & r \leq a \\
C_{ce} K_n(P_{cr} r)[A_{ce} \cos(n\phi) + B_{ce} \sin(n\phi)]e^{-j\beta_z} & r > a 
\end{cases} \\
H_z (r, \phi, z) &= \begin{cases} 
C_d J_n(P_d r) \cos(n\phi) e^{-j\beta_z} & 0 < r \leq a \\
C_{ce} K_n(P_{cr} r) \cos(n\phi) e^{-j\beta_z} & r > a 
\end{cases}
\end{align*}
\]

where \( \omega^2 \mu_d \epsilon_d \leq \beta^2 \leq \omega^2 \mu_d \epsilon_d \)

Since cylindrical waveguides are often used for optical communications we can write this in terms of the optical index of refraction

\[
\frac{\omega n_d \epsilon_d}{c} \leq \beta \leq \frac{\omega n_a \epsilon_d}{c}
\]

The propagation constant will lie between that of the core and the cladding.

These solutions must be continuous at \( r = a \)

One way to do this is to normalize the fields so the radial dependence is 1 at \( r = a \), i.e.

\[
H_z = \begin{cases} 
C \frac{J_n(P_{cr} r)}{J_n(P_{da} r)} \cos(n\phi) e^{-j\beta_z} & 0 < r \leq a \\
C \frac{K_n(P_{cr} r)}{K_n(P_{da} r)} \cos(n\phi) e^{-j\beta_z} & r > a 
\end{cases}
\]
Similarly

\[
E_z = \begin{cases} 
  c' \frac{J_n(hd)}{J_n(hda)} \sin(n\phi) e^{-j\beta z} & \text{for } 0 < r \leq a \\
  c' \frac{K_n(hd r)}{K_n(hd a)} \sin(n\phi) e^{-j\beta z} & \text{for } r > a
\end{cases}
\]

Note: we arbitrarily picked the origin for \( \phi \) so that we can have \( H_z \propto \cos(n\phi) \).

This choice forces \( E_z \) to be proportional to \( \sin(n\phi) \)

Once the axial components are specified the transverse field components can be found using Maxwell's equations.

Inside the dielectric core

\[
\begin{align*}
E_r &= \frac{j \omega \mu}{\varepsilon_0} \frac{\partial H_z}{\partial r} - \frac{j \beta}{h_d} \frac{\partial E_z}{\partial \phi} \\
E_\phi &= -\frac{j \beta}{h_d} \frac{\partial E_z}{\partial r} - \frac{j \omega \mu}{\varepsilon_0} \frac{\partial H_z}{\partial \phi} \\
H_r &= -\frac{j \beta}{h_d} \frac{\partial H_z}{\partial \phi} + \frac{j \omega \mu}{\varepsilon_0} \frac{\partial E_z}{\partial r} \\
H_\phi &= \frac{j \beta}{h_d} \frac{1}{r} \frac{\partial H_z}{\partial \phi} + \frac{j \omega \mu}{\varepsilon_0} \frac{\partial E_z}{\partial r}
\end{align*}
\]

In the cladding

\[
\begin{align*}
E_r &= -\frac{j \omega \mu}{\varepsilon_0} \frac{\partial H_z}{\partial r} + \frac{j \beta}{\varepsilon_0} \frac{\partial E_z}{\partial \phi} \\
H_\phi &= \frac{j \beta}{\varepsilon_0} \frac{1}{r} \frac{\partial H_z}{\partial \phi} + \frac{j \omega \mu}{\varepsilon_0} \frac{\partial E_z}{\partial r}
\end{align*}
\]

plus the other components.
However, we still don't know the value of $\bar{\beta}$

This comes from equation $H\Phi$ and $E\Phi$ at $r=a$.

This is pretty complex [See Romo, Whinnery & Van Duzer, Ch. 14]

Continuity of $E\Phi$ at $r=a$ gives

$$C \frac{\omega P_0}{s} f_n(s) - C' \frac{\sqrt{s}}{s^2} = -C \frac{\omega P_0}{t} g_n(t) + C' \frac{\sqrt{s}}{t^2}$$

Continuity of $H\Phi$ at $r=a$ gives

$$C \frac{\sqrt{s}}{s^2} - C' \omega e \omega f_n(s) = -C \frac{\sqrt{s}}{t^2} + C' \omega e \omega g_n(t)$$

where $s = \rho_d a$

t = \rho_{ce} a

$$f_n(\rho_d a) = \frac{1}{\rho_d} \left[ \frac{d}{dr} \left( \frac{J_n(\rho_d r)}{J_n(\rho_d a)} \right) \right]_{r=a} = \rho_d \frac{J_n'(\rho_d a)}{J_n(\rho_d a)}$$

$$g_n(\rho_{ce} a) = \frac{1}{\rho_{ce}} \left[ \frac{d}{dr} \left( \frac{K_n(\rho_{ce} r)}{K_n(\rho_{ce} a)} \right) \right]_{r=a} = \rho_{ce} \frac{K_n'(\rho_{ce} a)}{K_n(\rho_{ce} a)}$$

Each equation gives a solution for $\bar{\beta}$ in terms of $\frac{C}{C'}$

However, since the $\frac{C}{C'}$ must be the same for each equation we can equate $\frac{C}{C'}$ and get

$$\omega^2 P_0 \left( \frac{f_n(s)}{s} + \frac{g_n(t)}{t} \right) \left( \varepsilon_d \frac{f_n(s)}{s} + \varepsilon_c \frac{g_n(t)}{t} \right) = \left( \frac{\sqrt{s}}{s^2} + \frac{\sqrt{t}}{t^2} \right)^2$$
The solutions for $\beta$ can be determined numerically and result in hybrid modes in which neither $E_2$ or $H_2 = 0$.

For $n=0$ these can be broken into TE and TM modes.

For this specialized case

$$\frac{1}{s} f_n(s) = -\frac{1}{t} g_n(t)$$

and

$$s^2 + t^2 = \omega^2 \mu_0 (\varepsilon_d - \varepsilon_a) a^2 = u^2$$

Similarly for TM modes

$$\frac{\varepsilon_d}{s} f_n(s) = -\frac{\varepsilon_a}{t} g_n(t)$$

Other specialized cases occur when

$$\frac{|\varepsilon_d - \varepsilon_a|}{|\varepsilon_d|} < 1\%$$

or when $n$ gradually varies with $n$. 