4.1 Waves between parallel metal plates

Assume propagation of a wave in the +z direction.

We define \( \bar{\gamma} = \bar{x} + j \bar{\beta} \) and let the wave propagate as \( e^{-\bar{\gamma}z} \).

Note \( \bar{x}, \bar{\beta} \) are NOT \( \alpha, \beta \) for infinite extent waves and depend upon the dimensions of the guiding structure as well as \( \omega, \sigma, \varepsilon, \mu \).

If \( \bar{\beta} = 0 \) \( \Rightarrow \) \( \text{Re}\{e^{-\bar{x}z} e^{j\omega t}\} \rightarrow e^{-\bar{x}z} \cos(\omega t) \) evanescent wave

If \( \bar{x} = 0 \) \( \Rightarrow \) \( \text{Re}\{e^{-\bar{\beta}z} e^{j\omega t}\} \rightarrow \cos(\omega t - \beta z) \) propagating wave

In general \( \nabla^2 \bar{E} = -\omega^2 \mu \varepsilon \bar{E} \) and \( \nabla^2 \bar{H} = -\omega^2 \mu \varepsilon \bar{H} \)

\[
\frac{\partial^2 \bar{E}}{\partial x^2} + \frac{\partial^2 \bar{E}}{\partial y^2} + \frac{\partial^2 \bar{E}}{\partial z^2} = -\omega^2 \mu \varepsilon \bar{E}
\]

\[
\frac{\partial^2 \bar{H}}{\partial x^2} + \frac{\partial^2 \bar{H}}{\partial y^2} + \frac{\partial^2 \bar{H}}{\partial z^2} = -\omega^2 \mu \varepsilon \bar{H}
\]

Imposes boundary conditions

Assume \( \bar{E} \) dependence

\( i.e. \bar{H}(x, z) = \bar{H}_0^0(x) e^{-\bar{\gamma}z} \)

\( \therefore \frac{\partial}{\partial z} \bar{H}(x, z) \rightarrow -\bar{\gamma} \bar{H}_0(x) e^{-\bar{\gamma}z} \)
With this approach the wave equations become

\[
\frac{\partial^2 E}{\partial x^2} + \gamma^2 E = -\omega^2 \mu e \ E
\]
\[
\frac{\partial^2 H}{\partial x^2} + \gamma^2 H = -\omega^2 \mu e \ H
\]

Maxwell's equations also apply in the region between the plates

\[\nabla \times H = j\omega e \ E\]
\[\nabla \times E = -j\omega \mu H\]

where we assumed that \(\sigma = 0\), \(\varepsilon'' = 0\). Examining the components

\[\frac{\partial H_y}{\partial y} - \frac{\partial E_y}{\partial z} = j\omega e E_x\]
\[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega \mu H_x\]
\[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega e E_y\]
\[\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega \mu H_y\]
\[\frac{\partial E_z}{\partial x} - \frac{\partial E_y}{\partial y} = j\omega e E_z\]
\[\frac{\partial H_y}{\partial x} = -j\omega \mu H_z\]

and eliminating all partial derivatives wrt \(y\) and replacing partial derivatives wrt \(z\) by \(-\gamma\) we get

\[+\gamma H_y = j\omega e E_x\]
\[-\gamma E_y = -j\omega \mu H_x\]
\[-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega e E_y\]
\[-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\omega \mu H_y\]
\[\frac{\partial H_y}{\partial x} = j\omega e E_z\]
\[\frac{\partial E_y}{\partial x} = -j\omega \mu H_z\]

define \(h^2 = \gamma^2 + \omega^2 \mu e\) and rewrite these eqns:

\[-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega e E_y = j\omega e \left( -\frac{\partial H_x}{\gamma} \right)\]
\[-\gamma^2 H_x - \gamma \frac{\partial H_z}{\partial x} = +\omega^2 \mu e H_x\]
\[(-\gamma^2 - \omega^2 \mu e) H_x = +\gamma \frac{\partial H_z}{\partial x}\]

\[H_x = -\frac{\gamma}{h^2} \frac{\partial H_z}{\partial x}\]
\[
\vec{\gamma} H_y = j \omega E_x
\]

\[
\vec{\gamma} H_y = j \omega e \left( \frac{\partial E_x}{\partial x} - j \omega p H_y \right)
\]

\[
-\vec{\gamma}^2 H_y = j \omega e \frac{\partial E_x}{\partial x} + \omega^2 p e H_y
\]

\[
(\vec{\gamma}^2 - \omega^2 p e) H_y = j \omega e \frac{\partial E_x}{\partial x}
\]

\[
H_y = -j \omega e \frac{\partial E_x}{\partial x}
\]

\[
-\vec{\gamma} E_x - \frac{\partial E_x}{\partial x} = -j \omega p H_y = -j \omega p \left( \frac{j \omega e E_x}{\vec{\gamma}} \right)
\]

\[
-\vec{\gamma}^2 E_x - \vec{\gamma} \frac{\partial E_x}{\partial x} = + \omega^2 p e E_x
\]

\[
\left[ (\vec{\gamma}^2 - \omega^2 p e) \right] E_x = \vec{\gamma} \frac{\partial E_x}{\partial x}
\]

\[
E_x = -\frac{\vec{\gamma}}{\vec{\gamma}^2} \frac{\partial E_x}{\partial x}
\]

\[
\vec{\gamma} E_y = -j \omega p H_x
\]

\[
\vec{\gamma} E_y = -j \omega p \left( \frac{\partial H_x + j \omega p E_y}{\vec{\gamma}} \right)
\]

\[
-\vec{\gamma}^2 E_y = -j \omega p \frac{\partial H_x}{\partial x} + \omega^2 p e E_y
\]

\[
-\vec{\gamma}^2 E_y - \omega^2 p e E_y = -j \omega p \frac{\partial H_x}{\partial x}
\]

\[
E_y = \frac{j \omega p}{\vec{\gamma}^2} \frac{\partial H_x}{\partial x}
\]
4.1.1 Field Solutions for TE & TM Waves

Three categories of guided-wave solutions:

\[ \begin{align*}
E_z &= 0 \quad \text{transverse electric (TE) waves} \\
H_z &\neq 0 \\

E_z &\neq 0 \quad \text{transverse magnetic (TM) waves} \\
H_z &= 0 \\
E_z &= H_z = 0 \quad \text{transverse electromagnetic (TEM) waves}
\end{align*} \]

**TE Waves**

There is always and everywhere an electric field vector which is transverse to the direction of propagation & \( E_z = 0 \).

Use the wave equation to find \( E_y \)

\[ \frac{\partial^2 E_y}{\partial x^2} + \alpha^2 E_y = -\omega^2 \mu e E_y \quad \text{since } \frac{\partial E_y}{\partial y} = 0 \]

\[ \frac{\partial^2 E_y}{\partial x^2} = -(\omega^2 \mu e + \alpha^2) E_y = -\omega^2 E_y \]

Now write \( E_y \) as a product of functions:

\[ E_y(x, z) = E_y^0(x) e^{-\beta z} \]

\[ \frac{\partial^2 E_y^0}{\partial x^2} e^{-\beta z} = -\omega^2 E_y^0(x) e^{-\beta z} \]

\[ \frac{\partial^2 E_y^0}{\partial x^2} = -\omega^2 E_y^0(x) \]

Solutions are

\[ E_y^0(x) = C_1 \sin \beta x + C_2 \cos \beta x \]

The boundary conditions are

\[ E_y = 0 \text{ at } x = 0 \]
\[ E_y = 0 \text{ at } x = a \]
\[ \text{at } x=0 \quad E_y=0 \quad \text{requires } C_2=0 \]

\[ E_y(x, z) = C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-y z} \]

For \( E_y = 0 \) at \( x = a \), \( \frac{h a}{m \pi} = \frac{m \pi}{a} \), \( m = 1, 2, 3, \ldots \)

\( h \) is a "characteristic value" or eigenvalue.

\[ E_y(x, z) = C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-y z} \]

To find the \( H \) fields:

\[ \frac{\partial E_y}{\partial x} = -j \omega \mu H_z \]

\[ H_z = -\frac{1}{j \omega \mu} \frac{\partial E_y}{\partial x} = -\frac{m \pi}{j \omega \mu a} \cos \left( \frac{m \pi}{a} x \right) e^{-y z} \]

\[ \frac{\partial E_y}{\partial y} = -j \omega \mu H_x \]

\[ H_x = \frac{-y}{j \omega \mu} E_y = -\frac{y}{j \omega \mu} C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-y z} \]

Transverse magnetic (TM) fields.

There is always and everywhere a magnetic field vector transverse to the direction of propagation. \( H_z = 0 \)

Use the wave equation to find \( H_y \)

\[ \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} = -\omega^2 \mu \varepsilon H_y \quad \text{since} \quad \frac{\partial H_y}{\partial y} = 0 \]

\[ \frac{\partial^2 H_y}{\partial x^2} = -(\omega^2 \mu \varepsilon + \frac{\partial^2}{\partial y^2}) H_y = -\frac{\partial^2}{\partial x^2} H_y \]

Write \( H_y \) as a product of functions \( H_y = H_y^0(x) e^{-y z} \)

\[ \frac{\partial^2 H_y^0}{\partial x^2} e^{-y z} = -\frac{\partial^2}{\partial x^2} e^{-y z} \]

\[ H_y = H_y^0(x) e^{-y z} \]
\[
\frac{d^2 H_y}{dx^2} = -\Phi^2 H_y (x)
\]

Solutions are \( H_y = C_3 \sin \Phi x + C_4 \cos \Phi x \)

The boundary conditions do not directly apply to \( H_y \), but can be applied to \( E_z \) by using Maxwell's Equations

\[
E_z = \frac{1}{j\omega \epsilon} \frac{\partial H_y}{\partial x} \quad (\text{p. 2})
\]

\[
= \frac{1}{j\omega \epsilon} \left[ (C_3 \sin \Phi x + C_4 \cos \Phi x) e^{-k_z z} \right]
\]

\[
= \frac{k}{j\omega \epsilon} \left( C_3 \cos \Phi x - C_4 \sin \Phi x \right) e^{-k_z z}
\]

\[
E_z = \frac{\Phi}{j\omega \epsilon} \left( C_3 \cos \Phi x - C_4 \sin \Phi x \right) e^{-k_z z}
\]

We now apply B.C. that \( E_z = 0 \) at \( x = 0, a \)

\( E_z = 0 \) at \( x = 0 \) requires \( C_3 = 0 \)

\( E_z = 0 \) at \( x = a \) requires \( \Phi = \frac{m \pi}{a} \), \( m = 0, \pm 1, \pm 2, \ldots \)

Transverse electromagnetic (TEM) waves.

This is similar to the previous solutions EXCEPT there are no \( z \) fields \( H_z = E_z = 0 \)

The TE modes vanish since \( H_z = 0 \)

All but the \( m = 0 \) TM mode vanishes

Since \( E_z \) vanishes when \( \Phi = 0 \) we still have a TM\(_0\) mode, which is the TEM mode.

\[
H_y = C_4 e^{-k_z z} \quad \Rightarrow \quad H_y = C_4 \cos \left( \frac{m \pi}{a} x \right) e^{-k_z z}
\]

\[
E_x = \frac{k}{j\omega \epsilon} C_4 e^{-k_z z} \quad \text{goes to } 1 \text{ if } m = 0
\]

\( E_z = 0 \) so we have an \( E_x, H_y \) but \( E_z \rightarrow 0 \)

\[
E_z = j \frac{m \pi}{\omega \epsilon a} C_4 \sin \left( \frac{m \pi}{a} x \right) e^{-k_z z} \rightarrow 0
\]
\[ E_y = C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-\gamma z} \]

\[ H_z = -\frac{1}{j \omega \mu} \frac{\partial E_y}{\partial x} = -\frac{m \pi}{j \omega \mu a} C_1 \cos \left( \frac{m \pi}{a} x \right) e^{-\gamma z} \]  

\[ H_x = -\frac{\gamma}{j \omega \mu} E_y = -\frac{\gamma}{j \omega \mu} C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-\gamma z} \]  

[4.12]  

**Parallel-plate TE\(_m\) modes**  
\( m = 0, \pm 1, \pm 2, \ldots \)  

**TE\(_m\) modes have**  
\( H_z \neq 0, \ E_z = 0 \)

---

**FIGURE 4.2.**  
\( \text{TE}_1 \) and \( \text{TE}_2 \) modes. The electric and magnetic field distributions for the \( \text{TE}_1 \) and the magnetic field distribution for the \( \text{TE}_2 \) modes in a parallel-plate waveguide. Careful examination of the field structure for the \( \text{TE}_1 \) mode indicates that the magnetic field lines encircle the electric field lines (i.e., displacement current) in accordance with [4.1a] and the right-hand rule. The same is true for the \( \text{TE}_2 \) mode, although the electric field distribution for this mode is not shown.
Parallel-plate TMₘ modes, 
\( m = 0, \pm 1, \pm 2, \ldots \)

\[
H_x = C_4 \cos \left( \frac{m\pi}{a} x \right) e^{-\gamma z} \\
E_x = \frac{j}{j \omega \epsilon} H_x = \frac{j}{j \omega \epsilon} C_4 \cos \left( \frac{m\pi}{a} x \right) e^{-\gamma z} \\
E_z = \frac{j m \pi}{\omega \epsilon a} C_4 \sin \left( \frac{m\pi}{a} x \right) e^{-\gamma z}
\]

\[\text{[4.13]}\]

TMₘ modes have \( E_z \neq 0, H_z = 0 \)

The magnetic field lines increase near the plates.
The electric field lines wrap around the magnetic fields.
Note that \( E_z \) vanishes on the plates. This is the important BC.

FIGURE 4.3. TM₁ and TM₂ modes. The electric and magnetic field distributions for the TM₁ and TM₂ modes in a parallel-plate waveguide. Note that for TM₁, the electric field lines encircle the magnetic field lines (Faraday's law) in accordance with [4.1c]. The same is true for TM₂, although the magnetic field distribution for TM₁ is not shown.
Transverse Electromagnetic (TEM) Waves

Note that contrary to the TE case, the TM solutions do not all vanish for \( m = 0 \). Since \( E_z \) is zero for \( m = 0 \), the TM_0 mode is actually a transverse electromagnetic (or TEM) wave. In this case, we have

\[
\begin{align*}
H_y &= C_4 e^{-\gamma z} \\
E_x &= \frac{\gamma}{j\omega \epsilon} C_4 e^{-\gamma z} \\
E_z &= 0
\end{align*}
\]

This is actually a TM_0 mode since there is a \( E_x \) but \( E_z = 0 \).
4.1.2. Cutoff Frequency, Phase Velocity, Wavelength.

TE and TM modes have similar characteristics

(i) \( E \) and \( H \) have sinusoidal standing wave distributions in the \( x \) direction.

(ii) \( xy \) planes are equiphasic planes, i.e., surfaces of constant phase.

(iii) the equiphasic surfaces propagate along the waveguide with velocity \( v_p = \frac{\omega}{\beta} \)

Consider \( E_y \) for TE waves

\[
E_y = C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-\beta z}
\]

Assume \( \beta = 0 \) so \( \beta \to j \beta \). Then write the real wave as

\[
\frac{E_y}{c_0} (x, z, t) = C_1 \sin \left( \frac{m \pi}{a} x \right) \cos (\omega t - \beta z)
\]

transverse standing wave

By definition \( \beta^2 = \gamma^2 + \omega^2 \mu e \). But \( \beta^2 = \frac{m^2 \pi^2}{a^2} \)

Solving for \( \gamma \)

\( \gamma = \sqrt{\frac{m^2 \pi^2}{a^2} - \omega^2 \mu e} = \sqrt{\left( \frac{m \pi}{a} \right)^2 - \omega^2 \mu e} \)

There is a cutoff frequency \( f_{cm} \) for which \( \gamma = 0 \) or

\[
\frac{m^2 \pi^2}{a^2} = 4 \pi^2 f_{cm}^2 \mu e
\]

Solving for \( f \)

\[
f_{cm} = \frac{m}{2a \sqrt{\mu e}} = \frac{m v_p}{2a}
\]

For \( f > f_{cm} \)

\[
\gamma = j \beta_m = j \sqrt{\omega^2 \mu e - \left( \frac{m \pi}{a} \right)^2} = j \sqrt{\omega^2 \mu e - \left( \frac{2 f_{cm} \pi}{v_p} \right)^2}
\]

Using the expression for \( f_{cm} \)
\[ \chi = \sqrt{\omega^2 \mu_e - \left( \frac{2\pi f_{cm}}{\nu_p} \right)^2} \]

\[ \chi = j \sqrt{\beta^2 - \left( \frac{2\pi f_{cm}}{\nu_p} \right)^2} \]

\[ \chi = j \beta \sqrt{1 - \left( \frac{2\pi f_{cm}}{\nu_p} \right)^2} \]

\[ \chi = j \beta \sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2} \]

where \( f > f_{cm} \)

\[ \chi = j \beta_m \sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}, \quad f > f_{cm} \]

\( \beta_m \) emphasizes that this is for mode \( m \).

Note that for \( f < f_{cm} \), 
\[ \left( \frac{m\pi}{a} \right)^2 - \omega^2 \mu_e > 0 \quad \text{and} \]

\[ \chi = \lambda_m \sqrt{\left( \frac{m\pi}{a} \right)^2 - \omega^2 \mu_e} = \beta \sqrt{\left( \frac{f_{cm}}{f} \right)^2 - 1}, \quad f < f_{cm} \]

This is known as an evanescent wave.

Attenuation NOT due to energy loss but from boundary conditions.

\[ \lambda_m = \frac{2\pi}{\beta_m} = \frac{2\pi}{\beta_m \sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}} = \frac{2\pi/\beta}{\sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}} \]

\[ \lambda_m = \frac{\lambda}{\sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}} \]

Similarly

\[ \nu_{pm} = \frac{\omega}{\beta_m} = \frac{\nu_p}{\sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}} \]

We see that \( \lambda_m \) and \( \nu_{pm} \) vary as a function of the mode frequency.
4.1.3 TE & TM modes as superpositions of TEM waves

\[ E_y(x,z) = C_1 \sin \left( \frac{m \pi}{a} x \right) e^{-jBz} \]

Rewrite the \( \sin \) as \( \frac{e^{j \frac{m \pi x}{a}} - e^{-j \frac{m \pi x}{a}}}{2j} \)

\[ E_y(x,z) = \frac{C_1}{2j} \left[ e^{j \frac{m \pi x}{a}} - e^{-j \frac{m \pi x}{a}} \right] e^{-jBz} \]

\[ E_y(x,z) = \frac{C_1}{2} e^{-j \frac{\pi}{2} \left[ e^{j \frac{m \pi x}{a} - \frac{\beta z}{z} - j \frac{\pi}{2}} - e^{-j \frac{m \pi x}{a} + \beta z + j \frac{\pi}{2}} \right] } \]

\[ \perp \text{ polarized wrt xz plane} \]
\[ \text{TEM wave} \]
\[ \text{propagating in the} \]
\[ k_1 = -\hat{x} \cos \Theta_{im} + \hat{z} \sin \Theta_{im} \]
\[ \text{direction} \]

\[ k_2 = \hat{x} \cos \Theta_{im} + \hat{z} \sin \Theta_{im} \]

\[ \perp \text{ polarized wrt xz plane} \]
\[ \text{TEM wave} \]
\[ \text{propagates in the} \]
\[ k_2 = \hat{x} \cos \Theta_{im} + \hat{z} \sin \Theta_{im} \]
\[ \text{direction} \]

\[ \hat{k}_1 = -\frac{\beta_x \hat{x} + \beta_z \hat{z}}{\beta} \]

Identify \[ \beta_x = \beta \cos \Theta_{im} = \frac{m \pi}{a} \]
\[ \beta_z = \beta \sin \Theta_{im} = \frac{\sqrt{\beta^2 - \left(\frac{m \pi}{a}\right)^2}}{\omega \sqrt{\mu \varepsilon}} \]

From (4) \[ \cos \Theta_{im} = \frac{\frac{m \pi}{a}}{\beta} = \frac{\frac{m \pi}{a}}{\frac{2\pi}{\lambda}} = \frac{\frac{m}{2\pi}}{\frac{1}{\lambda}} = \frac{f_{cm} \sqrt{\mu \varepsilon}}{\omega_c} = \frac{f_{cm}}{\omega_c} = \frac{f_{cm}}{c} \]

where we used \[ f_{cm} = \frac{m}{2\pi \sqrt{\mu \varepsilon}} \]
\[ \theta_{im} = \cos^{-1}\left[ \frac{m \lambda}{2a} \right] \quad m = 0, 1, 2, \ldots \]

Waves only propagate for discrete \( \theta_{im} \)

\[ \text{FIGURE 4.9. Representation of propagating waveguide modes as} \]
"rays" reflecting between the parallel plates. The dashed lines represent the phase fronts, and the solid lines represent the ray paths.

The different \( \theta_{im} \) look like waves propagating at different \( \theta_{im} \)

We can do a geometric wave analysis.

\( BB' \) represents a constant phase front for a wave from \( A \) to \( B \)

For the wave to propagate

\( CC' \) must also be a constant phase front.

\[ \Rightarrow \text{phase change along } BC \text{ and } B'C' \text{ must be multiple of } 2\pi \]

\[ \beta(BC - B'C') = m \cdot 2\pi \]

\[ \overline{BC} = \frac{a}{\cos \theta_{im}} \]
\[
\sin \theta_{im} = \frac{\overline{B'C'}}{BC'}
\]

\[
\overline{BC'} = a \tan \theta_{im} - \frac{a}{\tan \theta_{im}}
\]

Combining:
\[
\beta (\overline{BC} - \overline{B'C'}) = m \cdot 2\pi
\]
\[
\beta \left( \frac{a}{\cos \theta_{im}} - \overline{BC'} \sin \theta_{im} \right) = m \cdot 2\pi
\]
\[
\beta \left( \frac{a}{\cos \theta_{im}} - \left( a \tan \theta_{im} - \frac{a}{\tan \theta_{im}} \right) \sin \theta_{im} \right) = m \cdot 2\pi
\]
\[
\beta \left( \frac{a}{\cos \theta_{im}} - a \frac{\sin \theta_{im}}{\cos \theta_{im}} \sin \theta_{im} + a \frac{\sin \theta_{im}}{\sin \theta_{im}} \cos \theta_{im} \right) = m \cdot 2\pi
\]
\[
\beta \alpha \left( 1 - \frac{\sin^2 \theta_{im}}{\cos \theta_{im}} + \cos \theta_{im} \right) = m \cdot 2\pi
\]
\[
\beta \alpha \left( \frac{\cos^2 \theta_{im}}{\cos \theta_{im}} + \cos \theta_{im} \right) = m \cdot 2\pi
\]

2\beta \alpha \cos \theta_{im} = m \cdot 2\pi

2 \left( \frac{2\pi}{\lambda} \right) a \cos \theta_{im} = m \cdot 2\pi

\cos \theta_{im} = \frac{m \lambda}{2a} \quad m = 0, 1, 2, \ldots
A simple physical model is to break the TE mode into \( \perp \) polarized TEM waves and the TM mode into II polarized TEM waves.

For both \( \lambda_{c,m} = \frac{2a}{m} \) cutoff wavelength

at \( \lambda_{c,m} \theta_{im} = 0 \)

Because of this zig-zag path of the wave, the energy propagates at a velocity less than that in free space, called the group velocity

\[ V_g = V_p \sin \theta_{im} \]

4.1.4. Attenuation in Parallel Plate Waveguides

Practical waveguides made of copper or brass usually coated with silver

Assume losses are very small so that they have a negligible effect on the field distribution, i.e., small perturbation approach.

Assume fields vary as \( e^{-\alpha z} \) so power goes as \( e^{-2\alpha z} \)

\[ -\frac{\partial P_{AV}}{\partial z} = +2\alpha c P_{AV} \]

Power lost per unit length

\[ \frac{\text{Power transmitted}}{\text{Power transmitted}} = \frac{2\alpha c P_{AV}}{2\alpha c} = 2\alpha \]

\[ \alpha_c = \frac{\text{Power lost per unit length}}{2 \times \text{Power Transmitted}} = \frac{P_{loss}}{P_{AV}} \]
Consider losses due to conduction currents for TEM waves.

For TEM waves

\[ H_y = C_4 e^{-j\beta y} \]
\[ E_x = \frac{B}{\omega e} C_4 e^{-j\beta z} \]

Surface current density on each plate is

\[ J_s = \hat{x} \times \hat{H} = \hat{x} \times C_4 e^{-j\beta y} \hat{y} = \frac{1}{2} C_4 e^{-j\beta y} \]

\[ |J_s| = C_4 \]

Total loss for a length of 1 meter is

\[ P_{\text{loss}} = 2 \int_0^b \left( \frac{1}{2} |J_s|^2 R_s \right) dy \cdot dz = C_4^2 R_s \cdot b \]

Surface resistance

\[ R_s = \frac{1}{\sigma a^2} \]

\[ |S_{AV}| = \frac{1}{2} |E_x|^2 \left( \frac{\gamma}{\eta} \right) = \frac{1}{2} \gamma |H_y|^2 = \frac{1}{2} \gamma C_4^2 \]

\[ P_T = |S_{AV}| ab = \frac{1}{2} \gamma C_4^2 ab \]

Using definition of \( \alpha_c \)

\[ \alpha_c = \frac{C_4 R_s b}{2 (\frac{1}{2} \gamma C_4^2 ab)} = \frac{R_s}{\gamma a} = \frac{1}{\eta a} \sqrt{\frac{\omega \mu_0}{2\pi}} \]
Now for parallel plate + E waves.

\[ E_y = \frac{c_1 \sin \left( \frac{m \pi}{a} x \right)}{j \omega \mu a} e^{-\frac{\beta z}{\gamma}} \]

\[ H_z = -\frac{m \pi}{j \omega \mu a} c_1 \cos \left( \frac{m \pi}{a} x \right) e^{-\frac{\beta z}{\gamma}} \]

\[ H_x = -\frac{\gamma}{j \omega \mu a} c_1 \sin \left( \frac{m \pi}{a} x \right) e^{-\frac{\beta z}{\gamma}} \]

Just as for TEM waves, there is a surface current due to the tangential \( H_z \) field.

\[ |J_{sy}| = |H_z| = \frac{m \pi c_1}{j \omega \mu a} \]

this is a y-directed current

Power loss is to direction of propagation

\[ P_{loss} = 2 \int_0^b \int_0^a \frac{1}{2} |J_{sy}|^2 R_s dydz = \mathcal{E}(1) \left( \frac{1}{2} \frac{m^2 \pi^2 c_1^2}{j \omega \mu a^2} \sqrt{\frac{\omega \mu_0}{2\gamma}} \right) \]

substitute expressions for \( |J_{sy}| \) and \( R_s \)

Now compute

\[ |S_{av}| = \frac{1}{2} \operatorname{Re} \left\{ \mathcal{E} \times \mathcal{H}^* \right\} \mathcal{E} = \frac{1}{2} E_y H_x^* = \frac{1}{z} \left[ c_1 \sin \left( \frac{m \pi}{a} x \right) e^{-\frac{\beta z}{\gamma}} \right] \left[ -\frac{\gamma}{j \omega \mu a} c_1 \sin \left( \frac{m \pi}{a} x \right) e^{-\frac{\beta z}{\gamma}} \right] \]

\[ \frac{\partial}{\partial x} \left( \frac{\gamma}{j \omega \mu a} \right) = -\dot{\mathcal{E}} \to \mathcal{E} + j\beta \equiv j\beta \]

assume losses are small

\[ |S_{av}| = \frac{1}{2} c_1^2 \sin \left( \frac{m \pi}{a} x \right) \frac{\gamma}{j \omega \mu a} \]

\[ |S_{av}| = \frac{1}{j} \frac{c_1 \mu_0}{2 \omega \mu a} \sin \left( \frac{m \pi}{a} x \right) \]

this is a power density

\[ P_{av} = \int_0^b \int_0^a \frac{\partial c_1^2}{2 \omega \mu a} \sin \left( \frac{m \pi}{a} x \right) dx dy = \frac{\partial c_1^2}{2 \omega \mu a} \left[ \frac{1}{2} - \frac{1}{4} \sin \left( \frac{m \pi}{a} x \right) \right] \]

\[ P_{av} = \frac{\partial c_1^2}{2 \omega \mu a} \left[ \frac{1}{2} ba \right] \]

by integrating over the surface of the plates we get power loss

\[ \alpha c_{TE} = \frac{P_{loss}}{2 P_{av}} = \frac{\sqrt{\frac{m^2 \pi^2 c_1^2}{j \omega \mu a^2} \sqrt{\frac{\omega \mu_0}{2\gamma}}}}{2 \frac{\beta c_1^2}{j \omega \mu a}} = \frac{2 m^2 \pi^2}{\beta \omega \mu_0 a^2} \sqrt{\frac{\omega \mu_0}{2\gamma}} \]

use definition and our expressions for TE waves to determine losses
We can re-write this in a more usable form

\[ \alpha_{c, \text{EM}} = \frac{2\pi^2 \rho^2}{\beta \omega \mu a^3} \sqrt{\frac{1}{2\sigma}} = \frac{2\pi^2 \rho^2 \frac{R_s}{\beta \omega \mu a^3}}{4a^2 \rho e} \]

\[ = \frac{2\pi^2 \rho^2 R_s}{\beta \omega \mu a} \frac{4a^2 \rho e}{f_{\text{cm}}} \]

\[ = \frac{8\pi^2 \rho^2 R_s \frac{2\pi f}{f_{\text{cm}}} \frac{R_s}{\beta \omega \mu a}}{16 \pi^3 \rho e f} \frac{4\pi^2 \beta a}{(\frac{f_{\text{cm}}}{f})} \]

\[ = \frac{4\pi^2 \rho e f}{a\beta (1 - (\frac{f_{\text{cm}}}{f})^2)} \frac{(\frac{f_{\text{cm}}}{f})^2 R_s}{\beta a} = \frac{4\pi^2 \rho e}{a\beta} \frac{1}{\sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}} \]

\[ = \frac{2\pi \rho e}{a\sqrt{\mu e \rho a}} \frac{(\frac{f_{\text{cm}}}{f})^2 R_s}{\sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}} \]

\[ = \frac{2}{\alpha} \sqrt{\frac{\epsilon}{\mu}} \frac{(\frac{f_{\text{cm}}}{f})^2 R_s}{\sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}} \]

\[ \alpha_{c, \text{TM}} = \frac{2 \frac{R_s}{\eta a \sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}}}{\frac{R_s}{\eta a \sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}}} \]

Similar calculations give for TM mode:

\[ \alpha_{c, \text{TM}} = \frac{2 \frac{R_s}{\eta a \sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}}}{\frac{R_s}{\eta a \sqrt{1 - (\frac{f_{\text{cm}}}{f})^2}}} \]
This is a graphical presentation of the mode losses.

**Cutoff frequencies for different modes**

![Graph showing cutoff frequencies for different modes](image)

**FIGURE 4.11.** Attenuation versus frequency for parallel-plate waveguide. Attenuation-versus-frequency characteristics of waves guided by parallel plates.

This figure shows the attenuation as a function of frequency for a few modes. Higher-order modes have higher losses.

**TM** modes have higher losses than **TE** modes since **TM** modes have a tangential $\mathbf{J}$ due to tangential $\mathbf{H}_y$.

\[ \mathbf{H}_y = C_4 \cos \left( \frac{m \pi}{a} x \right) e^{-\nu z} \]

**TE** modes have lower losses at higher frequencies.

\[ \mathbf{H}_z = -\frac{m \pi}{j \omega \mu a} C_1 \cos \left( \frac{m \pi}{a} x \right) e^{-\nu z} \]

since as $\omega$ increases, surface currents decrease very strongly with frequency dependence.
Attenuation due to dielectric losses.

Just like for TEM waves we have losses associated with the dielectric (i.e. polarization currents)

\[ \varepsilon_c = \varepsilon' - j\varepsilon'' \]

For TEM modes just compute α just like for a uniform plane wave:

\[ \alpha = \omega \sqrt{\frac{\mu_0}{\varepsilon}} \left[ \sqrt{1 + \left( \frac{\varepsilon'}{\varepsilon} \right)^2} - 1 \right] \frac{1}{2} \]

Use \[ \tan \delta_c = \frac{\varepsilon'}{\varepsilon} \rightarrow \frac{\omega \varepsilon''}{\omega \varepsilon'} \] for a dielectric

Just as we did for conduction losses we have to assume the losses are not too large so the field configurations don't change, i.e., a perturbation analysis.

However, the propagation constant will change.

For \( f \to \infty \) we know

\[ \bar{\gamma} = j \left[ \mu \varepsilon \left( \omega^2 - \omega_{m}^2 \right) \right] \frac{1}{2} = j \left[ \omega^2 \varepsilon \varepsilon - \left( \frac{m \pi}{a} \right)^2 \right] \frac{1}{2} \]

If we substitute \( \varepsilon \rightarrow \varepsilon' - j\varepsilon'' \)

\[ \bar{\gamma} = j \left[ \omega^2 \mu \left( \varepsilon' - j\varepsilon'' \right) - \left( \frac{m \pi}{a} \right)^2 \right] \frac{1}{2} \]

\[ = j \left[ \omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right] \frac{1}{2} \left[ \frac{\omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 - j \omega^2 \mu \varepsilon''}{\omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2} \right] \]

\[ = j \left[ \omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right] \frac{1}{2} \left[ 1 - \frac{j \omega^2 \mu \varepsilon''}{\omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2} \right] \]

\[ = j \left[ \omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right] \frac{1}{2} \left[ 1 - \frac{j \omega^2 \mu \varepsilon''}{2 \left[ \omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right]} + \ldots \right] \] binomial expansion

\[ \bar{\gamma} \approx \frac{\omega^2 \mu \varepsilon}{2 \left[ \omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right]} + j \left[ \omega^2 \mu \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right] \frac{1}{2} \]

keeping only the first term since \( \varepsilon'' \) is small
Now, if we recall that \( \omega_{c,m} = \frac{2\pi m}{2a \sqrt{\varepsilon \varepsilon'}} = \frac{\pi m}{a \sqrt{\varepsilon \varepsilon'}} \)

we define \( \omega_{c,m} \) for a lossy dielectric as

\[
\omega_{c,m} = \frac{\pi m}{a \sqrt{\varepsilon \varepsilon'}}
\]

Rewriting \( \bar{\gamma} \)

\[
\bar{\gamma} = \frac{\omega^2 \varepsilon''}{2 \left[ \omega \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right]^2} + j \left[ \frac{\left( \omega \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right)}{2 \left[ \omega \varepsilon' - \left( \frac{m \pi}{a} \right)^2 \right]^2} \right]^{1/2}
\]

\[
\bar{\epsilon} = \frac{\omega^2 \varepsilon''}{2 \left( \omega^2 - \left( \frac{m \pi}{a \sqrt{\varepsilon \varepsilon'}} \right)^2 \right)} + j \left[ \left( \omega^2 - \left( \frac{m \pi}{a \sqrt{\varepsilon \varepsilon'}} \right)^2 \right) \right]^{1/2}
\]

\[
\bar{\sigma} = \frac{\omega \sqrt{\varepsilon''}}{2 \sqrt{1 - \left( \frac{\omega_{c,m}}{\omega} \right)^2}} + j \left[ \frac{\mu \varepsilon' (\omega^2 - \omega_{c,m}^2)}{\omega} \right]
\]

\[
\bar{\gamma} = \alpha_d + j \beta_m
\]

\( \alpha_d \) attenuation constant due to dielectric losses

\[
\alpha_d = \frac{\omega \sqrt{\varepsilon''}}{2 \sqrt{1 - \left( \frac{\omega_{c,m}}{\omega} \right)^2}}
\]

you get TEM mode by substituting \( \omega_{c,m} = 0 \)
4.1.5. Voltage, Current & Impedance

TEM mode corresponds to voltage & current waves of transmission line analysis.

The voltage between the plates for a TEM wave is

\[
V_{\text{TEM}}(z) = -\int_{x=0}^{\infty} E_x dx = \int_{-\infty}^{0} E_x(x) dx = \frac{\beta}{\omega \epsilon} C_4 e^{-j\beta z}
\]

\[
V_{\text{TEM}}(z) = -\frac{\beta a C_4}{\omega \epsilon} e^{-j\beta z} = V^+ e^{-j\beta z}
\]

\[
I_{\text{TEM}}(z) = |J_s| (1m) = |\hat{n} \times \hat{H}| = \left| \pm \hat{x} \times \hat{y} \left[ H_x \right]_{x=0} \right| = \left| \hat{x} \times \hat{y} C_4 e^{-j\beta z} \right|
\]

\[
I_{\text{TEM}}(z) = C_4 e^{-j\beta z}
\]

\[
I_{\text{TEM}}(z) = \frac{V^+}{Z_{\text{TEM}}} e^{-j\beta z}
\]

\[
V^+ = -\frac{\beta a C_4}{\omega \epsilon} = -\eta a C_4
\]

\[
Z_{\text{TEM}} = \frac{E_x}{H_y} = \frac{\beta}{\omega \epsilon} C_4 e^{-j\beta z}
\]

\[
Z_{\text{TEM}} = \eta
\]

Voltage and current definitions are ambiguous for TE & TM modes.
Consider a TEₘ mode

\[ E_y = C_1 \sin \left( \frac{m\pi}{a} x \right) e^{-j\beta z} \]

\[ H_z = -\frac{m\pi}{j\omega \mu} C_1 \cos \left( \frac{m\pi}{a} x \right) e^{-j\beta z} \]

\[ H_x = -\frac{\beta}{\omega} C_1 \sin \left( \frac{m\pi}{a} x \right) e^{-j\beta z} \]

\[ \beta = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2} \]

Electric field is in \( \hat{y} \) direction so you would normally define voltage in \( \hat{y} \) direction. But

\[ \int_0^a \vec{E} \cdot d\vec{l} = -\int_0^a \hat{\gamma} \cdot \vec{E}_y \cdot \hat{\gamma} = 0 \]

Also, the current flows in the \( \hat{z} \)-direction

\[ J_z = \hat{\gamma} \times \hat{\gamma} \left[ \frac{H_z}{H_x} \right]_{x=0} = \hat{\gamma} \times \hat{\gamma} \left[ \frac{-\frac{m\pi \gamma}{j\omega \mu}}{-\frac{\beta}{\omega}} \right] e^{-j\beta z} = \hat{\gamma} \left[ \gamma \frac{m\pi \gamma}{j\omega \mu} \right] e^{-j\beta z} \]

Practically define \( V \) and \( I \) such that

(i) Line voltage \( \propto \) transverse electric field component

(ii) Line current \( \propto \) transverse \( H \) field component

(iii) \( P_{AV} = \frac{1}{2} \rho \omega \{VI^*\} \)

\[ Z_{TM_{m_{\text{TE}}}} \equiv \frac{E_x}{H_y} = \frac{-E_y}{H_x} \]

\[ \hat{\gamma} = (-\hat{\gamma}) \]

\[ Z_{TE_m} = -\frac{E_y}{H_x} = -\frac{C_1 \sin \left( \frac{m\pi}{a} x \right) e^{-j\beta x}}{-\frac{\beta}{\omega} C_1 \sin \left( \frac{m\pi}{a} x \right) e^{-j\beta z}} = \frac{\omega \mu}{\beta} = \frac{\omega}{\beta \sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}} \]

\[ Z_{TE_m} = \frac{\gamma}{\sqrt{1 - \left( \frac{f_{cm}}{f} \right)^2}} \]

resistive

larger than \( \gamma \)
For TM modes

\[ Z_{TM_m} = \frac{E_x}{H_y} = \frac{\beta}{\omega \varepsilon} \cdot \frac{c_4 \cos \left( \frac{m \pi}{a} x \right) e^{-j\beta z}}{c_4 \cos \left( \frac{m \pi}{a} x \right) e^{-j\beta z}} = \frac{\beta}{\omega \varepsilon} \]

\[ Z_{TM_m} = \eta \sqrt{1 - \left( \frac{f_{cem}}{f} \right)^2} \]
4.1.6 E&M field distributions

How do you plot field lines? Start with time domain expressions.

\[ \mathcal{E}_y(x, z, t) = Re \left\{ \mathcal{E}_0(x) e^{-j\beta z} \right\} = C_1 \sin \left( \frac{m\pi}{a} x \right) \cos \left( \omega t - \bar{\beta} z \right) \]

\[ \mathcal{H}_x(x, z, t) = -\frac{\beta}{j\omega} C_1 \sin \left( \frac{m\pi}{a} x \right) \cos \left( \omega t - \bar{\beta} z \right) \]

\[ \mathcal{H}_z(x, z, t) = -\frac{\omega}{\mu a} C_1 \cos \left( \frac{m\pi}{a} x \right) \sin \left( \omega t - \bar{\beta} z \right) \]

The magnetic field must vary as \( \frac{dx}{dz} \) where

\[ \frac{dx}{dz} = \frac{\mathcal{H}_x(x, z, t)}{\mathcal{H}_z(x, z, t)} = \left( \frac{\beta}{j\omega} \right) \frac{C_1 \sin \left( \frac{m\pi}{a} x \right) \cos \left( \omega t - \bar{\beta} z \right)}{C_1 \cos \left( \frac{m\pi}{a} x \right) \sin \left( \omega t - \bar{\beta} z \right)} \]

Rearranging,

\[ \left( \frac{\frac{m\pi}{a}}{\sin \left( \frac{m\pi}{a} x \right)} \right) dx = \frac{\beta \cos \left( \omega t - \bar{\beta} z \right)}{\sin \left( \omega t - \bar{\beta} z \right)} \frac{dz}{\sin \left( \frac{m\pi}{a} x \right)} \]

Recognizing

\[ \frac{\frac{m\pi}{a}}{\cos \left( \frac{m\pi}{a} x \right)} dx = d \left[ \sin \left( \frac{m\pi}{a} x \right) \right] \]

\[ \beta \cos \left( \omega t - \bar{\beta} z \right) dz = -d \left[ \sin \left( \omega t - \bar{\beta} z \right) \right] \]

And substituting,

\[ \frac{d \left[ \sin \left( \frac{m\pi}{a} x \right) \right]}{\sin \left( \frac{m\pi}{a} x \right)} = -d \left[ \sin \left( \omega t - \bar{\beta} z \right) \right] \]

Integrating

\[ \ln \left[ \sin \left( \frac{m\pi}{a} x \right) \right] = -\ln \left[ \sin \left( \omega t - \bar{\beta} z \right) \right] + \text{const}. \]

\[ \ln \left[ \sin \left( \frac{m\pi}{a} x \right) \right] + \ln \left[ \sin \left( \omega t - \bar{\beta} z \right) \right] = \text{const} \]

\[ \ln \left[ \sin \left( \frac{m\pi}{a} x \right) \sin \left( \omega t - \bar{\beta} z \right) \right] = \text{const}. \]

\[ \Rightarrow \sin \left( \frac{m\pi}{a} x \right) \sin \left( \omega t - \bar{\beta} z \right) = \text{const}. \]

Which can be used to plot \( H \).
4.2 Dielectric Waveguides

especially appropriate for optical wavelengths
1.3 μm (minimum distortion) and 1.5 μm (minimum loss)

Modal Theory.

want solutions with \( \vec{E} \) \( z \) variation

consider TM solutions written in terms of \( E_z(x, z) \)

\[
E_z(x, z) = E_z^0(x) e^{-\gamma z}
\]

where \( \frac{\partial}{\partial y} \rightarrow 0 \)

and \( E_z(x, z) = E_z^0(x) e^{-\gamma z} \)

We write a wave equation in the slab:

\[
\frac{\partial^2 E_z}{\partial x^2} + \gamma^2 E_z = -\omega^2 \mu_e E
\]

Rewriting for \( E_z \) component

\[
\frac{\partial^2 E_z}{\partial x^2} + \gamma^2 E_z = -\omega^2 \mu_e E_z
\]

use \( E_z^0(x) e^{-\gamma z} \)

\[
\frac{d^2 E_z^0(x)}{dx^2} + \gamma^2 E_z^0(x) = 0
\]

Giving

\[
\frac{d^2 E_z^0(x)}{dx^2} + \gamma^2 E_z^0(x) = 0
\]

Must solve in both the slab and outside

define \( \alpha_d^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 \)
The general solution to this equation inside the slab is

\[ E_z^0(x) = C_a \sin(\beta_x x) + C_b \cos(\beta_x x) \quad |x| \leq \frac{d}{2} \]

where \( \beta_x^2 = \omega^2 \mu_0 \epsilon_0 - \beta^2 = \frac{\omega^2}{\mu_0} \)

just like before

but just in the dielectric

assumed lossless

The coefficients \( C_a \) and \( C_b \) refer to odd and even respectively.

For the waves to be guided by the slab, the fields outside the slab should exponentially decay:

\[ E_z^0(x) = \begin{cases} 
C_a e^{-\alpha_x(x - \frac{d}{2})} & x > \frac{d}{2} \quad \text{above the slab} \\
C_b e^{\alpha_x(x + \frac{d}{2})} & x \leq \frac{d}{2} \quad \text{below the slab} 
\end{cases} \]

\[ \alpha_x^2 = \beta^2 - \omega^2 \mu_0 \epsilon_0 = -\frac{\rho_0^2}{\mu_0} \]

Note that the sign reverses since it decays.

\[ \alpha_x = \sqrt{\beta^2 - \omega^2 \mu_0 \epsilon_0} \]

Here it is assumed that \( \epsilon = \epsilon_0 \) outside the slab.

We require continuity of the field at the boundary:

\[ \frac{\partial E_z}{\partial x} \bigg|_{x = \pm \frac{d}{2}} = 0 \]

\[ f = 306 \text{MHz} \]
\[ d = 0.75 \text{cm} \]
\[ \epsilon_d = 2 \epsilon_0 \]

We use the even/odd property of the transverse \( E \) fields to classify the solutions. Other classification schemes are possible.
The other field components can be calculated from
\[ E_z(x, z) = E_z^0(x) e^{-jβz} \]

We assumed that \( H_z = 0 \) (this is a TM mode).

We can use the previously written wave equations
\[ \begin{align*}
H_y &= -j\omega e \frac{\partial E_x}{\partial x} \\
E_x &= -\frac{1}{j\omega} \frac{\partial H_y}{\partial x}
\end{align*} \]

The complete "odd" solutions are
\[ \begin{align*}
E_z^0(x) &= \left[ C_0 \sin(\frac{βx}{2}) \right] e^{-α_x (x - \frac{d}{2})} \\
E_x^0(x) &= -j\frac{α_x}{β_x} \left[ C_0 \sin(\frac{βx}{2}) \right] e^{-α_x (x - \frac{d}{2})} \\
H_y^0(x) &= j\frac{ω e θ_0}{α_x} \left[ C_0 \sin(\frac{βx}{2}) \right] e^{-α_x (x - \frac{d}{2})}
\end{align*} \]

Above slab, \( x > \frac{d}{2} \)
\[ \begin{align*}
E_z^0(x) &= C_0 \sin(β_x x) \\
E_x^0(x) &= -j\frac{β_x}{C_0} \cos(β_x x) \\
H_y^0(x) &= j\frac{ω e θ_0}{β_x} \cos(β_x x)
\end{align*} \]

This is what defines odd.

Dielectric slab, \( |x| \leq \frac{d}{2} \)
\[ \begin{align*}
E_z^0(x) &= \left[ C_0 \sin(\frac{βx}{2}) \right] e^{-α_x (x + \frac{d}{2})} \\
E_x^0(x) &= -j\frac{α_x}{β_x} \left[ C_0 \sin(\frac{βx}{2}) \right] e^{-α_x (x + \frac{d}{2})} \\
H_y^0(x) &= j\frac{ω e θ_0}{α_x} \left[ C_0 \sin(\frac{βx}{2}) \right] e^{-α_x (x + \frac{d}{2})}
\end{align*} \]

Below slab, \( x \leq -\frac{d}{2} \)
There are several observations:

The transverse $E_x$ and $H_y$ are in phase, so power flow is in $z$ direction.

$E_z$ and $H_y$ are out of phase so no average power flows in $x$ direction.

$\alpha_x$ and $\beta_x$ come from continuity of $H_y$ at $x = \pm \frac{d}{2}$ (Continuity of $E_z$ was already used).

\[
H_y^0(x = \pm \frac{d}{2}) = \frac{j \omega \varepsilon_0}{\alpha_x} \frac{\varepsilon_0 \sin(\frac{\beta_x d}{2})}{\varepsilon_d \cos(\frac{\beta_x d}{2})} \quad \text{free space above slab-}
\]

\[
H_y^0(x = \pm \frac{d}{2}) = \frac{j \omega \varepsilon_d}{\beta_x} \frac{\varepsilon_0 \cos(\frac{\beta_x d}{2})}{\varepsilon_d \cos(\frac{\beta_x d}{2})} \quad \text{within slab-}
\]

\[
\frac{\omega \varepsilon_0}{\alpha_x} \frac{\varepsilon_0 \sin(\frac{\beta_x d}{2})}{\varepsilon_d \cos(\frac{\beta_x d}{2})} = \frac{j \omega \varepsilon_d}{\beta_x} \frac{\varepsilon_0 \cos(\frac{\beta_x d}{2})}{\varepsilon_d \cos(\frac{\beta_x d}{2})}
\]

\[
\frac{\varepsilon_0}{\varepsilon_d} \frac{\sin(\frac{\beta_x d}{2})}{\cos(\frac{\beta_x d}{2})} = \frac{\alpha_x}{\beta_x}
\]

We also know that

\[
\beta_x^2 = \omega^2 \mu_d \varepsilon_d - \beta^2
\]

\[
\alpha_x^2 = \beta^2 - \omega^2 \varepsilon_0
\]

So that \( \alpha_x^2 + \beta_x^2 = \omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) \)

A good way to find $\alpha_x$ and $\beta_x$ is graphically.
For \( \varepsilon_d = 2\varepsilon_0 \)
\[ d = 1.25\lambda_0 \]

You can plot \( \alpha_x^2 + \beta_x^2 = \omega^2 (\nu_d \varepsilon_d - \nu_0 \varepsilon_0) \) as the circle shown above.

\[
\frac{\alpha_x^2}{4} + \frac{\beta_x^2}{4} = \omega^2 \left( 2\nu_0 \varepsilon_0 - \nu_0 \varepsilon_0 \right) \frac{d^2}{4}
\]

\[
\left( \frac{\alpha_x d}{2} \right)^2 + \left( \frac{\beta_x d}{2} \right)^2 = \frac{\omega^2 d^2}{4} = \left( \frac{2\pi \lambda}{4} \right)^2 \frac{25\lambda^2}{4} = \frac{4\pi^2}{\alpha^2} \frac{25\lambda^2}{4}
\]

\[
\left( \frac{\alpha_x d}{2} \right)^2 + \left( \frac{\beta_x d}{2} \right)^2 = \left( \frac{5\pi}{4} \right)^2
\]

S1, S2 and S3 are the only possible solutions for these parameters.
The even modes are very similar to the odd mode solutions except

$$E_x^0(x) = C_e \cos(\beta_x x) \quad |x| \leq \frac{d}{2}$$

and the mode expression slightly changes to

$$\frac{\alpha_x}{\beta_x} = -\frac{\varepsilon_0}{\varepsilon_d} \cot\left(\frac{\beta_x d}{2}\right)$$

cutoff frequencies

$$\omega \sqrt{\mu_0 \varepsilon_0} < \beta < \omega \sqrt{\mu_d \varepsilon_d}$$

as $\beta \to \omega \sqrt{\mu_0 \varepsilon_0}$

$\alpha_x \to 0$

so the slab does no confining.

This represents cutoff since no bound modes exist.

Then

$$\frac{\beta_x d}{2} = (m-1) \frac{\pi}{2} \quad \text{where } m = 1, 3, 5, \text{etc.}$$

"odd" modes

$$\omega_c \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} d = (m-1) \frac{\pi}{2}$$

$$\frac{2 \pi f_c \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} d}{2} = (m-1) \frac{\pi}{2}$$

$$f_{c,m} = \frac{m-1}{2 \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$
FIGURE 4.17. Variation of field distributions for various modes with respect to thickness and frequency. Dependence of the dielectric waveguide mode structure on frequency and slab thickness. The plots show the variation of the axial field component $E_z$ over the vertical cross section of the dielectric slab. For all cases, $\epsilon_d = 2\epsilon_0$. Note that the numerical values are the same as those for Example 4-6.

**Observations**

1. Increasing the operating frequency increases the number of propagating modes in the guide.
2. Increasing the slab thickness also increases the number of propagating modes.
FIGURE 4.18. Even TM$_2$ mode field distribution. The electric field lines are shown as solid lines, while the magnetic field lines are orthogonal to the page and are indicated alternately with circles or crosses. Figure taken (with permission) from H. A. Haus, Waves and Fields in Optoelectronics, Prentice Hall, Englewood Cliffs, New Jersey, 1984.
TE Modes  

Look very similar to TM modes

Fundamental solution inside slab is

\[ H_\nu^0(x) = C_0 \sin(\beta_x x) + C_0 \cos(\beta_x x) \]

The exact form is usually not necessary.

We apply the boundary conditions at \( x = \pm d/2 \) to get

\[ \frac{\alpha_x}{\beta_x} = \frac{\nu_0}{\mu_d} \tan \left( \frac{\beta_x d}{2} \right) \quad \text{odd TE modes} \]

\[ \frac{\alpha_x}{\beta_x} = -\frac{\nu_0}{\mu_d} \cot \left( \frac{\beta_x d}{2} \right) \quad \text{even TE modes} \]

Cutoff occurs when \( \alpha_x = 0 \), i.e. no attenuation with distance.

The cutoff frequencies for the TE modes are the same as for the TM modes

\[ f_{c,TE_m} = \frac{(m-1)}{2d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}} \quad m = 1, 3, 5 \text{ odd modes} \]

\[ f_{c,TE_m} = \frac{(m-1)}{2d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}} \quad m = 2, 4, 6 \text{ even modes} \]
Dielectric covered ground plane

\[ E_y = 0 \]
\[ E_x = \frac{j \omega \mu}{\beta_x} c_0 \sin(\beta_x x) e^{-j \beta_x z} \]
\[ E_z = c_0 \cos(\beta_x x) e^{-j \beta_x z} \]

Odd TM
\[ 0 \leq x \leq \frac{d}{2} \]
\[ E_y = 0 \]
\[ E_x = -\frac{j \omega}{\beta_x} c_0 \cos(\beta_x x) e^{-j \beta_x z} \]
\[ E_z = c_0 \sin(\beta_x x) e^{-j \beta_x z} \]

Not allowed since \( E_z \) must be zero.

Even TE
\[ 0 \leq x \leq \frac{d}{2} \]
\[ E_y = -\frac{j \omega \mu}{\beta_x} c_0 \sin(\beta_x x) e^{-j \beta_x z} \]
\[ E_x = 0 \]
\[ E_z = 0 \]

Not allowed since \( E_y \) must be zero.

Odd TE
\[ 0 \leq x \leq \frac{d}{2} \]
\[ E_y = \frac{j \omega \mu}{\beta_x} c_0 \cos(\beta_x x) e^{-j \beta_x z} \]
\[ E_x = 0 \]
\[ E_z = 0 \]

Only Even TM and odd TE modes are allowed.
For these "surface wave" modes continuity of fields requires

\[ \frac{\alpha_x}{\beta_x} = \frac{\varepsilon_0}{\varepsilon_d} \tan \left( \frac{\beta x d}{2} \right) \quad \text{odd TM} \]

\[ \frac{\alpha_x}{\beta_x} = -\frac{\varepsilon_0}{\mu_d} \cot \left( \frac{\beta x d}{2} \right) \quad \text{Even TE} \]

For all modes

\[ \alpha_x^2 + \beta_x^2 = \omega^2 (\mu_d\varepsilon_d - \varepsilon_0\varepsilon_0) \]

Cutoff frequencies

\[ f_{c_{TM\sigma TE}} = \frac{(m-1)}{2d\sqrt{\mu_d\varepsilon_d - \varepsilon_0\varepsilon_0}} \]

\[ m = 1, 3, 5 \quad \text{odd TM} \]

\[ m = 2, 4, 6 \quad \text{Even TE} \]

If you make the dielectric thick then \( \beta \rightarrow \beta_d \).

In this case

\[ \alpha_x = \sqrt{\beta^2 - \omega^2 \varepsilon_0\varepsilon_0} \approx -\sqrt{\omega^2 (\mu_d\varepsilon_d - \varepsilon_0\varepsilon_0)} \]

or \( \alpha_x \approx \beta \sqrt{\frac{\mu_d\varepsilon_d}{\varepsilon_0\varepsilon_0} - 1} \) which is large and

which indicates the transverse fields attenuate rapidly.

If you make the dielectric thin then \( \beta \rightarrow \beta = \omega \sqrt{\varepsilon_0\varepsilon_0} \)

\[ \beta_x^2 = \omega^2 \mu_d\varepsilon_d - \beta^2 \approx \omega^2 \mu_d\varepsilon_d - \omega^2 \varepsilon_0\varepsilon_0 \]

In this thin limit \( \frac{1}{\lambda_x} \rightarrow 0 \) and

\[ \frac{\alpha_x}{\beta_x} = \frac{\varepsilon_0}{\varepsilon_d} \tan \left( \frac{\beta x d}{2} \right) \approx \frac{\varepsilon_0}{\varepsilon_d} \left( \frac{\beta x d}{2} \right) + \ldots \]

\[ \alpha_x = \frac{\varepsilon_0}{\varepsilon_d} \frac{\beta_x^2 d}{2} = \frac{1}{2} \frac{\varepsilon_0}{\varepsilon_d} \omega^2 \mu_d \varepsilon_d \left( \frac{\varepsilon_d}{\varepsilon_0 \mu_d} - 1 \right) = 2 \pi \beta \left[ \frac{\mu_d}{\varepsilon_0 \mu_0} \right] \frac{d}{2\lambda} \]

so \( \alpha_x \) is small.
Something very unusual occurs in dielectric slab waveguides.

For the TM mode for both the dielectric slab and for the dielectric above a conducting ground plane:

\[ f_c = 0 \]

For the TE mode for a dielectric slab in free space:

\[ f_c, \text{TM} \neq f_c, \text{TE} = \frac{(m-1)}{2d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}} \]

However, this is not really true that you can go down to DC. As \( f \) decreases \( \alpha_x \) also decreases:

\[ \alpha_x = \sqrt{\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - \beta_x^2} \]

\[ \rightarrow \omega \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} \text{ as } \omega \text{ gets large.} \]

As \( \omega \rightarrow \infty \) \( \alpha_x \rightarrow 0 \)

and \( \frac{1}{\alpha_x} \rightarrow 0 \) This is the depth of penetration outside the slab so the field is essentially confined to the slab as \( \omega \rightarrow \infty \).

However, at \( \omega = \omega_c \) we already know that \( \alpha_x = 0 \) since at cutoff the fields do not decay. This means that \( \frac{1}{\alpha_x} \rightarrow \infty \) as \( \omega \rightarrow \omega_c \). The physical consequence is that the field spreads out to infinity at \( \omega = \omega_c \).
4.2.3 Dielectric slab waveguides: ray theory

un-guided wave since \( \theta_i < \theta_e \) and wave refracts out of guide

guided wave since \( \theta_i > \theta_e \) gives total internal reflection

However, not any angle can propagate

(a) These phase fronts

(b) and these must be in phase so

(c) This distance must be \( n \cdot 2\pi \)

\[ \beta_d (BC - B'C') - 2\phi_r = (m-1) \cdot 2\pi \quad m = 1, 2, 3 \]

\[ \beta_d = \frac{2\pi}{\lambda_d} = \frac{2\pi}{\lambda} \sqrt{\frac{\varepsilon_d}{\varepsilon_0}} \]

\( \phi_r \) is the phase shift from TIR at either B or C

Geometry gives

\[ 2\beta_d \cos \theta_i - 2\phi_r = (m-1) \cdot 2\pi \]

**Note:** \( \tan \theta_i = \frac{\beta}{\beta_x} \)

\[ \beta_d^2 = \beta_x^2 + \beta_y^2 = \omega^2 \mu_d \varepsilon_d. \]
Assume + polarization (E out of plane)

\[ \Pi = \frac{\cos \Theta_i + j \sqrt{\frac{\sin^2 \Theta_i - \varepsilon_0}{\varepsilon_d}}}{\cos \Theta_i - j \sqrt{\sin^2 \Theta_i - \varepsilon_0 \varepsilon_d}} = 1 e^{j \phi} \]

where \( \phi = 2 \tan^{-1}\left(\frac{\sqrt{\sin^2 \Theta_i - \varepsilon_0 \varepsilon_d}}{\cos \Theta_i}\right) \)

\[ 2 \beta_d \cos \Theta_i - 4 \tan^{-1}\left(\frac{\sqrt{\sin^2 \Theta_i - \varepsilon_0 \varepsilon_d}}{\cos \Theta_i}\right) = (m-1) 2\pi \]

\[ \tan \left(\frac{\beta_d \cos \Theta_i}{2} - \frac{(m-1) \pi}{2}\right) = \frac{\sqrt{\sin^2 \Theta_i - \varepsilon_0 \varepsilon_d}}{\cos \Theta_i} \]

Do graphically. Plot LHS and RHS.

**Example 4.9**

Even \( m = 2, 4, 6, \ldots \)

Odd \( m = 1, 3, 5, \ldots \)

Read \( \theta_i \)'s from graph as:

- TE1 \( \theta_i = 75.03^\circ \)
- TE2 \( \theta_i = 59.47^\circ \)
- TE3 \( \theta_i = 43.86^\circ \)

Parameters: \( f = 30 \text{ GHz}, d = 1 \text{ cm}, \varepsilon_d = 2.25 \varepsilon_0 \) (glass) surrounded by air.
4.3 Wave velocities and waveguide dispersion

for a parallel plate waveguide

\[ \beta = \omega \sqrt{\mu \varepsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \]

\[ \text{or} \quad \nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \varepsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \]

Note that \( \nu_p \) inside the waveguide is always greater than

\[ \nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{in the unbounded dielectric} \]

Note further that \( \nu_p \to \infty \) as \( f \to f_c \)

The phase velocity of uniform plane waves in lossless unbounded (unguided) media does not change.

\[ \beta = \omega \sqrt{\mu \varepsilon} \]

and \( \nu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \varepsilon}} \)

4.3.1 Group velocity

The envelope (modulation) of a wave travels at

\[ v_g = \frac{d\xi}{dt} = \frac{\Delta \omega}{\Delta \beta} \to \frac{d\omega}{d\beta} \]

For lossless media

\[ v_g = \frac{d\omega}{d\beta} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\sqrt{\mu \varepsilon}} = \nu_p. \]

wave packets of closely related frequencies travel at \( v_g \)
4.3.2 Dispersion ($\bar{\beta}$-$\omega$) diagrams

Consider the plot of $\bar{\beta}$ versus $\omega$ for TE, TM modes in a parallel plate waveguide.

\[ \bar{\beta} = \omega \sqrt{\mu \varepsilon} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \]

\[ \bar{v}_p = \bar{v}_p(\omega) = \frac{\omega}{\bar{\beta}} = \frac{v_p}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \quad v_p = \frac{1}{\sqrt{\mu \varepsilon}} \]

\[ \bar{v}_g = \frac{1}{\frac{d\bar{\beta}}{d\omega}} = v_p \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \]

The velocity of energy flow is the group velocity

\[ v_E = \bar{v}_g = v_p \sqrt{1 - \left(\frac{m\lambda}{2a}\right)^2} \]

which is also the component of each mode's velocity in the z-direction.