Maxwell's Equations

Ampere's Law:
\[ \nabla \times \mathbf{H} = \mathbf{J} \]
\[ \oint_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{J} \cdot d\mathbf{s} + \frac{d}{dt} \int_s \mathbf{D} \cdot d\mathbf{s} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \]

Faraday's Law:
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_s \mathbf{B} \cdot d\mathbf{s} \]
\[ \nabla \cdot \mathbf{D} = \rho \]
\[ \int_s \mathbf{D} \cdot d\mathbf{s} = \int_v \rho d\mathbf{v} \]

\[ \mathbf{D} = \varepsilon \mathbf{E} \]
\[ \mathbf{B} = \mu \mathbf{H} \]
\[ \mathbf{J} = \sigma \mathbf{E} \]
Electric fields from charge distributions

Electric field from a point charge

1) Field has spherical symmetry, so \( \mathbf{E} = E_r \hat{r} \)

2) Use Gauss' Law

\[
\oint_S \mathbf{D}_i \cdot d\mathbf{s} = \int_V \rho d\mathbf{V}
\]

\[
\oint_S \mathbf{D}_i \cdot d\mathbf{s} = \int_{S_r} \mathbf{D}_r \cdot \hat{r} \cdot r^2 \sin \theta d\theta d\phi
\]

\[
= D_r r^2 \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\phi = 4\pi D_r r^2
\]

\[
\int_V \rho d\mathbf{V} = Q
\]

\[
\therefore D_r 4\pi r^2 = Q \Rightarrow D_r = \frac{Q}{4\pi r^2}
\]

\[
E_r = \frac{Q}{4\pi \varepsilon_0 r^2}
\]

**NOTE**

**cylindrical coordinates**

\[
dS = r \, dr \, d\phi
\]

\[
dV = r \, dr \, d\phi \, dz
\]

**spherical coordinates**

\[
dS = r^2 \sin \theta \, dr \, d\theta
\]

\[
dV = r^2 \sin \theta \, dr \, d\theta \, d\phi
\]
Example 4.12  Line charge.

Whenever there is symmetry use Gauss Law.

$$\oint \mathbf{D} \cdot d\mathbf{s} = \oint \varepsilon_0 \mathbf{E} \cdot d\mathbf{s}$$

$$= \varepsilon_0 \int_0^{2\pi} E_r \, l r \, d\phi = Q_{\text{enc}}$$

$$Q_{\text{enc}} = \rho_e l$$

$$\therefore \varepsilon_0 E_r \, l \, r \, 2\pi = \rho_e l$$

$$E_r = \frac{\rho_e}{2\pi \varepsilon_0 r}$$
Example 4.13  Spherical cloud of charge

(1) Field has spherical symmetry whether $r > r_0$ or $r < r_0$

(2) Use Gauss' Law

\[
\oint_{S_2} \mathbf{D} \cdot d\mathbf{s} = \iint_{S_1 \to S_2} \mathbf{E} \cdot r^2 \sin \theta \, d\theta \, d\phi = \begin{cases} 
\int_0^{2\pi} \int_{r_0}^a \rho \, 4\pi \rho \, r \sin \theta \, d\theta \, dr & \text{for } r > a \\
\rho \cdot \frac{4}{3} \pi a^3 & \text{for } r < a
\end{cases}
\]

\[4\pi \varepsilon_0 E_r r^2\]

for $r > a$

\[4\pi \varepsilon_0 E_r r^2 = \rho \cdot \frac{4}{3} \pi a^3 \implies E_r = \frac{\rho r \cdot \frac{4}{3} \pi a^3}{4\pi \varepsilon_0 r^2} = \frac{Q}{4\pi \varepsilon_0 r^2}\]

for $r < a$

\[Q_{mc} = \frac{4}{3} \pi r^3 \rho \]

\[
\frac{4\pi \varepsilon_0 E_r r^2}{4\pi \varepsilon_0} = \frac{4}{3} \pi r^3 \rho
\]

\[o \cdot E_r = \frac{r \rho r}{3 \varepsilon_0} = \frac{r Q}{4\pi \varepsilon_0 a^3 \cdot 3 \varepsilon_0} = \frac{Q r}{4\pi \varepsilon_0 a^3}\]

\[\frac{Q}{4\pi \varepsilon_0 a^2}\]
When we don't have symmetry we use electric potential

electric potential at any point \( P \) is given by

\[
\Phi(P) = \left[ \frac{W}{q} \right] = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l}
\]

work per unit charge in putting unit charge at \( P \)

\[
\Phi(P) = \Phi(x, y, z) = -\int_{\infty}^{P} \left( \frac{q}{4\pi\varepsilon_0 |r|^3} \right) \cdot \mathbf{r} \, d\mathbf{r} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r}
\]

the work done in moving a charge from \( a \rightarrow b \) is the electrostatic potential difference between the two points

\[
\left[ \frac{W}{q} \right]_{a \rightarrow b} = \Phi_{ab} = \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = \Phi(b) - \Phi(a)
\]

Electrostatic potential is very useful as it is a scalar.

It can be readily shown that

\[
(dW)_x = q \left( \frac{\partial \Phi}{\partial x} \right) dx - q \Phi(x, y, z) = q \frac{\partial \Phi}{\partial x} \Delta x
\]

In three dimensions

\[
dW = q \left( \frac{\partial \Phi}{\partial x} \Delta x + \frac{\partial \Phi}{\partial y} \Delta y + \frac{\partial \Phi}{\partial z} \Delta z \right) = -q \mathbf{E} \cdot \Delta \mathbf{l}
\]

where \( \Delta \mathbf{l} = \left( \hat{x} \Delta x + \hat{y} \Delta y + \hat{z} \Delta z \right) \)

\[
\Rightarrow \mathbf{E} = -\left( \hat{x} \frac{\partial \Phi}{\partial x} + \hat{y} \frac{\partial \Phi}{\partial y} + \hat{z} \frac{\partial \Phi}{\partial z} \right)
\]

or \( \mathbf{E} = -\nabla \Phi \)
4.4.3. **Electrostatic Potential resulting from multiple point charges.**

\[
\Phi = \frac{1}{4\pi \varepsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\mathbf{r} - \mathbf{r}_k'|}
\]

The electric dipole

Summing the potentials

\[
\Phi = \frac{+Q}{4\pi \varepsilon_0 r_+} + \frac{-Q}{4\pi \varepsilon_0 r_-} = \frac{Q}{4\pi \varepsilon_0 \left( \frac{1}{r_+} - \frac{1}{r_-} \right)}
\]

Now use law of cosines

\[
\begin{align*}
\frac{d}{2} & \left\{ \begin{array}{c}
\theta \\
\end{array} \right. \\
\mathbf{r}_+ & = r^2 + \left( \frac{d}{2} \right)^2 - 2 \left( r \right) \left( \frac{d}{2} \right) \cos \theta \\
r_+ & = \sqrt{r^2 + \left( \frac{d}{2} \right)^2 - rd \cos \theta} \\
\mathbf{r}_- & = r^2 + \left( \frac{d}{2} \right)^2 - 2 \left( r \right) \left( \frac{d}{2} \right) \cos (\pi - \theta) \\
r_- & = \sqrt{r^2 + \left( \frac{d}{2} \right)^2 + rd \cos \theta}
\end{align*}
\]
In almost every case $r \gg d$, i.e. P is far away.

\[ \Phi = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{r_+} - \frac{1}{r_-} \right) \]

\[ = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta}} - \frac{1}{\sqrt{r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta}} \right) \]

Rewrite denominators and expand as a Taylor series

\[ \frac{1}{r\sqrt{1 + \left(\frac{d}{2r}\right)^2 - \left(\frac{d}{2r}\right)\cos\theta}} \approx \frac{1}{r} \left[ 1 - \frac{1}{2r} \frac{d}{2r} \cos\theta + \ldots \right] \]

\[ (1 + u)^{-\frac{1}{2}} = 1 - \frac{1}{2} u + \ldots \]

Then

\[ \Phi = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{r} \left[ 1 + \frac{d}{2r} \cos\theta \right] - \frac{1}{r} \left[ 1 - \frac{d}{2r} \cos\theta \right] \right) \]

\[ = \frac{Q}{4\pi\varepsilon_0 r} \left( \frac{d}{r} \cos\theta \right) \]

\[ \Phi = \frac{Qd \cos\theta}{4\pi\varepsilon_0 r} \]

\[ \mathbf{E} = -\nabla \Phi \]

\[ = - \left[ \hat{r} \frac{\partial \Phi}{\partial r} + \hat{\theta} \frac{\partial \Phi}{\partial \theta} \right] \]

\[ \text{in spherical coordinates} \quad \text{no } \phi \text{ dependence} \]

\[ = \hat{r} \frac{Qd \cos\theta}{2\pi\varepsilon_0 r^3} + \hat{\theta} \frac{Qd \sin\theta}{4\pi\varepsilon_0 r^3} \]

\[ \mathbf{E} = \frac{Qd}{4\pi\varepsilon_0 r^3} \left[ \hat{r} 2\cos\theta + \hat{\theta} \sin\theta \right] \]
In general we can extend discrete charge distributions to continuous ones.

\[\Phi = \frac{1}{4\pi \varepsilon_0} \sum \frac{Q}{|\mathbf{r} - \mathbf{r}_k|} + \frac{1}{4\pi \varepsilon_0} \int \frac{p(r') d\nu'}{|\mathbf{r} - \mathbf{r}'|} \text{ etc.}\]
Potential of a disk of charge at a point \( P \) on axis

\[
\Phi = \frac{1}{4\pi \varepsilon_0} \int_S \frac{p_s(r') \, ds'}{|r - r'|}
\]

\[
\Phi (r=0) = \frac{1}{4\pi \varepsilon_0} \int_0^{2\pi} \int_0^a \frac{p_s \, r' \, dr' \, d\phi}{\sqrt{z^2 + (r')^2}}
\]

\[
= \frac{p_s}{2\varepsilon_0} \left. \left( r'^2 + z^2 \right)^{1/2} \right|_0^a + C
\]

\[
= \frac{p_s}{2\varepsilon_0} \left[ (a^2 + z^2)^{1/2} - |z| \right] + C
\]

we require \( \Phi \to 0 \) as \( z \to \infty \) \( \Rightarrow C = 0 \)

\[
\Phi (r=0) = \frac{p_s}{2\varepsilon_0} \left[ \sqrt{a^2 + z^2} - |z| \right].
\]

\[
E_z (P) = -\frac{\partial \Phi}{\partial z} = \begin{cases} 
\frac{p_s}{2\varepsilon_0} \left[ 1 - z (a^2 + z^2)^{-1/2} \right] & z > 0 \\
\frac{p_s}{2\varepsilon_0} \left[ 1 + z (a^2 + z^2)^{-1/2} \right] & r < 0
\end{cases}
\]
Poisson's & Laplace's Equations

Gauss' Law \( \nabla \cdot \mathbf{D} = \rho \) or \( \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \)

But \( \varepsilon = -\nabla \Phi \) so \( \nabla \cdot (\nabla \Phi) = \frac{\rho}{\varepsilon_0} \) or \( \nabla \cdot \nabla \Phi = -\frac{\rho}{\varepsilon_0} \)

In rectangular coordinates

\[
\left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \Phi = 0
\]

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = -\frac{\rho}{\varepsilon_0}
\]

\[\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \quad \text{Poisson's equation}\]

In charge-free regions this reduces to

\[\nabla^2 \Phi = 0 \quad \text{Laplace's Equation}\]
Example 4-23: Two parallel plates

Find $\Phi(x, y, z)$ and $E(x, y, z)$ between the plates.

In rectangular coordinates Laplace's equation is

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \text{ since change free}$$

From symmetry no dependence on $x$ or $y$

$$\therefore \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Rightarrow \Phi(z) = C_1 z + C_2 \quad \text{where } C_1, C_2 \text{ come from the B.C.'s}$$

$$\Phi(0) = 0 \quad \Phi(d) = +V_0$$

$$\Phi(0) = 0 = C_2$$

$$\Phi(d) = +V_0 = C_1 d \quad \therefore C_1 = \frac{V_0}{d}$$

$$\Phi(z) = \frac{V_0}{d} z$$

$$E = -\nabla \Phi(z) = -\dot{z} \frac{\partial \Phi}{\partial z} = -\frac{V_0}{d}$$

To find the charge densities on the plates we use the boundary conditions.
Consider the case of E fields on different sides of a surface.

\[
\begin{align*}
& E_2 \quad D_2 = \varepsilon_0 E_2 \\
& E_1 \quad D_1 = \varepsilon_0 E_1 + P = \varepsilon E_1
\end{align*}
\]

Choose a contour about \((x, y, z)\) as

\[
\hat{z} \text{ from } 1 \text{ to } 2
\]

Since \( E \) is conservative, \( \oint E \cdot d\mathbf{r} = 0 \) (no time-dependent B field)

Doing contour integral

\[
E_1 \Delta \mathbf{h} + E_2 \mathbf{n} \cdot \Delta \mathbf{h} = E_2 \Delta \mathbf{w} - E_2 \mathbf{n} \cdot \Delta \mathbf{h} - E_1 \Delta \mathbf{h} + E_{1t} \Delta \mathbf{w} = 0
\]

Assume \( \Delta \mathbf{h} \) and \( \Delta \mathbf{w} \) are small enough that \( E_1 \) and \( E_2 \) are constant on \( C \)

This leaves

\[
E_{1t} \Delta \mathbf{w} - E_{2t} \Delta \mathbf{w} = 0
\]

or

\[
E_{1t} = E_{2t}
\]

Mathematically

\[
\hat{n}_2 \times (E_2 - E_1) = 0
\]
For the normal component \( \mathbf{D} \in \mathbf{E} \) choose a small volume centered on \( (x, y, z) \)

\[
D_z = E_0 E + \mathbf{p}
\]

Using Gauss' Law

\[
\int_{S} \mathbf{D} \cdot d\mathbf{s} = \rho_f \Delta \mathbf{A} \Delta S
\]

\[
\int_{\text{top}} \mathbf{D} \cdot d\mathbf{s} + \int_{\text{sides}} \mathbf{D} \cdot d\mathbf{s} + \int_{\text{bottom}} \mathbf{D} \cdot d\mathbf{s} = \rho_f \Delta \mathbf{A} \Delta S
\]

Consider the integral over the sides as \( \Delta S \to 0 \)

\[
\int_{\text{top view}} \mathbf{E}_{\tan} \cdot \mathbf{d}s = \mathbf{E}_{\tan} \cdot \mathbf{\hat{n}}
\]

since \( \mathbf{E}_{\tan} \) approx. constant and \( \mathbf{\hat{n}} \) rotates as it goes around the surface

then \( \int \mathbf{D} \cdot d\mathbf{s} \to 0 \)

This leaves

\[
\int_{\text{top}} \mathbf{D} \cdot d\mathbf{s} + \int_{\text{bottom}} \mathbf{D} \cdot d\mathbf{s} = \rho_f \Delta \mathbf{A} \Delta S
\]

Since \( \Delta \mathbf{A} \Delta S \to 0 \) we can assume \( \mathbf{D} \) is constant on each surface

\[
\mathbf{D}_{2n} \Delta S - \mathbf{D}_{1n} \Delta S = \rho_f \Delta \mathbf{A} \Delta S
\]

\[
\mathbf{D}_{2n} - \mathbf{D}_{1n} = \rho_f \Delta \mathbf{A} \to \mathbf{\rho}_s
\]

\[
\mathbf{\hat{n}}_{12} \cdot (\mathbf{D}_{2} - \mathbf{D}_{1}) = \mathbf{\rho}_s
\]
Polarization

dielectric - insulating material in which charge separation occurs at the microscopic level due to an applied \( \mathbf{E} \) field.

Consider a water molecule

alignment with applied fields

notice also that field can stretch the atoms (this creates electric dipoles)

define a macroscopic average

\[
P \equiv \lim_{\Delta V \to 0} \frac{\sum \mathbf{P}_i}{\Delta V} = N \mathbf{P}
\]

\( \mathbf{P} \) - number density of dipoles

in bulk materials

randomly distributed

align with applied field

This gives rise to surface charge

\[
\frac{-d \Delta \mathbf{P}}{\Delta x} = \mathbf{E} \Delta y
\]

net bound change is zero!
The change on each surface is given by
\[ dq = -(Nq_d) \left. \frac{\partial y \partial z}{\partial x} \right|_{x_0} \]

Relate this change to polarization

We have to relate this to bound change
\[ dq = + (Nq_d) \left. \frac{\partial y \partial z}{\partial x} \right|_{x_0} \]

\[ P_x(x_0) = Nq_d \left. \frac{\partial y \partial z}{\partial x} \right|_{x_0} \]

\[ P_x(x_0 + \Delta x) = Nq_d \left. \frac{\partial y \partial z}{\partial x} \right|_{x_0 + \Delta x} \]

The net bound change is then
\[ dq_{\text{total}} = \left. \frac{P_x(x_0) \partial y \partial z}{\Delta x} \right| - \left. \frac{P_x(x_0 + \Delta x) \partial y \partial z}{\Delta x} \right| \]

The bound change density is then given by
\[ P_b = \lim_{\Delta v \to 0} \frac{dq_{\text{total}}}{\Delta v} = \lim_{\Delta v \to 0} - \left. \frac{P_x(x_0 + \Delta x) - P_x(x_0)}{\Delta x} \right| \]

Extending to three dimensions
\[ P_b = -\nabla \cdot \mathbf{P} \]
This can be written at the macroscopic level

\[ D = \varepsilon_0 E + P \]

\[ = \varepsilon_0 E + \varepsilon_r \varepsilon_0 E \]

\[ = \varepsilon E \]
Now let's go back to the parallel plate capacitor with a dielectric, polarizable block between the plates.

![Diagram of capacitor with and without dielectric block]

Since normal $D$ is continuous, but $E$ is not continuous.

The difference between $E$ and $D$ is $P$.

$\Phi$ is the integral of $E$, $E = -\frac{d\Phi}{dx}$.

$E$ gives surface charge density.

E goes down in this region since $\varepsilon > \varepsilon_0$. 

Slope decreases.
Example 4-29. Capacitance of a Two-Wire Line

Very useful problem for telephony, radio transmission lines, and power transmission.

General solution is complicated because "proximity effect" causes current densities to be larger on facing sides.

We will solve for $d>a$ to avoid this effect.

This is a very difficult problem to solve for the potential directly.

See Inom & Inam, Engineering Electromagnetics, Example 4-11

The potential from a finite length of charge from $-l$ to $+l$

is given by

$$\Phi (P) = -\frac{P_e}{4\pi \varepsilon_0} \ln \left[ \frac{z-a + \sqrt{r^2 + (z-a)^2}}{z+a + \sqrt{r^2 + (z+a)^2}} \right]$$

You could compute $E = -\nabla \Phi$ from this function, but it is very complex as shown in this example.

Furthermore, we are interested in infinite length lines where $l \to \infty$

See Paris & Hard, Basic Electromagnetic Theory, Example 3-3

As $l \to \infty$, the argument of the $\ln \left[ \right]$ function increases without limit and $\Phi$ is undefined. The problem is that our previous expressions for $\Phi$, i.e., $\Phi = \frac{1}{4\pi \varepsilon_0} \int \frac{P}{r} \, dr$, work only for bounded charge distributions.

The best method to find the potential associated with an infinite line change is the integral form of Gauss' Law.
If we use this expression for the E-field between two conductors, we can write an expression for the E-field between the two conductors as:

$$E_x (x, 0, 0) = -\frac{\rho L}{2\pi \varepsilon_0 x} \sqrt{\frac{\rho L}{2\pi \varepsilon_0 (d-x)}}$$

ignores a.

We will need to use the most general definition of capacitance:

$$C = \frac{Q}{\Phi_{12}} = \frac{\int \mathbf{D} \cdot d\mathbf{s}}{-\int \mathbf{E} \cdot d\mathbf{e}}$$

Gauss Law

since these are infinite lines, Q is readily given as $\rho L$, the charge per unit length.

$\Phi_{12}$ can be integrated as follows (remember we can use any path):

$$\Phi_{12} = -\frac{1}{2\pi \varepsilon_0} \int_a^d \left[ -\frac{\rho L}{x} - \frac{\rho L}{d-x} \right] dx$$

$$= \frac{\rho L}{2\pi \varepsilon_0} \left[ -\frac{1}{x} - \frac{1}{d-x} \right] \bigg|_{x=a}^{x=d-a}$$

$$= \frac{\rho L}{2\pi \varepsilon_0} \left[ \ln \left( \frac{d-a}{d} \right) \right]_{x=a}^{x=d-a}$$

$$= \frac{\rho L}{2\pi \varepsilon_0} \left[ \ln \left( d-a \right) - \ln \left( \frac{a}{d-a} \right) \right] = \frac{\rho L}{2\pi \varepsilon_0} 2\ln \left( \frac{d-a}{a} \right)$$

and, for $d \gg a$,

$$\Phi_{12} \approx \frac{\rho L}{\pi \varepsilon_0} \ln \left( \frac{d}{a} \right)$$
\[ \int_D \hat{n} \, ds = \int \rho \, dx = Q_{\text{enclosed}} \]

By symmetry, \( D \cdot 2\pi r \cdot l = \rho_l l \) where \( \rho_l \) is the linear charge density.

\[ D_r = \frac{\rho_l}{2\pi r} \]

\[ D = \frac{\rho_l}{2\pi r} \hat{r} \]

\[ E = \frac{\rho_l}{2\pi \epsilon_0 r} \hat{r} \]

Since \( E = -\nabla \Phi \) and the field is circularly symmetric, i.e.

\[ \frac{\partial \Phi}{\partial r} \rightarrow 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial z} = 0 \]

we put our origin at

\[ \frac{\rho_l}{2\pi \epsilon_0} \hat{r} = -\nabla \Phi = -\frac{\partial \Phi}{\partial r} \hat{r} \]

\[ \Phi(r) = -\frac{\rho_l}{2\pi \epsilon_0} \ln r + C \]

If we pick a reference potential \( \Phi(r=r_0) = 0 \)

\[ 0 = \Phi(r=r_0) = -\frac{\rho_l}{2\pi \epsilon_0} \ln r_0 + C \]

and \( \Phi(r) = \frac{\rho_l}{2\pi \epsilon_0} \ln \left( \frac{r}{r_0} \right) \)
We can compute the capacitance per unit length as

\[ C \approx \frac{P_l}{\frac{d}{a} \text{ln}(\frac{d}{a})} = \frac{\pi \varepsilon_0}{\text{ln}(\frac{d}{a})} \]

For example, a 115kV transmission line uses two 1.407 cm aluminum conductors separated by 3 meters:

\[ C \approx \frac{\pi \left( 8.854 \times 10^{-12} \text{ F/m} \right)}{\text{ln}(\frac{3}{0.01407})} \approx 5.19 \text{ nF/km} \]

**Hint for future problems**

You can compute \( C \) for conductor configurations for which you have derived \( \varepsilon = \Phi \). For example, a single cylindrical conductor above a ground plane is that of a twin line with an infinitely large conducting sheet between the two conductors. You can also use the "method of images."
Laplace's Equation solved using separation of variables

\[ \nabla^2 \Phi = 0 \]

Assume \( \Phi = f(u) g(v) \)

Then \( \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = f \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0 \)

Divide by \( f g \) to get

\[ \frac{1}{f} \frac{\partial^2 f}{\partial u^2} + \frac{1}{g} \frac{\partial^2 g}{\partial v^2} = 0 \]

For this to be true, the first term must equal the second term independent of variables, i.e.

\[ \frac{1}{g} \frac{\partial^2 g}{\partial v^2} = -\frac{1}{f} \frac{\partial^2 f}{\partial u^2} \quad \text{independent of } u \text{ and } v \]

This can be written as

\[ \frac{1}{f} \frac{\partial^2 f}{\partial u^2} = k_u^2 \quad \text{where } k_u \text{ is a constant} \]

and

\[ \frac{1}{g} \frac{\partial^2 g}{\partial v^2} = k_v^2 \quad \text{where } k_v^2 = -k_u^2 \]

These are differential equations which need to have the boundary conditions specified. You can specify either \( \Phi \) or \( \frac{\partial \Phi}{\partial n} \)

\( \Phi \) Dirichlet boundary condition

\( \frac{\partial \Phi}{\partial n} \) Neumann boundary condition

Since \( D_{\text{normal}} = \varepsilon E_{\text{normal}} = -\varepsilon \frac{\partial \Phi}{\partial n} \)

equivalent to specifying charge density.

mixed combination of Dirichlet & Neumann
The solution is dependent upon the actual value of $k_w$.

If $k_w^2 > 0$
\[
\frac{d^2f}{du^2} - k_w^2 f = 0
\]
\[f(u) = c_1 e^{k_w u} + c_2 e^{-k_w u}\]

If $k_w^2 < 0$
\[
\frac{d^2f}{du^2} + k_w^2 f = 0
\]
\[f(u) = c_1 \sin(k_w u) + c_2 \cos(k_w u)\]

If $k_w^2 = 0$
\[f(u) = c_1 + c_2 u\]
Use these boundary conditions
\[ \Phi (0, y) = 0 \]
\[ \Phi (x, 0) = 0 \]
\[ \Phi (a, y) = 0 \]
\[ \Phi (x, b) = V \sin \left( \frac{\pi x}{a} \right) \]

Let \( \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \) and further let \( \overline{f}(x, y) = f(x) g(y) \)

As before \( g \frac{\partial f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0 \)

and dividing \( \frac{1}{f} \frac{\partial^2 f}{\partial x^2} + \frac{1}{g} \frac{\partial^2 g}{\partial y^2} = 0 \)

Separate to get
\[ \frac{1}{f} \frac{\partial^2 f}{\partial x^2} = k_x^2 \quad \text{and} \quad \frac{1}{g} \frac{\partial^2 g}{\partial y^2} = 0 \] where \( k_x^2 + k_y^2 = 0 \)

Which one is positive depends upon the boundary conditions.
Since \( \Phi (x, b) = V \sin \left( \frac{\pi x}{a} \right) \), we want the \( x \) solution to be sinusoidal, or \( k_x^2 < 0 \).

\[ \frac{d^2 f}{dx^2} - k_x^2 f = 0 \Rightarrow f = c_1 \sin k_x x + c_2 \cos k_x x \]

This requires \( k_y^2 > 0 \) and
\[ \frac{d^2 g}{dy^2} + k_y^2 g = 0 \Rightarrow g = c_3 e^{+k_y y} + c_4 e^{-k_y y} \]
$$\Phi(x, y) = (c_1 \sin k_x x + c_2 \cos k_x x)(c_3 e^{k_y y} + c_4 e^{-k_y y})$$

Since we want $$\Phi(x, b) = V \sin \frac{\pi x}{a}$$
this allows us to set $$c_2 = 0$$ and pick $$k_x = \frac{\pi}{a}$$.

Then

$$\Phi(x, y) = (\sin k_x x)(c_3 e^{k_y y} + c_4 e^{-k_y y})$$

At this point the problem becomes interesting.

You cannot use $$c_3$$ and $$c_4$$ to make $$\Phi(0, y) = 0$$ and $$\Phi(\pi, y) = 0$$ using exponentials.

Solution use the sine function.

For $$x = 0$$, $$\sin k_x x = 0$$ always.

For $$x = a$$, $$\sin k_x a = 0$$ when $$k_x a = n \pi$$, $$n = 1, 2, 3$$.

Using these results we can write $$\Phi(x, y)$$ as

$$\Phi(x, y) = \sum_{n=1}^{\infty} \sin(n \frac{\pi}{a} x) \left[c_3 e^{-n \frac{\pi}{a} y} + c_4 e^{n \frac{\pi}{a} y}\right]$$

since $$k_y = \frac{n \pi}{a}$$.

Note that my sign is correct since if $$k_x^2 < 0$$ for sinusoidal solutions, then $$k_y^2 > 0$$.

At $$y = 0$$, $$\Phi(x, 0) = \sum_{n=1}^{\infty} \sin(n \frac{\pi}{a} x) (c_3 + c_4)$$

Since this is independent of $$x$$, $$c_3 = -c_4$$.

$$\Phi(x, y) = \sum_{n=1}^{\infty} c \sin(n \frac{\pi}{a} x) \left[e^{-n \frac{\pi}{a} y} - e^{n \frac{\pi}{a} y}\right]$$
Now look at $y=b$ where $\Phi(x,b) = V \sin \frac{\pi x}{a}$

$$\Phi(x,b) = \sum_{n=1}^{\infty} c \sin \left( \frac{n\pi x}{a} \right) \left[ e^{-\frac{n\pi b}{a}} - e^{\frac{n\pi b}{a}} \right] = V \sin \left( \frac{\pi x}{a} \right)$$

Now this is only true if $n=1$ where

$$c \sin \left( \frac{\pi x}{a} \right) \left[ e^{-\frac{\pi b}{a}} - e^{\frac{\pi b}{a}} \right] = V \sin \left( \frac{\pi x}{a} \right)$$

$$c = \frac{V}{e^{-\frac{\pi b}{a}} - e^{\frac{\pi b}{a}}} = \frac{V}{2 \sinh \left( \frac{\pi b}{a} \right)}$$

Final solution,

$$\Phi(x,y) = c \sin \left( \frac{\pi x}{a} \right) \left[ e^{-\frac{\pi y}{a}} - e^{\frac{\pi y}{a}} \right]$$