Example 4-29. Capacitance of a Two-Wire Line

\[ x = d \quad \text{O} \quad + \rho e \]

\[ x = d - a \quad \text{O} \quad - \rho e \]

Very useful problem for telephony, radio transmission lines, and power transmission.

General solution is complicated because "proximity effect" causes change densities to be larger on facing sides. We will solve for \( d > a \) to avoid this effect.

This is a very difficult problem to solve for the potential directly.

See Inom & Inam, Engineering Electromagnetics, Example 4-11

The potential from a finite length of charge \(-l\) to \(+l\)

is given by

\[
\Phi(x) = \frac{-\rho e}{4 \pi \varepsilon_0} \ln \left[ \frac{z-a + \sqrt{r^2 + (z-a)^2}}{z+a + \sqrt{r^2 + (z+a)^2}} \right]
\]

You could compute \( E = -\nabla \Phi \) from this function but it is very complex as shown in this example.

Furthermore, we are interested in infinite length lines where \( l \to \infty \)

See Paris & Hard, Basic Electromagnetic Theory, Example 3-3

As \( l \to \infty \) the argument of the \( \ln \) function increases without limit and \( \Phi \) is undefined. The problem is that our previous expressions for \( \Phi \), i.e., \( \Phi = \frac{1}{4 \pi \varepsilon_0} \int_{V} \frac{\rho}{r} \, dv \), only work for bounded charge distributions.

The best method to find the potential associated with an infinite line charge is the integral form of Gauss’ Law.
If we use this expression for the electric field between the two conductors, we can write an expression for the electric field between the two conductors as

\[ E_x(x, 0, 0) = \frac{-\rho_e}{2\pi \varepsilon_0 x} \int_{d-x}^{d} \frac{\rho_e}{2\pi \varepsilon_0 (d-x)} \text{ direction} \]

We will need to use the most general definition of capacitance

\[ C = \frac{Q}{\Phi_{12}} = \frac{\int \mathbf{D} \cdot d\mathbf{s}}{-\int \mathbf{E} \cdot d\mathbf{e}} \quad \text{Gauss Law} \]

since there are infinite lines, \( Q \) is readily given as \( \rho_e \), the charge per unit length.

\( \Phi_{12} \) can be integrated as follows (remember we can use any path)

\[
\Phi_{12} = -\frac{1}{2\pi \varepsilon_0} \int_{a}^{d-a} \left[ \frac{-\rho_e}{x} - \frac{\rho_e}{d-x} \right] dx
\]

\[
= \frac{\rho_e}{2\pi \varepsilon_0} \int_{a}^{d-a} \left[ \frac{1}{x} - \frac{1}{d-x} \right] dx = \frac{\rho_e}{2\pi \varepsilon_0} \left[ \ln x - \ln (d-x) \right]_{x=a}^{x=d-a}
\]

\[
= \frac{\rho_e}{2\pi \varepsilon_0} \ln \left( \frac{x}{d-x} \right)_{x=a}^{x=d-a}
\]

\[
= \frac{\rho_e}{2\pi \varepsilon_0} \left[ \ln \left( \frac{d-a}{a} \right) - \ln \left( \frac{a}{d-a} \right) \right] = \frac{\rho_e}{2\pi \varepsilon_0} \frac{2 \ln (d-a)}{a}
\]

and, for \( d \gg a \),

\[ \Phi_{12} \approx \frac{\rho_e}{\pi \varepsilon_0} \ln \left( \frac{d}{a} \right) \]
\[ \oint_{S} \mathbf{D} \cdot \mathbf{n} \, ds = \int \rho \, d\mathbf{r} = \Phi_{\text{enclosed}} \]

By symmetry, \( D \cdot 2\pi r \cdot l = P \cdot l \) where \( P \) is the line charge density

\[ D = \frac{P}{2\pi r} \]

\[ E = \frac{P}{2\pi \varepsilon_0 r} \]

Since \( E = -\nabla \Phi \) and the field is circularly symmetric, i.e.

\[ \frac{\partial \Phi}{\partial b} \rightarrow 0 \quad \text{and} \quad \frac{\partial \Phi}{\partial z} = 0 \quad \text{since it does not matter where} \]

we put our origin as

\[ \frac{P}{2\pi \varepsilon_0} \mathbf{r} = E = -\nabla \Phi = -\frac{\partial \Phi}{\partial r} \mathbf{r} \]

so,

\[ \Phi(r) = -\frac{P}{2\pi \varepsilon_0} \ln r + c \]

If we pick a reference potential \( \Phi(r=r_0) = 0 \)

\[ 0 = \Phi(r_0) = -\frac{P}{2\pi \varepsilon_0} \ln r_0 + c \]

and \( \Phi(r) = \frac{P}{2\pi \varepsilon_0} \ln \left( \frac{r_0}{r} \right) \)
We can compute the capacitance per unit length as

\[ C = \frac{\frac{P_l}{A_l \ln(d/a)}}{\frac{\pi\varepsilon_0}{\ln(d/a)}} = \frac{\pi\varepsilon_0}{\ln(d/a)} \]

For example, a 115kV transmission line uses two 1.407cm aluminum conductors separated by 3 meters.

\[ C \approx \frac{\pi (8.854 \times 10^{-12} \text{ F/m})}{\ln \left(\frac{3}{0.01407}\right)} = 5.19 \text{ nF/km} \]

**Hint for future problems**

You can compute \( C \) for conductor configurations for which you have derived \( \varepsilon_0 \Phi \). For example, a single cylindrical conductor above a ground plane is that of a twin line with an infinitely large conducting sheet between the two conductors. You can also use the "method of images."
Vector potential

Others have proven that Maxwell's Equations are satisfied if

$$\nabla \cdot A = -\mu e \frac{\partial \Phi}{\partial t}$$

For static (time-independent) fields $\frac{\partial}{\partial t} \to 0$ and $\nabla \cdot A = 0$

Then, the vector potential is defined by

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} = \mu \mathbf{J}$$

We get three component equations

$$\nabla^2 A_x = -\mu J_x$$
$$\nabla^2 A_y = -\mu J_y$$
$$\nabla^2 A_z = -\mu J_z$$

which is similar to Poisson's Equation.

The general solution is $A_x = \int \frac{\mu J_x dv}{4\pi r}$

etc.

In general $A = \int \frac{\mu J d\mathbf{v}}{4\pi r}$

This is known as the Green's function solution for $A$
Determine the inductance per unit length of a two-wire transmission line in air as shown below, designed for an amateur radio transmitter, with conductor radius \( a = 1 \) mm and spacing \( d = 6 \) cm.

Superimpose the field from each conductor modeling the fields as those from two infinitely long parallel lines.

\[
B_y = -\frac{\mu_0 I}{2\pi (x + \frac{d}{2})} + \frac{\mu_0 I}{2\pi (x - \frac{d}{2})}
\]

Positive

Negative

The flux linked by this circuit of two conductors over a length \( l \) is approximated by:

\[
\Phi = \int B_y \, ds = \int_{-\frac{d}{2}}^{\frac{d}{2}} \left( \frac{1}{x + \frac{d}{2}} - \frac{1}{x - \frac{d}{2}} \right) dx
\]

\[
= \frac{\mu_0 I l}{2\pi} \left[ \ln \left( x + \frac{d}{2} \right) - \ln \left( x - \frac{d}{2} \right) \right]_{-\frac{d}{2}}^{-\frac{d}{2}}
\]

\[
= \frac{\mu_0 I l}{2\pi} \left[ \ln \left( a + \frac{d}{2} \right) - \ln \left( a - \frac{d}{2} \right) \right]
\]

\[
= \frac{\mu_0 I l}{2\pi} \left[ \ln \left( d - a \right) - \ln \left( a - d \right) \right]
\]

\[
= \frac{\mu_0 I l}{2\pi} \left( \ln \left( d - a \right) - \ln \left( a - d \right) \right)
\]

\[
= \frac{\mu_0 I l}{2\pi} \ln \left( \frac{d - a}{a} \right)
\]
Since $d \gg a$, \[ \Phi = \frac{\mu_0 I l}{\pi} \ln \left( \frac{d}{a} \right) \]

\[ \therefore L = \frac{\Phi}{I} = \frac{\mu_0 l}{\pi} \ln \left( \frac{d}{a} \right) \]

(a) \[ \frac{L}{l} = \frac{\mu_0}{\pi} \ln \left( \frac{d}{a} \right) = \frac{4\pi \times 10^{-7}}{\pi} \ln \left( \frac{60 \text{mm}}{1 \text{mm}} \right) = 1.64 \times 10^{-6} \ \frac{H}{m} \]

(b) Repeat part (a) if the conductor spacing is doubled (i.e.) $d = 12 \text{cm}$.

\[ \frac{L}{l} = \frac{\mu_0}{\pi} \ln \left( \frac{d}{a} \right) = \frac{4\pi \times 10^{-7}}{\pi} \ln \left( \frac{120 \text{mm}}{1 \text{mm}} \right) = 1.915 \times 10^{-6} \ \frac{H}{m} \]