Power Flow on a transmission line

You usually want to maximize the time-average power delivered to the load.

You can calculate the time-average power as

\[ P_{av}(z) = \frac{1}{T_p} \int_0^{T_p} V(z,t) I(z,t) \, dt \]

where \( T_p = \frac{2\pi}{\omega} \) is the period of the signal.

In terms of phasors \( P_{av}(z) = \frac{1}{2} \text{Re} \{ V(z) I^*(z) \} \)

We can re-write this in terms of transmission line parameters as

\[ V(z) = V^+ e^{-j\beta z} + \pi_L V^+ e^{j\beta z} \]

\[ I(z) = \frac{V^+}{Z_0} e^{-j\beta z} - \pi_L \frac{V^+}{Z_0} e^{j\beta z} \]

\[ \pi_L \pi_L^* = \rho^2 \quad \text{if power is going in opposite direction to } V(z) \text{ and } I(z) \]

Net power in forward direction

\[ P_{av} = P^+ + P^- = \frac{|V^+|^2}{2Z_0} - \rho^2 \frac{|V^+|^2}{2Z_0} = (1-\rho^2) \frac{|V^+|^2}{2Z_0} \]

\( \Rightarrow \) you maximize power flow by minimizing \( \rho \).

i.e. \( Z_L = Z_0 \)

\[ \pi_L = 0 \]

\[ S = 1 \]
Consider what happens if the load is NOT matched to the load.

\[ \frac{P_L}{P^+} = \frac{\text{power dissipated in load}}{\text{power delivered to matched load}} \]

\[ P_L = \frac{1}{2} \text{Re} \left\{ V_L I_L^* \right\} = \frac{1}{2} \text{Re} \left\{ V^+(1+P_L) \left[ \frac{V^+(1-P_L)}{Z_0} \right]^* \right\} \]

\[ = \frac{1}{2} \text{Re} \left\{ \frac{V^+(V^*)^*(1+P_L)(1-P_L^*)}{Z_0} \right\} \]

\[ = \frac{1}{2} \text{Re} \left\{ \left| V^+ \right|^2 \left[ \frac{1 + (P_L - P_L^*) - P_L P_L^*}{Z_0} \right] \right\} \]

\[ = \frac{1}{2} \frac{\left| V^+ \right|^2}{Z_0} (1 - P^2) \]

\[ \Rightarrow \frac{P_L}{P^+} = \frac{\frac{1}{2} \frac{\left| V^+ \right|^2}{Z_0} (1 - P^2)}{\frac{1}{2} \frac{|V^+|^2}{Z_0}} = 1 - P^2 \]

\[ \frac{P_L}{P^+} = 1 - P^2 = 1 - \left( \frac{S-1}{S+1} \right)^2 = \frac{(S+1)^2 - (S-1)^2}{(S+1)^2} = \frac{(S+2S+1)-(S^2-2S+1)}{(S+1)^2} \]

\[ \frac{P_L}{P^+} = \frac{4S}{(S+1)^2} \quad \text{this is a maximum when } S=1, \text{ i.e. matched.} \]
Example 3-12

A VHF transmitter operating at 125 MHz and developing \( V_s = 100 \, e^{j0} \) volts with a source resistance of \( R_s = 50 \Omega \) feeds an antenna with a feed-point impedance of \( Z_L = 100 - j60 \) through a 50 \( \Omega \), polyethylene-filled coaxial line that is 17 m long.

(a) Find the voltage \( V(z) \) on the line.

\[
\lambda = \frac{\sqrt{c/\mu}}{f} = \frac{20 \, \text{cm} \times 5 \times 10^8}{125 \times 10^6} = \frac{2 \times 10^8}{125 \times 10^6} = 1.6 \, \text{m}
\]

\[
\lambda = \frac{17 \text{m}}{1.6 \frac{\text{m}}{\lambda}} = 10.625 \lambda
\]

\[
\tan \beta l = \tan \left( \frac{2\pi}{\lambda} \cdot 10.625 \lambda \right) = 1
\]

\[
Z_m = \frac{Z_0}{Z_0 + jZ_L \tan \beta l} = \frac{50}{(100-j60)+j50(1)} = \frac{50}{50+j(100-j60)(1)}
\]

\[
Z_m = 50 \frac{100-j10}{110+j100} = 22.62 - j25.1 \, \Omega.
\]

We can first solve for \( V_s \) using a voltage divider.

\[
V_s = \frac{22.62-j25.1}{22.62-j25.1+50} \, 100e^{j0} = \frac{22.62-j25.1}{100e^{j0}} = 38.51-j21.27 \, \text{Volts}
\]

\[
V_s = 43.99 \, e^{-j0.504}
\]

But \( V_s \) also can be written using the wave expressions for \( V(z) \).

\[
V_s = V(z) = V^+ \, e^{-j\beta z} \left( 1 + \frac{\Gamma_1}{Z_0} \, e^{-j2\beta z} \right) \quad \text{remember} \ z = -l \, e^{-j2\beta z}
\]

\[
\beta l = \frac{2\pi}{\lambda} \cdot 10.625 \lambda = 21.25 \pi = \frac{5}{4} \pi = -\frac{3}{4} \pi \quad \text{note sign change}
\]

\[
2\beta l = \frac{2\cdot2\pi}{\lambda} \cdot 10.625 \lambda = 42.5 \pi = \frac{5}{2} \pi = \frac{1}{2} \pi
\]

\[
\Gamma_1 = \frac{Z_L-Z_0}{Z_L+Z_0} = \frac{(100-j60)-50}{(100-j60)+50} = 0.42-j0.23 = 0.483e^{-j28.4^0}
\]

\[
V_s = V(z) = V^+ \, e^{-j(\frac{3}{4} \pi)} \left( 1 + (0.42-j0.23)e^{-j\frac{1}{2} \pi} \right)
\]

\[
V_s = V^+ \, e^{-j\frac{3}{4} \pi} \left( 1 + (-0.23-j0.42) \right) = V^+ \, e^{-j\frac{3}{4} \pi} (0.77-j0.42)
\]
These expressions for \( V_z \) can be equated and solved for \( V^+ \)

\[
V^+ e^{-j\frac{3\pi}{4}} (0.77-j0.42) = 43.99 e^{-j0.504}
\]

\[
V^+ = \frac{43.99 e^{-j0.504}}{(0.77-j0.42)} e^{j\frac{3\pi}{4}} = (50.15-0.23) e^{j\frac{3\pi}{4}}
\]

\[
V^+ = 50.15 e^{j\frac{3\pi}{4}} e^{-j\frac{3\pi}{4}} \approx 50.1 e^{j\frac{3\pi}{4}}
\]

\[
\Rightarrow V(z) = V^+ e^{-j\beta z} (1 + \Gamma(z))
\]

\[
V(z) = 50.1 e^{j\frac{3\pi}{4}} e^{-j\frac{3\pi}{4}} (1 + 1.483 e^{-j28.4^\circ - j\frac{3\pi}{4}})
\]

(b) Find the load voltage \( V_L \).

\[
V_L = V(z=0) = 50.1 e^{j\frac{3\pi}{4}} (1 + 1.483 e^{-j28.4^\circ})
\]

\[
V_L = 50.1 e^{j\frac{3\pi}{4}} (1.426-j0.23) = 50.1 e^{j\frac{3\pi}{4}} 1.444 e^{-j9.17^\circ} = 72.3 e^{j125.8} \text{ volts}.
\]

(c) Find the time average power absorbed by the VHF antenna.

\[
P_L = \frac{1}{2} |I_L|^2 R_L = \frac{1}{2} \left| \frac{V_L}{Z_L} \right|^2 R_L \quad \text{since it is absorbed}
\]

\[
Z_L = 100 + j60 = 116.6 e^{j31.0^\circ}
\]

\[
P_L = \frac{1}{2} \left( \frac{(72.3)^2}{(116.6)^2} \right) (100) = 19.2 \text{ watts}.
\]

(d) Find the power absorbed by the source impedance \( R_s \).

To find this we need to know \( I_s \)

\[
I_s = \frac{V_0}{R_s + Z_{in}} = \frac{100 e^{j0^\circ}}{50 + 22.62 - j25.1} = 1.301 e^{j19.1^\circ}
\]

\[
P_{RS} = \frac{1}{2} |I_s|^2 R_s = \frac{1}{2} (1.301)^2 (50) = 42.3 \text{ watts}.
\]
3.5 Impedance matching

Why do impedance matching

(i) reduce reflections & standing waves that jeopardize the power-handling capabilities of the line

(ii) maximize power delivered to the load

(iii) improve signal-to-noise ratio of system

(iv) reduce amplitude and phase errors of system

3.5.1. Matching using Lumped Reactive Elements

To impedance match connect \( Y_s \) at \( z = -l \) such that

\[
Y_2(z = -l) = Y_1(z = -l) + Y_s = Y_0
\]

Do in terms of normalized admittance:

\[
\overline{Y}(z) = \frac{Y(z)}{Y_0} = \frac{1 - \frac{\pi}{l} e^{j2\beta z}}{1 + \frac{\pi}{l} e^{j2\beta z}}
\]

\( \Rightarrow \) find \( z = -l \) such that \( \overline{Y}_1 = \overline{Y}(z = -l) = 1 - j\overline{B} \)

\( \uparrow \) want real part to be 1

\( \overline{B} \) is whatever \( Y_L \) and \( Y_0 \) and \( l \) cause it to be.

Choose \( \overline{Y}_s = +j\overline{B} \) so that \( \overline{Y}_2 = \overline{Y}_1 + \overline{Y}_s = 1 - j\overline{B} + j\overline{B} = 1 \)
We can calculate the distance \( l \) from the load we wish to place \( Y_l \).

\[
Y_l = Y(z = -l) = \frac{1 - \rho e^{-j2\beta l}}{1 + \rho e^{-j2\beta l}} = 1 - jB
\]

from the problem.

Since \( \frac{Z_L - Z_0}{Z_L + Z_0} = \rho e^{j\psi} \), we can substitute it into \( Y_l \).

\[
Y_l = \frac{1 - \rho e^{j\psi} e^{-j2\beta l}}{1 + \rho e^{j\psi} e^{-j2\beta l}} = \frac{1 - \rho e^{j\theta}}{1 + \rho e^{j\theta}}
\]

where \( \theta = \psi - 2\beta l \)

we can rationalize this and separate it into real and imaginary parts

\[
Y_l = \frac{1 - \rho e^{j\theta}}{1 + \rho e^{j\theta}} = \frac{1 + \rho e^{-j\theta} - \rho^2}{1 + \rho e^{-j\theta} + \rho^2}
\]

\[
Y_l = \frac{1 + \rho (-2j\sin \theta) - \rho^2}{1 + 2\rho \cos \theta + \rho^2} = \frac{1 - \rho^2}{1 + 2\rho \cos \theta + \rho^2} - j \frac{2\rho \sin \theta}{1 + 2\rho \cos \theta + \rho^2}
\]

For matching we want to pick \( \lambda \) such that the first term is 1.

\[
\frac{1 - \rho^2}{1 + 2\rho \cos \theta + \rho^2} = 1
\]

\[
\lambda - \rho^2 = \lambda + 2\rho \cos \theta + \rho^2
\]

\[
2\rho \cos \theta + 2\rho^2 = 0
\]

\[
\cos \theta + \rho = 0
\]

\[
\cos \theta = -\rho
\]

\[
\theta = \cos^{-1}(-\rho)
\]

\[
\theta = \psi - 2\beta l = \cos^{-1}(-\rho)
\]

Solving for \( l \) gives

\[
l = \frac{\psi - \theta}{2\beta} = \frac{\psi - \cos^{-1}(-\rho)}{2\beta} = \frac{\lambda}{4\pi} \left[ \psi - \cos^{-1}(-\rho) \right]
\]

this will actually have two solutions.
This relationship has two solutions  \( \theta = \cos(-\rho) \) since \( \cos \) is an even function

\[
\frac{\pi}{2} \leq \theta_1 \leq \pi
\]

and \( -\pi \leq \theta_2 \leq -\frac{\pi}{2} \}

since \( -\rho < 0 \) these solutions must
lie in the 2nd and 3rd quadrants
as shown.

This specifies \( l \) by

\[
l = \frac{\lambda}{4\pi} \left[ \psi - \cos^{-1}(-\rho) \right]
\]

Now we assumed that \( z = -l \). If \( l \) comes out negative
add \( \frac{\lambda}{2} \) since this simply adds
\( 2\pi \left( \frac{\lambda}{2} \right) = 2 \left( \frac{2\pi}{\lambda} \right) \left( \frac{\lambda}{2} \right) = 2\pi \)
to the argument.

Once the \( l \) at which \( \text{Re} \left\{ Y_2 \right\} = 1 \) is determined we can
calculate the corresponding susceptance \( B \)

\[
B = -\sin \left\{ \left| Y_1 \right| \right\}_{\cos \theta = -\rho} = \frac{2\rho \sin \theta}{1 + 2\rho \cos \theta + \rho^2} \bigg|_{\cos \theta = -\rho}
\]

since \( \cos \theta = -\rho \)

\[
\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm \sqrt{1 - \rho^2}
\]

\[
B = \pm 2\rho \sqrt{1 - \rho^2} \over 1 - 2\rho^2 + \rho^2 = \pm 2\rho \sqrt{1 - \rho^2} \over 1 - \rho^2 = \pm \frac{2\rho}{\sqrt{1 - \rho^2}}
\]

The sign of \( B \) comes from the angle \( \theta \)

if \( \theta \) in the 2nd quadrant \( \sin \theta > 0 \) and we use + (capacitor)
if \( \theta \) in the 3rd quadrant \( \sin \theta < 0 \) and we use - (inductor)

We did this for parallel (shunt) matching.

You can also do this with series elements, except you
would calculate \( Z_i = Z(z = -l) = 1 - jX \) and match
with a series element.

\[
l = \frac{\psi - \cos^{-1}(\rho)}{2\beta} = \frac{\lambda}{4\pi} \left[ \psi - \cos^{-1}(\rho) \right]
\]

\[
x = \pm \frac{2\rho}{\sqrt{1 - \rho^2}}
\]
Example 3-15

An antenna having a feed-point impedance of 110 \Omega is to be matched to a 50 \Omega coaxial line with \( V_p = 2 \times 10^8 \) m/s using a single shunt lumped reactive element as shown below. Find the position (nearest to the load) and the appropriate value of the reactive element for operation at 30 MHz using (a) a capacitor, and (b) an inductor.

\[ Z_L = 110 \Omega \]

(a) \[ \pi = \rho e^{-V_p} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{110 - 50}{110 + 50} = 0.375 \quad \text{(real)} \]

The location of the capacitor will be

\[ \lambda_1 = \frac{\pi - \theta}{2\rho} = \frac{\lambda}{4\pi} \left[ \psi - \cos^{-1} (\rho) \right] = -\frac{\lambda}{4\pi} \cos^{-1} (0.375) = -\frac{1.955}{4\pi} \]

For a capacitor the sign of \( B \) is positive so this is a 2nd quadrant angle. \( \frac{\pi}{2} < \theta < \pi \) (since \( B = j\omega C \))

\[ \text{since} \ \lambda = \frac{V_p}{f} = \frac{2 \times 10^8 \text{m/s}}{30 \times 10^6} = 0.67 \text{m}. \]

Since \( \lambda_1 \) is negative add \( \frac{\lambda}{2} \)

\[ \lambda_1 = -0.156 \lambda + 0.5 \lambda = +0.344 \lambda = 0.344(0.67 \text{m}) = 2.29 \text{m} \]

The capacitance is given by

\[ (j \overline{B_1}) = j\omega C (z_0) \]

\[ \overline{B_1} = \frac{2\rho}{\sqrt{1-\rho^2}} = \frac{2(0.375)}{\sqrt{1-(0.375)^2}} = 0.809 \]

Where I chose the + sign to correspond to a shunt capacitance.

\[ (0.809) = 2\pi(30 \times 10^6) C (50) \]

\[ C = 85.8 \text{ pf}. \]
(b) For an inductor we much choose a 3rd quadrant angle.

i.e. \( \cos^{-1}(-0.375) = -1.955 \)

\[ \uparrow \text{ different sign} \]

The corresponding \( l \) is given by

\[ l_2 = \frac{\pi - \theta_2}{2\beta} = \frac{\lambda}{4\pi} \left[ \pi - \cos^{-1}(-\rho) \right] = -\frac{\lambda}{4\pi} (-1.955) = +0.1555 \lambda \]

\[ l_2 = +0.1555(0.67) = 1.04 \text{m} \]

The value of the susceptance is \( \overline{B_2} = -\frac{2\rho}{1-\rho^2} = -0.809 \)

The corresponding value is

\[ -\frac{j}{\omega L_5} Z_0 = j\overline{B_2} = -j0.809 \]

\[ L_5 = \frac{-jZ_0}{\omega (-j0.809)} = \frac{50}{2\pi (30 \times 10^6)(0.809)} = 0.328 \mu\text{H} \]
3.5.2 Matching using Series or Shunt Stubs

At microwave frequencies we commonly use open- or short-circuited stubs (short lengths of transmission line) connected in series or parallel to match.

**Short-circuited stub** - used for coax and waveguides because a short is less sensitive to pick-up and radiates less than an open.

**Open-circuit stub** - used for microstrips and striplines because it is easier to fabricate.

A shunt (parallel) stub is usually better than a series stub since breaking the line to add the stub may create discontinuities and lead to reflections.

![Diagram of series and shunt stubs]
From the previous expressions for $l$ and $B$, we only need to determine $l_s$ of the stub to give $Y_s = +jB$ at the junction.

For a short-circuited line, we know

$$Z_{in} = jZ_0 \tan(\beta l)$$

Inverting

$$Y_s = jB = \frac{Z_0}{jZ_0 \tan(\beta l_s)} = \frac{1}{j \tan(\beta l_s)}$$

so you can readily calculate $\tan(\beta l_s) = -\frac{1}{B}$ for a short-circuited stub.

You can measure the stub location $l$ relative to the position $z_{max}$ of the nearest voltage maximum toward the load since

$$l + m\frac{\lambda}{2}$$

You can specify

$$\theta = \psi - 2\beta l = \psi + 2\beta z_{max} - 2\beta \Delta l_{max}$$

$$\theta = -m2\pi - 2\beta \Delta l_{max}$$

$$\Delta l_{max} = -\frac{\theta + m2\pi}{2\beta} = -\frac{1}{2\beta} \left[ \cos^{-1}(\rho) + m2\pi \right]$$

which can be measured.
Example 3-16
Design a single stub system to match a load consisting of a resistance $R_L = 200 \Omega$ in parallel with an inductance $L_L = 200 \frac{\mu H}{\pi}$ to a transmission line with characteristic impedance $Z_0 = 100 \Omega$ and operating at 500 MHz. Connect the stubs in parallel with the line.

Express the load as an admittance

$$Y_L = \frac{1}{R_L} - j\frac{1}{\omega L_L} = \frac{1}{200} - j\frac{1}{2\pi(j(500 \times 10^6))(\frac{200}{\pi} \times 10^{-6})} = 0.005 - j0.005$$

The reflection coefficient at the load is

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = \frac{0.01 - (0.005 - j0.005)}{0.01 + (0.005 - j0.005)} = \frac{0.005 + j0.005}{0.015 - j0.005} = \frac{1 + j}{3 - j} = 0.2 + j0.4$$

$$\Gamma_L = 0.447 e^{j63.43^\circ} = 0.447 e^{j1.107}$$

The location of the stub is given as

$$l = \frac{\lambda}{4\pi} \left[ 1 - \cos^2(-\rho) \right] = \frac{\lambda}{4\pi} \left[ 1.107 + 2.034 \right] = \left\{ \begin{array}{l} -0.074 \lambda + 5\lambda = 426\lambda \\
+0.250 \lambda \\
\end{array} \right.$$

The angle can be 2nd or 3rd quadrant angle

i.e. $\cos \phi$

$$\overline{B} = \pm \frac{2\rho}{\sqrt{1 - \rho^2}} = \pm \frac{2(0.447)}{\sqrt{1 - (0.447)^2}} = \pm 1$$

The length of the first stub is

$$\tan(\beta L_1) = -\frac{1}{\overline{B}} = -1$$

for a short-circuited line

$$\beta L_1 = \frac{2\pi}{\lambda} L_1 = -0.7853$$

$$l_1 = -0.125 \lambda + 0.5 \lambda = 0.375 \lambda$$

The length of the second stub is

$$\tan(\beta L_2) = -\frac{1}{\overline{B}} = -1$$

$$\beta L_2 = \frac{2\pi}{\lambda} L_2 = +0.7854$$

$$l_2 = +0.125 \lambda$$

Preferred solution.
One thing we have NOT discussed is the frequency dependence of a solution.

Consider these two designs:

$$Y_L(f) = \frac{1}{200} - j \frac{1}{2\pi f \left( \frac{200}{\pi} \times 10^{-9} \right)} = \frac{1}{200} - j \frac{10^9}{400f}$$

On the line $\beta l = \frac{2\pi}{\lambda}$, $j = \frac{2\pi f \lambda}{\epsilon}$.

Then just to the right of the stub:

$$\gamma_1(f) = \gamma_0 \frac{Y_L + j \gamma_0 \tan \left( \frac{2\pi f \lambda}{c} \right)}{\gamma_0 + j \gamma_L \tan \left( \frac{2\pi f \lambda}{c} \right)}$$

The total line admittance just to the left of the stub is then:

$$\gamma_2(f) = \gamma_0 + \gamma = -j \gamma_0 \frac{1}{\tan \left( \frac{2\pi f \lambda}{c} \right)} + \gamma_0 \frac{Y_L + j \gamma_0 \tan \left( \frac{2\pi f \lambda}{c} \right)}{\gamma_0 + j \gamma_L \tan \left( \frac{2\pi f \lambda}{c} \right)}$$

We can now compute:

$$\Pi_2(f) = \frac{\gamma_0 - \gamma_2(f)}{\gamma_0 + \gamma_2(f)} \quad \text{and} \quad S_2(f) = \frac{1 + |\Pi_2(f)|}{1 - |\Pi_2(f)|}$$

Most engineers would consider solution #2 to be better since it works over a broader range of frequencies.
3.5.3. Quarter Wave Transformer Matching

A common and powerful technique for matching a load to a transmission line is to use a quarter-wave length long transmission line.

For \( l = \frac{\lambda}{4} \) the input impedance is

\[
Z_{\text{in}} \bigg|_{l = \frac{\lambda}{4}} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \bigg|_{l = \frac{\lambda}{4}} = Z_0 \frac{jZ_0 \tan(\beta l)}{jZ_L \tan(\beta l)} = \frac{Z_0^2}{Z_L}
\]

\[
\beta l = \frac{2\pi}{\lambda}, \quad l = \frac{\pi}{2}
\]

In terms of normalized impedances

\[
Z_m = \frac{Z_0^2}{Z_L}
\]

or

\[
\frac{Z_{\text{in}}}{Z_0} = \frac{Z_L}{Z_0}
\]

\[
\bar{Z}_{\text{in}} = \frac{1}{Z_L}
\]

We have two cases, \( Z_L = R_L \) and \( Z_L = R_L + jX_L \).

Assume that the impedance of the matching section is \( Z_Q \).

(a) For \( Z_L = R_L \)

\[
Z_{\text{in}} = \frac{Z_Q^2}{R_L}
\]

If we want \( Z_{\text{in}} \) to match \( R_L \) (a causal impedance)

\[
Z_Q = \sqrt{R_L R_L}
\]

(b) For \( Z_L = R_L + jX_L \)

in this case

\[
\bar{Z}_{\text{in}} = \frac{Z_Q^2}{R_L + jX_L}
\]

This is better written in terms of admittances:

\[
Y_{\text{in}} = \frac{R_L + jX_L}{Z_Q^2} = \frac{R_L}{Z_Q^2} + j\frac{X_L}{Z_Q^2}
\]

If \( Z_L \) is \( R_L \) then \( Y_{\text{in}} \) is positive susceptance which is a capacitance.

 conductance
Example 3-17

Design a quarter-wavelength section to match a thin monopole antenna of length 0.24λ having a purely resistive feed-point impedance of $R_L = 30\,\Omega$ to a transmission line having a characteristic impedance of $Z_0 = 100\,\Omega$.

$Z_0 = 100 \quad Z_a = 54.8 \quad R_L = 30\,\Omega$

$S = 1 \quad S = 1.82$

The matching section is $\frac{λ}{4}$ so $Z_{in} = \frac{Z_Q^2}{R_L} = 100$ for matching.

$Z_Q = \sqrt{(100)(30)} = \sqrt{3000} = 54.77\,\Omega$

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{30 - 54.77}{30 + 54.77} = \frac{-24.77}{84.77} = -0.292 = 0.29\, e^{j\pi}$

$S = \frac{1 + \rho}{1 - \rho} = \frac{1 + 0.29}{1 - 0.29} = \frac{1.29}{0.71} = 1.82$
Example 3-18
A thin wire have wave dipole antenna has an input impedance of \( Z_L = 73 + j42.5 \, \Omega \). Design a quarter-wave transformer to match this antenna to a transmission line with characteristic impedance \( Z_0 = 100 \, \Omega \).

At \( z = 0 \):
\[
\rho_C j^\psi = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{73 + j42.5 - 100}{73 + j42.5 + 100} = -0.09 + j0.26 = 0.283 e^{j108.6^\circ}
\]
\( \phi = 108.6^\circ = 1.896 \) radians

The load is inductive since \( j42.5 > 0 \)

For an inductive load the voltage minimum is more than \( \frac{\lambda}{4} \) away from the load. The maximum occurs when
\[
\psi + 2 \beta z_{\text{max}} = 0 \quad (1 + \rho \text{aligns})
\]
\[
\frac{\psi}{2\beta} = -\frac{1.896}{2(2\pi/\lambda)} = -0.151\lambda
\]

We picked a maximum since \( \psi_m(Z_{\text{in}}) = 0 \) at the voltage maximum.

We need to calculate \( Z(z = -0.151\lambda) \) and switch to normalized impedances
\[
\bar{Z}(z = -0.151\lambda) = \frac{Z_L - j\tan \beta z}{1 - jZ_L\tan \beta z}
\]
\[
= \frac{73 + j42.5}{1 - j(73 + j42.5)(-1.395)}
\]
\[
\tan(\beta z) = \tan \left( \frac{2\pi}{\lambda} \cdot -0.151\lambda \right) = -\tan \left( -302\pi \right) = -1.395
\]
\[
\bar{z} = (z = -0.151\lambda) = \frac{73 + j1.82}{107 + j1.018} = 1.7936 \quad \text{(normalized impedance)}
\]

We insent a \( \frac{\lambda}{4} \) section at \( z = -0.151\lambda \). The impedance \( Z_0 \) this section will be
\[
\bar{Z}_0 = \sqrt{1.7936(1)} = 1.339 \quad Z_0 = 133.9 \, \Omega
\]

\[
\begin{array}{c}
\begin{array}{c}
Z_0 = 100 \, \Omega \\
Z_Q = 133.9 \, \Omega \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
Z_L = 73 + j42.5 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\lambda = 0.151\lambda
\end{array}
\end{array}
\]
3.6 The Smith Chart

$$Z(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$ where \( \Gamma(z) = pe^{j(y+2\beta z)} \)

Let \( z = R + jX \) and \( \Gamma = u + jV \)

The relationship between these variables can be calculated by equating these expressions

$$z = \bar{R} + j\bar{X} = \frac{1 + (u + jv)}{1 - (u + jv)} = \frac{(1+u) + jv}{(1-u) - jv}$$

$$\frac{(1-u) + jv}{(1-u)^2 + v^2} = \frac{(1-u^2) + j\sqrt{v^2 - v^2}}{(1-u)^2 + v^2}$$

$$\bar{R} = \frac{1-u^2-v^2}{(1-u)^2 + v^2}$$

$$\bar{R}(1-u^2) + \bar{R}v^2 = 1 - u^2 - v^2$$

$$-1 + \bar{R}(1-u^2) + u^2 + \bar{R}v^2 + v^2 = 0$$

$$\bar{R}(1-u^2) - (1-u^2) + (\bar{R}+1)v^2 = 0$$

$$(1-u^2)(\bar{R}+1) + (\bar{R}+1)v^2 = 0$$

$$1 + (1-u^2)(\bar{R}+1) + (\bar{R}+1)v^2 = 1$$

$$1 + (1-u^2)(\bar{R}+1) + v^2 = \frac{1}{(\bar{R}+1)^2}$$

$$1 + \frac{(\bar{R}^2 - u^2\bar{R} - 1 + u^2)}{(\bar{R}+1)^2} + v^2 = \frac{1}{(\bar{R}+1)^2}$$

$$-u^2\bar{R}^2 + u^2 + \bar{R}^2 + v^2 = \frac{1}{(\bar{R}+1)^2}$$

$$(U - \frac{R}{R+1})^2 + v^2 = \frac{1}{(1+R)^2}$$

There are equations of curves in UV plane centered at \( u = \frac{R}{R+1}, v = 0 \) radius \( \frac{1}{1+R} \)

centered at \( u = 1, v = \frac{1}{X} \) radius \( \frac{1}{X} \)
These are the circles of $\frac{R}{\theta}$, the real part of $\frac{1}{\theta}$.

$$\left( u - \frac{\bar{R}}{1 + \bar{R}} \right)^2 + v^2 = \left( \frac{1}{\sqrt{1 + \bar{R}}} \right)^2$$

The circles of $\frac{1}{\theta}$, given $\bar{R}$, (constant $\bar{R}$ circles)

The circles are tangent to the $u = 1$ line

$$(u-1)^2 + \left( v - \frac{1}{\bar{x}} \right)^2 = \frac{1}{\bar{x}^2}$$

This allows $\pm V$ symmetry

The circles of $\frac{1}{\theta}$, given $\bar{x}$, (constant $\bar{x}$ circles)

There are two circles
rotation is understood by remembering that $\Gamma = \Gamma e^{j\phi} - pe$

as you move away from the load, $\phi$ is increasing negative reducing the angle $\phi$
What is very easy with a Smith chart is locating voltage maxima and minima.

Recall that $S = \frac{1 + \rho}{1 - \rho}$

At a voltage maxima $\bar{Z} = \bar{R}_{\text{max}} = S$

so the $S$ circles are the same as the $\rho$ circles.
Example 3-20

Find the input impedance of a lossless transmission line with the parameters \( z_0 = 100 \Omega \), \( z_L = 50 + j0 \Omega \), \( l = 86.25 \text{cm} \), and \( \lambda = 1.5 \text{m} \).

Electrical length of line is \( \frac{0.8625}{1.5} = 0.575 \lambda \)

1. \( \overline{Z}_L = \frac{z_L}{z_0} = \frac{50}{100} = 0.5 \) and enter on chart

2. Draw constant \( \rho \) circle

3. Impedance goes as \( 2\pi \rho \) so a full cycle every \( \frac{\lambda}{2} \) so examine line length \( 0.575\lambda - 0.5\lambda = 0.075\lambda \) move \( 0.075\lambda \) away from load (towards source)

4. Read off \( \overline{Z}_{in} \approx 0.59 + j \cdot 3.6 \)

\[ Z_{in} = 59 + j \cdot 36 \Omega \]
Example 3-22

Find the input impedance \( z_\text{in} \) of a lossless transmission line given the following parameters: \( Z_0 = 100 \Omega, Z_L = 100 + j100 \), line length \( l = 0.676\lambda \) (i.e., \( 0.5\lambda + 0.176\lambda \)).

1. The normalized load impedance is \( \bar{Z}_L = \frac{100 + j100}{100} = 1 + j1 \).

2. Draw circle of constant \( \rho \) (centered at origin) through this point.

3. Note that the intersection of this circle with the real axis gives \( \bar{R} = 2.62 \). This is also the value of \( S \).

4. To find \( Z_{\text{in}} \), which goes as a \( 2\beta \) we move along this circle (clockwise) towards source. Remember that it repeats every \( \frac{\lambda}{2} \) so we go \( 0.176\lambda \). We start at \( 0.162\lambda \) from chart and add \( 0.176\lambda \) to get \( 0.338\lambda \). This corresponds to \( \bar{Z}_{\text{in}} = 1 - j1 = 100 - j100 \).
Example 3-23  Find the normalized load impedance on a transmission line with the following measured parameters: standing wave ratio $S = 3.6$ and first voltage minimum $Z_{\text{min}} = -0.166 \lambda$.

1. Draw constant $\rho$ circle corresponding to $S = 3.6$

2. $Z_{\text{min}}$ is where this circle intersects negative $\lambda$ axis.

3. Start with $Z_{\text{min}}$ and move towards load (counterclockwise) a distance $+0.166 \lambda$.

4. This location gives unknown (now known) load impedance to be $\bar{Z}_L = 0.89 - j1.13$
Example 3-24

Given a characteristic impedance $Z_0 = 80\Omega$ and a load admittance $Z_L = 160 - j80$, match the line to the load by using a short circuited stub.

1. Compute $Z_L = \frac{160 - j80}{80} = 2 - j1$

   and plot on chart.

2. Draw a circle of constant $\rho$ through $Z_L$.

3. Convert $Z_L$ to $Y_L$. $Y_L = \frac{1}{Z_L} = 0.4 + j2$ and plot.

   Note: this corresponds to reflection through the origin.

4. Determine $Y_{in} = 1 + j\beta$ by moving on constant $\rho$ circle towards source (clockwise)

   until you reach $\rho = 1$ (Res = $1$) circle.

   The amount of rotation determines $\beta$ which is $0.162 - 0.04 = 0.122\lambda$.

   The intersection is at $\Gamma = 1 + j1$. So we require stub with $Y_s = -j1$.

5. A short circuit is the right most point on the admittance chart.

6. Move on circle of constant $\rho$, i.e. outer edge of chart until you reach $\rho = 1$. This is what you wanted and gives $\lambda = 0.125\lambda$.

7. A second solution exists at $\Gamma = 1 - j1$. This solution is at $\lambda = 0.338 - 0.04 = 0.298\lambda$.

   This requires $Y_s = -1$ and requires a much longer stub, i.e., $\lambda = 0.375\lambda$. 
3.25 Given a transmission line with a characteristic impedance $Z_0 = 120 \Omega$ and line impedance $Z_L = 120 \Omega$ and load impedance $Z_L = 72 + j96 \Omega$, match the line to the given load using a quarter-wave transformer.

![Diagram showing impedance matching](image)

This is different than stub-matching where we find $l_1$ such that $\sqrt{Z_1}$ would be $1 + j8$. With quarter-wave transformers, find $l_1$ where $\sqrt{Z_1}$ is entirely real, i.e., $\alpha = 0$.

1. $\bar{Z}_L = \frac{72 + j96}{120} = 0.6 + j0.8$
2. Draw constant $\rho$ circle thru $\bar{Z}_L$.
3. Move along circle away from load (clockwise) to intersection with horizontal axis, i.e., $\alpha = 0$.
   At this point $\bar{Z}_1 = S_1 = 3$.
4. Quarter-wave transformer gives $Z_Q = \sqrt{R_1 R_2} = \sqrt{(Z_0) (3Z_0)}$
   $Z_Q = 1.732 \times 120 = 207.8$. This moves up from circle of $S = 3$ to circle of $S = 1.732$, i.e., $3/1.732 = 1.73$
5. $\lambda/4$ corresponds to a 180° rotation clockwise (towards source).
6. $\bar{Z}_{in} = 0.577$. This is referred to $Z_Q = 207.8$. Converting back to $Z_0 = 120$ we get $\bar{Z}_{in} = 0.577 (\frac{207.8}{120}) = 1$ so we are matched.