Consider writing the following voltage current relationships between voltage and current.

\[ V_1 = Z_{11} i_1 + Z_{12} i_2 \]
\[ V_2 = Z_{21} i_1 + Z_{22} i_2 \]

In matrix form

\[ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = [Z][I] \]

Note that each element in this matrix can be found by measuring the voltage \( V_i \) while port \( i \) is being driven by current \( i_j \), and all other currents are zero.

For example, let \( i_2 = 0 \), then the matrix equations reduce to

\[ V_1 = Z_{11} i_1 \quad \text{or} \quad Z_{11} = \frac{V_1}{i_1} \]
\[ V_2 = Z_{21} i_1 \quad \text{or} \quad Z_{21} = \frac{V_2}{i_1} \]

Similarly, if \( i_1 = 0 \)

\[ V_1 = Z_{12} i_2 \quad \text{or} \quad Z_{12} = \frac{V_1}{i_2} \]
\[ V_2 = Z_{22} i_2 \quad \text{or} \quad Z_{22} = \frac{V_2}{i_2} \]

Z-parameters can be represented by the following equivalent circuit.
Very commonly, if engineers use $y$-parameters instead of $z$-parameters.

$y$-parameters are defined by the following matrix equation

$$[I] = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [Y] [V]$$

We can find the various $y$-parameters by shorting one of the outputs sequentially, i.e.

if $v_1 = 0$
$$i_1 = y_{12} v_2$$
$$i_2 = y_{22} v_2$$

$$y_{12} = \frac{i_1}{v_2}$$
$$y_{22} = \frac{i_2}{v_2}$$

if $v_2 = 0$
$$i_1 = y_{11} v_1$$
$$i_2 = y_{21} v_1$$

$$y_{11} = \frac{i_1}{v_1}$$
$$y_{21} = \frac{i_2}{v_1}$$

The $y$-parameters can be electrically modeled by the following equivalent circuit

```
\[ \begin{array}{c}
\begin{array}{c}
| \begin{array}{c}
\downarrow y_{12} \\
\downarrow y_{21} \\
\downarrow y_{22}
\end{array}
\end{array}
\end{array} \]
```

NOTE: $[Z] = [Y]^{-1}$
Example: $z$-parameters of a Pi-network.

For the pi-network shown below with generic impedances $Z_A$, $Z_B$ and $Z_C$ find the impedance and admittance matrices.

To find $Z_{ij}$ drive the input with $i_i$ and let the output open, i.e., $i_2 = 0$

\[
\begin{bmatrix}
    v_1 \\
    v_2 
\end{bmatrix} =
\begin{bmatrix}
    Z_{11} & Z_{12} \\
    Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
    i_1 \\
    i_2
\end{bmatrix}
\]

If $i_2 = 0$ then $v_1 = Z_{11}i_1$ and $Z_{11} = \frac{v_1}{i_1}$

$v_2 = Z_{21}i_1$

$Z_{21} = \frac{v_2}{i_1}$

by inspection $v_2 = \frac{Z_C v_1}{Z_B + Z_C}$

\[i_A = \frac{v_1}{Z_A} \]
\[i_B = \frac{v_1}{Z_B + Z_C} \]

\[Z_{11} = \frac{v_1}{i_1} = \frac{v_1}{\frac{Z_A}{Z_A} + \frac{v_1}{Z_B + Z_C}} = \frac{(Z_B + Z_C)(Z_A)}{Z_A + Z_B + Z_C} \]
\[i_1 = i_A + i_B = \frac{v_1}{Z_A} + \frac{v_1}{Z_B + Z_C} \]

\[Z_{21} = \frac{v_2}{i_1} = \frac{\frac{Z_C}{Z_B + Z_C} v_1}{\frac{Z_A}{Z_A} + \frac{v_1}{Z_B + Z_C}} = \frac{\frac{Z_C}{Z_B + Z_C} Z_A}{Z_A (Z_B + Z_C)} = \frac{Z_A Z_C}{Z_B + Z_C + Z_A} \]

We get the remaining $z$-parameters by repeating the process for $i_i = 0$. 

if \( i_l = 0 \) (input open)

\[
\begin{align*}
\nu_1 &= Z_{12} i_2 \\
\nu_2 &= Z_{22} i_2
\end{align*}
\]

\[
Z_{12} = \frac{\nu_1}{i_2}
\]

\[
Z_{22} = \frac{\nu_2}{i_2}
\]

For the given circuit:

\[

\begin{align*}
\frac{\nu_1}{Z_A} &= \frac{Z_A}{Z_A + Z_B} \\
\frac{\nu_2}{Z_B} &= \frac{Z_B}{Z_A + Z_B}
\end{align*}
\]

\[

\begin{align*}
\nu_1 &= \frac{Z_A}{Z_A + Z_B} \\
\nu_2 &= \frac{Z_B}{Z_A + Z_B}
\end{align*}
\]

\[

\begin{align*}
\nu_1 - \nu_2 &= \frac{Z_A}{Z_A + Z_B} - \frac{Z_B}{Z_A + Z_B} \\
&= \frac{Z_A - Z_B}{Z_A + Z_B}
\end{align*}
\]

Using these results in the above formulas gives:

\[
Z_{12} = \frac{\nu_1}{i_2} = \frac{\frac{Z_A}{Z_A + Z_B} \nu_2}{Z_A + Z_B}
\]

\[
Z_{12} = \frac{Z_C}{Z_A + Z_B + Z_C}
\]

\[
Z_{22} = \frac{\nu_2}{i_2} = \frac{\nu_2}{Z_A + Z_B + Z_C}
\]

\[
Z = \begin{bmatrix}
\frac{Z_A}{Z_A + Z_B + Z_C} & \frac{Z_A Z_C}{Z_A + Z_B + Z_C} \\
\frac{Z_A Z_B}{Z_A + Z_B + Z_C} & \frac{Z_C (Z_A + Z_B)}{Z_A + Z_B + Z_C}
\end{bmatrix}
\]

You can find the \( Y \) parameters the same way.
Example 4.2

Describe the common-emitter BJT transistor in terms of its $h$ (hybrid) parameters for the low-frequency, small-signal model shown below:

\[
\begin{bmatrix}
    v_1 \\
    i_2
\end{bmatrix} =
\begin{bmatrix}
    h_{11} & h_{12} \\
    h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
    i_1 \\
    v_2
\end{bmatrix}
\]

To evaluate $h_{11}$, short the output, i.e. $v_2 = v_{ce} = 0$

\[
i_B = \frac{v_{BE}}{r_{BE}} + \frac{v_{BE}}{r_{BC}} = \frac{v_{BE}}{r_{BC}}
\]

\[
i_C = -i_{rb} + \beta i_B' = -\frac{v_{BE}}{r_{BC}} + \beta \frac{v_{BE}}{r_{BE}}
\]

from matrix if $v_2 = 0$

\[
v_1 = h_{11} i_1 \Rightarrow h_{11} = \frac{v_1}{i_1} = \frac{v_{BE}}{i_B}
\]

\[
i_2 = h_{21} i_1 \Rightarrow h_{21} = \frac{i_2}{i_1} = \frac{i_C}{i_B}
\]

\[
h_{11} = \frac{v_{BE}}{i_B} = \frac{v_{BE}}{v_{BE}\left(\frac{r_{BE} + r_{BC}}{r_{BE}r_{BC}}\right)} = \frac{r_{BE}r_{BC}}{r_{BE} + r_{BC}}
\]

\[
h_{21} = \frac{i_C}{i_B} = \frac{v_{BE}\left(\frac{-r_{BE} + \beta r_{BC}}{r_{BC}r_{BE}}\right)}{v_{BE}\left(\frac{r_{BE} + r_{BC}}{r_{BE}r_{BC}}\right)} = \frac{-r_{BE} + \beta r_{BC}}{r_{BE} + r_{BC}} = \frac{\beta r_{BC} - r_{BE}}{r_{BE} + r_{BC}}
\]
from matrix if $i_1 = i_B = 0$

\[ v_1 = \frac{r_{12}}{v_2} \quad \hat{v}_{12} = \frac{v_1}{v_2} \]

\[ i_2 = \frac{r_{22}}{v_2} \quad \hat{i}_{22} = \frac{i_2}{v_2} \]

\[ \hat{v}_{12} = \frac{r_{BE}}{v_{CE}} = \frac{v_{CE}}{r_{BE} + r_{BC}} \]

\[ \hat{v}_{22} = (1+\beta) \frac{v_{CE}}{r_{BC} + r_{BE}} + \frac{v_{BE}}{v_{CE}} = \frac{(1+\beta)}{r_{BC} + r_{BE}} + \frac{1}{r_{CE}} \]

\[ \hat{h}_{22} = \frac{(1+\beta) r_{CE} + r_{BC} + r_{BE}}{r_{CE} (r_{BC} + r_{BE})} \]

\[
[\hat{h}] = \begin{bmatrix}
\frac{r_{BE}}{r_{BE} + r_{BC}} & \frac{r_{BE}}{r_{BE} + r_{BC}} \\
\frac{r_{BC} - r_{BE}}{r_{BE} + r_{BC}} & \frac{(1+\beta) r_{BC} + r_{BE} + 1}{r_{BC} + r_{BE}}
\end{bmatrix} \approx \begin{bmatrix}
r_{BE} & 0 \\
\beta & \frac{1}{r_{CE}} + \frac{\beta}{r_{BC}}
\end{bmatrix}
\]

for real transistors $\beta > 1$, $r_{BC} > r_{BE}$

The hybrid network representation is a very popular way to characterize the BJT, and $h$-parameter coefficients are reported in many data sheets.
Define the chain or ABCD parameters as

\[
\begin{bmatrix}
  v_1 \\
  i_1
\end{bmatrix} =
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  v_2 \\
  -i_2
\end{bmatrix}
\]
Cascading networks analyzed using ABCD matrices

For the first network
\[
\begin{bmatrix} i_1' \\ i_1'' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} v_1' \\ -i_1' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i_1'' \\ i_1'' \end{bmatrix} = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{bmatrix} v_2'' \\ -i_2'' \end{bmatrix}
\]

For the second network

The key thing about cascading networks is that \( i_1'' = -i_2' \) and \( v_1'' = v_2' \)

Then
\[
\begin{bmatrix} v_1' \\ i_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} v_2' \\ -i_2' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} v_1'' \\ i_1'' \end{bmatrix}
\]
\[
\begin{bmatrix} v_1' \\ i_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{bmatrix} v_1'' \\ i_1'' \end{bmatrix}
\]
Example 4-6 \( \text{ABCD-matrix coefficient representation of a transmission line section.} \)

\[
\begin{bmatrix}
\nu_1 \\
i_1
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\nu_2 \\
i_2
\end{bmatrix}
\]

If we short port 2, i.e., \( \nu_2 = 0 \) (a shorted line)

\[
\begin{align*}
\nu_1 &= -B i_2 \\
i_1 &= -D i_2
\end{align*}
\]

\[
\begin{align*}
B &= -\frac{\nu_1}{i_2} \\
D &= -\frac{i_1}{i_2}
\end{align*}
\]

If we open port 2, i.e., \( i_2 = 0 \) (an open line)

\[
\begin{align*}
\nu_1 &= A \nu_2 \\
i_1 &= C \nu_2
\end{align*}
\]

\[
\begin{align*}
A &= \frac{\nu_1}{\nu_2} \\
C &= \frac{i_1}{\nu_2}
\end{align*}
\]

For an open line

\[
\begin{align*}
V(d) &= 2V^+ \cos(\beta d) \\
I(d) &= \frac{2V^+}{Z_0} \sin(\beta d)
\end{align*}
\]

current defined as towards load

For a short line

\[
\begin{align*}
V(d) &= 2j V^+ \sin(\beta d) \\
I(d) &= \frac{2V^+}{Z_0} \cos(\beta d)
\end{align*}
\]

current defined as towards load.

Using these results for a transmission line:

\[
A = \left. \frac{\nu_1}{\nu_2} \right|_{i_2 = 0} = \frac{2V^+ \cos(\beta l)}{2V^+ \cos(\beta 0)} = \frac{\cos \beta l}{\cos \beta 0}
\]

\[
B = -\left. \frac{\nu_1}{i_2} \right|_{\nu_2 = 0} = \frac{2j V^+ \sin(\beta l)}{2V^+ \cos(\beta 0)} = j \frac{Z_0 \sin(\beta l)}{\cos(\beta 0)}
\]
\[ c = \frac{i_1}{v_2} \bigg|_{v_2 = 0} = \frac{2j \frac{V^+}{Z_0} \sin(\beta l)}{2 \frac{V^+}{Z_0} \cos(\beta')} = j Y_0 \sin(\beta l) \]

\[ d = -\frac{i_1}{i_2} \bigg|_{v_2 = 0} = -\frac{2 \frac{V^+}{Z_0} \cos(\beta l)}{2 \frac{V^+}{Z_0} \cos(\beta')} = -\cos(\beta l) \]

\[ \text{For a transmission line} \]

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos \beta l & jZ_0 \sin \beta l \\ jY_0 \sin \beta l & -\cos \beta l \end{bmatrix} \]
4.4 Scattering parameters

Almost all microwave engineers use s-parameters (scattering parameters) at microwave frequencies because it is very difficult to achieve a true open or short at r.f./microwave frequencies. Furthermore, you do not want to introduce large reflection coefficients which can lead to oscillations and/or destroy a semiconductor device.

\[ a_1 = \frac{1}{2\sqrt{Z_0}} (v_1 + Z_0 I_1) \quad \text{where} \quad I_1, v_1 \text{ are at the input} \]

\[ a_2 = \frac{1}{2\sqrt{Z_0}} (v_2 + Z_0 I_2) \quad \text{I}_2, v_2 \text{ are at the output} \]

and a reflected normalized power

\[ b_1 = \frac{1}{2\sqrt{Z_0}} (v_1 - Z_0 I_1) \]

\[ b_2 = \frac{1}{2\sqrt{Z_0}} (v_2 - Z_0 I_2) \]

If we solve these equations for \( V \) and \( I \) we get

\[ V_1 = \sqrt{Z_0} (a_1 + b_1) \quad \text{(1)} \]

\[ V_2 = \sqrt{Z_0} (a_2 + b_2) \quad \text{(2)} \]

\[ I_1 = \frac{1}{\sqrt{Z_0}} (a_1 - b_1) \quad \text{(3)} \]

\[ I_2 = \frac{1}{\sqrt{Z_0}} (a_2 - b_2) \quad \text{(4)} \]

These look like strange definitions but consider these in terms of traveling waves and power.
If you simply examine (1) & (3) \( a_i \) is simply the forward traveling wave and \( b_i \) is the backward traveling wave.

\[
\begin{align*}
    a_1 &= \frac{V_1^+}{\sqrt{Z_0}} = \sqrt{Z_0} I_1^+ \\
    b_1 &= \frac{V_1^-}{\sqrt{Z_0}} = -\sqrt{Z_0} I_1^-
\end{align*}
\]

Note that \( V^+ = I_1^+ Z_0 \) and \( V^- = -Z_0 I_1^- \)

\[
\begin{align*}
    a_2 &= \frac{V_2^+}{\sqrt{Z_0}} = \sqrt{Z_0} I_2^+ \\
    b_2 &= \frac{V_2^-}{\sqrt{Z_0}} = -\sqrt{Z_0} I_2^-
\end{align*}
\]

The \( S \)-parameters are closely related to power.
\[
\begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix} =
\begin{bmatrix}
  S_{11}  & S_{12}  \\
  S_{21}  & S_{22}
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix}
\]

\( S_{11} = \frac{b_1}{a_1} \bigg|_{a_2 = 0} \) \( \frac{\text{reflected power wave at port #1}}{\text{incident power wave at port #1}} \) \( \frac{\text{no input power}}{\text{at port #2}} \)

\( S_{12} = \frac{b_1}{a_2} \bigg|_{a_1 = 0} \) \( \frac{\text{transmitted power wave at port #1}}{\text{incident power wave at port #2}} \) \( \frac{\text{no input power}}{\text{at port #1}} \)

\( S_{21} = \frac{b_2}{a_1} \bigg|_{a_2 = 0} \) \( \frac{\text{transmitted power wave at port #2}}{\text{incident power wave at port #1}} \) \( \frac{\text{no input power}}{\text{at port #2}} \)

\( S_{22} = \frac{b_2}{a_2} \bigg|_{a_1 = 0} \) \( \frac{\text{reflected power wave at port #2}}{\text{incident power wave at port #1}} \) \( \frac{\text{no input power}}{\text{at port #1}} \)

These conditions are true when input \( (a_1) \) or output \( (a_2) \) are matched to port impedance.
Recall that \( P_R = \frac{1}{2} \text{Re} \left\{ V I^* \right\} \)

At the input port \( P_i = \frac{1}{2} \text{Re} \left\{ V_i I_i^* \right\} = \frac{1}{2} \text{Re} \left\{ \frac{V_i^+ (1 + \Gamma_{in})}{Z_0} \frac{V_i^+}{Z_0} (1 - \Gamma_{in}) \right\} \)

\[ P_i = \frac{1}{2} \left| \frac{V_i^+}{Z_0} \right|^2 (1 - |\Gamma_{in}|^2) = P_i^+ + P_i^- \]

using our definitions that \( V_i = \sqrt{Z_0} (a_i + b_i) \)
\( I_i = \frac{1}{\sqrt{Z_0}} (a_i - b_i) \)

we can also compute the power at port #1 as
\[ P_1 = \frac{1}{2} \text{Re} \left\{ V_1 I_1^* \right\} = \frac{1}{2} \text{Re} \left\{ \frac{V_1^+}{\sqrt{Z_0}} (a_1 + b_1) \frac{1}{\sqrt{Z_0}} (a_1^* - b_1^*) \right\} \]

\[ P_1 = \frac{1}{2} \text{Re} \left\{ |a_1|^2 - |b_1|^2 \right\} = \frac{1}{2} |a_1|^2 \left\{ 1 - \frac{|b_1|^2}{|a_1|^2} \right\} \]

but \( a_1 = \frac{V_i^+}{\sqrt{Z_0}} \) so we can re-write this equation as
\[ P_1 = \frac{1}{2} \left| \frac{V_i^+}{\sqrt{Z_0}} \right|^2 \left\{ 1 - \left| \frac{b_1}{a_1} \right|^2 \right\} = \frac{1}{2} \left| V_i^+ \right|^2 \left\{ 1 - |S_{ii}|^2 \right\} \]

Comparing these two expressions for \( P_1 \) we quickly see that
\[ \Gamma_{in} = \frac{V_i^-}{V_i^+} = \frac{b_1}{a_1} \bigg|_{a_2=0} = S_{ii} \]

This allows us the write the SWR at port #1 as
\[ S = \frac{1 + |S_{ii}|}{1 - |S_{ii}|} \]

Note also that \( \frac{1}{2} |a_1|^2 = \frac{1}{2} \left| \frac{V_i^+}{\sqrt{Z_0}} \right|^2 = P_{incident} \)

You can do the same analysis at the output to get at port #2
\[ P_2 = \frac{1}{2} \left\{ |a_2|^2 - |b_2|^2 \right\} = \frac{|a_2|^2}{2} (1 - |R_{out}|^2) \]
4.4.2. Meaning of s-parameters

The output \( Z_L \) is matched to \( Z_0 \) so that no \( V_2^+ \) is created at the output.

Under the above output matched conditions \( a_2 = 0 \).

Since \( S_{11} = \Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \) since \( S_{11} = \frac{b_1}{a_1} \), \( b_2 = 0 \)

we have just developed a method for measuring \( S_{11} \)

also since \( a_2 = 0 \)

\( S_{21} = \frac{b_2}{a_1} \bigg|_{b_2 = 0} = \frac{V_2^-}{\frac{1}{2 Zi_0} (V_1 + Z_0 I_1)} \bigg|_{I_2^+ = V_2^- = 0} \)

Note that \( V_2^+ I_2^+ \) are going into the output just like a two-port

Substituting \( V_1 = V_{g1} - Z_0 I_1 \)

\( S_{21} = \frac{2V_2'}{V_{g1} - Z_0 I_1 + Z_0 I_1} = \frac{2V_2'}{V_{g1}} = \frac{2V_2}{V_{g1}} \)

This is the forward voltage gain \( G_0 \) of the network

\( G_0 = |S_{21}|^2 = \left| \frac{V_2}{V_{g1/2}} \right|^2 \)

is the forward power gain
You can measure \( S_{22} \) and \( S_{12} \) by matching at port #1 and using a generator at port #2.

![Circuit Diagram](image)

Line is matched to the impedance \( Z_0 \).
To make sure no \( V^-_1 \) is created at the load.

The results at the output are identical to the input

\[
S_{22} = \frac{V^-_1}{V_2} = \frac{Z_{out} - Z_0}{Z_{out} + Z_0} \quad \text{since} \quad S_{22} = \frac{b_2}{a_2} \bigg|_{a_1=0}
\]

Since \( a_1 = 0 \)

\[
S_{12} = \frac{b_1}{a_2} \bigg|_{a_1=0} = \frac{V^-_1}{\sqrt{Z_0}} = \frac{1}{2\sqrt{Z_0}} \left( V_2 + Z_0 I_2 \right) \bigg|_{I_1^+ = V_1^+ = 0}
\]

Note again that \( I_1^+, V_1^+ \) are going into the input just like a two-port.

Cancelling terms and substituting \( V_2 = V_{g2} - Z_0 I_2 \).

\[
S_{12} = \frac{2V^-_1}{V_{g2} - Z_0 I_2 + Z_0 I_2} = \frac{2V^-_1}{V_{g1} + V_{g1}} = \frac{2V^-_1}{V_{g1}} = \frac{2V_1}{V_{g1}}
\]

This is the reverse voltage gain.
Example 4-7  Find the $s$-parameters and the resistive elements for the 3dB attenuator network shown below assuming that the network is placed in a transmission line section with a characteristic line impedance $z_0 = 50\Omega$

The attenuator must be matched to the line impedance $z_0$

For a terminated output ($z = 50\Omega$) we have

$$Z_{in} = R_1 + \frac{R_3(R_2 + 50)}{R_3 + R_2 + 50} = 50$$

For the terminated input ($z = 50\Omega$) we have the same circuit.

If matched, no reflection.

For the voltage relationship we have the output voltage given by

$$V_2 = \frac{R_{3||}(R_2 + 50)}{R_{3||}(R_2 + 50) + R_2} \cdot \frac{50}{R_1 + 50} \cdot \frac{V_1}{V_2}$$

Voltage at $A$  Voltage at $50\Omega$ input termination

For 3dB attenuation (no input power at port #2)

$$S_{21} = \left. \frac{V_2}{V_1} \right|_{I_2^+ = V_2^+ = 0} = \frac{1}{\sqrt{2}} = \frac{2V_2}{V_{61}}$$

This is where we bring in the $s$-parameters.
This gives two equations.
Since we want the network to be symmetric $R_1 = R_2$, leaving two equations in two unknowns.
These can be solved to give

\[ R_1 = R_2 = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \] 
\[ Z_0 = 8.58 \Omega \]

\[ R_3 = 2\sqrt{2} Z_0 = 141.4 \Omega \]
4.4.3. Chain Scattering Matrix

We can rearrange the S matrix to group input and output terms together. This is for cascading networks just like the ABCD matrix:

\[
\begin{bmatrix}
    a_1^A \\
    b_1^A
\end{bmatrix}
= \begin{bmatrix}
    T_{11} & T_{12} \\
    T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
    b_2^A \\
    a_2^A
\end{bmatrix}
\]

Network A is described by:

\[
\begin{bmatrix}
    a_1^A \\
    b_1^A
\end{bmatrix}
= \begin{bmatrix}
    T_{11}^A & T_{12}^A \\
    T_{21}^A & T_{22}^A
\end{bmatrix}
\begin{bmatrix}
    b_2^A \\
    a_2^A
\end{bmatrix}
\]

inputs

Network B is described by:

\[
\begin{bmatrix}
    a_1^B \\
    b_1^B
\end{bmatrix}
= \begin{bmatrix}
    T_{11}^B & T_{12}^B \\
    T_{21}^B & T_{22}^B
\end{bmatrix}
\begin{bmatrix}
    b_2^B \\
    a_2^B
\end{bmatrix}
\]

outputs

since

\[
\begin{bmatrix}
    b_2^A \\
    a_2^A
\end{bmatrix}
= \begin{bmatrix}
    a_1^B \\
    b_1^B
\end{bmatrix}
\]

Note change in direction reverses a and b.

we can rewrite as

\[
\begin{bmatrix}
    a_1^A \\
    b_1^A
\end{bmatrix}
= \begin{bmatrix}
    T_{11}^A & T_{12}^A \\
    T_{21}^A & T_{22}^A
\end{bmatrix}
\begin{bmatrix}
    T_{11}^B & T_{12}^B \\
    T_{21}^B & T_{22}^B
\end{bmatrix}
\begin{bmatrix}
    b_2^B \\
    a_2^B
\end{bmatrix}
\]

T_{ij} and S_{ij} can be algebraically related.

S_{ij} can also be related to E_{ij} parameters
4.4.5. Signal Flow Chart Modeling

```
\[ \begin{align*}
  & a \\
  \rightarrow & \text{source node } a \text{ which launches a wave} \\
  \rightarrow & b \\
  \text{sink node } b \text{ which receives a wave} \\
\end{align*} \]
```

branch which connects source and sink \( b = \Gamma a \)

Simple system

```
\begin{align*}
  & z_G \\
  & \text{signal generator} \\
  \text{at input of line generator} \\
  & b' \\
  & a' \\
  & b \\
  & \Gamma_s \\
  & b_s \\
  & \Gamma_L \\
\end{align*}
```

Note: Don't interpret variables as static voltages but waves propagating either \( L \rightarrow R \text{ or } R \rightarrow L \) by inspection

\[ b' = b_s + a' \Gamma_s \]

giving the source as \( b_s = b' - a' \Gamma_s \)

We can compare this with the wave based equivalent representation:

\[ V_S^+ + V_S^- = V_G + Z_G \left[ \frac{V_S^+ - \frac{1}{2} V_S^-}{Z_0} \right] \]

voltage drop across \( Z_G \)
Note that \( a' = \Gamma_L b' \),

so that \( b' = b_s + a' \Gamma_L = b_s + \Gamma_L \Gamma_s b' \)

\[ b' = \frac{b_s}{1-\Gamma_L \Gamma_s} \]

In signal flow chart terms, this can be reduced to a single branch.

---

**Basic signal flowchart elements**

- **Node assignment**
  \[ \frac{a}{Z_0} \frac{b}{Z_0} \]

- **Branch**
  \[ \frac{a}{Z_0} \frac{b}{Z_0} \]

- **Series connection**
  \[ \frac{S_{ba}}{a} \frac{S_{eb}}{b} \frac{c}{S_{eb}} \]

- **Splitting of branches**
  \[ \frac{b}{S_2} \frac{S_3}{c} \frac{S_1}{a} \]

- **Parallel connection**
  \[ \frac{S_1}{a} \frac{S_2}{b} \frac{S_1+S_2}{c} \]

- **Self-loop**
  \[ \frac{a}{1-\Gamma_L} \frac{b}{\Gamma_L} \frac{c}{S_1+S_2} \]
Example 4-8  For the network shown below find the ratio $\frac{a_1}{b_5}$.
Assume unity for the multiplication factor of the transmission line segments.

Initial signal flow chart

Step 1: split right-most loop

Step 2: collapse (decompose) the self-loop between $b_2$ and $a_2$

Step 3:

then combine series elements

and parallel elements
step 4: split loop into a self-loop

\[ b_5 \xrightarrow{1} a \]

\[ (s_{11} + \frac{s_{12} s_{21}}{1-s_{22} \Gamma_L}) \Gamma_3 \]

step 5: decompose the self-loop

\[ b_5 \xrightarrow{1} a \]

\[ \frac{1}{1-(s_{11} + \frac{s_{12} s_{21}}{1-s_{22} \Gamma_L}) \Gamma_3} \]

\[ a_1 = \frac{1}{1-(s_{11} + \frac{s_{12} s_{21}}{1-s_{22} \Gamma_L}) \Gamma_3} \cdot b_5 \]

\[ a_1 = \frac{1-s_{22} \Gamma_L}{1-(s_{11} \Gamma_3 + s_{22} \Gamma_L + s_{12} s_{21} \Gamma_3) + s_{11} s_{22} \Gamma_3 \Gamma_L} \cdot b_3 \]
4.4.7. Practical Measurement of S-parameters.

Basic concept is relatively simple. The ratio $A/R$ gives $S_{11}$; $S_{21}$ comes from $B/R$.

You can measure $S_{12}$ and $S_{22}$ by reversing the DUT.

Real system is much more complex because of cable lengths, impedances, non-ideal external components, etc.

Practical system for measurement
A lot of research involves using a computer and three known loads (open, short, and matched) to estimate \( E_{11}, E_{12}, E_{22}, E_X, E_R \) and \( E_T \).

Another popular method is the Through-Reflect-Line (TRL) method:

Through: directly connect ports 1 and 2 of the DUT (a through short, not an end short)

Reflect: use a load with high reflectivity and the same reflection coefficient for both input and output ports of the DUT.

Through: connect ports 1 and 2 by a transmission line, matched to the impedance of the error boxes.