$\pm a$ (corresponding to points aligned with but beyond the rails) it can be shown that $\partial A_y/\partial x \to 0$, and H_z is finite.

The expressions derived here can be used to compute the electromagnetic pulse produced at a point in space by an electric gun of given characteristics. It is necessary, of course, to make preliminary calculations of armature velocity and geometric parameters as functions of time. An example calculation has been made of the electric field pulse produced by a gun with characteristics similar to those of EMACK, namely, rail length: 5 m, muzzle velocity: 4.2 km/s, rail spacing: 7 cm, and current: 2.1×10^6 A. A space point in the plane of the rails has been selected, 5 m distant from, and abreast of, the muzzle. Equations (14) and (15) have been used.

The magnitude of the E-vector is plotted versus time in Fig. 4. The direction of the vector changes with time, from an orientation at 45° to the rails at the start, to a position parallel to the rails at maximum field. The abrupt termination of the pulse, which gives the pulse broad-band capability for interference, is caused in this simple model by the instantaneous switch-off of current when the armature leaves the rails. This condition may be modified in real life by factors such as arcing at the muzzle.

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A Procedure for Calculating the Atmospheric Mutual Coherence Function Via the Statistical Fourier-Optical Method

JEFFREY J. SITTERLE, MEMBER, IEEE,
PAUL C. CLASPY, SENIOR MEMBER, AND
FRANCIS L. MERAT, MEMBER, IEEE

***Abstract*—An algorithm for computing the atmospheric mutual coherence function from flux measurements taken at the focal plane of a reflector antenna is presented. The procedure consists of first computing the inverse Abel transform of the flux, taking the Fourier transform of the result, and then dividing by the aperture pupil function. It is shown that when flux measurements contain additive noise, the Abel inversion is an ill-posed problem. Therefore, calculation of the inverse Abel transform is accomplished via a Kalman filtering algorithm. Results of the mutual coherence function estimator are given for simulated flux measurements.**

Manuscript received October 7, 1987; revised May 2, 1988. This work was supported by the U.S. Army Research Office under Contract DAAG-29-81-K-0172.

J. J. Sitterle is with the Georgia Tech Research Institute, Georgia Institute of Technology, Atlanta, GA 30332.

P. C. Claspy and F. L. Merat are with the Department of Electrical Engineering and Applied Physics, Case Western Reserve University, Cleveland, OH 44106.

IEEE Log Number 8824126.

I. Introduction

The mutual coherence function (MCF) of a propagating electromagnetic wave is a measure of the long-term spatial coherence of the complex fields in a plane transverse to the direction of propagation. It is an extremely useful quantity that arises in propagation studies [1] (e.g., it provides the degree of coherence of the field at two points, the angular distribution of the field, and the field's mean energy) and it determines the signal-to-noise ratio (SNR) in an optical heterodyne receiver [2], [3].

The experimental determination of the MCF can be accomplished through two methods; the statistical Fourier-optical method (SFOM) [4] and the long baseline interometric method (LBIM) [5]. The LBIM has been used extensively at microwave and millimeter wave frequencies and consists of a multiple-phased receiving array. By employing a variable phase shifter in each antenna output and mixing various combinations of antenna outputs together, measurements of both the amplitude and phase fluctuation, along with the amplitude correlation of the wave front, are made. However, because the antennas are at fixed positions during the measurements, the MCF can only be determined for those spatial separations.

The SFOM, first introduced by Land [4] for visible wavelengths, is the two-dimensional continuous analogue of the LBIM with each point of a large spherical reflector acting as a separate point antenna. Using the basic principles of optics for reflector antennas, we know that the instantaneous intensity distribution appearing at the focal plane of the reflector is equal to the magnitude-squared spatial Fourier transform of the electromagnetic field at the antenna's aperture plane. Therefore, the position and shape of the focal plane distribution is directly related to the angle-of-arrival and intensity fluctuations, and the MCF is computed directly from the temporal average of these fluctuations [6]. Here, the MCF is given over a continuous interval which is limited by the aperture size of the reflector.

In this communication we present a method for computing the MCF at optical and millimeter wavelengths from a receiver based on the SFOM. The receiver consists of a parabolic reflector (or lens) with a spatial sampling device located at the focal plane. The spatial sampling device scans the focal plane diffraction pattern and allows measurement of the spatial and temporal fluctuations of the pattern.

II. The Statistical Fourier-Optical Method

For the case of a parabolic reflector antenna (or lens) of radius R and focal length f, the average intensity $\langle I(\mathbf{q})\rangle$ at a point $\mathbf{q}$ in the focal plane is given by [6]

$$\langle I(\mathbf{q})\rangle = \left(\frac{k}{2\pi f}\right)^2 \int_{-\infty}^{\infty} \Gamma(\rho) M_L(\rho) \exp\left[-i\frac{k}{f}\mathbf{q}\cdot\rho\right] d^2\rho \tag{1}$$

where the MCF, $\Gamma(\mathbf{r}_1, \mathbf{r}_2) = \langle E(\mathbf{r}_1)E^*(\mathbf{r}_2)\rangle$, is taken to be homogeneous (i.e., the MCF depends only on the difference coordinate $\rho = \mathbf{r}_1 - \mathbf{r}_2$), $\mathbf{r}_1$ and $\mathbf{r}_2$ are coordinates in the aperture plane, $E(\mathbf{r})$ is the stochastic aperture plane field, $\langle\cdot\rangle$ denotes an ensemble average, $k = 2\pi/\lambda$ where λ is the wavelength, and $M_L(\rho)$, defined as the convolution of the receiver aperture function, $W(\mathbf{r})$, with itself, is given by

$$M_L(\rho) = \int_{-\infty}^{\infty} W(r)W(\mathbf{r}, \mathbf{r}-\rho)\, d^2\mathbf{r}. \tag{2}$$

Since it is assumed that the ergodic theorem holds, the ensemble average in (1) is replaced by an average over time where the statistic is taken over a period much larger than the characteristic time of the atmosphere: $l/V_\perp$ where l is the smaller of the initial transmitted beam diameter or the transverse coherence length[1] of the wave and $V_\perp$ is the wind velocity transverse to the direction of wave propagation [2].

Assuming that the atmospheric fluctuations are not only homogeneous, but also isotropic (this assumption is questionable for propagation near the ground), we can convert (1) into plane polar coordinates. Performing the angular integration yields

$$\langle I(q)\rangle = \frac{1}{2\pi}\left(\frac{k}{f}\right)^2 \int_0^\infty \Gamma(\rho) M_L(\rho) J_0\left(\frac{kq\rho}{f}\right)\rho\, d\rho \quad (3)$$

which is the fundamental equation of the SFOM relating the measured average intensity to the MCF for the case of homogeneous and isotropic fluctuations of the electric field in the aperture plane. Changing variables, one can see that (3) is simply the Hankel transform of the function $\Gamma(f\rho/k)M_L(f\rho/k)$. Solving for the MCF in terms of the measured average intensity gives

$$\Gamma\left(\frac{f\rho}{k}\right) = \frac{2\pi}{M_L\left(\frac{f\rho}{k}\right)} \int_0^\infty \langle I(q)\rangle J_0(q\rho)\, q\, dq \quad (4)$$

where

$$M_L(\rho) = \begin{cases} 2R^2\left[\cos^{-1}\left(\frac{|\rho|}{2R}\right) - \left(\frac{|\rho|}{2R}\right)\left(1 - \left(\frac{|\rho|}{2R}\right)^2\right)^{1/2}\right], & |\rho| \le 2R \\ 0, & |\rho| > 2R. \end{cases} \quad (5)$$

III. Calculation of the Mutual Coherence Function

It is well known [7] that the Hankel transform of a function can be computed as the one-dimensional Fourier transform of the Abel transform of the function (the projection-slice theorem). The Abel transform $A\Gamma(P)$ of the two-dimensional, circularly symmetric intensity distribution is

$$A\Gamma(P) = 2\int_P^\infty \frac{\langle I(q)\rangle\, q\, dq}{\sqrt{q^2 - P^2}} \quad (6)$$

where $A\Gamma(P)$ is an even function. The MCF is then computed as

$$\Gamma\left(\frac{f\rho}{k}\right) = \frac{1}{M_L\left(\frac{f\rho}{k}\right)} \int_{-\infty}^{\infty} A\Gamma(P) \exp\{-i2\pi\rho P\}\, dP. \quad (7)$$

Calculation of (6) can be performed by any one of the many Abel transform algorithms given in the literature [7]–[9] and (7) is computed with a fast Fourier transform (FFT) algorithm.

For the case of the focal plane spatial sampling device being a radially scanning iris, the quantity that is measured is not the average intensity, but the average flux passing through an iris of radius q. Expressing the intensity in terms of the flux, we get

$$\langle I(q)\rangle = \frac{1}{2\pi q}\left\langle \frac{d}{dq}\Phi(q)\right\rangle = \frac{\frac{d}{dq}\langle\Phi(q)\rangle}{2\pi q}. \quad (8)$$

[1] The authors are using the spherical wave coherence length given by Fante [2] as

$$\rho_0 = \left[1.46k^2 x \int_0^1 d\xi (1-\xi)^{5/3} C_n^2(\xi x)\right]^{-3/5}.$$

It is possible to interchange the averaging operator and the differential operator in (8) since the average is over time and the derivative is over space. Substituting (8) into (6) results in

$$A\Gamma(P) = \frac{1}{\pi}\int_P^\infty \frac{\frac{d}{dq}\langle\Phi(q)\rangle}{\sqrt{q^2 - P^2}}\, dq \quad (9)$$

which is the negative of the inverse Abel transform. Therefore, computation of the MCF from the flux consists of calculating the inverse Abel transform of the flux, negating, taking the Fourier transform, and dividing by the aperture pupil function. One difficulty in applying this solution to experimental measurements is that, for noisy data, the Abel inversion is ill-posed; i.e., taking the derivative of the noise in the flux results in a new additive white noise sequence whose variance is inversely proportional to the iris radial step size. Hence, to avoid the calculation of the derivative in (9), a slightly modified version of the Kalman filter described by Hansen and Law [8] is used to compute the Abel inversion.

For the Abel inversion of (9), the Kalman filter consists of assuming a form for the projection $A\Gamma(P)$, computing the forward Abel transform of $A\Gamma(P)$ to obtain an estimate of the flux, computing the error between the estimated and measured average flux, and correcting the initial estimate of the projection. The forward Abel transform [9] is written in terms of a linear shift-variant system and then approximated by a state variable model. The advantage of this algorithm is not only very fast computational speeds, but also the fact that the recursive state equations can be easily converted into the Kalman filter.

The basis of the Kalman filter that we use here is the inwardly recursive augmented state variable model given by

$$\zeta(n-1) = \mathbf{F}(n)\zeta(n) + \mathbf{G}w(n) \quad (10a)$$

$$\langle\Phi(n)\rangle = \mathbf{H}\zeta(n) + v(n) \quad (10b)$$

where $v(n)$ is the zero-mean white Gaussian measurement noise, $w(n)$ is a similar process noise (independent of $v(n)$) and

$$\zeta(n) = \begin{bmatrix} A\Gamma(n) \\ A\Gamma(n-1) \\ \hline \mathbf{x}(n) \end{bmatrix} \quad \mathbf{F}(n) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline \mathbf{B}_0(n) & \mathbf{B}_1(n) & & \Xi(n) & \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ 1 \\ - \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{H}^T = \begin{bmatrix} 0 \\ 0 \\ - \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (11)$$

Here, $\mathbf{x}(n)$ is the state vector and $\mathbf{B}_0(n), \mathbf{B}_1(n)$, and $\Xi(n)$ are Abel transform dependent matrices given in [9].

Suppose now that a minimum variance, unbiased linear estimate of $\zeta(n)$ has been obtained using measurements $\langle\Phi(N)\rangle$ down to and including $\langle\Phi(n)\rangle$ (measurements are made recursively inward from the iris open position $n = N$ down to the closed position $n = 1$). The estimate and its estimation error covariance matrix are represented by $\hat{\zeta}(n|n)$ and $\mathbf{\Lambda}(n|n)$, respectively. A prediction of the next state vector $\zeta(n-1)$, without using further measurements, can be obtained from

$$\hat{\zeta}(n-1|n) = \mathbf{F}(n)\hat{\zeta}(n|n) \tag{12}$$

where $\hat{\zeta}(n-1|n)$ means that $\zeta(n-1)$ has been estimated with measurements down to only n. The error covariance matrix for the estimate is

$$\mathbf{\Lambda}(n-1|n) = \mathbf{F}(n)\mathbf{\Lambda}(n|n)\mathbf{F}^T(n) + \mathbf{G}\sigma_w^2\mathbf{G} \tag{13}$$

where σ_w^2 is the variance of $w(n)$. The filter's estimate of the next data point $\langle\Phi(n-1)\rangle$ is then

$$\langle\hat{\Phi}(n-1|n)\rangle = \mathbf{H}\hat{\zeta}(n-1|n). \tag{14}$$

The second step of the filtering process is to compare the predicted output $\langle\hat{\Phi}(n-1|n)\rangle$ to the actual measured flux $\langle\Phi(n-1)\rangle$ and compute the error. This error is then combined with the estimated state vector to produce a new filtered flux estimate $\langle\hat{\Phi}(n-1|n-1)\rangle$.

The error between the predicted flux value and the actual measured flux is

$$\hat{\epsilon}(n-1|n) = \langle\Phi(n-1)\rangle - \mathbf{H}\hat{\zeta}a(n-1|n). \tag{15}$$

Multiplying the estimation error by the Kalman gain vector $\mathbf{K}(n)$ and adding the result to the predicted flux, the new filtered estimate becomes

$$\hat{\zeta}(n-1|n-1) = \hat{\zeta}(n-1|n) + \mathbf{K}(n)\hat{\epsilon}(n-1|n) \tag{16}$$

and

$$\mathbf{\Lambda}(n-1|n-1) = \mathbf{\Lambda}(n-1|n) - \mathbf{K}(n)\mathbf{H}\mathbf{\Lambda}(n-1|n) \tag{17}$$

where the Kalman gain matrix is

$$\mathbf{K}(n) = \mathbf{\Lambda}(n-1|n)\mathbf{H}^T(\mathbf{H}\mathbf{\Lambda}(n-1|n)\mathbf{H}^T + \sigma_v^2)^{-1}. \tag{18}$$

The quantity in parenthesis in (18) is a scalar quantity, therefore the matrix inversion reduces to a division. The projection estimate, $A\Gamma(n-1)$ can now be calculated by

$$A\Gamma(n-1) = \mathbf{G}\hat{\zeta}(n-1|n-1). \tag{19}$$

Initial conditions for the Kalman filter are $\hat{\zeta}(N|N) = 0$ and $\mathbf{\Lambda}(N|N) = \mathbf{G}\sigma_v^2\mathbf{G}^T$. The variance σ_v^2 is computed from the measurement noise of the receiver, while the quantity σ_w^2 is computed as in [8].

IV. Numerical Results

Results of the Kalman filter MCF estimator on simulated average flux signals are given in Figs. 1 and 2. The simulation model, based on a submillimeter wave experiment [10], assumes a collimated beamwave (initial beam diameter is 0.6 m) with a 0.89 mm wavelength propagating through an atmosphere characterized by weak turbulence. The receiver, located 1.6 km away from the transmitter, has a radius of 0.79 m and a focal length of 0.66 m. Average focal plane flux estimates are computed from 1000 samples taken at 512 iris radii with an iris step size of 0.01 mm. A complete scan takes approximately 20 min.

One advantage of the Kalman filter form of the MCF estimator is that calculations can be performed in real time. At each iris radius the average flux is computed from the 1000 data samples received at that radius and the projection $A\Gamma(n)$ is determined before starting the next iris radius. Therefore, at the end of each scan, the MCF is simply computed by performing an FFT on the values of $A\Gamma$ and

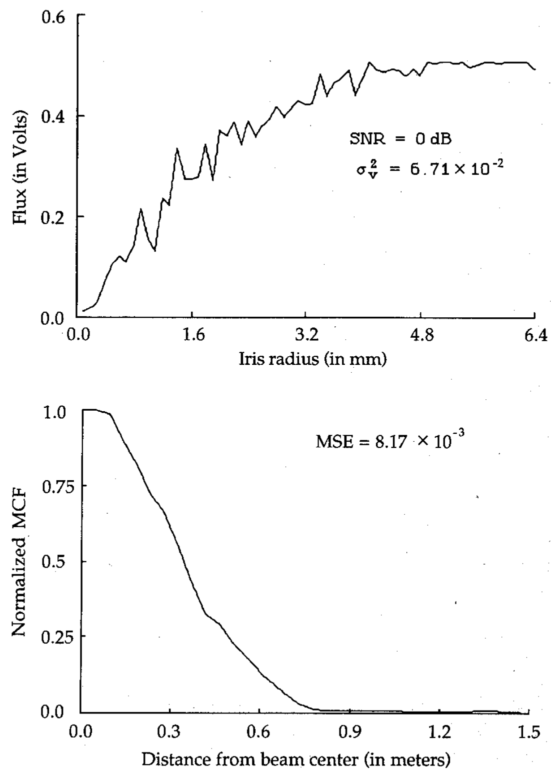

Fig. 1. Average flux and estimated mutual coherence function for a simulated signal with 0 dB additive noise.

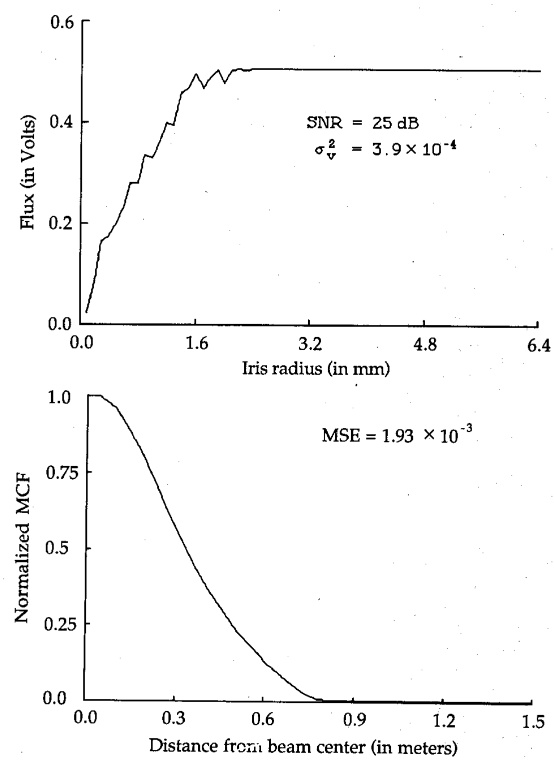

Fig. 2. Average flux and estimated mutual coherence function for a simulated signal with 25 dB additive noise.

dividing by the aperture function. Some points that are exploited in implementing the estimator are 1) the matrix $\mathbf{F}(n)$ is quite sparse and its values are precomputed and stored in memory [8], 2) the variances σ_v^2 and σ_w^2W are fairly constant over the course of an experiment, therefore, the covariance matrices, and hence, the Kalman gain matrix are precomputed and stored in memory, and 3) the aperture function, given in (5), is also precomputed and stored in memory.

Fig. 1 gives the results for the flux plus a 0 dB SNR additive instrumentation noise and Fig. 2 gives the results for 25 dB additive noise (SNR is defined as peak signal to average noise). In both figures, the turbulence level is $C_{n_R}^2 = 2 \times 10^{-11}$ m$^{-2/3}$, $C_{n_I}^2 = 10^{-17}$ m$^{-2/3}$, $C_{n_r n_i} = 0$, $n_{\mathrm{OR}} = 1.0$, $n_{\mathrm{OI}} = 10^{-6}$, and for completeness the transverse wind velocity $V_\perp$, across the propagation path is 2 m/s. The error is computed as the mean-squared difference between the estimated MCF and the true MCF which is calculated from the theoretical equations given in [11]. In both cases, agreement between the estimated and true values is excellent. Although this example assumes clear-air turbulence, it should be noted that this method is valid for computing the MCF caused by other atmospheric phenomena.

V. Conclusion

A procedure for calculating the atmospheric mutual coherence function of a propagating beamwave from measurements of the noisy flux at the focal plane of a reflective antenna has been derived. The algorithm consists of first computing the inverse Abel transform of the average flux, negating the result, taking the one-dimensional Fourier transform, and dividing by the aperture pupil function. Since the average measured flux is a noisy process, the Abel inversion is an ill-posed problem. Therefore, to compute the Abel inversion, a modified version of the Kalman filter given by Hansen was derived. Results of the MCF estimator on simulated signals was shown to give a mean-square error on the order of 10^{-3} for signal-to-noise ratios above 0 dB. The advantages of the Kalman filtering algorithm over other methods for computing (4) are: 1) the algorithm is very robust in the presence of noise (numerically integrating the flux plus noise using the same conditions as in Figs. 1 and 2 gives a MSE of 2.72×10^{-3}W for 25 dB SNR and 6.42×10^{-2} for 0 dB SNR) and 2) it is computationally efficient (the algorithm requires only $O[81N + N \log_2 N]$ multiplies compared to $O[N^2]$ required for direct numerical integration).

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