Microwave Engineering - Take Home Quiz, May 1972

Name FRANCIS L. MERAT

1. [redacted]9
2. 9
3. 9

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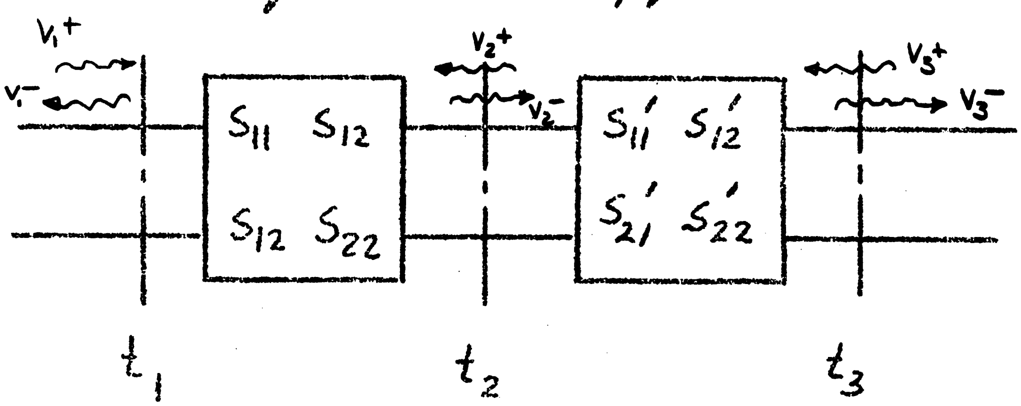
1. Two lossless two-ports are connected in cascade as shown. They are characterized by scattering matrices with parameters S_{11}, S_{22}, S_{12} and $S'_{11}, S'_{22}, S'_{12}$ respectively where $S_{11} = S_{22} = j/4$, $S'_{11} = 1/2$, $S'_{22} = -1/2$. Find the scattering matrix parameters for the overall network.

Hint: With a matched load connected at t_3 the first two-port sees a termination at t_2 with a reflection coefficient S'_{11}.



$V_1^- = \frac{j}{4} V_1^+ + S_{12} V_2^+$

$V_2^- = S_{12} V_1^+ + \frac{j}{4} V_2^+$

$V_2^+ = \frac{1}{2} V_2^- + S'_{12} V_3^+$

$V_3^- = S'_{12} V_2^- + \frac{1}{2} V_3^+$

$$S_{11} = \frac{2j+4}{8-j}\,\frac{8+j}{8+j} = \frac{16j+32-2+4j}{65} = \frac{20j+30}{65} = \frac{4j+6}{13}$$

$$S_{22} = \frac{2j-4}{8-j} \cdot \frac{8\overset{+}{-}j}{8+j} = \frac{16j-32+2+4j}{65} = \frac{20j-30}{65} = \frac{4j-6}{13}$$

$$\frac{-34+12j}{65}$$

$S_{12} = S_{21}$ for a lossless junction

$$|S_{11}| = \frac{16+36}{169} = \frac{52}{169}$$

$$|S_{22}| = \frac{16+36}{169} = \frac{52}{169}$$

Procedure OK

$$\angle S_{11} = \text{Arctan}\left(\tfrac{4}{6}\right) = \text{Arctan}\left(\tfrac{2}{3}\right)$$

$$\angle S_{22} = \text{Arctan}\left(-\tfrac{2}{3}\right) = -\text{Arctan}\left(\tfrac{2}{3}\right)$$

$\angle S_{22}$ is $\pi - \arctan \frac{2}{3}$ for your value of S_{22}

sign error above makes results for S_{12} wrong

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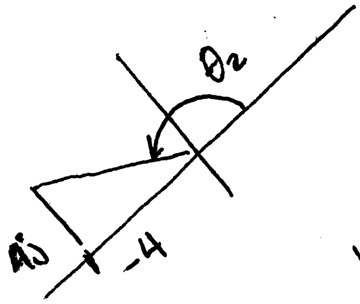
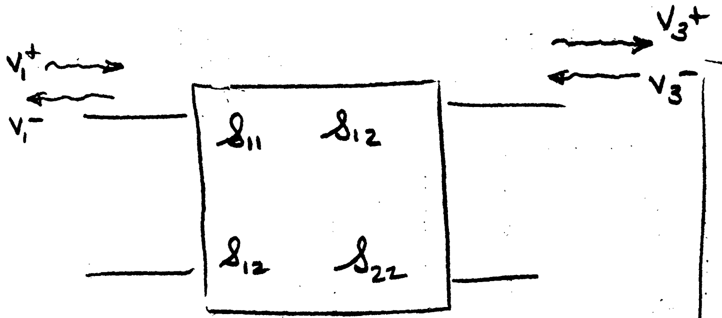
thus $\angle S_{12} = \dfrac{\angle S_{11} + \angle S_{22}}{2} + \dfrac{\pi}{2} \mp n\pi = \dfrac{\pi}{2} \mp n\pi$

$$e^{j\frac{\pi}{2}} e^{\mp jn\pi} = j e^{\mp jn\pi} = \pm j$$

$$|S_{12}|^2 = 1 - |S_{11}|^2 = 1 - \frac{52}{169} = \frac{117}{169}$$

$$S_{12} = S_{21} = \pm j\,\frac{117}{169}$$





$$S_{11} = \frac{4j+6}{13}$$

$$S_{12} = \pm j\,\frac{117}{169}$$

$$S_{22} = \frac{4j-6}{13}$$

assume a matched load at t_3 then $V_3^+ \equiv 0$ if $V_1^- = \mathcal{S}_{11} V_1^+ + \mathcal{S}_{12} V_3^+ = \mathcal{S}_{11} V_1^+ ; V_3^+ \equiv 0$

where $\mathcal{S}_{11}$ & $\mathcal{S}_{12}$ are elements of the scattering matrix for the overall network

then $\mathcal{S}_{11} = \dfrac{V_1^-}{V_1^+}$

$$V_1^- = \frac{j}{4} V_1^+ + S_{12} V_2^+$$

$$\frac{1}{S_{12}} \left(V_1^- - \frac{j}{4} V_1^+ \right) = V_2^+ \quad \therefore \quad S_{12} V_1^+ = -\left(\frac{j}{4} V_2^+ - V_2^- \right) = -\left(\frac{j}{4} V_2^+ - 2V_2^+ \right) = -\frac{j-8}{4} V_2^+$$

$$\therefore V_2^+ = -\frac{4}{j-8} V_2^+$$

$$\frac{1}{S_{12}} \left(V_1^- - \frac{j}{4} V_1^+ \right) = -\frac{4}{j-8} S_{12} V_1^+$$

$$\mathcal{S}_{11} - \frac{j}{4} = -\frac{4 S_{12}^2}{j-8} \qquad \mathcal{S}_{11} = \frac{j}{4} - \frac{4 S_{12}^2}{j-8}$$

$$S_{12}^2 = |S_{12}|^2 e^{j2\phi} \quad \text{where } \phi = \frac{\theta_1 + \theta_2}{2} + \frac{\pi}{2} \pm n\pi \qquad \begin{matrix} S_{11} = |S_{11}| e^{j\theta_2} \\ S_{22} = |S_{22}| e^{j\theta_2} \end{matrix} \quad \therefore \begin{matrix} S_{11} = \frac{1}{4} e^{j\frac{\pi}{2}} \\ S_{22} = \frac{1}{4} e^{j\frac{\pi}{2}} \end{matrix}$$

$$= \frac{\frac{\pi}{2} + \frac{\pi}{2}}{2} + \frac{\pi}{2} \pm n\pi = \pi \pm n\pi = (1 \pm n)\pi$$

$$e^{j2(1 \pm n)\pi} = 1$$

$$|S_{12}|^2 = 1 - |S_{11}|^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\mathcal{S}_{11} = \frac{j}{4} - \frac{4\left(\frac{15}{16}\right)}{j-8} = \frac{-1 - 8j - 15}{4(j-8)} = -\frac{8j + 16}{4(j-8)} = \frac{2j+4}{8-j}$$

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assume a matched load at t_1 then $V_1^+ \equiv 0$ $\quad V_3^- = \mathcal{S}_{12} \cancel{V_1^+} + \mathcal{S}_{22} V_3^+ \quad \therefore \mathcal{S}_{22} = \dfrac{V_3^-}{V_3^+}$

$$V_2^+ = \frac{1}{2} V_2^- + S_{12}' V_3^+ \quad \therefore \left(V_2^+ - \frac{1}{2} V_2^- \right) = S_{12}' V_3^+$$

$$V_3^- = S_{12}' V_2^- - \frac{1}{2} V_3^+ \quad \therefore \quad S_{12} V_2^- = V_3^- + \frac{1}{2} V_3^+$$

as $V_1^+ = 0$ $\quad V_1^- = S_{12} V_2^+$ & $V_2^- = \dfrac{j}{4} V_2^+$

$$V_2^+ - \frac{1}{2}\left(\frac{j}{4}\right) V_2^+ = S_{12}' V_3^+ \qquad S_{12}' \frac{j}{4} V_2^+ = V_3^- + \frac{1}{2} V_3^+$$

$$V_2^+ \left(1 - \frac{j}{8}\right) = S_{12}' V_3^+ \qquad V_2^+ \left(\frac{j}{4} S_{12}'\right) = V_3^- + \frac{1}{2} V_3^+$$

$$\therefore \frac{S_{12}' V_3^+}{\frac{8-j}{8}} \frac{j}{4} S_{12}' = V_3^- + \frac{1}{2} V_3^+ \quad \therefore \left[\frac{j}{4} S_{12}'^2 V_3^+ = \left(\frac{8-j}{8}\right) V_3^- + \left(\frac{8-j}{16}\right) V_3^+ \right] 16$$

$$4j S_{12}'^2 = 2(8-j) \mathcal{S}_{22} + (8-j) \quad \therefore \quad \mathcal{S}_{22} = \frac{4j S_{12}'^2 - (8-j)}{2(8-j)} = \frac{4j S_{12}'^2 + j - 8}{2(8-j)}$$

again $\quad \begin{matrix} S_{11}' = \frac{1}{2} \\ S_{22}' = \frac{1}{2} e^{j\pi} \end{matrix} \quad \therefore$ identify $\begin{matrix} \theta_1' = 0 \\ \theta_2' = \pi \end{matrix} \qquad \phi' = \pi \mp n\pi$

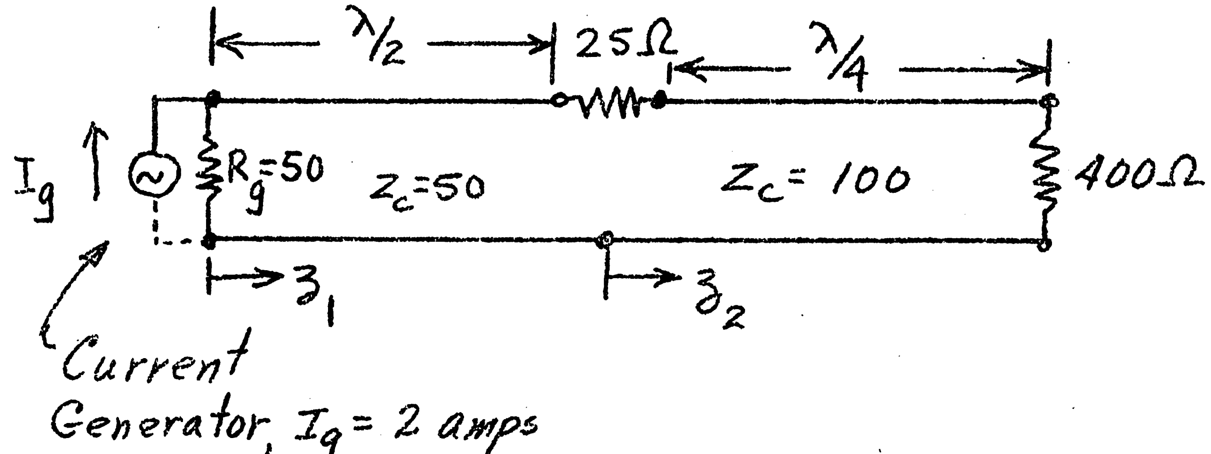
$$\Rightarrow S_{12}'^2 = |S_{12}'|^2 = \frac{1}{4}$$

as $\quad S_{12}'^2 = |S_{12}'|^2 = 1 - |S_{11}'|^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$$\mathcal{S}_{22} = \frac{4j\left(\frac{3}{4}\right) + j - 8}{2(8-j)} = \frac{4j - 8}{2(8-j)} = \frac{2j-4}{8-j}$$

note that $|\mathcal{S}_{11}| = |\mathcal{S}_{22}|$ as expected for a lossless junction

2. For the illustrated transmission line circuit find the total voltage and current at <u>all points</u> on each line section. Find the power delivered to the 25Ω and 400Ω resistors.

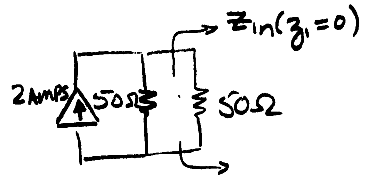


$$Z_{in} = Z_c \frac{Z_L + jZ_c \tan\beta l}{Z_c + jZ_L \tan\beta l}$$

$$Z_{in}(z_2=0) = Z_c \frac{jZ_c}{jZ_L} = \frac{Z_c^2}{Z_L} = \frac{(100)^2}{400} = 25\Omega$$

$$Z_{in}(z_1=\tfrac{\lambda}{2}) = 25\Omega + Z_{in}(z_2=0) = 25+25 = 50\Omega$$

$$Z_{in}(z_1=0) = 50\Omega$$



$V = IZ = (2\text{ amps})(25\Omega) = 50$ volts.

on the first section $\quad V = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$

$$I = \frac{V^+}{Z_c} e^{-j\beta z} - V^- e^{+j\beta z}$$

at $z_1 = \frac{\lambda}{2}$ $\quad \Gamma = \frac{Z_L - Z_c}{Z_L + Z_c} = 0 \quad$ but $\Gamma = \frac{V_1^- e^{+j\beta\frac{\lambda}{2}}}{V_1^+ e^{-j\beta\frac{\lambda}{2}}} = \frac{V_1^-}{V_1^+}$

$$\Rightarrow V_1^- = 0$$

at $z_1=0$ $\quad V = V_1^- + V_1^+ = 50 \quad \therefore \quad V_1^+ = 50$ ✓

and $V = 50 e^{-j\beta z_1}$ $I = e^{-j\beta z_1}$

at $z_1 = \frac{\lambda}{2}$, $V = 50 e^{-j\beta\frac{\lambda}{2}} = -50$

$I = -1$

across the 25Ω resistor

$I = 1$ amp

$V_{drop} = IR = (1)(25) = 25$ volts

$P_R = \frac{1}{2}|I|^2 R = \frac{1}{2}|1|^2 25 = 12.5$ watts ✓

at $z_2 = \frac{\lambda}{4}$ $\Gamma = \frac{400-100}{400+100} = \frac{3}{5}$ & $\Gamma = \frac{V^-}{V^+} e^{2j\beta\frac{\lambda}{4}} = \frac{V^-}{V^+} e^{j\beta\frac{\lambda}{2}} = -\frac{V^-}{V^+}$

$\therefore$ $\frac{3}{5} = -\frac{V^-}{V^+}$ or $V^- = -\frac{3}{5} V^+$

at $z_2 = 0$, $25 = V^- + V^+$

$25 = -\frac{3}{5} V^+ + V^+$

Sign error

$V^+ = \frac{125}{2}$ VOLTS

$V^- = -\frac{3}{5}\left(\frac{125}{2}\right) = -\frac{75}{2}$ VOLTS

$$V = \frac{125}{2} e^{-j\beta z_2} - \frac{75}{2} e^{+j\beta z_2}$$

$$I = \frac{5}{8} e^{-j\beta z_2} + \frac{3}{8} e^{+j\beta z_2}$$ $\left[\frac{125}{2} \frac{1}{100} = \frac{5}{8}, \text{etc}\right]$

at $z_2 = \frac{\lambda}{4}$ $I = -\frac{5}{8} - \frac{3}{8} = -1$ amp

$V = -\frac{25}{2}$ volt

$$P_L = \frac{1}{2} I V^* = \frac{1}{2}(-1)\left(-\frac{25}{2}\right) = 12.5 \text{ watts}$$

To summarize: at the generator $I = 2$ amps $V = 50$ volts

$0 \le z_1 \le \frac{\lambda}{2}$ $I = e^{-j\beta z_1}$ $V = 50 e^{-j\beta z_1}$

voltage drop across 25Ω resistor is 25 volts

power delivered to it is 12.5 watts

✓

$0 \le z_2 \le \frac{\lambda}{4}$ – $I = \frac{5}{8} e^{-j\beta z_2} + \frac{3}{8} e^{+j\beta z_2}$

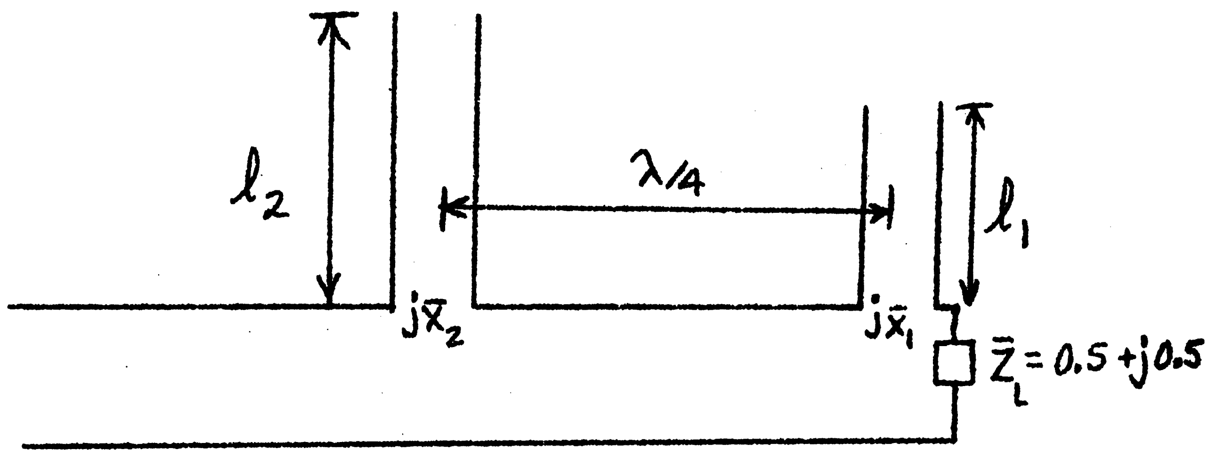
– $V = \frac{125}{2} e^{-j\beta z_2} - \frac{75}{2} e^{+j\beta z_2}$

at the load

$I = -1$ amp $V = -\frac{25}{2}$ volt

Power delivered to load (400Ω resistor) is 12.5 watts

3. A normalized load $\bar{Z}_L = 0.5 + j0.5$ is to be matched by means of two open-circuited series stubs spaced $\lambda/4$ apart. Use a Smith chart to find the required stub reactances and their lengths l_1 and l_2.



$\bar{X}_1 = ?$ $\bar{X}_2 = ?$ $l_1 = ?$ $l_2 = ?$

$$2\beta d = 2\,\frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \pi$$

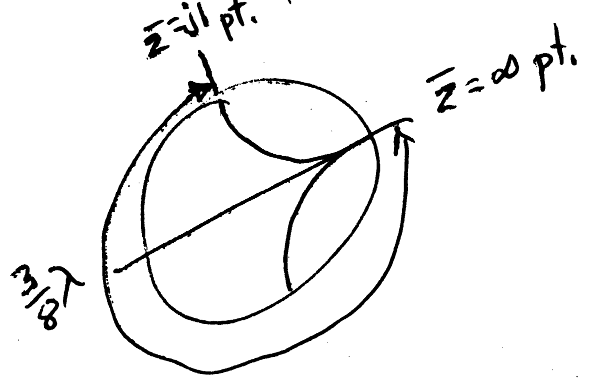
as $\bar{Z}_L$ lies on the displaced $\bar{Z} = 1$ circle $\bar{X}_1 = 0$ ✓

from the chart $l_1 = 0.25\lambda$

$\bar{X}_2 = 1$ so that $\bar{Z} + j\bar{X}_2 = (1-j) + j = 1$ indicating a match

$\beta l_2 = \beta(.125\lambda)$ $\Rightarrow$ $l_2 = 0.125\lambda$

$\frac{3}{8}\lambda$



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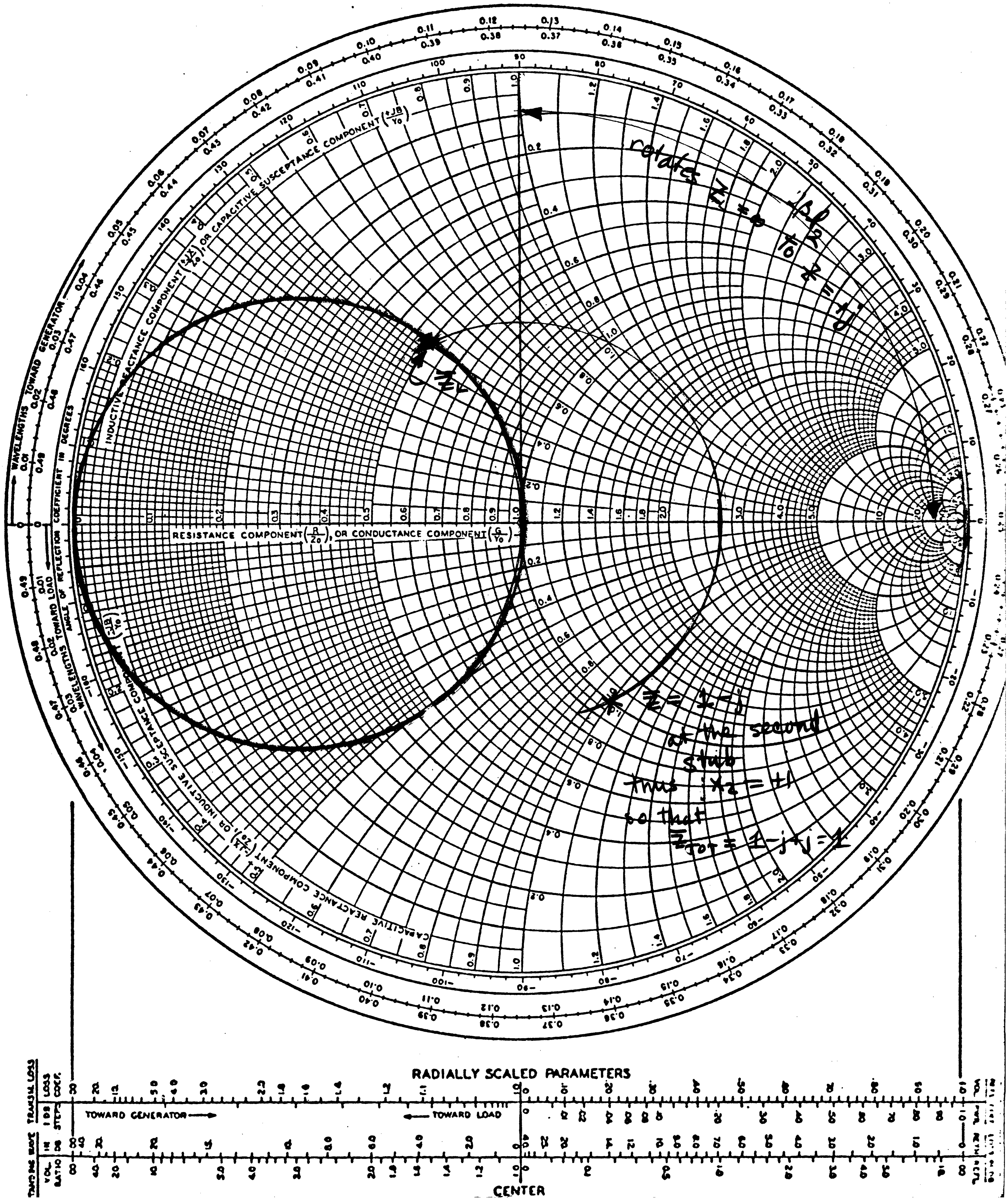
SMITH CHART Form 756-N	GENERAL RADIO COMPANY, WEST CONCORD, MASSACHUSETTS	DATE

IMPEDANCE OR ADMITTANCE COORDINATES



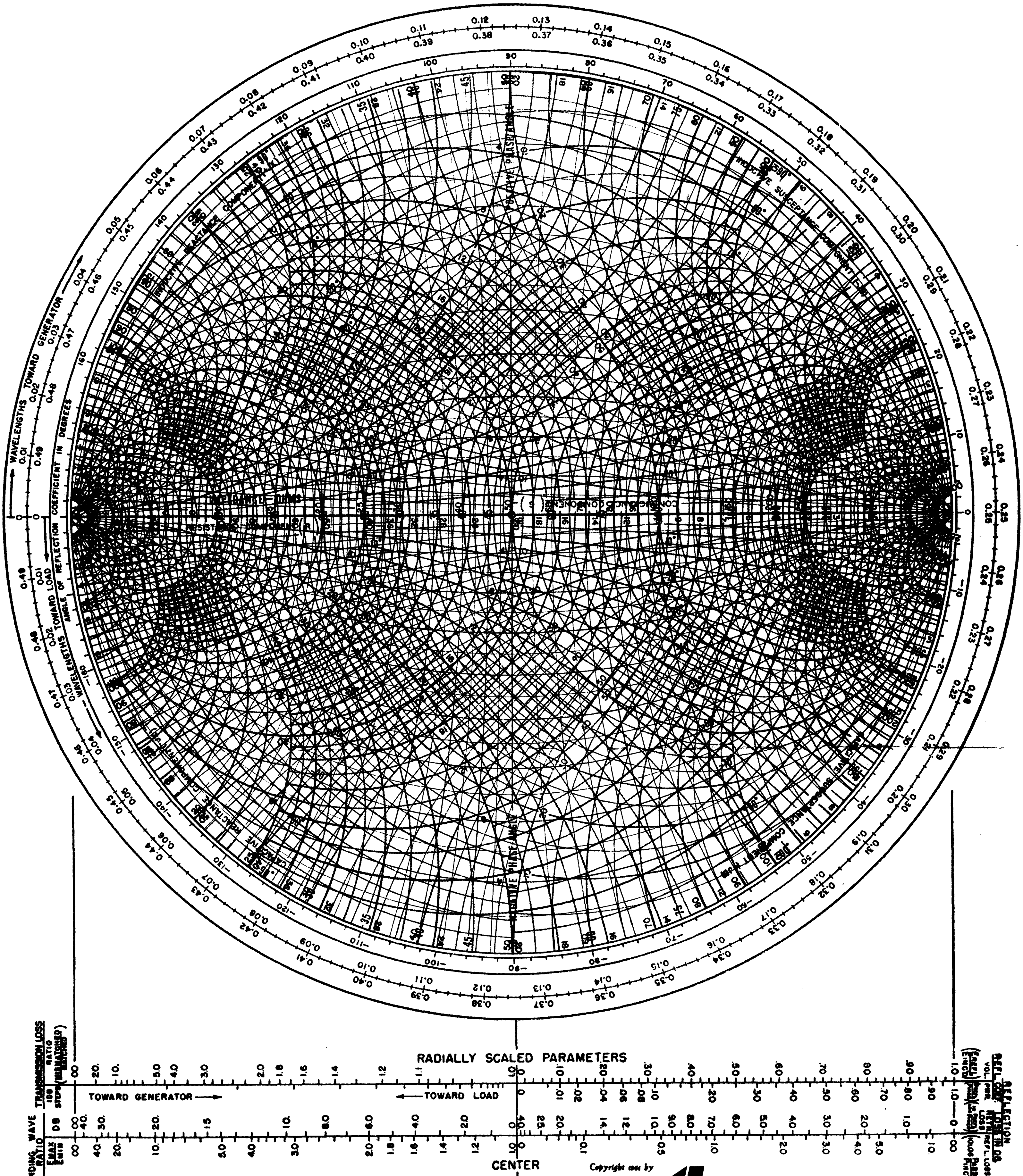
TITLE

DATE

IMMITTANCE CHART

IMPEDANCE COORDINATES — 50 OHM CHARACTERISTIC IMPEDANCE

ADMITTANCE COORDINATES — 20 MILLIMHO CHARACTERISTIC ADMITTANCE



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Microwave Engineering Exam, April 1972

1. 9

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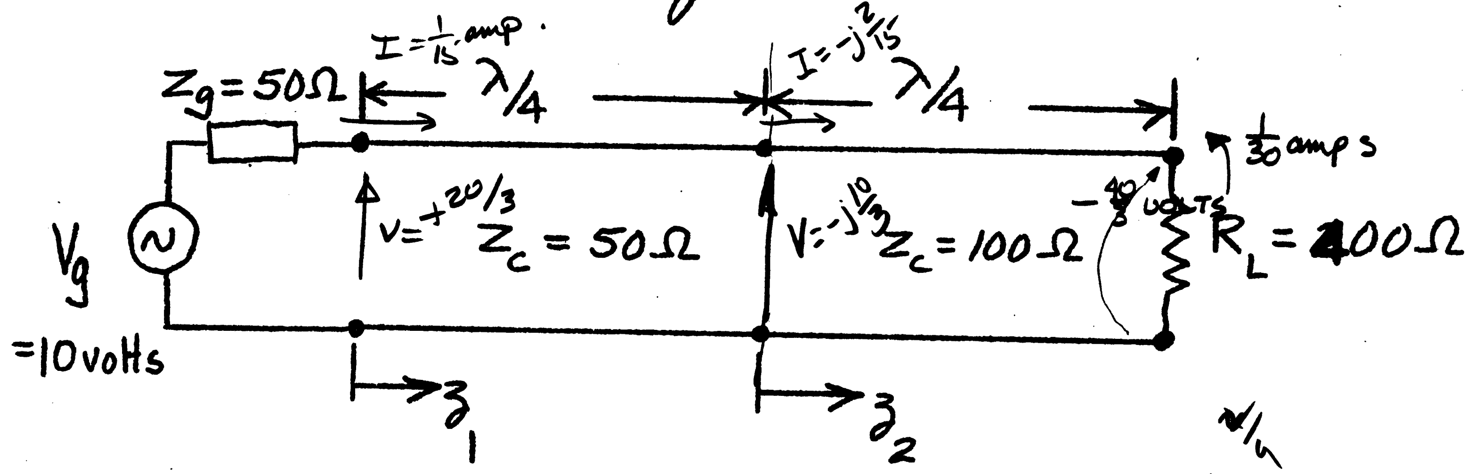
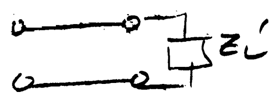
1. For the illustrated transmission line circuit find the total voltage and current at all points on both line sections. Find the power delivered to the load. z_1 and z_2 measure distances along the two sections.



$P = \frac{1}{2}\frac{20}{45} = \frac{2}{9}$

$P = \frac{1}{2}\left|\frac{1}{30}\right|\left|\frac{40}{3}\right| = \frac{2}{9}$



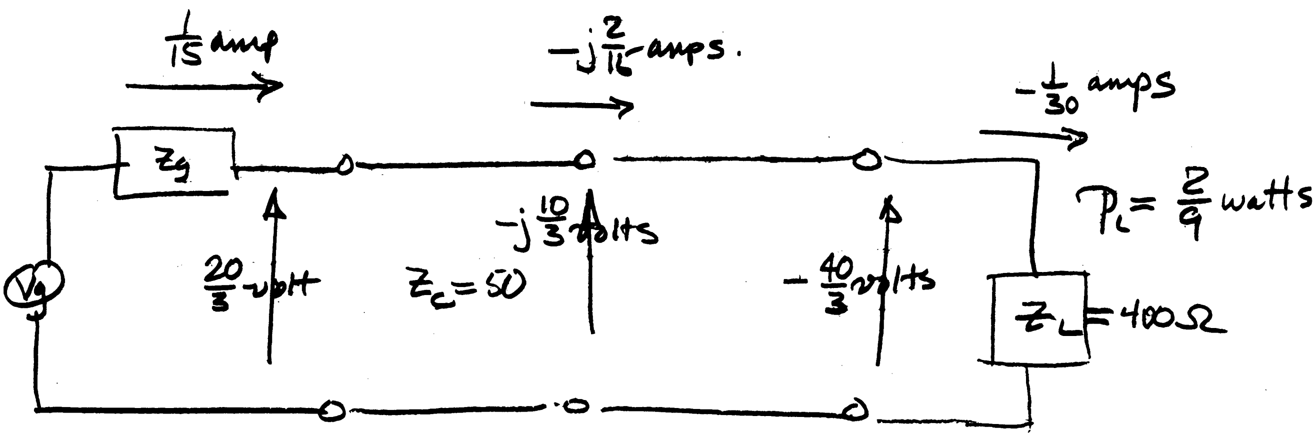
$$Z_L'' = Z_g \frac{Z_L' + jZ_c \tan\beta l}{Z_c + jZ_L' \tan\beta l}$$

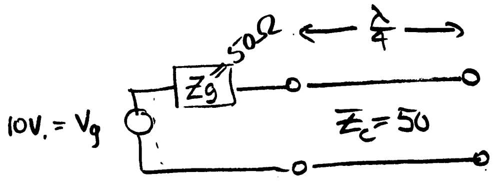
$$= 50 \frac{25 + j50 \tan\frac{\pi}{2}}{50 + j25 \tan\frac{\pi}{2}}$$

$$= 50 \cdot 2 = 100$$

$$I_L' = \frac{V_g}{Z_L'' + Z_g} = \frac{10}{100 + 50} = \frac{10}{150} = \boxed{\frac{1}{15} \text{ amp}}$$

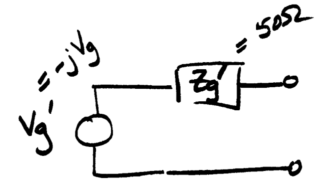
$$V_L'' = V_g \frac{Z_L''}{Z_L'' + Z_g} = 10 \frac{100}{100+50} = 10 \frac{10}{15} = \boxed{\frac{20}{3} \text{ volts}}$$





$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$

THEVENIZE THIS SECTION. TO



Hard way to do this problem

$$Z_g' = Z_c \frac{Z_g + jZ_c \tan\beta l}{Z_c + jZ_g \tan\beta l}$$ from text p. 167 eq. 4.49

$$= 50 \; \frac{50 + j50 \tan\frac{\pi}{2}}{50 + j50 \tan\frac{\pi}{2}}$$

$$= 50$$

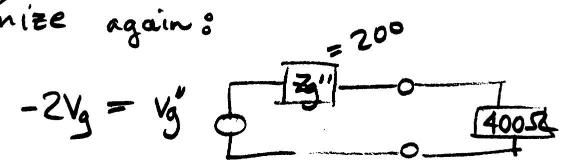
$$V_{oc} = \frac{2V_g e^{-j\beta l}}{(1+Y_cZ_g) + (1-Y_cZ_g)e^{-2j\beta l}}$$

$$e^{-j\beta l} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j \qquad 1+Y_cZ_g = 1+\frac{1}{50}\cdot 50 = 2$$

$$e^{-2j\beta l} = e^{-j\pi} = \cos\pi - j\sin\pi = -1 \qquad 1-Y_cZ_g = 0$$

$$\Rightarrow V_{oc} = \frac{2V_g(-j)}{2} = -jV_g \; ✓$$

thevenize again:



$$Z_g'' = Z_c \frac{Z_g' + jZ_c\tan\beta l}{Z_c + jZ_g'\tan\beta l} = 100 \; \frac{50 + j100\tan\frac{\pi}{2}}{100 + j50\tan\frac{\pi}{2}}$$

$$= 100 \; \frac{j100}{j50} = 200 \; ✓$$

(9)

$$V_{oc} = \frac{2V_g' e^{-j\beta l}}{(1+Y_cZ_g) + (1-Y_cZ_g)e^{-2j\beta l}} = \frac{2(-j)V_g'}{\frac{3}{2} + \frac{1}{2}(-1)} = \frac{2(-j)V_g'}{\frac{3}{2} - \frac{1}{2}} = \frac{2(-j)V_g'}{1}$$

$$1 + Y_cZ_g = 1 + \frac{1}{100}\cdot 50 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$1 - Y_cZ_g = 1 - \frac{1}{100}\cdot 50 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$= -2jV_g'$$

$$= -2j(-jV_g)$$

$$= -2V_g \; ✓$$

at load
$$I_L = \frac{V_g''}{Z_g'' + 400\Omega} = \frac{-2V_g}{200+400} = -\frac{2}{600}V_g = -\frac{1}{300}(10) = \boxed{-\frac{1}{30} \text{ amps}}$$

$$V_L = V_g'' \frac{400}{Z_g'' + 400} = -2V_g \frac{400}{600} = -\frac{4}{3}(10) = \boxed{-\frac{40}{3} \text{ VOLTS}}$$

$$P = \frac{1}{2}|I_L|^2 Z_L = \frac{1}{2}\left|-\frac{1}{30}\right|^2 400 = \frac{200}{900} = \boxed{\frac{2}{9} \text{ WATTS}} \; ✓$$

at junction between waveguide sections

$$Z_L' = Z_c \frac{Z_L + jZ_c\tan\beta l}{Z_c + jZ_L\tan\beta l} = 100 \; \frac{400 + j100\tan\frac{\pi}{2}}{100 + j400\tan\frac{\pi}{2}} = 25$$

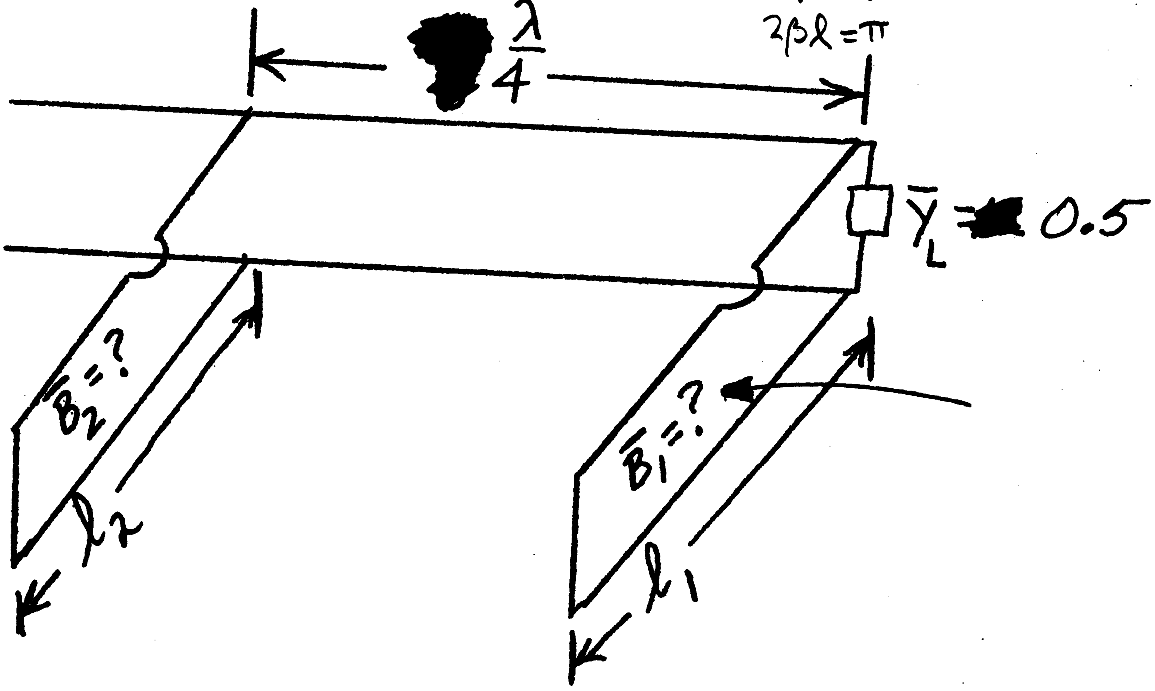
$$I_L' = \frac{V_g'}{Z_g' + Z_L'} = \frac{-jV_g}{50+25} = -\frac{j}{75}10 = \boxed{-j\frac{2}{15}} \; ✓$$

$$V_L' = V_g' \frac{Z_L'}{Z_g' + Z_L'} = -j10 \; \frac{25}{75} = \boxed{-j\frac{10}{3}}$$

2. For the illustrated double stub tuner find the lengths and susceptances that each stub must have to match the load admittance $\bar{Y}_L =$ [redacted] 0.5 to the main line. Show all constructions on the attached Smith Chart.

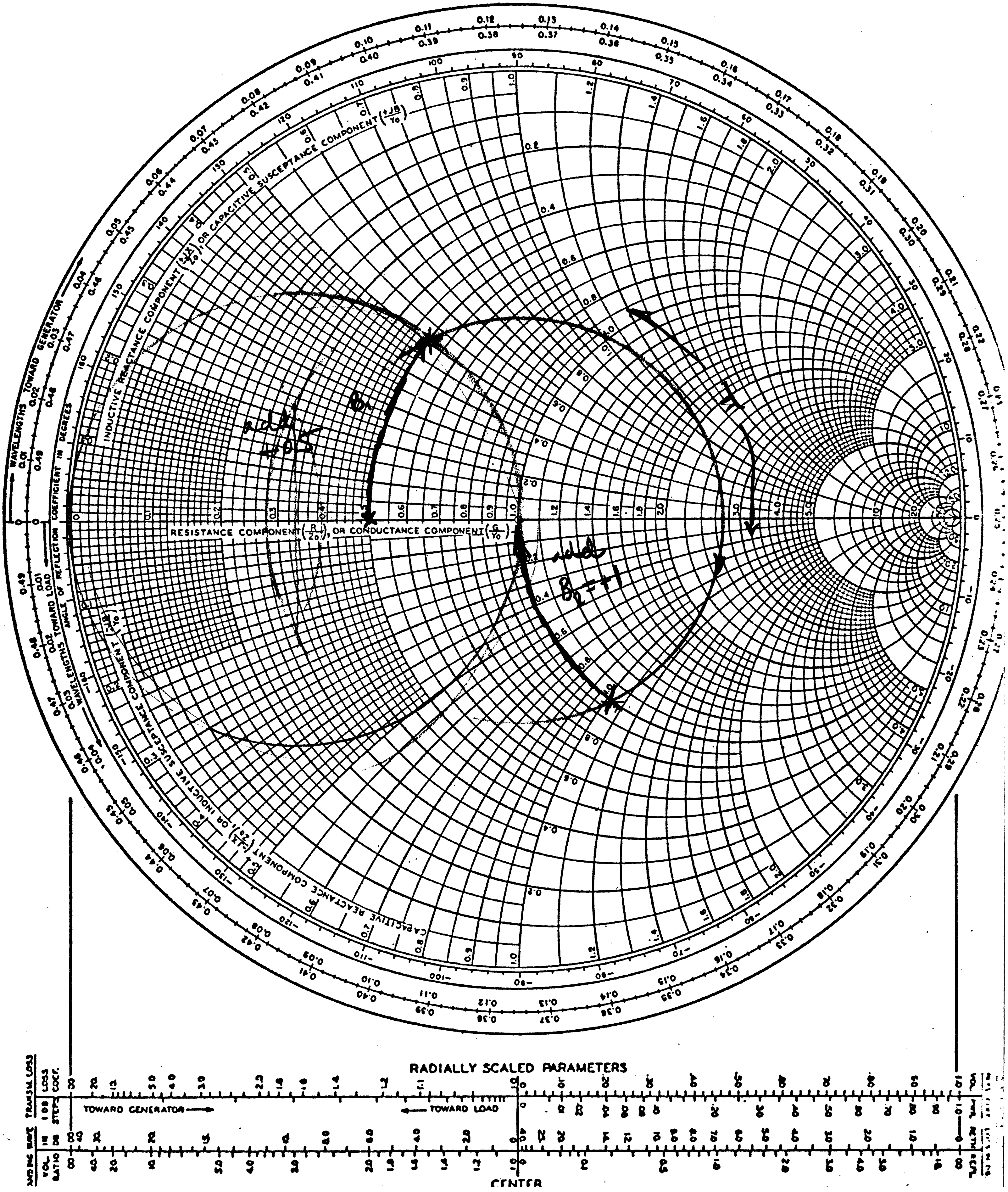
$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$

$2\beta l = \pi$



NAME	TITLE	DWG. NO.
SMITH CHART Form 756-N	GENERAL RADIO COMPANY, WEST CONCORD, MASSACHUSETTS	DATE

IMPEDANCE OR ADMITTANCE COORDINATES



from chart $\boxed{B_1 = +0.5}$

want $\bar{Y}_L' = \bar{Y}_L + j\bar{B}_1 = 0.5 + j0.5$

want B_2 to be -1 $\boxed{B_2 = +1.0}$

$Y(l) = -j\cot\beta l$

$\cancel{j}\frac{1}{2} = -\cancel{j}\cot\beta l_1$

$\boxed{-\frac{1}{2} = \cot\beta l_1}$ look up in table Use chart to find l_1, l_2

$j1 = -j\cot\beta l_2$

$-1 = \cot\beta l_2$

$\boxed{\beta l_2 = \frac{3\pi}{8}}$

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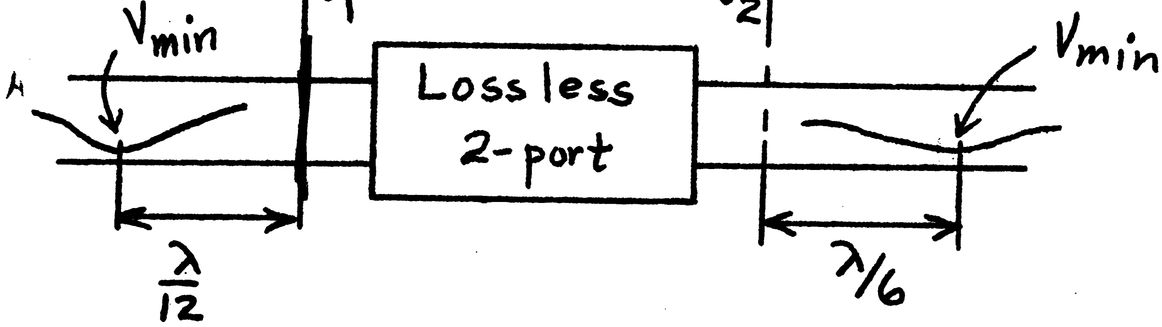
3. Consider the illustrated <u>lossless</u> 2-port junction. With the output terminated in a matched load the VSWR on the input line was 3 and a voltage minimum occured at $\lambda/12$ from reference plane t_1. With the input side terminated in a matched load it was found that a voltage minimum occured at $\lambda/6$ from the reference plane t_2 on the output line. In terms of this data find the scattering matrix parameters S_{11}, S_{12} and S_{22} for the junction between reference planes t_1 and t_2.



$$S_{12} = |S_{12}|e^{j\theta}$$

$$|S_{12}| = \sqrt{1-|S_{11}|^2} = \sqrt{1-(\tfrac{1}{2})(\tfrac{1}{2})} = \frac{\sqrt{3}}{2} \checkmark$$

$$S_{11}S_{12}^* + S_{12}S_{22}^* = 0$$

$$-\tfrac{1}{2}e^{+j\frac{\pi}{3}} \cancel{\tfrac{\sqrt{3}}{2}} e^{-j\theta} + \cancel{\tfrac{\sqrt{3}}{2}} e^{+j\theta} (\cancel{-\tfrac{1}{2}}) e^{-j\frac{2\pi}{3}} = 0$$

$$e^{j(\frac{\pi}{3}-\theta)} + e^{j(\theta-\frac{2\pi}{3})} = 0$$

$$e^{j(\frac{\pi}{3}-\theta)} = -e^{j(\theta-\frac{2\pi}{3})} \quad \text{try } e^{-j\pi} = -1$$

$$= e^{-j\pi} e^{j(\theta-\frac{2\pi}{3})}$$

$$\frac{\pi}{3} = \frac{180}{\pi} = 60^\circ$$

$$e^{j(\frac{\pi}{3}-\theta)} = e^{j(\theta + \frac{5\pi}{3})} \qquad e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{j\frac{\pi}{3}} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3}$$

$$= \tfrac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$\frac{2\pi}{3} \cdot \frac{180}{\pi} = 120^\circ = -60^\circ$$

$$\frac{\pi}{3} - \theta = \theta - \frac{5\pi}{3}$$

$$\frac{\pi}{3} + \frac{5\pi}{3} = 2\theta$$

$$2\pi = 2\theta$$

$$\pi = \theta$$

$$\Rightarrow \quad S_{12} = \frac{\sqrt{3}}{2}e^{j\pi} = -\frac{\sqrt{3}}{2}$$

$$\boxed{\begin{aligned} S_{11} &= -\tfrac{1}{4} - j\tfrac{\sqrt{3}}{4} \\ S_{12} &= -\tfrac{\sqrt{3}}{2} \checkmark \\ S_{22} &= -\tfrac{1}{4} + j\tfrac{\sqrt{3}}{4} \end{aligned}}$$

$$S_{12} = |S_{12}|e^{j\phi}$$

$$S_{ii} = |S_{ii}|e^{j\theta_i}$$

$$\phi = \frac{\theta_1+\theta_2}{2} + \frac{\pi}{2} \pm n\pi$$

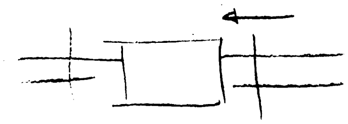
$$= \frac{\frac{4\pi}{3} + \frac{5\pi}{3}}{2} + \frac{\pi}{2} \pm n\pi$$

$$= \frac{3\pi}{2} + \frac{\pi}{2} \pm n\pi$$

$$= \pm n\pi$$

or zero phase is OK

output terminated in matched load



$$V_1^- = S_{11} V_1^+ + S_{12} \cancel{V_2^+}^{\,0}$$

$$S_{11} = \frac{V_1^-}{V_1^+} = \Gamma \text{ at plane } t_1$$

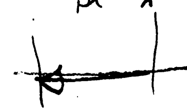
VSWR = 3 $\quad \frac{\lambda}{12}$ from t_1

but $Y_{IN} = \frac{1-\Gamma}{1+\Gamma} = \frac{1+\rho}{1-\rho} = S$ at voltage minimum

$$1-\Gamma = 3(1+\Gamma)$$
$$1-\Gamma = 3+3\Gamma$$
$$-2 = 4\Gamma$$
$$\Gamma = -\tfrac{1}{2} \text{ at } \tfrac{\lambda}{12} \text{ from } t_1$$

$2\beta l = 60^\circ$

$\beta l = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{12} = \frac{\pi}{6} = 30^\circ$

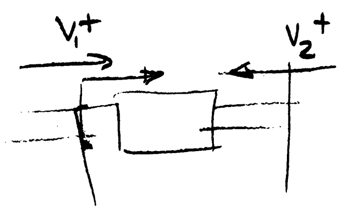


Γ_L

transform $\quad \Gamma(l = \tfrac{\lambda}{12}) = \Gamma_L e^{-2j\beta l}$

$$-\tfrac{1}{2} = \Gamma_L e^{-2j\beta l} = \Gamma_L e^{-j\frac{\pi}{3}}$$

$$-\tfrac{1}{2} e^{+j\frac{\pi}{3}} = \Gamma_L = S_{11}$$



$$\boxed{S_{11} = -\tfrac{1}{2} e^{+j\frac{\pi}{3}}} \checkmark$$
$$= \tfrac{1}{2} e^{j4\pi/3}$$

$$V_2^- = S_{21} V_1^+ + S_{22} V_2^+$$

if input is matched $V_1^+ \to 0$

$$|S_{11}| e^{j\theta} = \frac{V_2^-}{V_2^+} = S_{22}$$

lossless junction

$$-\tfrac{1}{2} e^{j\theta} = S_{22} = \Gamma_L$$

$2\beta l = 2\frac{2\pi}{\lambda}\cdot\frac{\lambda}{6} = \frac{2\pi}{3}$

$e^{j\phi} = \cos\phi + j\sin\phi$

at 0

but $\quad \Gamma(l) = \Gamma_L e^{-2j\beta l}$

$$\Gamma(\tfrac{\lambda}{6}) = -\tfrac{1}{2} e^{j\theta} e^{-j\frac{2\pi}{3}} = -\tfrac{1}{2} e^{j(\theta - \frac{2\pi}{3})}$$

at a voltage minimum Γ is negative and real

$\Rightarrow$ pick first time $\quad \theta - \frac{2\pi}{3} = n\pi$

i.e. pick $n = 0$

$e^{j\pi} = \cancel{\cos\pi}^{\,0} + j\cancel{\sin\pi}^{\,0}$

$\theta - \frac{2\pi}{3} = 0$

$\theta = \frac{2\pi}{3}$

$$\Rightarrow \boxed{S_{22} = -\tfrac{1}{2} e^{+j\frac{2\pi}{3}}} \checkmark$$
$$= \tfrac{1}{2} e^{j\frac{5\pi}{3}}$$

$5\pi/3 \cdot 180/\pi = 300^\circ = -60^\circ$

let $S_{12} = A e^{jB}$

$\cancel{S_{11}S_{12}^* + S_{12}S_{22}^* = 0}$

$\cancel{-\frac{1}{2}e^{+j\frac{\pi}{3}} A e^{-jB} + A e^{jB} (-\tfrac{1}{2}) e^{-j\frac{5\pi}{3}} = 0}$

$\cancel{e^{-j(\frac{\pi}{3}-B)} e^{j(\frac{\pi}{3}-B)} + e^{j(B-5\frac{\pi}{3})} e^{-j(\frac{\pi}{3}-B)} = 0}$

$\cancel{1 + e^{j(B-\frac{5\pi}{3}-\frac{\pi}{3}+B)} = 0}$

$S_{12} = A + jB$

$S_{11} = -\frac{1}{2}e^{j60^\circ} = -\frac{1}{2}\cos 60 - \frac{1}{2}j\sin 60$

$= -\frac{1}{2}\frac{1}{2} - \frac{1}{2}j\frac{\sqrt{3}}{2}$

$= -\frac{1}{4} - \frac{1}{4}j\sqrt{3}$

$S_{22} = -\frac{1}{2}e^{-j60} = -\frac{1}{4} + \frac{1}{4}j\sqrt{3}$

$$S_{11}S_{12}^* + S_{12}S_{22}^* = 0$$

$$\cancel{(-\tfrac{1}{4} - \tfrac{1}{4}j\sqrt{3})}(A - jB) + (A + jB)(-\tfrac{1}{4} \cancel{+\tfrac{1}{4}j\sqrt{3}}) = 0$$

$$A - jB + A + jB = 0$$

$$2A = 0$$

$$|S_{12}| = \sqrt{1 - |S_{11}|^2} = \sqrt{1 - S_{11}S_{11}^*} = \sqrt{1 - (-\tfrac{1}{2})e^{+j60}(-\tfrac{1}{2})e^{-j60}}$$

$$= \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}$$

$S_{12} = A + jB$

$|S_{12}| = \sqrt{A^2 + B^2}$ but $A = 0$

$= \sqrt{B^2} = B$

$\Rightarrow B = \frac{1}{2}\sqrt{3}$

$$\boxed{\begin{aligned} S_{11} &= -\tfrac{1}{4} - \tfrac{1}{4}j\sqrt{3} \\ S_{12} &= j\tfrac{1}{2}\sqrt{3} \\ S_{22} &= -\tfrac{1}{4} + \tfrac{1}{4}j\sqrt{3} \end{aligned}}$$