Radiation and Antennas

Vector and Scalar Potentials

$\nabla\cdot\vec{B} = 0$ so $\vec{B} = \nabla\times\vec{A}$. Then $\nabla\times\vec{E} = -j\omega\vec{B}$

or $\nabla\times(\underbrace{\vec{E}+j\omega\vec{A}}) = 0$ so $\vec{E}+j\omega\vec{A} = -\nabla\Phi$. From

most general thing that can satisfy this is $\nabla\Phi$ because $\nabla\times\nabla\Phi = 0$

$\nabla\times\vec{H} = j\omega\epsilon_0\vec{E}+\vec{J}$ we get $\nabla\times(\nabla\times\vec{A}) = j\omega\epsilon_0\mu_0\vec{E}+\mu_0\vec{J}$

$= j\omega\mu_0\epsilon_0(-j\omega\vec{A}-\nabla\Phi)+\mu_0\vec{J}$ or

$\nabla\nabla\cdot\vec{A} - \nabla^2\vec{A} = k_0^2\vec{A} - j\omega\mu_0\epsilon_0\nabla\Phi + \mu_0\vec{J}$. Since $\nabla\cdot\vec{A}$ and Φ are not specified as yet let

$$\nabla\cdot\vec{A} = -j\omega\mu_0\epsilon_0\Phi \quad (\text{Lorentz condition}) \qquad (1)$$

Then

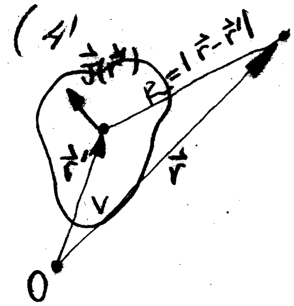
$$\nabla^2\vec{A}+k_0^2\vec{A} = -\mu_0\vec{J} \qquad (\nabla^2+k_0^2)\Phi = -\frac{\rho}{\epsilon_0} \qquad (2)$$

and

$$\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} \qquad (3)$$

$$\vec{E} = -j\omega\vec{A}-\nabla\Phi = -j\omega\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{j\omega\mu_0\epsilon_0}$$

$$= \frac{k_0^2\vec{A}+\nabla\nabla\cdot\vec{A}}{j\omega\mu_0\epsilon_0} \qquad (4)$$



2. Radiation From Current Element

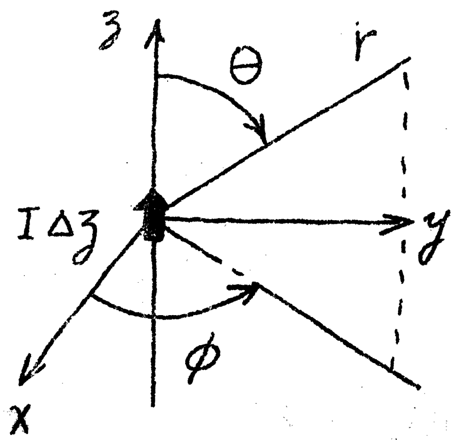
The solution for $\vec{A}(\vec{r})$ is given by

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \vec{J}(\vec{r}')\frac{e^{-jk_0R}}{R}\,dV', \quad R = |\vec{r}-\vec{r}'| \qquad (5)$$

$J = \frac{I}{\Delta s}$ $dV = \Delta s \Delta z$  surface area Δs

$J\,dV = I \Delta z$

$\vec{J}(\vec{r}')\,dV$



$$\vec{A} = \vec{a}_z \frac{\mu_0 \boxed{I \Delta z}}{4\pi r} e^{-jk_0 r}$$

Using (3) and (4) gives

$Z_0 \triangleq$ intrinsic impedance of free space.

$$\vec{E} = \frac{-jI\Delta z\, Z_0}{2\pi k_0} \cos\theta \left(\frac{jk_0}{r^2} + \frac{1}{r^3} \right) e^{-jk_0 r}\, \vec{a}_r$$

$$- \frac{jI\Delta z\, Z_0}{4\pi k_0} \sin\theta \left(\frac{-k_0^2}{r} + \frac{jk_0}{r^2} + \frac{1}{r^3} \right) e^{-jk_0 r}\, \vec{a}_\theta \qquad (6a)$$

$$\vec{H} = \frac{I\Delta z}{4\pi} \sin\theta \left(\frac{jk_0}{r} + \frac{1}{r^2} \right) e^{-jk_0 r}\, \vec{a}_\phi \qquad (6b)$$

The part which varies as $1/r$ is dominant when $r \to \infty$ and is the radiation field. Thus

in the radiation zone the radial component is always zero

Radiation Field

important result

$$\vec{E} = jk_0\, I\Delta z\, Z_0 \frac{e^{-jk_0 r}}{4\pi r} \sin\theta\, \vec{a}_\theta \qquad (7a)$$

$$\vec{H} = jk_0 I \Delta z \frac{e^{-jk_0 r}}{4\pi r} \sin\theta\, \vec{a}_\phi \qquad (7b)$$

Only the radiation field contributes to power flow at infinity (i.e. to radiated power).

$\bar{H} = \frac{\bar{a}_r \times \bar{E}}{Z_0}$ in radiation zone

$$\bar{E} \times \bar{H}^* \cdot \bar{a}_r = \frac{\vec{E} \times (\vec{a}_r \times \vec{E}^*) \cdot \vec{a}_r}{Z_0} = \frac{\vec{E} \cdot \vec{E}^*}{Z_0} \vec{a}_r$$

The power density per unit area in the radial direction is $\frac{1}{2} \mathrm{Re}\, \vec{E} \times \vec{H}^* \cdot \vec{a}_r = \frac{dP}{d\Omega}$ or

$$\frac{dP}{d\Omega} = \frac{k_0^2 Z_0 \, II^* (\Delta z)^2}{32\pi^2 r^2} \sin^2\theta \qquad (8)$$

The total radiated power is obtained by integrating (8) over the surface of a sphere surrounding the radiating current element and is

$$P_{tot} = \int_0^{\pi} \int_0^{2\pi} \frac{k_0^2 Z_0 \, II^* (\Delta z)^2}{32\pi^2 r^2} \sin^2\theta \quad r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{(k_0 \Delta z)^2 II^*}{12\pi} Z_0 \qquad (9)$$

The radiation resistance R_0 is defined by $\frac{1}{2} R_0 II^* = P_{tot}$ which gives

$$R_0 = \frac{P_{tot}}{\frac{1}{2} II^*} = \frac{(k_0 \Delta z)^2 Z_0}{6\pi} = 80\pi^2 \left(\frac{\Delta z}{\lambda_0}\right)^2 \qquad (10)$$

If $\Delta z << \lambda_0$ then R_0 is very small. For $\Delta z = \lambda_0/100$ we get $R_0 = 0.079\,\Omega$.

Power radiated per unit solid angle in the direction θ, ϕ is given by $r^2 P$ or

$$\frac{dP}{d\Omega} = \frac{k_0^2 Z_0 I I^* (\Delta z)^2}{32\pi^2} \sin^2\theta \text{ watts/sterradian} \quad (11)$$

The average power radiated per sterradian is $P_{tot}/4\pi$. The directivity of the antenna is defined by

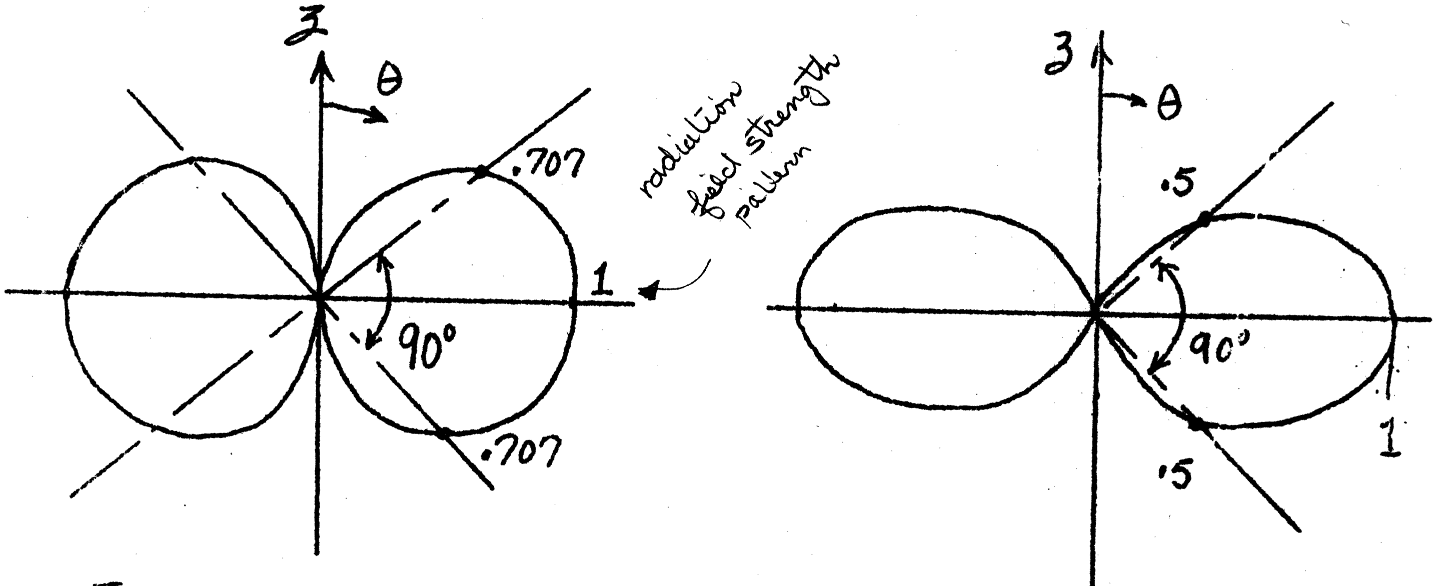
$$D(\theta,\phi) = \frac{\text{Power per unit solid angle in direction } \theta,\phi}{\text{Average radiated power per unit solid angle}}$$

$$= \frac{dP/d\Omega}{P_{tot}/4\pi} = 4\pi \frac{dP/d\Omega}{P_{tot}} \quad (12)$$

For the current element

$$D(\theta,\phi) = 1.5 \sin^2\theta \quad (13)$$

The maximum directivity is 1.5. A plot of $D(\theta,\phi)$ in polar form or as a three dimensional surface gives the power radiation pattern. Plots of $|\vec{E}|$ and $|H|$ for the radiation field gives the field strength radiation patterns. Usually normalized plots are used to show relative values as a function of θ and ϕ only. For a current element there is no dependence on ϕ.



E_θ vs θ, Beamwidth = 90°

$D(\theta)$ vs θ

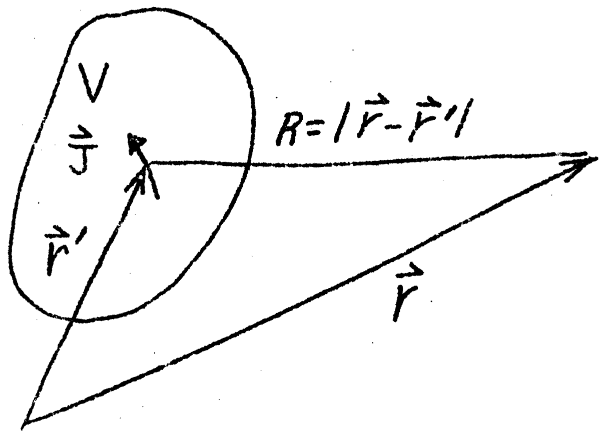
<u>Normalized plots of E_θ and $D(\theta)$</u>

The gain $G(\theta,\phi)$ of an antenna is defined by

$$G(\theta,\phi) = 4\pi \frac{\text{Power per unit solid angle in direction } \theta, \phi}{\text{Total input power to antenna}} \qquad (14)$$

G differs from D by a factor which is the efficiency of the antenna. If a fraction η of the input power is radiated then $G = \eta D$. Antennas are less than 100% efficient because of ohmic losses.

Radiation Fields From Current Distributions



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{J}(\vec{r}\,') \frac{e^{-jk_0 R}}{R} dV'$$

$$\vec{E} = -j\omega\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{j\omega\mu_0\epsilon_0}$$

$$\vec{H} = \mu_0^{-1} \nabla\times\vec{A}$$

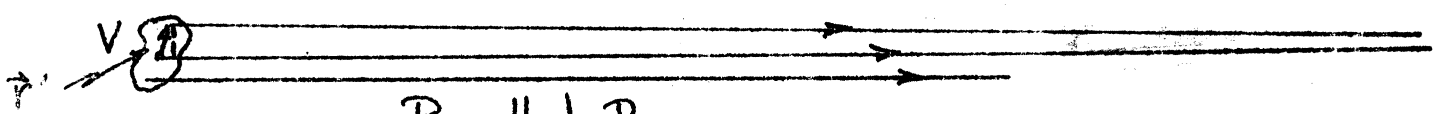
In the radiation zone $|\vec{r}|$ is so large compared with the maximum value of $|\vec{r}\,'|$ so that in the integral for $\vec{A}$ we can approximate $1/R$ by $1/r$, i.e. error is 1% or less if $|\vec{r}| > |\vec{r}\,'|_{max} \times 10^2$.

In the exponential we must use a more accurate expression since a small change in k_0R corresponding to an angle of π or so is significant even if k_0R equals several hundred multiples of π. Thus we use $|\vec{r} - \vec{r}\,'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,')^{1/2} \approx \left[r^2\left(1 - \frac{2\vec{r}\cdot\vec{r}\,'}{r^2}\right)\right]^{1/2}$

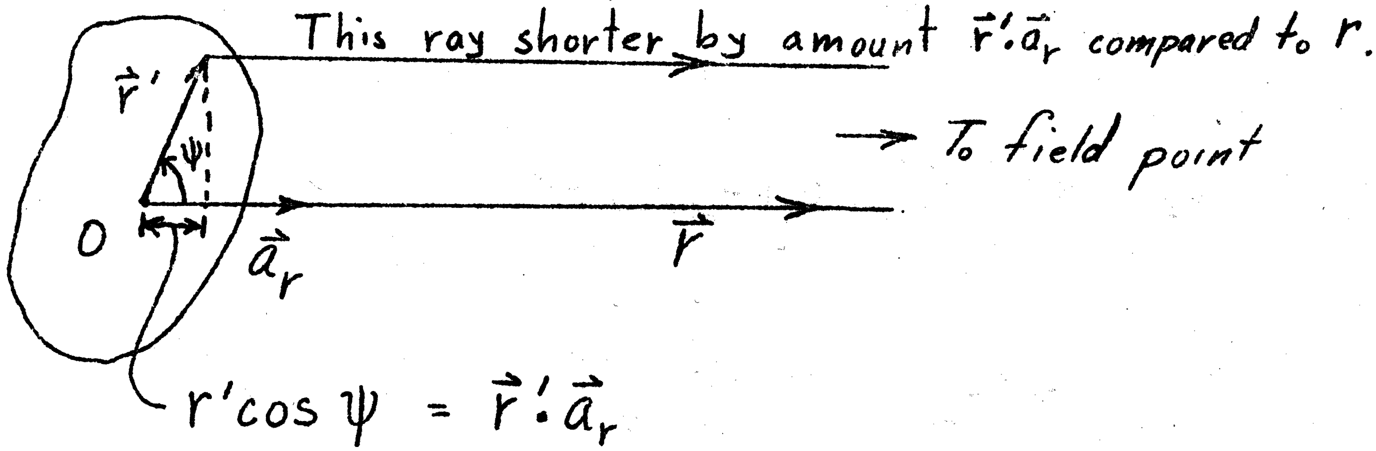
$\approx r - \frac{\vec{r}\cdot\vec{r}\,'}{r} = r - \vec{a}_r\cdot\vec{r}\,'$ to get

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi r} e^{-jk_0 r} \int_V e^{jk_0\vec{a}_r\cdot\vec{r}\,'} \vec{J}(\vec{r}\,') dV' \quad (15)$$

Physically the approximation in (15) is that all rays from the volume V containing $\vec{J}$ to the far zone field point are essentially parallel (see figure).



Parallel Rays



Enlarged View

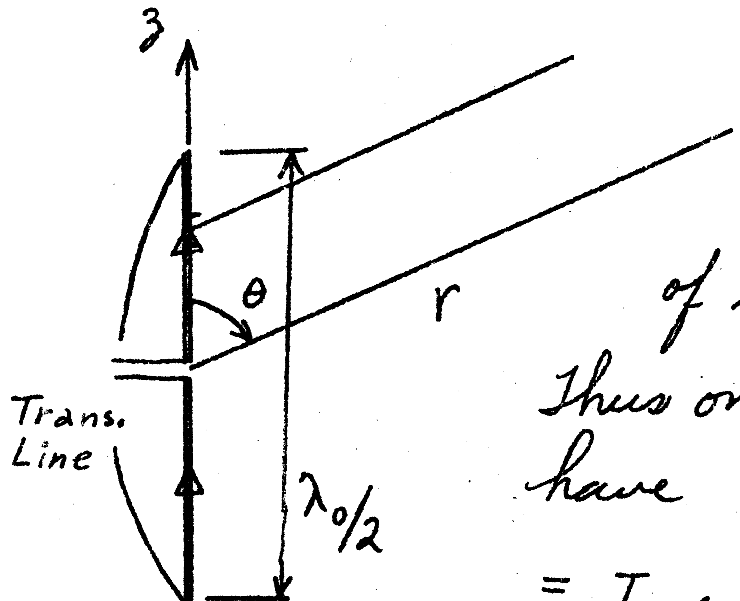
The electric and magnetic fields in the radiation zone are obtained from $\vec{A}$ by the operations given together with the approximation of keeping only the terms that vary as $1/r$. Terms varying as $1/r^2$ and $1/r^3$ are negligible by comparison. On this basis we find that

$$\vec{E}(\vec{r}) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r} \int_V \left[\vec{a}_r \cdot \vec{J}(\vec{r}')\,\vec{a}_r - \vec{J}(\vec{r}')\right] e^{jk_0 \vec{a}_r \cdot \vec{r}'} \quad (16a)$$

$$\vec{H}(\vec{r}) = Z_0^{-1}\, \vec{a}_r \times \vec{E}(\vec{r}) \quad (16b)$$

Equation (16b) states that $\vec{H}$ is perpendicular to $\vec{E}$ and $\vec{a}_r$ in the radiation zone while from (16a) we see that $\vec{E}$ depends only on J_θ and J_ϕ, i.e. $\vec{E}$ is also perpendicular to $\vec{a}_r$. Thus $\vec{E}$ and $\vec{H}$ correspond to outward propagating spherical waves whose amplitudes may depend on θ and ϕ. A spherically symmetric electromagnetic wave is not possible.

Half Wave Dipole Antenna



On thin wire antennas the current distribution is very nearly in the form of sinusoidal standing waves. Thus on a half-wave dipole we have $I(z) = -I_0 \sin k_0(|z| - \lambda_0/4)$

$$= I_0 \cos k_0 z = I_0 \cos 2\pi z/\lambda_0 .$$

To find the radiated electric field in the radiation zone we use (16a) specialized to a line source, i.e.

$$\vec{E}(\vec{r}) = \frac{jk_0 Z_0}{4\pi r} e^{-jk_0 r} \int_C \left[\vec{a}_r \cdot \vec{a}_z \, I(z')\vec{a}_r - \vec{a}_z \, I(z') \right] e^{jk_0 \vec{a}_r \cdot \vec{a}_z z'} dz'$$

Now $\vec{a}_z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta$ so we get

$$\vec{E} = \frac{jk_0 Z_0 I_0}{4\pi r} e^{-jk_0 r} \int_{-\lambda_0/4}^{\lambda_0/4} \vec{a}_\theta \sin\theta \cos k_0 z' e^{jk_0 z' \cos\theta} dz'$$

By writing $\cos k_0 z'$ as $\frac{1}{2}(e^{jk_0 z'} + e^{-jk_0 z'})$ we can easily evaluate the integral to obtain

$$\vec{E} = \frac{j I_0 Z_0}{2\pi r} e^{-jk_0 r} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \vec{a}_\theta \qquad (17a)$$

From (16b)

$$\vec{H} = \frac{j I_0 e^{-jk_0 r}}{2\pi r} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \vec{a}_\phi \qquad (17b)$$

The Poynting vector is $\vec{E}\times\vec{H}^* = E_\theta H_\phi^* \vec{a}_r$ so the power density per meter^2 is $\frac{1}{2} \mathrm{Re}\, E_\theta H_\phi^* = \frac{1}{2} Y_0 |E_\theta|^2$. The total radiated power is

$$P_{tot} = \frac{I_0^2 Z_0}{8\pi^2} \int_0^{2\pi}\int_0^{\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} d\theta\, d\phi$$

This integral can be evaluated in terms of the tabulated "cosine integral" $Ci\, x = -\int_x^\infty \frac{\cos u}{u} du$ (see Stratton, Electromagnetic Theory, Sec. 8.7, McGraw-Hill) to give

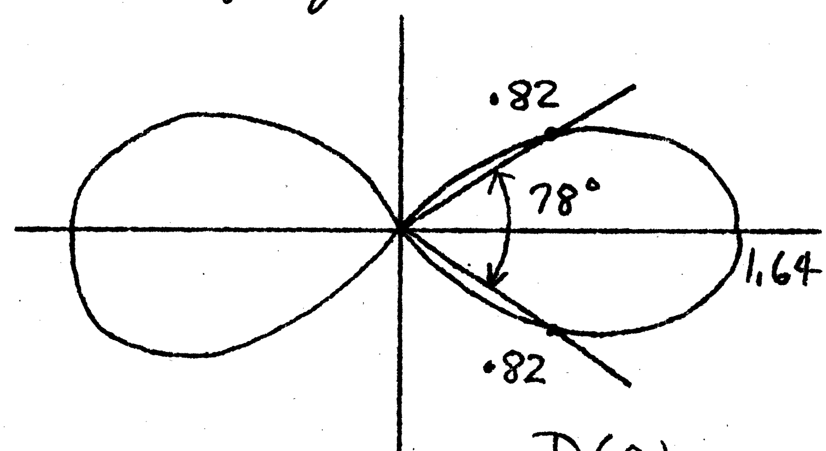
$$P_{tot} = 36.57\, I_0^2 \qquad (18)$$

We have assumed I_0 real so the radiation resistance of the half-wave dipole antenna is given by $\frac{1}{2} I_0^2 R_0 = P_{tot} = 36.57 I_0^2$ or

$$R_0 = 73.13 \text{ ohms} \qquad (19)$$

The directivity $D(\theta) = \frac{60}{36.57} \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)^2$

and has a maximum value of 1.64. For the short current element $D_{max} = 1.5$ so the half-wave dipole is only slightly more directive.



$D(\theta)$

Antenna Impedance

The input impedance to an antenna can be expressed as $Z_{in} = R + jX$. R consists of the radiation resistance R_0 plus ohmic loss resistance while jX accounts for the reactive energy stored in the near zone fields. R_o can be evaluated from the total radiated power but the evaluation of jX requires detailed expressions for the near zone fields and is much more difficult to evaluate. Usually R_o is much greater than the ohmic resistance for an antenna comparable to a wavelength in size. We can calculate the ohmic resistance by finding the power dissipated in the skin effect resistance $1/\sigma\delta_s$. The total current on the antenna is $I_o \cos k_o z$ so the current density is $\frac{I_o}{2\pi a}\cos k_o z = J_s$ where 'a' is the antenna radius. The power dissipated is

$$P_d = \frac{1}{2}\int_0^{2\pi} a\,d\phi \int_{-\lambda_o/4}^{\lambda_o/4} \left(\frac{I_o}{2\pi a}\right)^2 \frac{\cos^2 k_o z}{\sigma\delta_s}\,dz$$

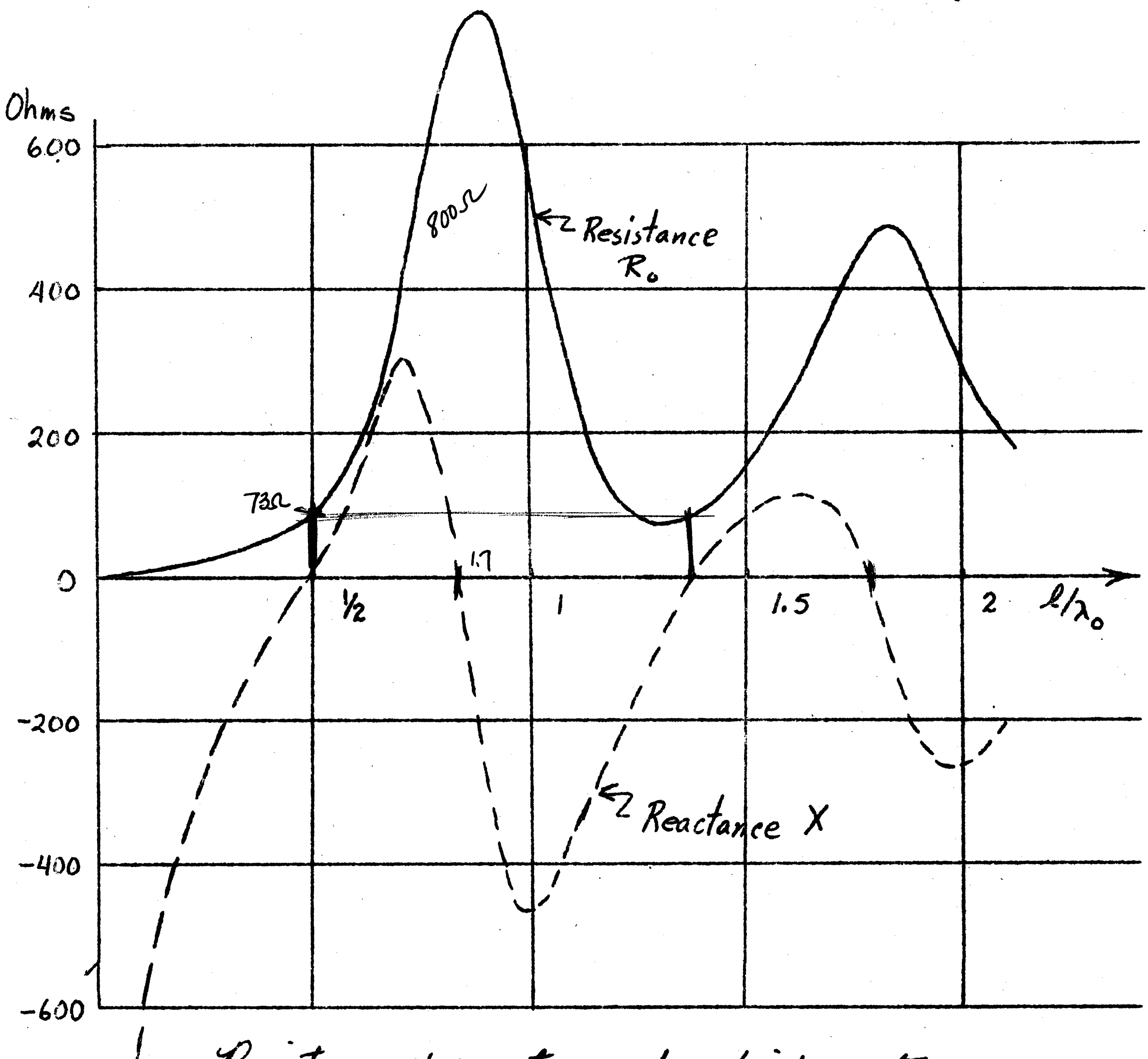
$$= 2\pi a \frac{\lambda_o}{8}\left(\frac{I_o}{2\pi a}\right)^2 \frac{1}{\sigma\delta_s} = \frac{1}{2} I_o^2 R_1$$

so

$$\text{Ohmic resistance } R_1 = \frac{\lambda_o}{8\pi a\sigma\delta_s} \qquad (20)$$

For $\lambda_o = 3$ m.
$\sigma = 5.8\times10^7$ (copper)
$\delta_s = 6.6\times10^{-6}$
$a = 1$ cm.
$R_1 = 0.031\,\Omega$
which is much less than $R_o = 73.13\,\Omega$

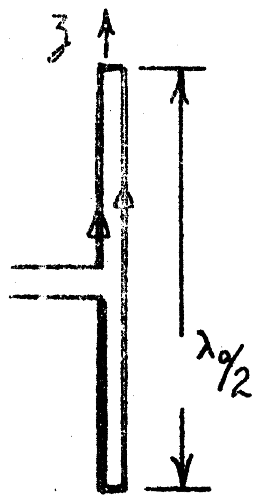
In the figure below we show the general behavior of the input radiation resistance and reactance of a thin dipole antenna as a function of l/λ_0 for $a = 0.0032\,l$ where l is the total antenna length.



Resistance & reactance of a dipole antenna

Note that when $l/\lambda_0 \approx 1/2$ the reactance is zero. This is the first resonant length and is the length at which $R_0 = 73\,\Omega$. At resonance there are equal amounts of reactive energy stored in the near zone electric and magnetic fields, i.e. $W_e = W_m$. Another resonance occurs at $l/\lambda_0 \approx 1.7$ and at this point the radiation resistance is about $800\,\Omega$. If the antenna is made thinner this second resonance point moves closer to $l/\lambda_0 = 2$ and the radiation resistance can reach values of several thousand ohms. For a thicker antenna the reactance and resistance are more nearly uniform with changes in l/λ_0, a feature which is desirable if the antenna is to be operated over a band of frequencies. Note also that an antenna with l/λ_0 much less than $1/2$ has a very small radiation resistance and a large capacitive reactance. The antenna can be tuned to resonance with a coil at the feed point but the additional ohmic losses reduces the efficiency.

Folded Dipole Antenna



The current in each arm is $I = I_0 \cos 2\pi z/\lambda_0$. Since the two arms are very closely spaced the currents can be considered as coincident. Thus the radiated field is twice as large as that from a half-wave dipole and the radiated power is four times greater.

Since the current on the transmission line at the feed point is the same as for the half-wave dipole the radiation resistance for the folded dipole is given by $\frac{1}{2} R_0 |I_0|^2 = 4 \times 36.57 |I_0|^2$ or $R_0 = 292.5\,\Omega$. The folded dipole also has less impedance variation with frequency about its resonant length. The improvement in impedance properties may be seen as follows:

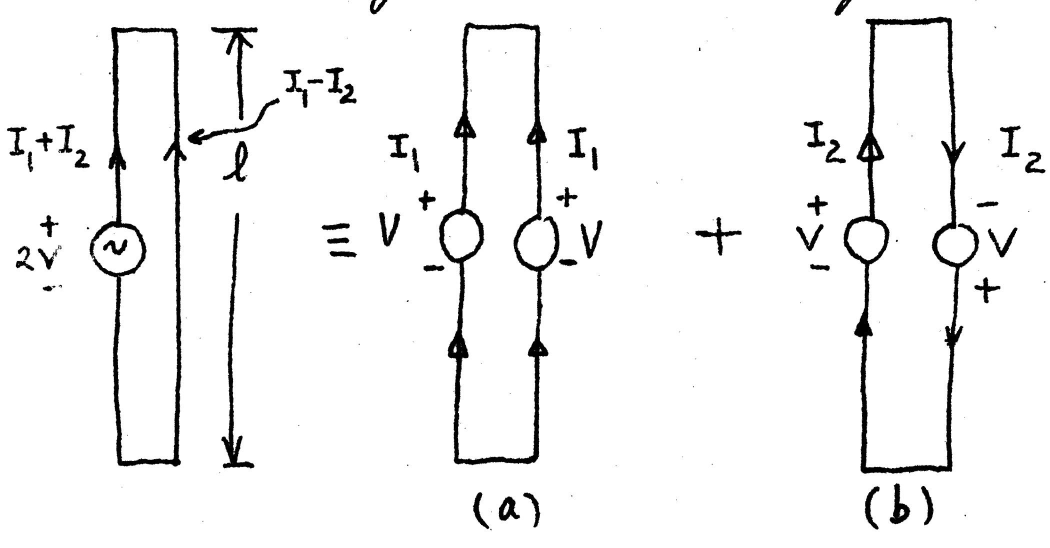
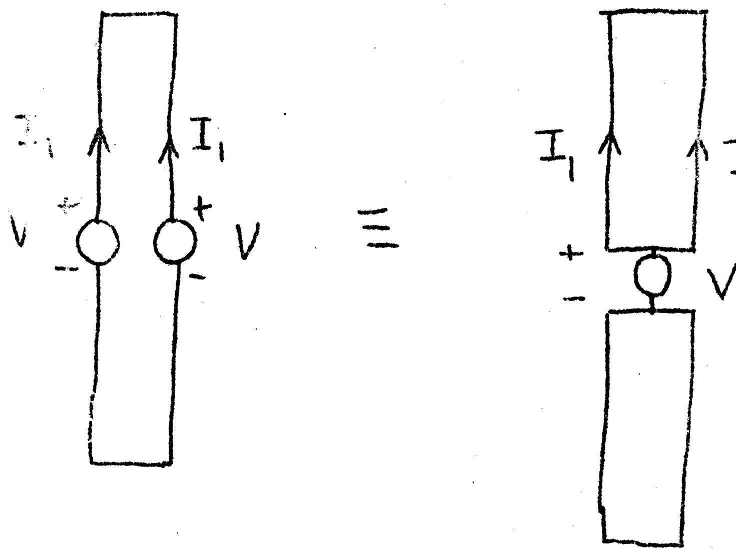


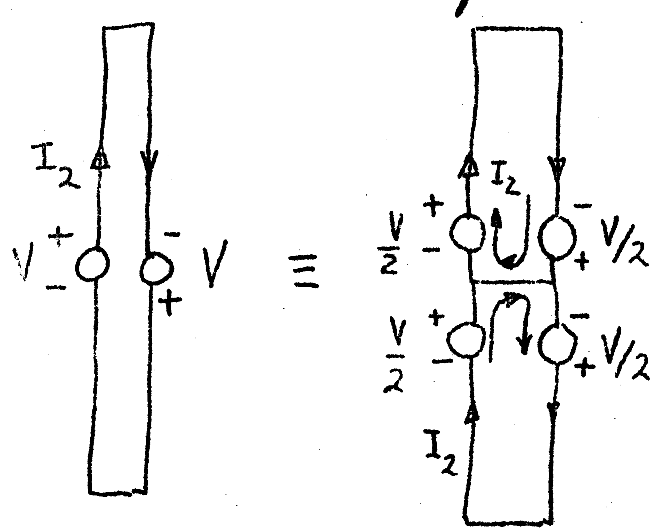
Figure 'a' is equivalent to two dipoles in parallel, i.e. to the following:

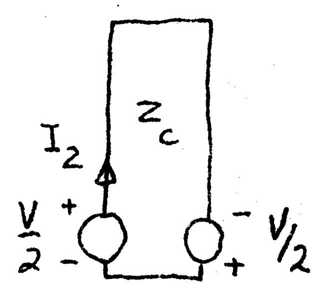


$$Y_1 = \frac{2 I_1}{V}$$

Y_1 = input admittance of a half-wave dipole made of two conductors that are spaced and connected in parallel - equivalent to one conductor of greater thickness.

Figure 'b' is a transmission line problem (no radiation since currents are oppositely directed in the two halfs).





$$I_2 = V\, Y_{in}$$

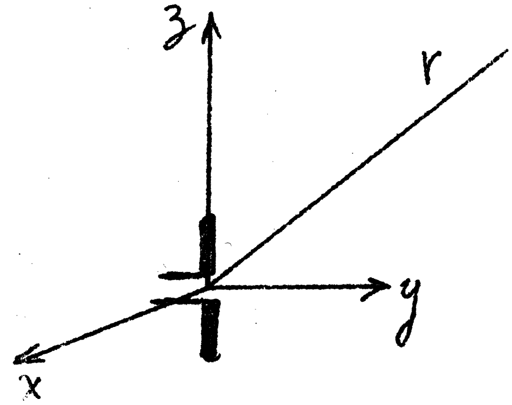
$$\frac{I_2}{V} = -j\, Y_c \cot k_0 \frac{l}{2}$$

For the folded dipole we can now write

$$Y_{in} = \frac{I_1 + I_2}{2V} = \frac{I_1}{2V} + \frac{I_2}{2V} = \frac{Y_1}{4} - \frac{jY_c}{2}\cot k_0 \frac{l}{2} \quad (21)$$

When $k_0 \frac{l}{2} \to \pi/2$, I_2 vanishes since $I_2 = -jY_c V \cot k_0 \frac{l}{2}$ and $Y_1 = (73.13)^{-1}$ mhos. For $k_0 l/2 \neq \pi/2$ we have $Y_1 = G_1 + jB_1$ with B_1 positive for $k_0 l/2 < \pi/2$ ($l < \lambda_0/2$) and hence the extra term $-\frac{jY_c}{2} \cot k_0 l/2$ is a compensating inductive reactance for $l < \lambda_0/2$. For $l > \lambda_0/2$, B_1 is negative but now $\cot k_0 l/2$ also changes sign. Hence the folded dipole may be viewed as having a piece of short-circuited transmission line as a tuning device to improve its impedance properties.

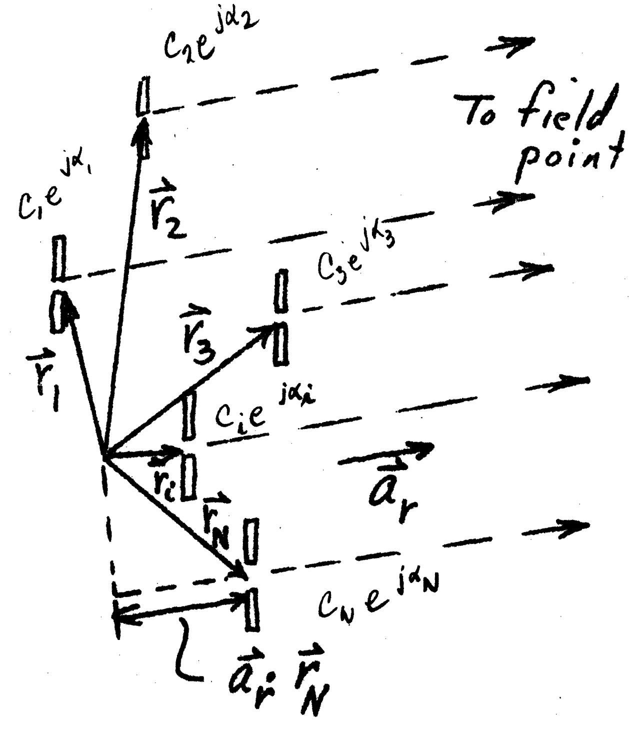
Antenna Arrays



Let the electric field radiated by an antenna at the origin be (in radiation zone, i.e. r large)

$$\vec{E} = \vec{f}(\theta,\phi)\, \underbrace{\frac{e^{-jk_0 r}}{4\pi r}}_{\text{represents a spherically propagating outward wave.}} \qquad (22)$$

$\vec{f}$ describes how E_θ and E_ϕ varies with the direction in space. Consider now an array of N identical antennas located at positions $\vec{r}_i$, $i = 1, 2, \cdots N$ and excited with relative amplitudes C_i and phase α_i. In the radiation



zone all rays from the antennas to the field point are essentially parallel so each antenna contributes a field given by (22) multiplied by $C_i e^{j\alpha_i}$ times a factor $e^{jk_0 \vec{a}_r \cdot \vec{r}_i}$ accounting for additional propagation delay or advance in phase

For the total field we can write

$$\vec{E}_{tot} = \sum_{i=1}^{N} C_i e^{j\alpha_i} \vec{f}(\theta,\phi) \frac{e^{-jk_0 r}}{4\pi r} e^{jk_0 \vec{a}_r \cdot \vec{r}_i}$$

$$= \vec{f}(\theta,\phi) \frac{e^{-jk_0 r}}{4\pi r} \sum_{i=1}^{N} C_i e^{j\alpha_i + jk_0 \vec{a}_r \cdot \vec{r}_i} \tag{23}$$

IMPORTANT:

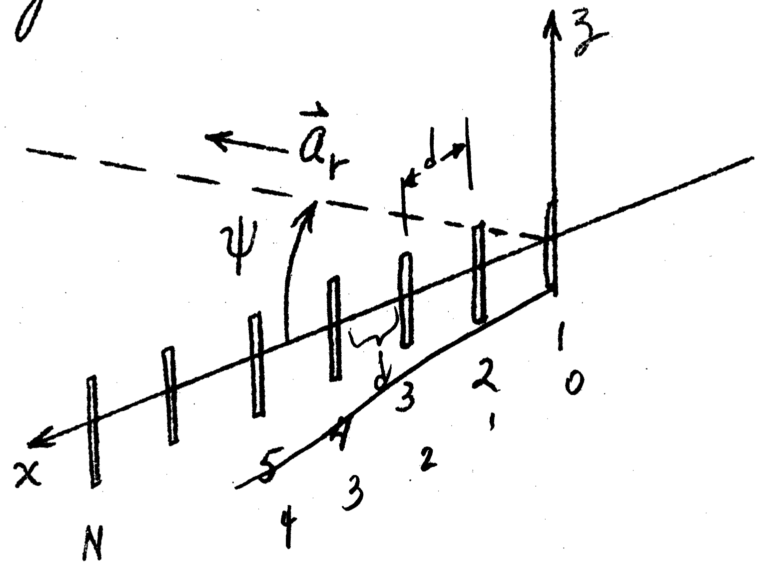
This field is the product of the field from the reference antenna (given by Eq. 22) and a factor due to the array only. The array factor $F(\theta,\phi)$ is

$$F(\theta,\phi) = \sum_{i=1}^{N} C_i e^{j\alpha_i + jk_0 \vec{a}_r \cdot \vec{r}_i} \tag{24}$$

The radiation pattern is proportional to $|\vec{f}(\theta,\phi)|\,|F(\theta,\phi)|$ which expresses the "principle of pattern multiplication". In the study off arrays it is usual practese to focus attention on the array factor only since normally each individual antenna in the array has little directivity.

Linear Arrays

Consider an array of $N+1$ antennas equispaced along the x axis as shown.



The array factor is

$$F = C \sum_{n=0}^{N} e^{j\alpha_n + jk_0 \vec{a}_r \cdot \vec{r}_n}$$

$$= C \sum_{n=0}^{N} e^{j\alpha_n + jk_0 nd \cos\psi} \qquad (25)$$

where we have assumed a constant excitation amplitude C and ψ is the angle between $\vec{a}_r$ and the x axis.

If we choose α_n to be a linear phase change of amount $n\alpha d$ for the n'th antenna we get

$$F = C \sum_{n=0}^{N} e^{jn(\alpha + k_0 \cos\psi)d} \qquad (26)$$

This geometric series can be summed to give

$$F = C \, \frac{1 - e^{j(N+1)(\alpha + k_0 \cos\psi)d}}{1 - e^{j(\alpha + k_0 \cos\psi)d}} \qquad (27a)$$

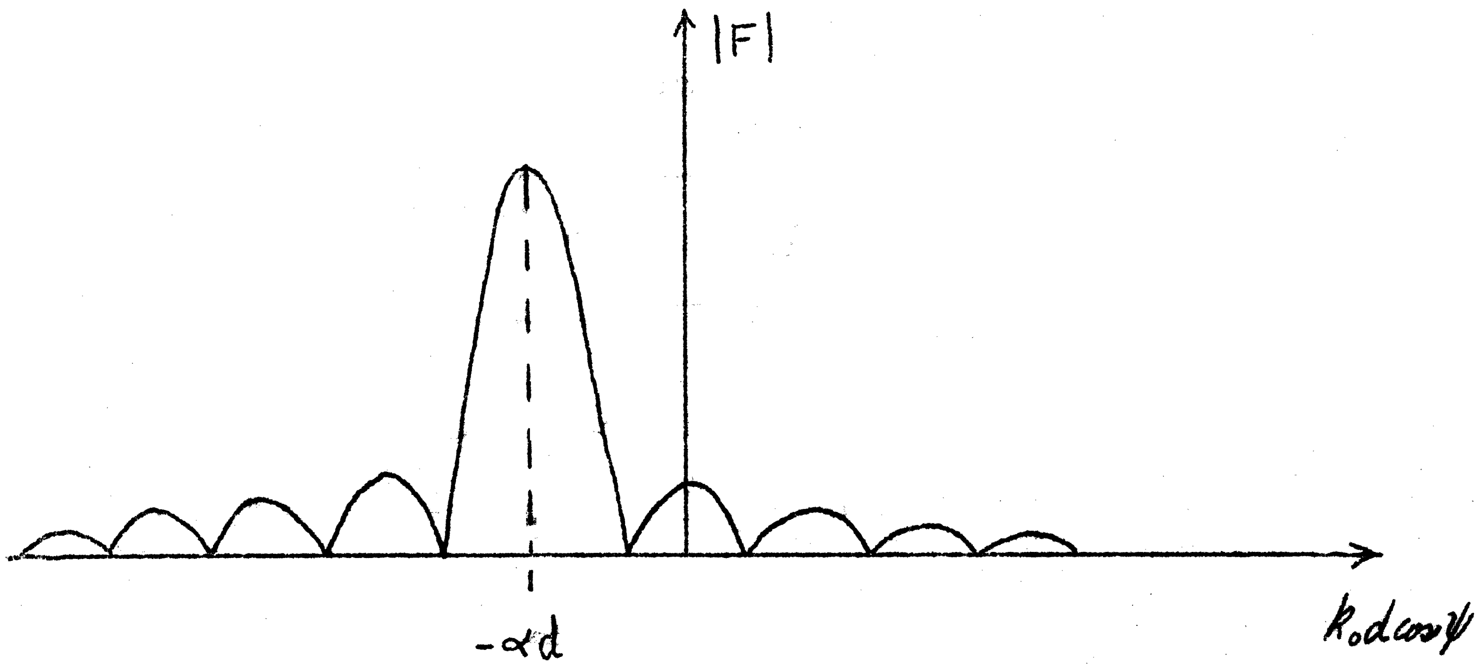
and $$|F| = C \left| \frac{\sin \frac{N+1}{2} d(\alpha + k_0 \cos\psi)}{\sin \frac{d}{2}(\alpha + k_0 \cos\psi)} \right| \qquad (27b)$$

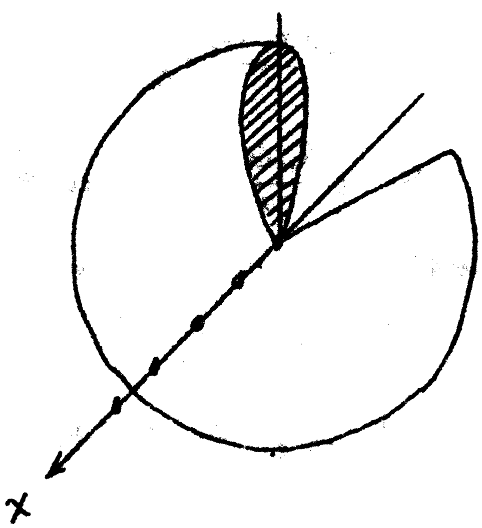
The array factor has a maximum at

$\alpha + k_0 \cos\psi = 0$ or

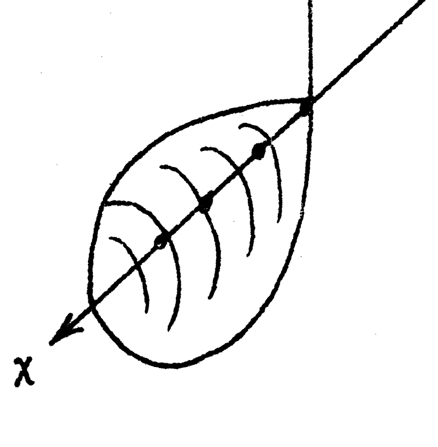
$$\cos\psi = -\frac{\alpha}{k_0} \qquad (28)$$

If we choose $\alpha = 0$ the maximum is at $\psi = \pi/2$ (broadside array) while for $\alpha = \pm k_0$ the maximum is along the direction of the array (endfire array)





Broadside array Pattern. Figure of revolution about x-axis.



End fire array pattern. Pencil beam along x-axis

For the broadside array $|F| = C \left| \dfrac{\sin\left(\frac{N+1}{2} k_0 d \cos\psi\right)}{\sin\left(\frac{k_0 d}{2}\cos\psi\right)} \right|$.

The nulls for the main beam occur when

$$\frac{N+1}{2} k_0 d \cos\psi = \pm\pi \quad \text{or} \quad \cos\psi = \frac{\pm 2\pi}{(N+1)(2\pi/\lambda_0)d} = \pm\frac{\lambda_0}{(N+1)d}$$

For N large, $(N+1)d$ can be replaced by the total array length $Nd = L$ and since the beam is very narrow $\psi = \frac{\pi}{2} \pm \Delta\psi$ with $\Delta\psi$ small. Thus $\cos\left(\frac{\pi}{2} + \Delta\psi\right) = -\sin\Delta\psi \approx -\Delta\psi = -\frac{\lambda_0}{L}$

and hence the full angular width of the main beam is

$$BW = 2\Delta\psi = \frac{2\lambda_0}{L} \qquad (29)$$

The beamwidth is inversely proportional to the array length measured in wavelengths.

In order to avoid more than one main beam $\sin\left(\frac{k_0 d}{2}\cos\psi\right)$ must have one zero only for $0 \le \psi \le \pi$. This requires $\frac{k_0 d}{2} < \pi$ or $d < \lambda_0$, i.e. the maximum element spacing should be less than λ_0.

If we choose $\alpha = -k_0$ to get a beam in the x direction we have $|F| = C \left| \dfrac{\sin \frac{N+1}{2} k_0 d (1-\cos\psi)}{\sin \frac{k_0 d}{2}(1-\cos\psi)} \right|$.

The main beam null occurs at $\frac{N+1}{2} k_0 d (1-\cos\psi) = \pi$ and for $\psi = \Delta\psi$, $\cos\Delta\psi \approx 1 - \frac{(\Delta\psi)^2}{2}$ so

$$\frac{(\Delta\psi)^2}{2} = \frac{2\pi}{(N+1)k_0 d} \approx \frac{\lambda_0}{L}$$ so we get

$$BW = \Delta\psi = \sqrt{\frac{2\lambda_0}{L}} \qquad (30)$$

Thus for the endfire case the beamwidth is inversely proportional to the square root of the array length measured in wavelengths.

To have only a single main beam we must have $\left| \frac{k_0 d}{2}(1-\cos\psi) \right| < \pi$

so for $\psi = \pi$, $k_0 d < \pi$ or $d < \lambda_0/2$, i.e. for an endfire array we must keep the spacing less than $\lambda_0/2$ to avoid a second main beam.

Receiving Antennas

Lorentz reciprocity theorem

Let $\vec{E}_1, \vec{H}_1, \vec{J}_1$ and $\vec{E}_2, \vec{H}_2, \vec{J}_2$ be two sets of solutions to Maxwell's equations for which

$$\nabla \times \vec{E}_1 = -j\omega\mu_0 \vec{H}_1 \,, \quad \nabla \times \vec{H}_1 = j\omega\epsilon_0 \vec{E}_1 + \vec{J}_1$$

$$\nabla \times \vec{E}_2 = -j\omega\mu_0 \vec{H}_2 \,, \quad \nabla \times \vec{H}_2 = j\omega\epsilon_0 \vec{E}_2 + \vec{J}_2$$

Consider $\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1)$ which may be expanded to give $(\nabla \times \vec{E}_1) \cdot \vec{H}_2 - \vec{E}_1 \cdot \nabla \times \vec{H}_2 - (\nabla \times \vec{E}_2) \cdot \vec{H}_1 + \vec{E}_2 \cdot \nabla \times \vec{H}_1$.

By using Maxwell's equations we obtain

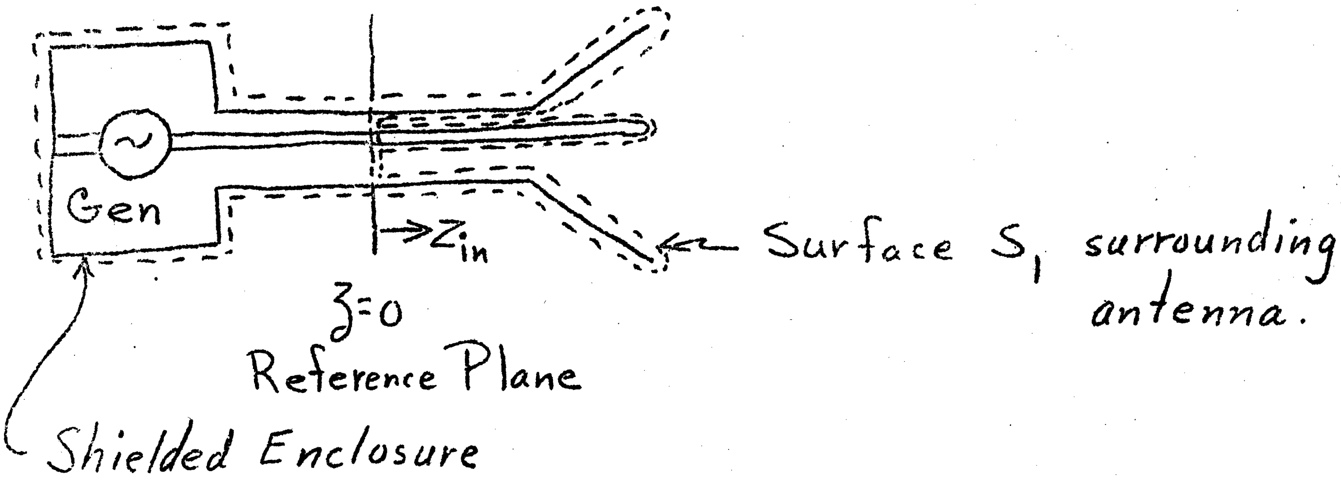
$\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = \vec{J}_1 \cdot \vec{E}_2 - \vec{J}_2 \cdot \vec{E}_1$ since all other terms cancel. By integrating over a volume V and using the divergence theorem we obtain

$$\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{S} = \int_V (\vec{J}_1 \cdot \vec{E}_2 - \vec{J}_2 \cdot \vec{E}_1)\, dV \quad (31)$$

which is our desired reciprocity theorem.

We will use this result to find the power received by an antenna.

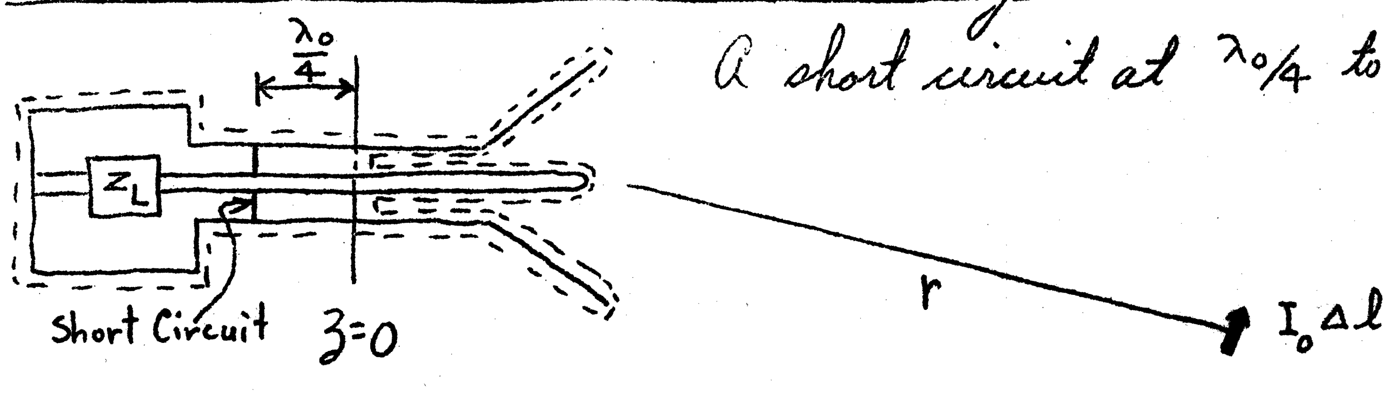
Conditions 1: Antenna is Transmitting



On the surface surrounding the antenna we assume $\vec{n} \times \vec{E}_1 = 0$ (perfect conductors) except on the reference plane in the coaxial feed line. On this plane $\vec{E}_1 = \dfrac{V \vec{a}_r}{r \ln b/a}$, $\vec{H}_1 = \dfrac{Y_{in} V \vec{a}_\phi}{2\pi r}$ where V is the total voltage at $z=0$ and Y_{in} is the input admittance to the antenna.

Conditions 2: Antenna is Receiving

A short circuit at $\lambda_0/4$ to



the left of the reference plane gives open circuit conditions at $z=0$. A current element $I_0 \Delta l$ a distance r from the antenna provides the incident field. Since we have an open circuit at $z=0$ the received voltage here is V_{oc} = open circuit voltage and the current at this point is zero. Hence on the reference plane

$$\vec{E}_2 = \frac{V_{oc}\,\vec{a}_r}{r \ln b/a}, \quad \vec{H}_2 = 0.$$

In order to apply (31) we choose for S the surface S_1 around the antenna and the surface S_∞ of a sphere of infinite radius. At infinity the fields $\vec{E}_1, \vec{H}_1$, and $\vec{E}_2, \vec{H}_2$ are spherical TEM waves for which $\vec{H}_1 = Y_0 \vec{a}_r \times \vec{E}_1$ and $\vec{H}_2 = Y_0 \vec{a}_r \times \vec{E}_2$. Hence on S_∞ we have

$$\vec{E}_1 \times \vec{H}_2 \cdot \vec{a}_r - \vec{E}_2 \times \vec{H}_1 \cdot \vec{a}_r = \vec{a}_r \times \vec{E}_1 \cdot \vec{H}_2 - \vec{a}_r \times \vec{E}_2 \cdot \vec{H}_1$$

$= Z_0 \vec{H}_1 \cdot \vec{H}_2 - Z_0 \vec{H}_2 \cdot \vec{H}_1 = 0$ so we get zero contribution from the integral over S_∞. From the integral over S_1 we only get a contribution from the reference plane in the coaxial feed line since $\vec{n} \times \vec{E}_1 = \vec{n} \times \vec{E}_2 = 0$ on the conductors. For conditions 1 we have $\vec{J}_1 = 0$ while $\vec{J}_2$ is the current density associated with the current element $I_0 \Delta l$. Thus (31) gives

$$\int_a^b \int_0^{2\pi} \left[-\frac{V_{oc}}{r \ln b/a} \vec{a}_r \times \frac{Y_{in} V}{2\pi r} \vec{a}_\phi \right] \cdot (-\vec{a}_z \, r\, d\phi\, dr)$$

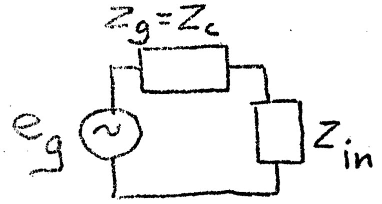
$$= \int_V -\vec{J}_2 \cdot \vec{E}_1 \, dV = Y_{in} V V_{oc} \qquad (32)$$

Let us assume that the antenna radiates a linearly polarized field and let us choose $I_0 \Delta l$ oriented parallel to $\vec{E}_1$. Then (32) yields

$$V_{oc} = -\frac{I_0 \Delta l \, E_1(\vec{r})}{V Y_{in}} \qquad (33)$$

We may choose I_0 to be real so

$$|V_{oc}| = I_0 \Delta l \left| \frac{\vec{E}_1(\vec{r})}{V Y_{in}} \right| = I_0 \Delta l \frac{|E_1(\vec{r})|}{|V Y_{in}|} \qquad (34)$$



Let the generator be matched to the feed line. Under transmitting conditions we then have $|V| = \left|\frac{Z_{in}}{Z_{in}+Z_c}\right| |e_g|$

and the power delivered to the antenna will be $\frac{1}{2}\,\mathrm{Re}\, VI^* = \frac{1}{2}\,\mathrm{Re}\,\frac{VV^*}{Z_{in}} = \frac{|V|^2}{2}\,\mathrm{Re}\, Y_{in}$

$$= \frac{|V|^2}{2}\,\mathrm{Re}\,\frac{Z_{in}^*}{Z_{in} Z_{in}^*} = \frac{|V|^2}{2}\,\mathrm{Re}\,\frac{Z_{in}^*}{|Z_{in}|^2} = \frac{|e_g|^2\,\mathrm{Re}\, Z_{in}^*}{2|Z_{in}+Z_c|^2}$$

If the input power were uniformly radiated it would produce a power density

$$\frac{|V|^2\,\mathrm{Re}\, Z_{in}^*}{2|Z_{in}|^2\, 4\pi r^2} \quad \text{watts}/m^2.$$

By definition of the gain $G(\theta,\phi)$ of the antenna the actual power density produced in the direction θ, ϕ will be (we are putting $\mathrm{Re}\,\frac{Z_{in}^*}{|Z_{in}|^2} = \mathrm{Re}\, Y_{in}$)

$$\frac{|V|^2\,\mathrm{Re}\, Y_{in}}{8\pi r^2}\, G(\theta,\phi) = \frac{1}{2}\, Y_0\, |E_1(\bar{r})|^2 \quad (35)$$

since the power density in a radiated spherical

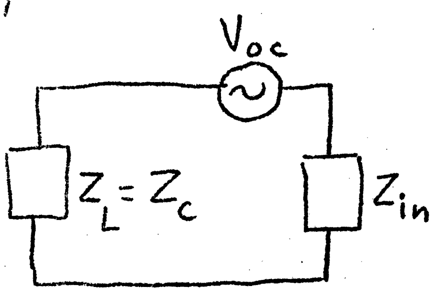
TEM wave is $\frac{1}{2} Y_0 |E_1(\bar{r})|^2$. Solving for $|E_1|$ yields

$$|E_1| = \frac{|V| \sqrt{G_1 \, \mathrm{Re}\, Y_{in}}}{\sqrt{4\pi Y_0} \; r}$$

which may be used in (34) to give

$$|V_{oc}| = \frac{I_0 \Delta l \sqrt{G_1 \, \mathrm{Re}\, Y_{in}}}{\sqrt{4\pi Y_0} \; |Y_{in}| \, r} \qquad (36)$$

If we choose a load $Z_L = Z_c$ the received power will be (we use Thevenin's theorem)



$$P_{rec} = \frac{1}{2} \mathrm{Re} \left| \frac{V_{oc}}{Z_c + Z_{in}} \right|^2 Z_c = \frac{1}{2} \frac{|V_{oc}|^2 Z_c}{|Z_c + Z_{in}|^2}$$

~~Using (36) gives~~

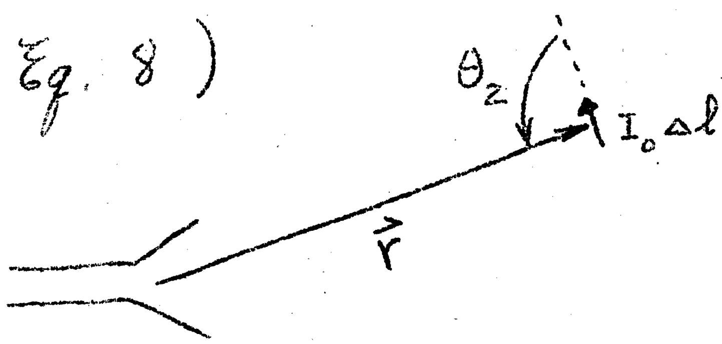
$$P_{rec} = \frac{(I_0 \Delta l)^2 G_1(\theta,\phi) Z_c |Z_{in}|^2 \mathrm{Re}\, Y_{in}}{8\pi r^2 Y_0 |Z_c + Z_{in}|^2}$$

$$= \frac{(I_0 \Delta l)^2 G(\theta,\phi) Z_c \, \mathrm{Re}\, Z_{in}^*}{8\pi r^2 Y_0 |Z_c + Z_{in}|^2} \qquad (37)$$

The received power is dependent only on the field incident on the antenna and not on the source of that field. Consider therefore the case when the antenna is matched to the line, i.e. $Z_{in} = Z_c$, in which case (37) becomes

$$P_{rec} = \frac{Z_0 (I_0 \Delta l)^2 G(\theta,\phi)}{32\pi r^2} \tag{38}$$

Now $Z_0(I_0 \Delta l)^2 k_0^2 / (32\pi^2 r^2)$ is the incident power density from the current element if we orient $I_0 \Delta l$ perpendicular to the radius vector $\vec{r}$ so that $\sin\theta_2 = 1$ (see figure and Eq. 8)



Hence we can express (38) in the form

$$P_{rec} = A_r P_{inc} = \frac{\pi G(\theta,\phi)}{k_0^2} P_{inc}$$

$$= \frac{\lambda_0^2}{4\pi} G(\theta,\phi) P_{inc} \tag{39}$$

This fundamental results states that an antenna, under matched impedance and polarization conditions, has an effective receiving cross section or area A_r given by

$$A_r = \frac{\lambda_0^2}{4\pi} G(\theta,\phi) \quad (40)$$

When $Z_{in} \neq Z_c$ we have, from (37),

$$P_{rec} = \frac{\lambda_0^2}{4\pi} G(\theta,\phi) P_{inc}\left(\frac{4Z_c \, \mathrm{Re}\, Z_{in}^*}{|Z_c + Z_{in}|^2}\right)$$

where the term in parenthesis reduces to 1 when $Z_{in} = Z_C$. For the mismatched antenna the reflection coefficient Γ is given by $\Gamma = \dfrac{Z_{in} - Z_c}{Z_{in} + Z_c}$ and $1 - |\Gamma|^2$ equals

$$(1 - \Gamma\Gamma^* = 1 - \frac{(Z_{in} - Z_c)(Z_{in}^* - Z_c)}{|Z_{in} + Z_c|^2}$$

$$= \frac{(Z_{in} + Z_c)(Z_{in}^* + Z_c) - (Z_{in} - Z_c)(Z_{in}^* - Z_c)}{|Z_{in} + Z_c|^2} = \frac{2Z_c(Z_{in} + Z_{in}^*)}{|Z_{in} + Z_c|^2}$$

or $$1-|\Gamma|^2 = \frac{4 Z_c \, Re \, Z_{in}}{|Z_{in}+Z_c|^2} \qquad (41)$$

Hence under unmatched conditions we have

$$P_{rec} = \frac{\lambda_0^2}{4\pi}\left(1-|\Gamma|^2\right) G_1(\theta,\phi)\, P_{inc}$$

$$= \left(1-|\Gamma|^2\right) A_r \, P_{inc} \qquad (42)$$

where $|\Gamma|^2$ is the power reflection coeffecient for the unmatched antenna.

In deriving the above results we assumed that the antenna radiated a linearly polarized field so that we could orient the current element $I_0 \Delta l$ along $\vec{E}_1$. This corresponds to matched polarization conditions. In general, if the incoming radiation is not properly polarized the received power will be less than that given by (39) or (42). We examine this aspect below.

Polarization Mismatch For Antennas

In general, in a given direction, an antenna will radiate an electric field with an E_θ and an E_ϕ component that are not in phase. Thus let the field radiated be $E_\theta = E_0 \frac{e^{-jk_0 r}}{4\pi r}$, $E_\phi = \tau e^{j\beta} E_0 \frac{e^{-jk_0 r}}{4\pi r}$

in a given direction, where τ and β are real. Thus $E_\phi = \tau e^{j\beta} E_\theta$. In the time domain the fields are $E_\theta = \frac{E_0}{4\pi r} \cos(k_0 r - \omega t)$ and $E_\phi =$

$\frac{\tau E_0}{4\pi r} \cos(k_0 r - \omega t - \beta)$ if we take E_0 real.

Let $k_0 r - \omega t = \alpha$, then $E_\theta = \frac{E_0}{4\pi r} \cos\alpha$ and

$E_\phi = \frac{\tau E_0}{4\pi r} (\cos\alpha \cos\beta + \sin\alpha \sin\beta)$.

To find the resultant total field magnitude we eliminate the time as follows:

$$\left(\frac{4\pi r E_\theta}{E_0}\right) = \cos\alpha, \quad 1 - \left(\frac{4\pi r E_\theta}{E_0}\right)^2 = \sin^2\alpha$$

From the expression for E_ϕ we can write

$$\left(\frac{4\pi r E_\phi}{\tau E_0} - \cos\beta \, \frac{4\pi r E_\theta}{E_0}\right)^2 = \sin^2\beta\left[1-\left(\frac{4\pi r E_\theta}{E_0}\right)^2\right]$$

which can also be expressed as

$$\left(\frac{4\pi r E_\phi}{\tau E_0}\right)^2 + \left(\frac{4\pi r E_\theta}{E_0}\right)^2 - 2\cos\beta \, \frac{(4\pi r)^2 E_\theta E_\phi}{\tau \, E_0^2} = \sin^2\beta$$

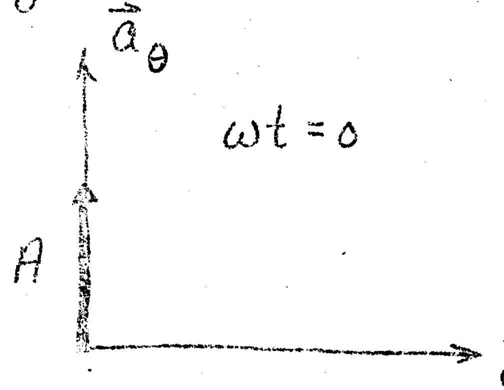
or

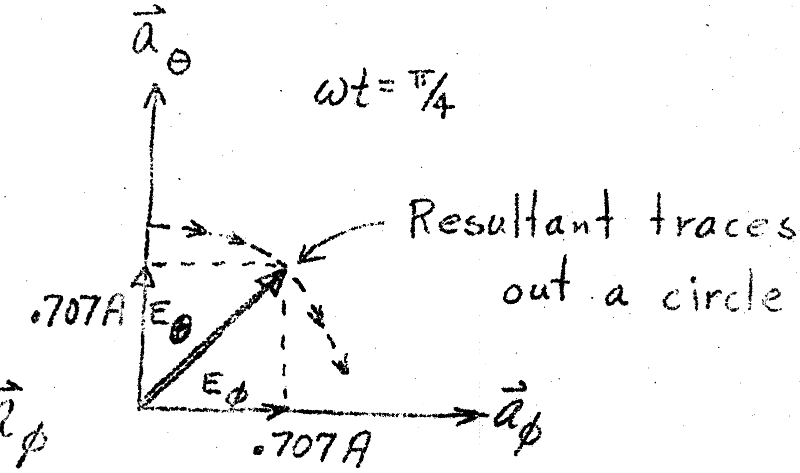
$$\left(\frac{E_\phi}{\tau}\right)^2 + E_\theta^2 - 2\frac{\cos\beta}{\tau} E_\theta E_\phi = \frac{E_0^2 \sin^2\beta}{(4\pi r)^2} \qquad (43)$$

This is the equation of an ellipse. At a given point in space the resultant field vector traces out an ellipse, once per period in time. If the direction of rotation is clockwise, looking in the direction of propagation, the field is said to be positive or right elliptical polarized. If the direction of rotation is anti-clockwise the field is negative or left elliptical polarized. If $\tau = 1$ and $\beta = \pm \pi/2$ then (43) reduces to

$$E_\phi^{\,2} + E_\theta^{\,2} = \frac{E_0^{\,2}}{(4\pi r)^2} \qquad (44)$$

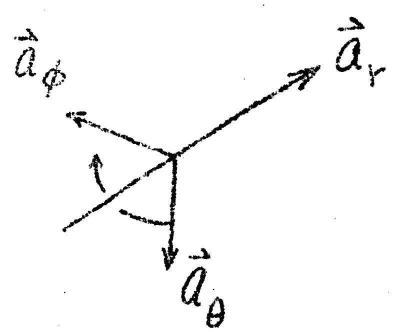
which is the equation of a circular. For this case the field is circularly polarized (see figures below)





$E_\theta = A\cos\omega t = A\,\mathrm{Re}\, e^{j\omega t}$

$E_\phi = A\sin\omega t = -A\,\mathrm{Re}\, j\, e^{j\omega t}$



Rotation from $\vec{a}_\theta$ into $\vec{a}_\phi$ is clockwise so above field is positive or right circular polarized. If $E_\theta = A\,\mathrm{Re}\, e^{j\omega t}$ and $E_\phi = A\,\mathrm{Re}\, j\, e^{j\omega t}$ the field is left circular polarized.

It is convenient to express the field radiated by an antenna relative to that which a unit current element would radiate. Thus let the radiated field be

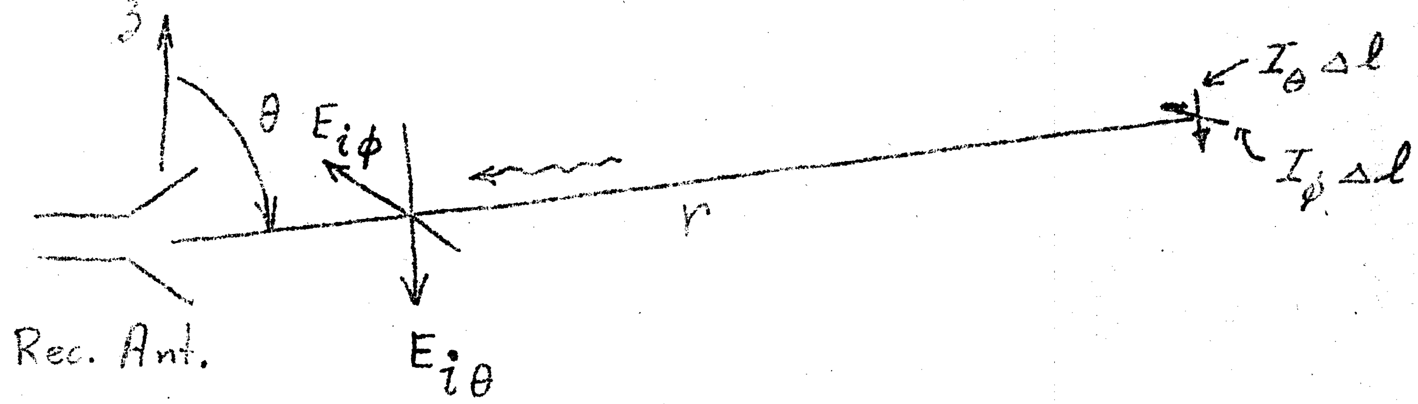
$$\vec{E} = \frac{j Z_0 k_0 I_{in}}{4\pi r} \vec{h} e^{-jk_0 r} \qquad (45)$$

where I_{in} is the input current to the antenna and equals $V Y_{in}$, while $\vec{h} = h_\theta \vec{a}_\theta + h_\phi \vec{a}_\phi$ is a complex vector called the effective complex length of the antenna (compare (45) with

$$E_\theta = \frac{j Z_0 k_0 e^{-jk_0 r}}{4\pi r} (I_0 \Delta l) \sin\theta$$

for the field from a current element). Note that $\vec{h}$ is a function of direction specified by the angles θ and ϕ.

In general the field incident on an antenna is also elliptically polarized. In order to utilize (32) for the received open circuit voltage it is convenient to think of the incident field as being produced by two current elements $I_\theta \Delta l \, \vec{a}_\theta$ and $I_\phi \Delta l \, \vec{a}_\phi$.



The field that current elements $I_\theta \Delta l$ and $I_\phi \Delta l$ produce at the receiving antenna are:

$$E_\theta = \frac{-jk_0 Z_0 I_\theta \Delta l}{4\pi r} e^{-jk_0 r}, \quad E_\phi = \frac{-jk_0 Z_0 I_\phi \Delta l}{4\pi r} e^{-jk_0 r}.$$

(the negative sign is due to the current orientations). To reproduce the incident field we must choose

$$I_\theta \Delta l = \frac{-4\pi r}{jk_0 Z_0} e^{jk_0 r} E_{i\theta} \qquad (46a)$$

$$I_\phi \Delta l = -\frac{4\pi r}{jk_0 Z_0} e^{jk_0 r} E_{i\phi} \qquad (46b)$$

where $E_{i\theta}$ and $E_{i\phi}$ are the actual incident fields.

We now apply (32) and use (45) and (46) to get

$$V_{oc}\, V Y_{in} = V_{oc} I_{in} = -\int_V \vec{J}_2 \cdot \vec{E}_1$$

$$= -\frac{j Z_0 k_0 I_{in}}{4\pi r} e^{-jk_0 r}\, \vec{h} \cdot (I_\theta \Delta l\, \vec{a}_\theta + I_\phi \Delta l\, \vec{a}_\phi)$$

$$= I_{in} \vec{h} \cdot \vec{E}_i \qquad \text{or}$$

$$V_{oc} = \vec{h} \cdot \vec{E}_i \qquad (47)$$

This equation further illuminates why $\vec{h}$ is called the effective length of the antenna since it shows that V_{oc} can be thought of as the voltage induced in a wire of length h when $\vec{h}$ and $\vec{E}_i$ are linearly polarized. The maximum value that $\vec{h} \cdot \vec{E}_i$ can have is $|\vec{h}||\vec{E}_i|$ and this occurs when $\vec{h}$ equals a real constant times the complex conjugate of $\vec{E}_i$. To the extent that $|\vec{h} \cdot \vec{E}_i| < |\vec{h}||\vec{E}_i|$ the antenna polarization is mismatched to that of the incident field. The polarization mismatch factor p is given by

$$p = \frac{|\vec{h} \cdot \vec{E}_i|^2}{|\vec{h}|^2 |\vec{E}_i|^2} \qquad (48)$$

and the received power is now given by

$$\boxed{P_{rec} = \underbrace{(1-|\Gamma|^2)}_{\text{at receiving end}} \frac{\lambda_0^2}{4\pi} p \underbrace{G}_{\text{receiving antenna}} P_{inc}} \qquad (49)$$

i.e. reduced by the factor p.

Examples

Let $\vec{h} = h_0(\vec{a}_\theta - j\vec{a}_\phi)$ which corresponds to a right circular polarized antenna.
Let the incident field be $\vec{E}_i = E_0(\vec{a}_\theta - j\vec{a}_\phi)$

The electric vector of the incident field rotates from $\vec{a}_\theta$ into $\vec{a}_\phi$ but because it corresponds to a wave propagating toward the antenna it represents a left circular polarized field. From (48) we get

$$p = \frac{(h_0 E_0)^2 (\vec{a}_\theta - j\vec{a}_\phi)\cdot(\vec{a}_\theta - j\vec{a}_\phi)}{4 h_0^2 E_0^2} = 0$$

Thus a right circular polarized antenna will not receive a left circular polarized wave.

If $\vec{E}_i = E_0(\vec{a}_\theta + j\vec{a}_\phi)$ (right circular polarized wave) then $\vec{E}_i = \frac{E_0}{h_0}\vec{h}^*$ and

$$p = \frac{(h_0 E_0)^2 (\vec{a}_\theta - j\vec{a}_\phi)\cdot(\vec{a}_\theta + j\vec{a}_\phi)}{4 h_0^2 E_0^2} = 1$$

This corresponds to perfectly matched polarization.

If $\vec{E}_i = E_0\vec{a}_\theta$ (linearly polarized incident field), then

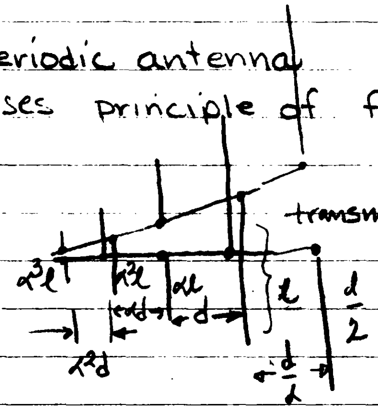
$$p = \frac{(h_0 E_0)^2 (\vec{a}_\theta - j\vec{a}_\phi)\cdot\vec{a}_\theta}{2 h_0^2 E_0^2} = \frac{1}{2}$$

so a 3 db. loss occurs in receiving linear polarization with a circular polarized antenna.

Microwaves: may 3, 1972

log periodic antenna

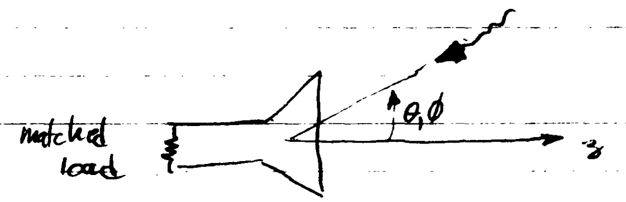
uses principle of frequency scaling



usually used alphas in the range 0.7–0.9

let $l = \frac{\lambda_1}{4}$

you can cut this antenna at both ends so that you can restrict yourself to a frequency band



$P_{inc} = \frac{1}{2} Y_0 \, |\bar{E}_{inc}|^2$

$P_{rec} = P_{inc} \, A_e$ ← effective area

$P_{rec} = P_{inc} \, \frac{\lambda_0^2}{4\pi} \, G(\theta,\phi)$ [under matched impedance & polarization conditions]

May 5, 1972

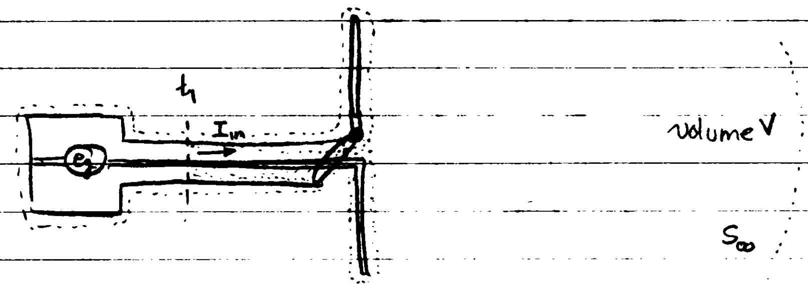
$\bar{E}_1, \bar{H}_1, \bar{J}_1$ $\qquad$ $\nabla \times \bar{E}_1 = -j\omega\mu_0 H_1$

$\bar{E}_2, \bar{H}_2, \bar{J}_2$ $\qquad$ $\nabla \times \bar{H}_1 = j\omega\epsilon_0 \bar{E}_1 + \bar{J}_1$

$$\nabla \cdot \left[\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1\right] = -\bar{E}_1 \cdot \bar{J}_2 + \bar{E}_2 \cdot \bar{J}_1$$

$$(\nabla \times \bar{E}_1) \cdot \bar{H}_2 - (\nabla \times H_2) \cdot E_1 + \cdots \quad -j\omega\mu_0 \cancel{\bar{H}_1 \cdot \bar{H}_2} - j\omega\epsilon_0 \cancel{\bar{E}_2 \cdot \bar{E}_1} - \cancel{\bar{J}_2 \cdot \bar{E}_1}$$

$$\oint (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \bar{n}\, ds = \int_V (\bar{E}_2 \cdot \bar{J}_1 - \bar{E}_1 \cdot \bar{J}_2)\, dV$$



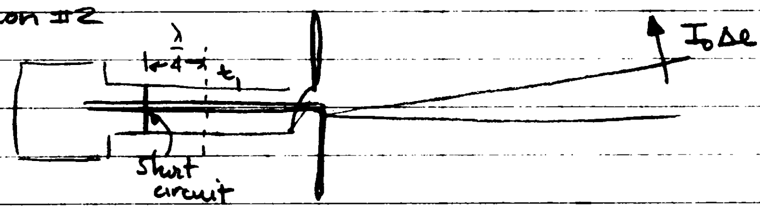
condition #1: antenna transmitting

at t_1 $\quad \bar{E}_1 = \dfrac{\bar{a}_r V}{r \ln b/a}$

$$\bar{H}_1 = \frac{\bar{a}_\phi I_{in}}{2\pi r}$$

$\bar{J}_1$ in volume is zero.

condition #2



on t_1 $\qquad \bar{E}_2 = \dfrac{\bar{a}_r V_{oc}}{r \ln b/a}, \quad H_2 = 0$

$\bar{J}_2$ is current $I_0 \Delta l$

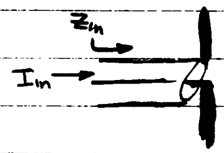
$$\oint_S \to \int_{t_1}$$

$$\int_0^{2\pi}\int_a^b \left[-\frac{V_{oc}\bar{a}_r}{r \ln b/a} \times \frac{\bar{a}_\phi I_{in}}{2\pi r}\right](-\bar{a}_z)\, r\, dr\, d\phi = \int_0^{2\pi}\int_a^b \frac{V_{oc} I_{in}}{2\pi r \ln b/a}\, dr\, d\phi$$

$$= V_{oc} I_{in} = -\int_V \bar{E}_1 \cdot \bar{J}_2\, dV$$

let $I_0 \Delta l$ be oriented parallel to $\bar{E}_1$, $\bar{E}_1$ is linearly polarized

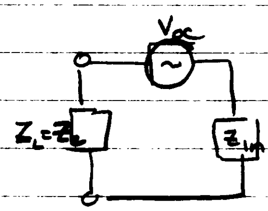
$$V_{oc} = -\frac{E_1 I_0 \Delta l}{I_{in}}$$



$$P_{in} = \frac{1}{2} \mathrm{Re}\, Z_{in} |I_{in}|^2$$

$$\frac{P_{in}}{4\pi r^2} G(\theta,\phi) = \frac{1}{2} Y_0 |\bar{E}_1|^2$$

$$|E_1|^2 = \frac{G |I_{in}|^2 \mathrm{Re}\, Z_{in}}{4\pi r^2 Y_0}$$

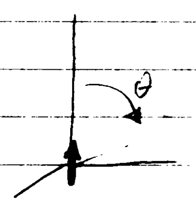


$$\frac{1}{2} \left| \frac{V_{oc}}{Z_c + Z_{in}} \right|^2 Z_c = P_{rec}.$$

$$P_{rec} = \frac{Z_c |E_1|^2 I_0^2 \Delta l^2}{2(Z_c + Z_{in})^2 |I_{in}|^2} \frac{|I_{in}|^2 \mathrm{Re}\, Z_{in}}{4\pi r^2 Y_0}$$

Let $Z_{in} = Z_c$ (matched antenna)

$$P_{rec} = \frac{Z_0 I_0^2 \Delta l^2 G}{32\pi r^2}$$



$$E_\theta = \frac{j k_0 Z_0 I_0 \Delta l}{4\pi r} e^{-jk_0 r} \sin\theta$$

$$P = \frac{k_0^2 Z_0 I_0^2 \Delta l^2}{32\pi r^2} \sin^2\theta$$

$P = P_{inc}$

$$P_{rec} = \left[\frac{k_0^2 Z_0 I_0^2 \Delta l^2}{32\pi^2 r^2} \right] \frac{G\pi}{k_0^2}$$

$[\,] = P_{inc}$

assumed matched impedance condition
matched polarization conditions

$\Rightarrow P_{rec} = P_{inc} A_e$

$$A_e = \frac{G\pi}{k_0^2} = \frac{G\pi \lambda_0^2}{4\pi^2} = \frac{\lambda_0^2}{4\pi} G$$

when $Z \neq Z_c$ $\quad (1 - |\Gamma_{in}|)^2$

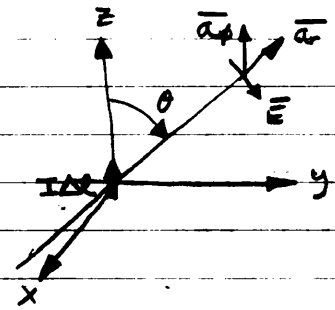
$$P_{rec} = P_{inc} \frac{\lambda_0^2}{4\pi} G \frac{4 Z_c \mathrm{Re}\, Z_{in}}{|Z_c + Z_{in}|^2}$$

FINAL.
problems similar to quizzes
basis ideas behind antenna array; received power for an antenna system, quarter wave transformers,

Scattering matrix, ABCD matrix
Transmission matrix

Microwaves: May 8, 1972

$$V_{oc} = \frac{-\int \bar{J}_2 \cdot \bar{E}_1 \, dV}{I_{in}}$$



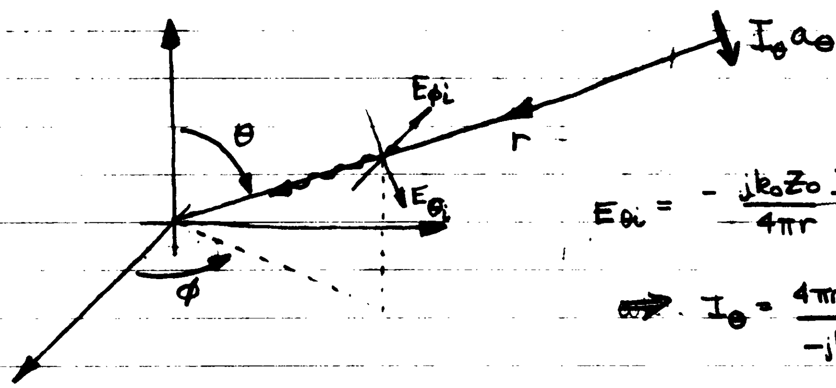
$$\bar{E} = jk_0 Z_0 \, I \, (\Delta l \sin\theta) \, \frac{e^{-jk_0 r}}{4\pi r} \, \bar{a}_\theta$$

arbitrary antenna produces a field

$$\bar{E} = jk_0 Z_0 \frac{e^{-jk_0 r}}{4\pi r} I_{in} \vec{h}$$

$\vec{h} \triangleq$ complex effective length

$$\vec{h} = h_\theta(\theta,\phi)\,\bar{a}_\theta + h_\phi(\theta,\phi)\,\bar{a}_\phi$$



$$E_{\theta i} = -\frac{jk_0 Z_0 I_\theta}{4\pi r} e^{-jk_0 r}$$

$$\Rightarrow I_\theta = \frac{4\pi r e^{-jk_0 r}}{-jk_0 Z_0 \Delta l} E_{\theta i}$$

and

$$I_\phi = \frac{4\pi r e^{-jk_0 r}}{-jk_0 Z_0 \Delta l} E_{\phi i}$$

$$V_{oc} = \vec{h} \cdot \bar{E}_i$$

$$|V_{oc_{max}}|^2 = |\vec{h}|^2 |\bar{E}_i|^2$$

$$\frac{|V_{oc}|^2}{|V_{oc_{max}}|^2} = \frac{|\vec{h} \cdot \bar{E}_i|^2}{|\vec{h}|^2 |E_i|^2} = p \triangleq \text{polarization mismatch factor}$$

modify earlier expression for received power

$$P_{rec} = (1 - |\Gamma|^2)\, P_{inc} \frac{\lambda_0^2}{4\pi} G \, p$$

Example:

let antenna radiate circularly polarized field

right circular polarization

then $\bar{h} = h(\theta,\phi)\,[\bar{a}_\theta \mp j\bar{a}_\phi]$

left circular polarization

Let $\bar{E}_i = \bar{a}_\theta E_i$ let $\Gamma = 0$

$$p = \frac{|\bar{h}\cdot E_i|^2}{|\bar{h}|^2 |E_i|^2} = \frac{|hE_i|^2}{|E_i|^2\, 2h^2} = \frac{1}{2}$$

$$\Rightarrow P_{rec} = \left[\frac{\lambda_0^2}{4\pi} G P_{inc}\right]\frac{1}{2}$$

left circular polarization

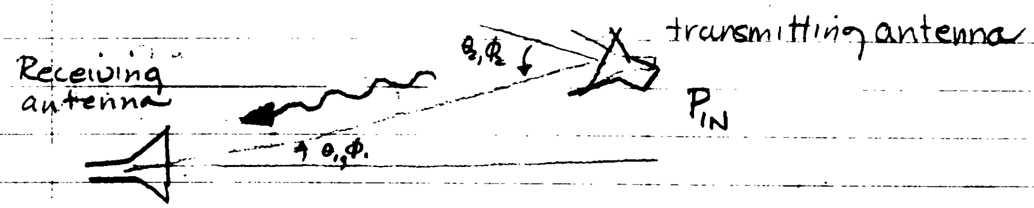
Let incident field by $\bar{E}_i = E_i\,(\bar{a}_\theta \mp j\bar{a}_\phi)$

right circular polarization

Let $\bar{h} = h(\bar{a}_\theta - j\bar{a}_\phi)$

$$\bar{E}_i = E_i(\bar{a}_\theta - j\bar{a}_\phi)$$

$$\bar{h}\cdot\bar{E}_i = hE_i - hE_i \quad \therefore\ p = 0$$



$P_{TOT} = \frac{(k_0 \Delta z)^2 I I^* Z_0}{12\pi}$

$$\underbrace{\frac{P_{IN}}{4\pi r^2}}_{\text{radial}}\ \underbrace{G(\theta_2,\phi_2)}_{\text{gain in direction of receiving antenna}} = P_{INC} \quad \text{incident power on the receiving antenna}$$

$$P_{rec} = \left[1 - |\Gamma_1|^2\right] \frac{\lambda_0^2}{4\pi} G_1(\theta_1,\phi_1)\ P_{IN} \frac{G_2(\theta_2,\phi_2)}{4\pi r^2}\ p$$