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$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot \bar{D} = \rho$$

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P}$$

$$\bar{B} = \mu_0 \bar{H} + \mu_0 \bar{M}$$

$$e^{j\omega t} = \cos\omega t + j \sin\omega t$$

$$\bar{\mathcal{E}}(\bar{r},t) = \mathrm{Re}\, \bar{E}(\bar{r}) e^{j\omega t}$$

$$\bar{E}(\bar{r}) = \bar{a}_x \left(E_{xr}(r) + jE_{xi}(r)\right) + \bar{a}_y \left(E_{yr}(r) + jE_{yi}(r)\right) + \dots$$

$$E_x = |E_x| e^{j\phi_x}$$

Thus

$$\nabla \times \bar{E} = -j\omega\mu \bar{H}$$

$$\nabla \times \bar{H} = j\omega\epsilon \bar{E}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot \epsilon \bar{E} = \rho$$

$$-j\bar{k} \times \bar{E} = -j\omega\mu\bar{H}$$

$$-j\bar{k} \times \bar{H} = j\omega\epsilon\bar{E}$$

$$-j\bar{k} \cdot \bar{B} = 0$$

$$-j\bar{k} \cdot \epsilon\bar{E} = \rho$$ where $\rho = 0$ for free space

plane waves

$$\bar{E} = \bar{E}_0 e^{-jk_x x - jk_y y - jk_z z}$$

$$\bar{\mathcal{E}} = \mathrm{Re}\, \vec{E}_0 e^{-j\bar{k}\cdot\bar{r} + j\omega t}$$

$$= \bar{E}_0 \cos(\omega t - \bar{k}\cdot\bar{r})$$

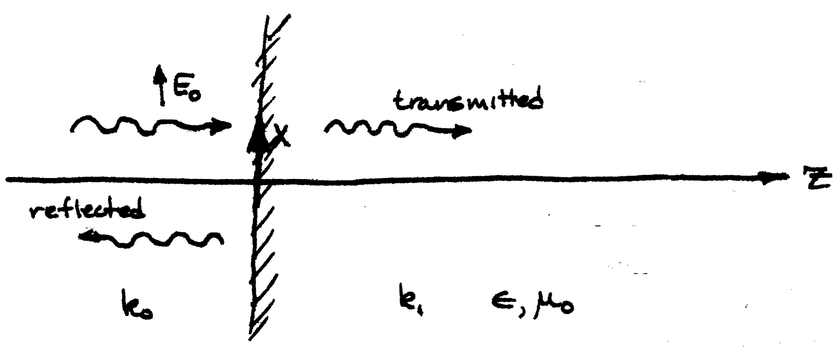
$$= \bar{E}_0 \cos\Big(t - \underbrace{\frac{\bar{k}\cdot\bar{r}}{\omega}}_{\text{time delay}}\Big)$$

$$-j\bar{k}\cdot\epsilon\bar{E}=0 \Rightarrow \bar{k}\cdot\bar{E}_o=0$$
$$-j\bar{k}\cdot\bar{B}=0 \Rightarrow \bar{k}\cdot\bar{H}_o=0$$

$$-j\bar{k}\times\bar{E}=-j\omega\mu_o\bar{H}_o \Rightarrow \bar{k}\times\bar{E}_o=\omega\mu_o\bar{H}_o$$
$$-j\bar{k}\times\bar{H}=j\omega\epsilon_o\bar{E}_o \Rightarrow \bar{k}\times\bar{H}_o=\omega\epsilon_o\bar{E}_o$$

$$\bar{k}\times(\bar{k}\times\bar{E}_o)=\omega\mu_o(\bar{k}\times\bar{H}_o)$$
$$=\omega\mu_o(-\omega\epsilon_o)\bar{E}_o$$
$$(\bar{k}\times\bar{E}_o)\bar{k}_o-k^2\bar{E}_o=-\omega^2\mu_o\epsilon_o\bar{E}_o$$
$$k=\omega\sqrt{\mu_o\epsilon_o}=\frac{\omega}{c}$$

$$\bar{k}=\frac{\omega}{c}\bar{a}_x=k_o\bar{a}_x$$



$$\bar{E}_i=E_o\,\bar{a}_x e^{-jk_o z} \qquad z<0$$
$$\bar{H}_i=H_o\,\bar{a}_y e^{-jk_o z} \qquad z<0$$
$$=\frac{E_o}{Z_o}\bar{a}_y e^{-jk_o z}$$

Note: $\bar{k}\times\bar{E}_o=\omega\mu_o\bar{H}_o$

$$\bar{H}_o=\frac{\bar{k}\times\bar{E}_o}{\omega\mu_o}=\frac{\frac{\bar{k}}{k_o}\times\bar{E}_o}{\frac{\omega\mu_o}{\omega\sqrt{\mu_o\epsilon_o}}}=\frac{(\frac{\bar{k}}{k_o})\times\bar{E}_o}{\sqrt{\frac{\mu_o}{\epsilon_o}}}$$

$$\therefore Z_o\triangleq\sqrt{\frac{\mu_o}{\epsilon_o}}$$ the intrinsic impedance of free space

transmission coefficent

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$$\bar{E}_t = TE_0 e^{-jk_1 z} \quad \text{where } k_1 = \omega\sqrt{\mu_0 \epsilon}$$

$$\bar{H}_t = \frac{TE_0}{Z_1} e^{-jk_1 z}, \quad Z_1 = \sqrt{\frac{\mu_0}{\epsilon_1}}$$

$$E_r = RE_0 \, e^{-jk_0 z}$$

reflection coefficent

$$H_r = -\frac{RE_0}{Z_0} e^{-jk_0 z}$$

boundary conditions:

tangential electric fields must be continuous across boundary

$$E_0 + RE_0 = TE_0 \implies 1 + R = T$$

$$\frac{E_0(1-R)}{Z_0} = \frac{TE_0}{Z_1} \implies 1 - R = T\left(\frac{Z_0}{Z_1}\right)$$

$$\frac{1+R}{1-R} = \frac{Z_1}{Z_0} \implies R = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

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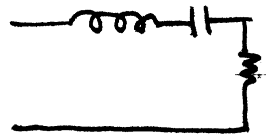
POYNTING'S VECTOR



$v = \mathrm{Re}\, V e^{j\omega t}$

$$I = \frac{V}{R + j\omega L + \frac{1}{j\omega C}}$$

complex input power is $P = \frac{1}{2} V I^* = \frac{1}{2} I I^* Z$

$$P = \frac{1}{2} I I^* R + j\omega \left[\frac{1}{2} L I I^* - \frac{1}{2} \frac{I I^*}{(\omega C)^2} C \right]$$

$\frac{1}{2} \mathrm{Re}\, V I^* =$ real power input

$\frac{1}{2} \mathrm{Im}\, V I^* =$ reactive input power

$$\frac{1}{2} I I^* R + 2j\omega \left[\underbrace{\frac{1}{4} L I I^*}_{\omega_m} - \underbrace{\frac{1}{4} \frac{I I^*}{(\omega C)^2} C}_{\omega_e} \right]$$

Thus $\frac{1}{2} \mathrm{Im}\, V I^* = 2j\omega \left[\omega_m - \omega_e \right]$

$$\nabla \times \bar{E} = -j\omega\mu \bar{H}$$
$$\nabla \times \bar{H} = j\omega\epsilon \bar{E} + \bar{J} \qquad \nabla \times \bar{H}^* = -j\omega\epsilon \bar{E}^* + \bar{J}^*$$

$$\nabla \cdot [\bar{E} \times \bar{H}^*] = (\nabla \times \bar{E}) \cdot \bar{H}^* - \bar{E} \cdot (\nabla \times \bar{H}^*)$$
$$= -j\omega\mu \bar{H} \cdot \bar{H}^* + j\omega\epsilon \bar{E} \cdot \bar{E}^* - \bar{E} \cdot \bar{J}^*$$

$$\underbrace{\frac{1}{2} \oint \bar{E} \times \bar{H}^* \cdot \bar{n}\, dS}_{\text{power flow into volume}} = \int_V \left[2j\omega \left(\frac{\mu \bar{H} \cdot \bar{H}^*}{4} - \frac{\epsilon \bar{E} \cdot \bar{E}^*}{4} \right) + \frac{1}{2} \bar{E} \cdot \bar{J}^* \right] dV$$

complex Poynting vector
$$\begin{cases} \frac{1}{2} E \times \bar{H}^* = \text{avg. complex power}/m^2 \\ \frac{1}{2} \mathrm{Re}\, \bar{E} \times \bar{H}^* = \text{avg. real power}/m^2 \end{cases}$$

Skin effect:

$$\nabla \times \overline{H}^* = j\omega\epsilon \overline{E} + \overline{J}$$
$$= (j\omega\epsilon + \sigma)\ \overline{E}$$
$$\approx \sigma E$$

$J = \sigma \overline{E}$ for conductor

$\sigma \approx 10^7 \frac{\text{mho}}{m}$ for metals

$\epsilon \approx 10^{-11}$

$\Rightarrow$ for $\omega \leq 10^{15}$ $\quad \omega\epsilon \ll \sigma$

$$\nabla \times \overline{E} = -j\omega\mu H$$
$$\nabla \times [\nabla \times \overline{E}] = -j\omega\mu\sigma \overline{E} = \nabla \overbrace{\cancel{\nabla \cdot \overline{E}}}^{0} - \nabla^2 E = -\nabla^2 E$$

since $\nabla \cdot E = \frac{\rho}{\epsilon} = 0$

$$\nabla^2 E - j\omega\mu\sigma \overline{E} = 0$$

Assume $\overline{E} = \overline{E}_0 f(z)$

Then $\quad \frac{\partial^2 \overline{E}}{\partial z^2} - j\omega\mu\sigma \overline{E} = 0$

assume $\overline{E}_0 f(z) = \overline{E}_0 e^{-\Gamma z}$

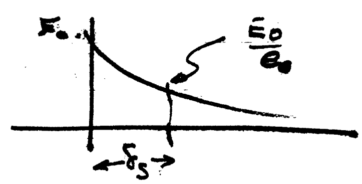
$$\Gamma^2 \overline{E}_0 e^{-\Gamma z} - j\omega\mu\sigma \overline{E}_0 e^{-\Gamma z} = 0$$

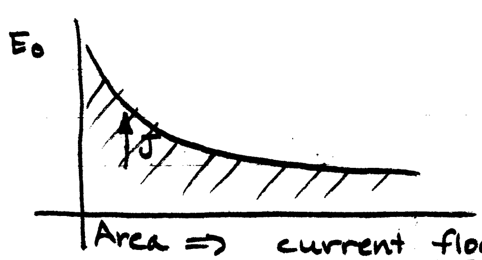
$$\Gamma^2 = j\omega\mu\sigma$$

$$\Gamma = \pm\sqrt{j\omega\mu\sigma} = \pm(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

thus $\overline{E} = \overline{E}_0 e^{-(1+j)\sqrt{\frac{\omega\mu\sigma}{2}}z}$

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}}$$





Area $\Rightarrow$ current flowing into conductor (J_s)

$$J_s = \sigma E_0 \delta_s \text{ amps/}m$$

$$|H_0| = |J_s|$$

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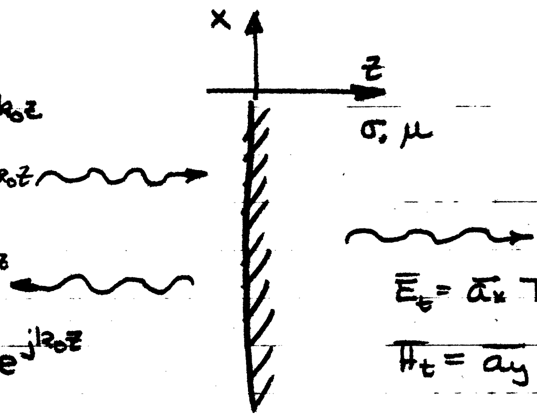
$\bar{E}_i = E_o \bar{a}_x e^{-jk_o z}$

$\bar{H}_i = \frac{E_o}{Z_o}\bar{a}_y e^{-jk_o z}$

$E_r = \Gamma E_o \bar{a}_x e^{jk_o z}$

$\bar{H}_r = \frac{\Gamma E_o}{Z_o}\bar{a}_y e^{jk_o z}$



$\bar{E}_t = \bar{a}_x T E_o e^{-\gamma z}$

$\bar{H}_t = \bar{a}_y \frac{T E_o}{Z_w} e^{-\gamma z}$

$$\nabla \times \bar{E} = -j\omega\mu \bar{H}$$
$$\nabla \times \bar{H} = j\omega\epsilon \bar{E} + \sigma \bar{E}$$
$$\text{let } \bar{E} = E_x(z)\,\bar{a}_x$$
$$\nabla \times \nabla \times \bar{E} = -j\omega\mu(j\omega\epsilon + \sigma)\bar{E}$$
$$\cancel{\nabla \nabla \cdot \bar{E}} - \nabla^2 E = \text{''}$$
$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon} = 0$$
$$\nabla^2 \bar{E} - j\omega\mu(j\omega\epsilon + \sigma)\bar{E} = 0$$
$$\frac{\partial^2 E_x(z)}{\partial z^2} - j\omega\mu(j\omega\epsilon + \sigma)E_x(z) = 0$$

in air $\sigma = 0 \quad \epsilon = \epsilon_o \quad \mu = \mu_o$

$$\text{let } \omega^2\mu_o\epsilon_o = \frac{\omega^2}{c^2} = k^2$$
$$\frac{\partial^2 E_x}{\partial z^2} + k_o^2 E_x = 0$$
$$E_x = Ae^{-jk_o z} + Be^{+jk_o z}$$
$$-j\omega\mu_o \bar{H} = \nabla \times \bar{E} = \bar{a}_y \frac{\partial E_x}{\partial z} = -jk_o(Ae^{-jk_o z} - Be^{+jk_o z})$$
$$\bar{H} = \bar{a}_y \frac{k_o}{\omega\mu_o}\left[E_o e^{-jk_o z} - \Gamma E_o e^{-jk_o z}\right]$$
$$\frac{\omega\mu_o}{k_o} = \frac{\omega\mu_o}{\omega\sqrt{\mu_o\epsilon_o}} = \sqrt{\frac{\mu_o}{\epsilon_o}} = Z_o$$

in metal
$$\frac{\partial^2 E_x}{\partial z^2} = -j\omega\mu\sigma E_x = 0$$
$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \qquad \gamma^2 = j\omega\mu\sigma$$
$$E_x = Ce^{-\gamma z} + De^{+\gamma z}$$
$$\gamma = \sqrt{j\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1+j}{\delta_o} \qquad \delta_o \triangleq \sqrt{\frac{2}{\omega\mu\sigma}}$$

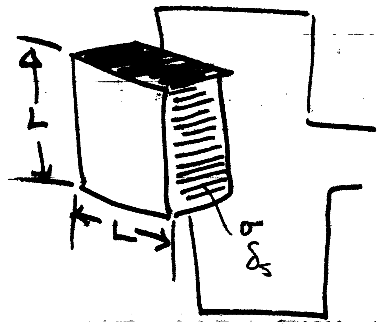
$$-j\omega\mu\bar{H} = \bar{a}_y \frac{\partial E_x}{\partial z}$$

$$\bar{H} = \frac{\gamma}{j\omega\mu} e e^{-\gamma z}$$

$$Z_m = \frac{j\omega\mu\,\delta_s}{(1+j)} = \frac{j\omega\mu}{\sqrt{j\omega\mu\sigma}}\frac{\sqrt{\sigma}}{\sqrt{\sigma}} = \frac{\sqrt{j\omega\mu\sigma}}{\sigma} = \frac{1+j}{\sigma\delta_s}$$

$$Z_m = \frac{1+j}{\sigma\delta_s}$$

example:



$$R = \frac{L}{L\delta_s\sigma} = \frac{1}{\delta_s\sigma}$$

from B.C.

$$(1+\Gamma)E_0 = TE_0$$

$$1+\Gamma = T$$

$$1-\Gamma = T\frac{Z_0}{Z_m}$$

$$\Gamma = \frac{Z_m - Z_0}{Z_m + Z_0}$$

$$T = \frac{2Z_m}{Z_m + Z_0}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi = 377\,\Omega$$

$$Z_m = \frac{1+j}{\delta_s\sigma} \qquad \left.\begin{array}{l}\delta_s \sim 10^{-6}\,m \\ \sigma \sim 5.8\times10^{7}\end{array}\right\} \text{for copper at 1 GHz.}$$

$$Z_m \sim 0.02\,(1+j)$$

Thus $\Gamma \approx -1$ which implies that most of the signal is reflected.

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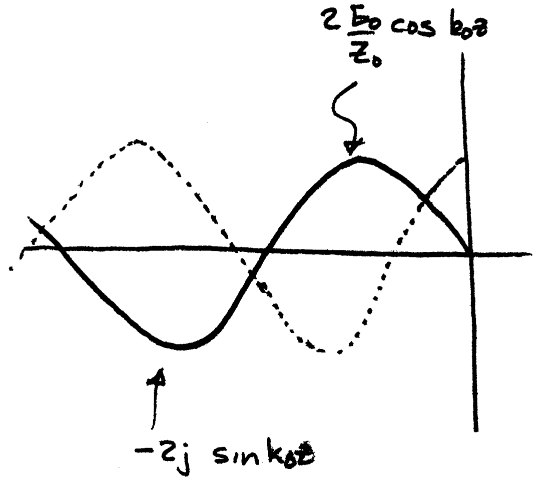
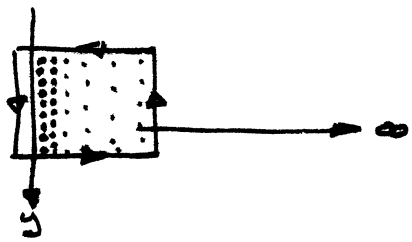


For $\Gamma = -1$

$$k_0\lambda_0 = 2\pi \;:\; k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c} = \frac{2\pi f}{c}$$



$$\oint_c \bar{H}\cdot d\ell = \int_0^\infty J_x \, dz$$

$$\frac{(1+\Gamma)E_0}{Z_0} = H_{tan} = \int_0^\infty J_x \, dz = J_s$$

$$E_{tan} = TE_0 = Z_m H_t$$

$$E_{tan} = J_s Z_m = Z_m H_{tan}$$

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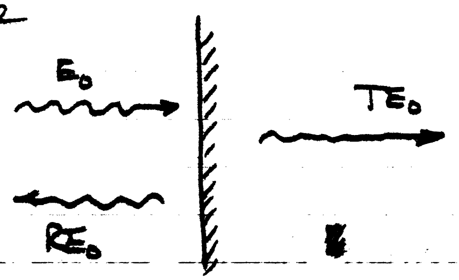


$$\Gamma = \frac{Z_m - Z_0}{Z_m + Z_0} \sim 1$$

$$\mathcal{Re}\ \tfrac{1}{2}\ \bar{E} \times \bar{H}^* \cdot \bar{a}_z = \text{power lost}/m^2 \text{ in the conductor}$$

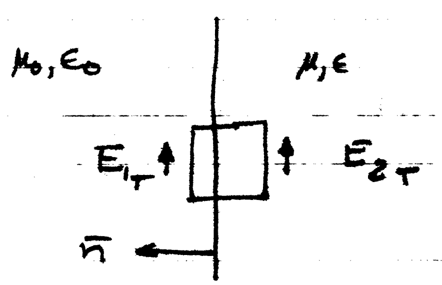
$$\mathcal{Re}\ \tfrac{1}{2} Z_m \bar{J}_s \times \bar{H}^* \cdot \bar{a}_z = \mathcal{Re}\ \tfrac{1}{2} Z_m (-\bar{a}_z \times \bar{H}) \times \bar{H}^* \cdot \bar{a}_z$$
$$= \mathcal{Re}\ \tfrac{1}{2} Z_m \bar{H} \cdot \bar{H}^* = \mathcal{Re}\ \tfrac{1}{2} \frac{\bar{J}_s \cdot \bar{J}_s^*}{Z_m}$$

1) calculate $\bar{E}$ & $\bar{H}$ for $\sigma = \infty$
then $\bar{n} \times \bar{E} = 0$ at metal surfaces
$\bar{n} \times \bar{H} = \bar{J}_s$

2) use this $\bar{H}$ in
$\tfrac{1}{2}\mathcal{Re}\ Z_m \bar{H} \cdot \bar{H}^*$ to get power loss/unit area

note: While this is the wrong $\bar{H}$ the error is negligible
$H = H_{inc} - \Gamma H_{inc} \approx 2H_{inc}$

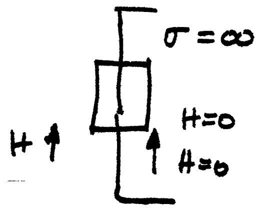
$$\oint_c \bar{E}\, d\ell = -\frac{\partial}{\partial t} \int \bar{B} \cdot d\bar{S}$$



tangential $\bar{E}$ must be continuous across the interface
$\bar{n} \times \bar{E}_1 = \bar{n} \times \bar{E}_2$

$$\oint_c \bar{H} \cdot d\ell = \frac{\partial}{\partial t} \int_s \epsilon \bar{E} \cdot d\bar{S} + \int_s \bar{J} \cdot d\bar{S}$$

$\bar{n} \times \bar{H}_1 = \bar{n} \times \bar{H}_2$ for $\sigma \approx 0$

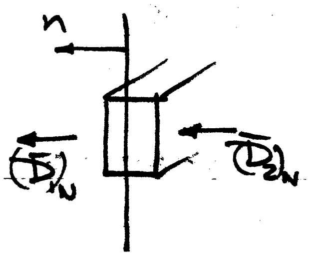


$$\bar{n} \times \bar{H} = \bar{J}_s$$

tangential $\bar{H}$ equals the current flowing on the ~~~~ surface

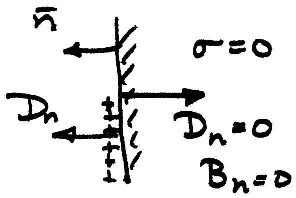
$$\nabla \cdot \bar{D} = \rho$$
$$\nabla \cdot \bar{B} = 0$$

$$\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho \, dv$$
$$\oint_S \bar{B} \cdot d\bar{s} = 0$$



$$(\bar{D}_1)_N = (\bar{D}_2)_N$$
$$(\bar{B}_1)_N = (\bar{B}_2)_N$$

i.e. $\bar{n} \cdot \bar{B}$ & $\bar{n} \cdot \bar{D}$ are continuous



$$\left.\begin{array}{l} \bar{n} \cdot \bar{B} = 0 \\ \bar{n} \cdot \bar{D} = \dfrac{\rho_s}{\epsilon_0} \end{array}\right\} \text{airside of conductor}$$

note: both are zero on a conductor

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classification of solutions

transmission lines — at least two conductors

wave guides — single conducting surfaces

let z-axis be that of line or guide

there are 3 classes of solutions or waves (modes)

1) TEM

$E_z = H_z = 0$

can exist only on transmission line down to zero frequency

2) TE or H

$E_z = 0, \quad H_z \neq 0$

3) TM or E

$E_z \neq 0, \quad H_z = 0$

exist on both transmission lines and wave guides — but frequency must be greater than f_c

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Maxwell's equations for E&M waves on transmission lines

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E}$$
$$\nabla \times \bar{E} = -j\omega\mu\bar{H}$$
$$\nabla \cdot \bar{B} = 0$$
$$\nabla \cdot \bar{D} = 0$$

$$\bar{E}(x,y,z) = \bar{E}_t(x,y,z) + \cancel{\bar{E}_z(x,y,z)} \quad 0$$

$$\bar{H} = \bar{H}_t + \bar{H}_z$$

want solutions to have $e^{\pm j\beta z}$ dependence on z

$$\bar{E}_t(x,y,z) = \bar{e}(x,y)\, e^{\pm j\beta z}$$
$$\bar{H}_t(x,y,z) = \bar{h}(x,y)\, e^{\pm j\beta z}$$

$$\nabla = \nabla_t + \bar{a}_z \frac{\partial}{\partial z} = \nabla_t \pm j\beta\bar{a}_z$$

$$\underbrace{(\nabla_t \pm j\beta\bar{a}_z) \times \bar{h}}_{\text{split into two parts}} = j\omega\epsilon\bar{e}$$

$$\nabla_t \times \bar{h} = 0 \qquad \pm j\beta\bar{a}_z \times \bar{h} = j\omega\epsilon\bar{e}$$

$$\nabla_t \times \bar{e} = 0 \qquad \pm j\beta\bar{a}_z \times \bar{e} = -j\omega\mu\bar{h}$$

since $\nabla_t \times \bar{e} = 0 \qquad \bar{e} = -\nabla_t \Phi(x,y)$

$$\nabla_t \times \nabla_t \Phi = 0 \qquad \epsilon\nabla \cdot \bar{E} = 0$$
$$\Rightarrow \nabla_t^2 \Phi = 0$$

assuming that we have found $\bar{e}$
then:

$$\bar{h} = \pm \frac{\beta}{\omega\mu} \bar{a}_z \times \bar{e} \qquad \bar{h} \perp \bar{e}$$

$$j\omega\epsilon\, \bar{a}_z \times \bar{e} = \pm j\beta\, \bar{a}_z \times (\bar{a}_z \times \bar{h}) = \pm j\beta\bar{h}$$

$$\bar{h} = \pm \frac{\omega\epsilon}{\beta} \bar{a}_z \times \bar{e}$$

$$\beta^2 = \omega^2\mu\epsilon$$
$$\beta = \omega\sqrt{\mu\epsilon}$$

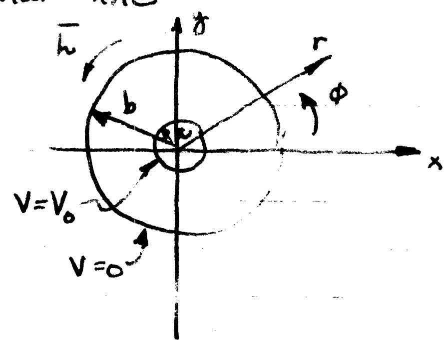
$\bar{e} = -\nabla_t \Phi$

$\nabla_t^2 \Phi = 0$

$\bar{E} = \bar{e}\, e^{\pm j\beta z} \qquad \beta = \omega\sqrt{\mu\epsilon}$

$$\bar{h} = \mp \frac{\beta}{\mu\omega}\, \bar{a}_z \times \bar{e} = \mp \frac{\bar{a}_z \times \bar{e}}{Z} \qquad Z \triangleq \sqrt{\frac{\mu}{\epsilon}}$$

coaxial line



0

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\frac{\partial \Phi}{\partial r} = \frac{C}{r}$$

$$\Phi = C \ln r + A$$

$V_0 = C \ln a + A$

$0 = C \ln b + A$

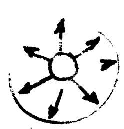
$$\Phi = \frac{V_0 \ln \frac{r}{b}}{\ln \frac{a}{b}}$$

wave in $+z$ direction

$$\bar{e} = V^+ e^{-j\beta z}\left(\frac{-\bar{a}_r}{r \ln a/b}\right) = V^+ e^{-j\beta z}\left(\frac{\bar{a}_r}{r \ln b/a}\right)$$

in $-z$ direction

$$\bar{e} = V^- e^{j\beta z}\left(\frac{-\bar{a}_r}{r \ln a/b}\right)$$



$$V = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

$$V = \int_a^b \bar{e} \cdot d\bar{l}$$

$$\bar{H} = \left(\frac{V^+}{Z} e^{-j\beta z} - \frac{V^-}{Z} e^{j\beta z}\right) \frac{\bar{a}_\phi}{r \ln \frac{b}{a}}$$

$$I = \left(\frac{V^+}{Z} e^{-j\beta z} - \frac{V^-}{Z} e^{j\beta z}\right) \frac{2\pi}{\ln b/a}$$

from $\int_0^{2\pi} h_\phi\, r\, d\phi = \oint_c \bar{h} \cdot d\bar{l}$

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$$I = \frac{V^+}{Z_c} e^{-j\beta z} - \frac{V^-}{Z_c} e^{j\beta z}$$

Characteristic impedance of line is $Z_c = \frac{Z \ln b/a}{2\pi}$

V^+ & V^- can be complex

$$V^+ = |V^+| e^{j\alpha_+}$$
$$V^- = |V^-| e^{j\alpha_-}$$

$$v(z,t) = \mathrm{Re}\, V e^{j\omega t} = |V^+| \cos(\omega t - \beta z + \alpha_+) + |V^-| \cos(\omega t + \beta z + \alpha$$

Note: $\omega t - \beta z = \omega(t - \frac{z}{c})$

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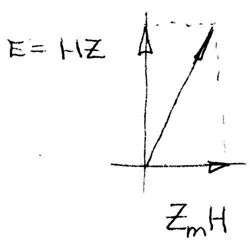
$$\bar{E} = (V^+ e^{-j\beta z} + V^- e^{j\beta z}) \frac{\bar{a}_r}{r \ln b/a}$$

$$\bar{H} = (V^+ e^{-j\beta z} - V^- e^{j\beta z}) \frac{\bar{a}_\phi}{Z\, r \ln b/a}$$

$\bar{E} = \bar{J}_s Z_m$; TEM wave solution valid only for $Z_m = 0$, $\sigma = \infty$

$$J_s = H_{tan}$$



$Z_m \ll Z$

consider a single wave

$$\bar{E} = V^+ e^{-j\beta z} \frac{\bar{a}_r}{r \ln b/a}$$

$$\bar{H} = \frac{V^+}{Z} e^{-j\beta z} \frac{\bar{a}_\phi}{r \ln b/a}$$

Note: $\bar{J}_s = \bar{n} \times \bar{H}$

at $r = a$ $\quad \cancel{J_s = V^+ e^{-j\beta z} \frac{\bar{a}_r}{r \ln b/a}}$ $\quad J_s = \dfrac{V^+ e^{-j\beta z}}{Z a \ln b/a}$

$r = b$ $\quad J_s = \dfrac{V^+ e^{-j\beta z}}{Z b \ln b/a}$

1) equivalent to taking the Poynting vector of power into the line
2) P_ℓ represents the power flowing into the wall (heat, etc.)

$$P_\ell = \frac{1}{2} \mathrm{Re}\, Z_m \left[\int_0^{2\pi} |J_s(a)|^2 a\, d\phi + \int_0^{2\pi} |J_s(b)|^2 b\, d\phi \right]$$

$$P_\ell = \frac{|V^+|^2}{Z \ln b/a} \left(\frac{\pi}{b \sigma \delta_s} + \frac{\pi}{a \sigma \delta_s} \right) \quad \text{watt}/m^2$$

for σ finite

$$\bar{E} = \bar{e} V^+ e^{-j\beta z - \alpha z}$$

where α = attenuation factor

$P = P_0 e^{-2\alpha z}$ = power flowing along the line P_0 = power flow @ $z=0$

$$-\frac{dP}{dz} = P_e = 2\alpha P$$

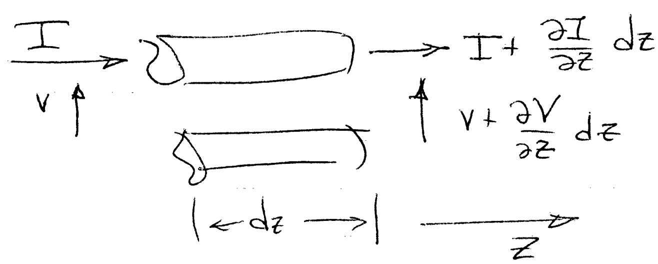
Note! power loss is twice that of the wave attenuation

$$\alpha = \frac{P_e}{2P}$$

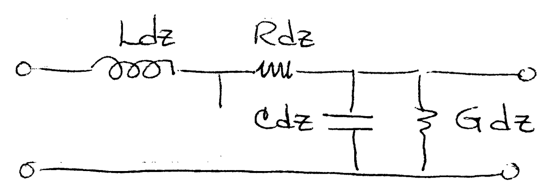
$$P = \frac{1}{2}\mathrm{Re}\int_a^b \int_0^{2\pi} E_r H_\phi^* \, r\,d\phi\,dr = \frac{|V^+|^2}{2Z_c} = \frac{|V^+|^2}{2\left(\frac{Z \ln b/a}{2\pi}\right)}$$

1) solve for $\sigma = \infty$
2) assume solution acceptable for finite σ
3) calculate power dissipated per unit length (find J_s)
4) attenuation exists
5) Set up power balance to find α

Note: for an air gap $\beta = k_0$



model line as



then $I - \left(\frac{\partial I}{\partial z}dz + I\right) = V\,(j\omega C + G)\,dz$

$$= -\frac{\partial I}{\partial z}\,dz$$

$$\Rightarrow \quad \frac{\partial I}{\partial z} = -(j\omega C + G)V$$

$$V - \left(\frac{\partial V}{\partial z}dz + V\right) = I\,(j\omega L + R)$$

$$\frac{\partial V}{\partial z} = -(j\omega L + R)\,I$$

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to solve

$$\frac{d^2V}{dz^2} = (j\omega L + R)(j\omega C + G)V = \gamma^2 V$$

$$\gamma^2 \triangleq (j\omega L + R)(j\omega C + G)$$

then $V = V^+ e^{-\gamma z} + V^- e^{\gamma z}$

for a perfect line

$R = 0 \; ; \; G = 0$

then $\gamma^2 = -\omega^2 LC$

$\gamma = j\omega\sqrt{LC} = j\beta$

if $V = V^+ e^{-j\beta z} + V^- e^{j\beta z}$

then $$I = \frac{1}{-j\omega L}\left(-j\beta V^+ e^{-j\beta z} + j\beta V^- e^{j\beta z}\right)$$

$$= \frac{1}{\sqrt{L/C}}\left[V^+ e^{-j\beta z} - V^- e^{j\beta z}\right]$$

where $Z_c \triangleq \sqrt{\frac{L}{C}}$

if one makes the calculations $\omega\sqrt{LC} = \omega\sqrt{\mu\epsilon}$

for a coaxial cable

$$Z_c = \sqrt{\frac{L}{C}} = \frac{Z \ln b/a}{2\pi}$$

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for a single wave

$$V = V^+ e^{-\gamma z} = V^+ e^{-j\beta z - \alpha z}$$

$$I = \frac{V^+}{Z_c} e^{-j\beta z - \alpha z}$$

$$\gamma^2 = (j\beta + \alpha)^2 = (j\omega L + R)(j\omega C + G)$$

$$Z_c = \sqrt{\frac{j\omega L + R}{j\omega C + G}}$$

for lossless line $\gamma = j\beta = j\omega\sqrt{LC}$

$$Z_c = \sqrt{\frac{L}{C}}$$

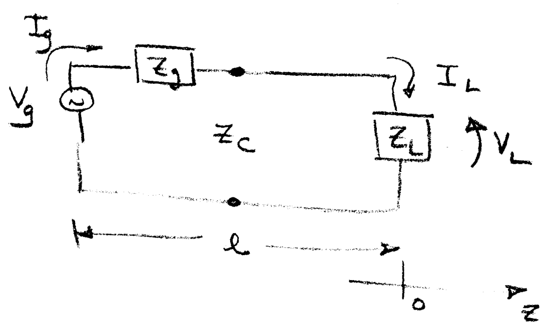
for good line $\omega L \gg R$, $\omega C \gg G$

$$Z_c \approx \sqrt{\frac{L}{C}}$$

$$\gamma^2 = -\omega^2 LC \left[1 + \frac{R}{j\omega L}\right]\left[1 + \frac{G}{j\omega C}\right]$$

$$\gamma = j\omega\sqrt{LC}\left[1 + \frac{R}{j2\omega L}\right]\left[1 + \frac{G}{j2\omega C}\right] = j\omega\sqrt{LC} + \sqrt{LC}\left[\frac{R}{2L} + \frac{G}{2C}\right]$$

terminated line



$$V = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I = \frac{V^+}{Z_c} e^{-j\beta z} - \frac{V^-}{Z_c} e^{j\beta z}$$

@ $z = 0$ $\quad V = V^+ + V^- = V_L$

$$I = \frac{V^+ - V^-}{Z_c} = I_L$$

then $$Z_L = \frac{V_L}{I_L} = \frac{V^+ + V^-}{V^- - V^-} Z_c$$

$$= \frac{1 + \frac{V^-}{V^+}}{1 - \frac{V^-}{V^+}} Z_c$$

where $\left.\frac{V^-}{V^+}\right|_{@\ load} = \Gamma_L =$ reflection coefficient of load

$$\frac{1+\Gamma_L}{1-\Gamma_L} = \frac{Z_L}{Z_c} = \bar{Z}_L \text{ (normalized load impedance)}$$

$$\Gamma_L = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{\bar{Z}_L - 1}{\bar{Z}_L + 1}$$

note: $\Gamma_L = 0$ if $Z_L = Z_c$ [matched termination]

@ $z = -l$

$$V = V^+ e^{j\beta l} + \Gamma_L V^+ e^{-j\beta l} = V_g - I_g Z_g$$

$$I = \frac{V^+ e^{j\beta l}}{Z_c} - \frac{\Gamma_L V^+}{Z_c} e^{-j\beta l} = I_g$$

can be solved for I_g and V^+

$$\left.\frac{V^-}{V^+}\right|_{at\ load} = \Gamma_L \qquad \frac{1+\Gamma_L}{1-\Gamma_L} = \frac{Z_L}{Z_c} = \frac{1}{Z_c} \frac{V_L}{I_L}$$

@ $z = -d$

$$\frac{V^- e^{-j\beta d}}{V^+ e^{+j\beta d}} = \Gamma(d) = \Gamma_L e^{-j2\beta d}$$

$Z_{in}(d) \Rightarrow$ impedance seen looking toward load at $z = -d$

$$Z_{in}(d) = \frac{V(d)}{I(d)} = \frac{V^+ e^{j\beta d} + V^- e^{-j\beta d}}{\frac{V^+ e^{j\beta d}}{Z_c} - \frac{V^- e^{-j\beta d}}{Z_c}} = \frac{1+\Gamma(d)}{1-\Gamma(d)} Z_c$$

$$Z_{in}(d) = \frac{1+\Gamma_L e^{-2j\beta d}}{1-\Gamma_L e^{-2j\beta d}}$$

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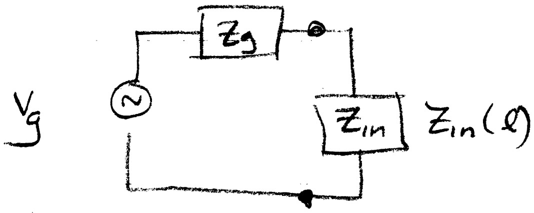
if we substitute $\Gamma_L = \dfrac{Z_L - Z_c}{Z_L + Z_c}$

$$Z_{in}(d) = Z_c \frac{Z_L + jZ_c \tan\beta d}{Z_c + jZ_L \tan\beta d}$$



$$I_g = \frac{V_g}{Z_g + Z_{in}(\ell)}$$

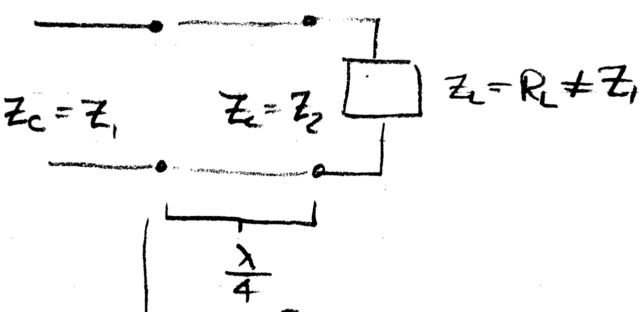
$$V = \frac{Z_{in}}{Z_{in} + Z_g} V_g$$

then $V^+ = \dfrac{V_g Z_{in}}{Z_{in} + Z_g} \left(\dfrac{1}{e^{+j\beta\ell} + \Gamma_\ell e^{-j\beta\ell}} \right)$

for $\ell = \dfrac{\lambda}{4}$ $\quad \beta = \dfrac{\omega}{v} = \omega\sqrt{\mu\epsilon} = \dfrac{2\pi}{\lambda}$

then $e^{-j\beta z}$ $\quad \beta\ell = \beta\dfrac{\lambda}{4} = \dfrac{\pi}{2}$

and $Z_{in}\left(\dfrac{\lambda}{4}\right) = \dfrac{Z_c^2}{Z_L}$ (quarterwave transformation)



$$Z_{in} = \frac{Z_2^2}{Z_L} = Z_1$$

$$Z_2 = \sqrt{R_2 Z_1}$$

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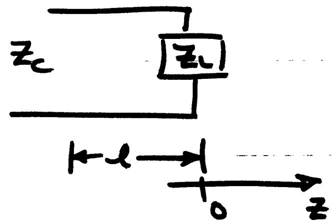


$$V = V^{+} e^{-j\beta z} + V^{-} e^{j\beta z}$$

$$V^{-} = \Gamma_L V^{+}$$

$$\Gamma_L = \frac{Z_L - Z_c}{Z_L + Z_c}$$

$$\Gamma(l) = \Gamma_L e^{-2j\beta l}$$

$$Z_{in} = \frac{1+\Gamma}{1-\Gamma} = Z_c \frac{Z_L + j Z_c \tan \beta l}{Z_c + j Z_L \tan \beta l}$$

for $l = \frac{\lambda}{4}$ $\quad Z_{in} = \frac{Z_c^2}{Z_L}$

$l = \frac{\lambda}{2}$

$Z_{in} = Z_L$

$\Gamma_L = \rho e^{+j\theta}$

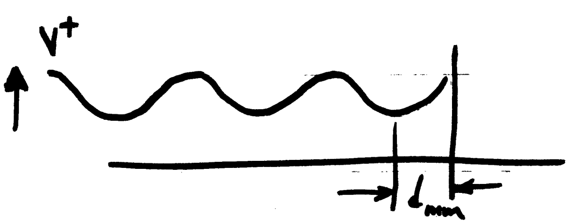
note: the input impedance will repeat every $\frac{\lambda}{2}$ down the line

then $V = V^{+}(e^{-j\beta z} + \rho e^{j\theta + j\beta z}) = V^{+} e^{-j\beta z}(1 + \rho e^{j\theta + 2j\beta z})$

thus $V_{max} = |V^{+}| \, (1+\rho)$

$V_{min} = |V^{-}| \, (1-\rho)$

$$\frac{V_{max}}{V_{min}} = S = \frac{1+\rho}{1-\rho}$$ Voltage Standing Wave Ratio (VSWR)



$$\theta \pm 2\beta d_{min} = \pi \pm n 2\pi$$

$$\theta = \pi - 2\beta d_{min} \pm 2n\pi$$

$$\Gamma_L = \frac{S-1}{S+1} e^{-j2\beta d_{min} + j\pi}$$

$$= \frac{1-S}{1+S} e^{-j2\beta d_{min}}$$

$$Z_L = \frac{1+\Gamma_L}{1-\Gamma_L}$$

at a voltage min.

$$\left.\begin{aligned} V &= V^+(1-\rho) \\ I &= \frac{V^+}{Z_c}(1+\rho) \end{aligned}\right\} Z_{in} = \frac{V}{I} = \frac{Z_c}{S}$$

at a voltage max:

$$Z_{in} = Z_c S$$

finding Z_{in} $(=Z_L)$ at a distance d from a voltage min

$$Z_L = \frac{\frac{Z_c}{S} - jZ_c \tan \beta d_{min}}{Z_c - j\frac{Z_c}{S} \tan \beta d_{min}}$$

Note: sign change due to trig function (-d instead of d)

$$Z_L = Z_c \frac{1 - Sj \tan \beta d_{min}}{S - j \tan \beta d_{min}}$$

in a lossy line

$$\Gamma(\ell) = \Gamma_L e^{-2j\beta\ell - 2\alpha\ell}$$

Note: $\Gamma(\ell)$ decays as one moves away from the load

$$V = V^+ e^{-2j\beta z - \alpha z} + V^- e^{2j\beta z + \alpha z}$$

and

$$Z_{in} = \frac{Z_L + Z_c \tanh (j\beta + \alpha)\ell}{Z_c + Z_L \tanh (j\beta+\alpha)\ell}$$