#1

ENGR 336
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1. (a) Using complex Poynting's theorem find the complex admittance of a system. Use an equivalent parallel RLC circuit as a model.

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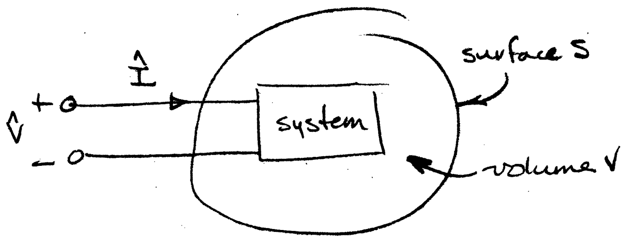
$$-\nabla \cdot \hat{\underline{P}} = \frac{1}{2}\hat{\underline{E}}\cdot\hat{\underline{J}}^* + 2j\omega\left[\frac{1}{4}\mu|\hat{\underline{H}}|^2 - \frac{1}{4}\epsilon|\hat{\underline{E}}|^2\right] \quad (1)$$



complex power input to the system

$$Y = \frac{\hat{I}}{\hat{V}} = \frac{\hat{I}\hat{V}^*}{|\hat{V}|^2} = 2\frac{\frac{1}{2}\hat{I}\hat{V}^*}{|V|^2} = 2\frac{\left(-\oint_S \vec{P}\cdot d\vec{s}\right)^*}{|V|^2} \quad (2)$$

integrating eq (1) over the volume V and using Gauss' theorem

$$-\oint_S \vec{P}\cdot d\vec{s} = \int_{vol}\frac{\hat{E}\cdot\hat{J}^*}{2}dv + 2j\omega\int_{vol}\left[\frac{1}{4}\mu|\hat{H}|^2 - \frac{1}{4}\epsilon|E|^2\right]dv \quad (3)$$

$$\left(\frac{1}{2}\hat{I}\hat{V}^*\right) = \langle P_d\rangle + 2j\omega\left[\langle u_m\rangle - \langle u_e\rangle\right] \quad (4)$$

where $\langle P_d\rangle \triangleq \int_{vol}\frac{\hat{E}\cdot\hat{J}^*}{2}dv$ average power dissipated (5)

$\langle u_m\rangle \triangleq \int_{vol}\frac{1}{4}\mu|\hat{H}|^2$ average stored magnetic energy (6)

$\langle u_e\rangle \triangleq$ average stored electric energy (7)

Note that $-\oint_S \hat{P}\cdot ds = \frac{1}{2}\hat{V}\hat{I}^*$ (8)

$$\Leftrightarrow \quad \frac{1}{2}\hat{I}\hat{V}^* = \left(\frac{1}{2}\hat{V}\hat{I}^*\right)^* = \left\{\langle P_d\rangle + 2j\omega\left[\langle u_m\rangle - \langle u_e\rangle\right]\right\}^* \quad (9)$$

$$= \langle P_d\rangle + 2j\omega\left[\langle U_e\rangle - \langle U_m\rangle\right] \quad (10)$$

good

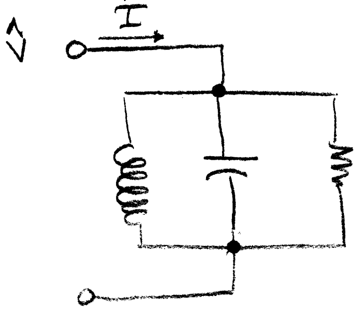
Substituting eq(10) into eq(2) we obtain

u - energy density

U - energy (11)

$\int_{vol} u\, dv$

$$Y = \frac{2}{|V|^2}\left[\langle P_d\rangle + 2j\omega\left[\langle u_e\rangle - \langle u_m\rangle\right]\right]$$



$$\langle u_m \rangle = \frac{VV^*}{4\omega^2 L} \quad (12) \qquad \langle u_e \rangle = \frac{CVV^*}{4} \quad (13) \qquad \langle P_d \rangle = \frac{VV^*}{2R} \quad (14)$$

using (12)(13)(14) in (11) we obtain

$$\boxed{Y = \frac{2}{|V|^2}\left[\frac{|V|^2}{2R} + 2j\omega\left[\frac{C|V|^2}{4} - \frac{|V|^2}{4\omega^2 L}\right]\right]} \quad (15)$$

(b) Find an equation for the admittance near the resonant frequency and determine the Q.

resonance occurs when $\frac{C|V|^2}{4} - \frac{|V|^2}{4\omega^2 L} = 0$ (16)

$$\omega_0^2 = \frac{1}{LC} \quad (17)$$

$$Y = Y(\omega_0) + (\omega-\omega_0)\left.\frac{\partial Y}{\partial \omega}\right|_{\omega=\omega_0} \quad (18)$$

$$= \frac{2}{|V|^2}\left[\frac{|V|^2}{2R} + 2j\omega_0\underbrace{\left[\frac{C|V|^2}{4} - \frac{|V|^2}{4\omega^2 L}\right]}_{0}\right] + (\omega-\omega_0)\frac{4j}{|V|^2}\left.\underbrace{\left[\frac{C|V|^2}{4} + \frac{|V|^2}{4\omega^2 L}\right]}_{\langle U_e\rangle + \langle U_m\rangle}\right|_{\omega=\omega_0} \quad (19)$$

$$Y = \frac{2}{|V|^2}\left[\frac{|V|^2}{2R}\right] + (\omega-\omega_0)\frac{4j}{|V|^2}\left[\frac{C|V|^2}{4} + \frac{|V|^2}{4\omega_0^2 L}\right] \quad (20)$$

$$\boxed{Y \simeq \frac{2}{|V|^2}\left[\frac{|V|^2}{2R} + 2j(\omega-\omega_0)\left\{\frac{C|V|^2}{4} + \frac{|V|^2}{4\omega_0^2 L}\right\}\right]} \quad (21)$$

$$\frac{|V|^2}{2R} + 2(\omega-\omega_0)\left[\frac{C|V|^2}{4} + \frac{|V|^2}{4\omega_0^2 L}\right] = 0 \quad (22)$$

$$\Delta\omega = 2(\omega_0-\omega) = \frac{\frac{|V|^2}{2R}}{\frac{C|V|^2}{4} + \frac{|V|^2}{4\omega_0^2 L}} \quad (23)$$

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\frac{\cancel{|V|^2}}{2R}} \left(\frac{C\cancel{|V|^2}}{4} + \frac{\cancel{|V|^2}}{4\omega_0^2 L} \right) \tag{24}$$

$$= 2R\omega_0 \left[\frac{C}{4} + \frac{1}{4\omega_0^2 L} \right] = RC\omega_0 \qquad \omega_0^2 = \frac{1}{LC} \tag{25}$$

$$\boxed{Q = \frac{\omega_0 RC}{2} + \frac{R}{2\omega_0 L}} \tag{26}$$

(c) Determine the change in resonant frequency for a small change in the magnetic permeability. Hint: Consider the equivalent inductance to be proportional to the permeability.

i.e. I wish to determine $\frac{\partial\omega_0}{\partial L}$

differentiating eq. (17) with respect to ω_0

$$2\omega_0 \frac{\partial\omega_0}{\partial L} = \frac{1}{C}\left(-\frac{1}{L^2}\right)$$

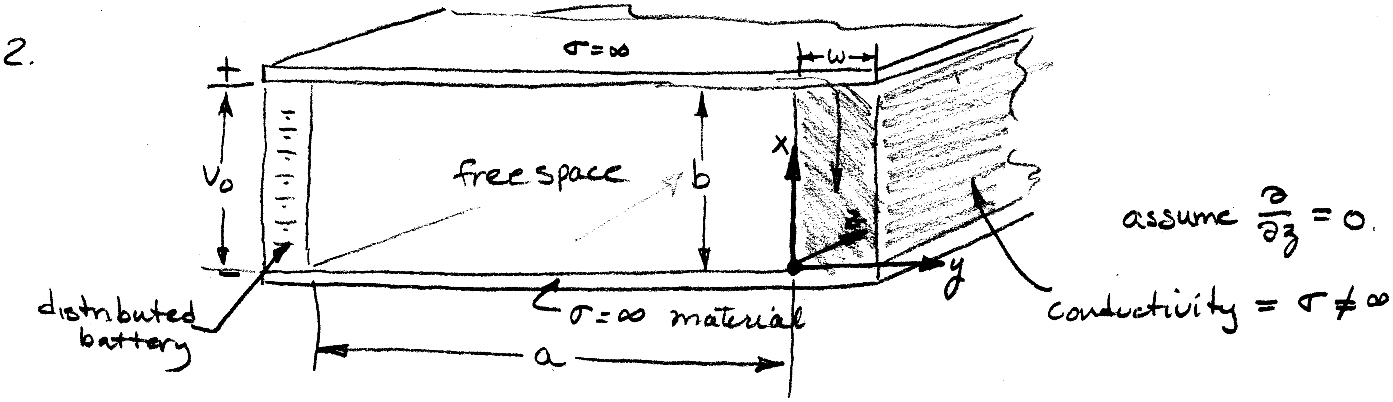
$$\frac{\partial\omega_0}{\partial L} = \frac{1}{2\omega_0}\left(-\frac{1}{L^2}\right) = -\frac{1}{2\omega_0 L^2}$$

See solutions

$$\Delta Y = \left.\frac{\partial Y}{\partial\omega}\right|_{\omega_0}(\omega-\omega_0) + \left.\left(\frac{\partial Y}{\partial L}\frac{\partial L}{\partial\mu}\right)\right|_{\omega_0}\Delta\mu \quad \text{which is set} = 0$$

$\Delta Y = 0$ since we change ω from ω_0 to return to the resonant condition ie $Y_0 + \Delta Y_{\omega_0} = 0$

2.

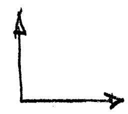


assume $\frac{\partial}{\partial z} = 0$.

Note: Work the problem per unit length in the z-direction.

a) Determine the electric & magnetic fields in the resistor (surrounding region has zero conductivity).

Neglecting fringing fields.



$$\phi = -\int E \cdot dl$$

but in $\sigma = \infty$ material there can be no electric fields

assume $\bar{E} = -E_x \bar{i}_x$

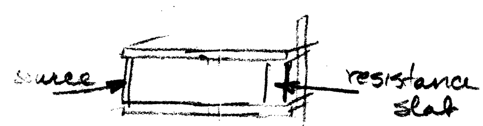
$$\therefore \quad \phi = V_o = -\int_b^0 (-E_x \bar{i}_x) \cdot (-i_x dx) = \int_0^b E_x\, dx$$

$$V_o = E_x b$$

$$E_x = \frac{V_o}{b}$$

$$\boxed{\bar{E} = -\frac{V_o}{b}\bar{i}_x}$$

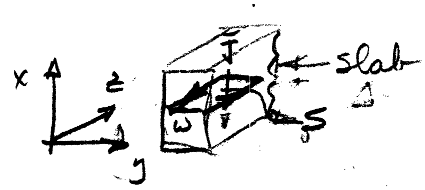
As this is a static problem



$$\oint_c H \cdot dl = \int_S \bar{J} \cdot d\bar{S}$$

pick a surface $S \perp (\bar{a}_z \times \bar{a}_y)$

$$\bar{J} = \sigma \bar{E} = -\sigma \frac{V_o}{b} \bar{i}_x$$



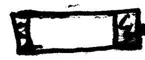
$$d\bar{S} = w\,dz\, \bar{i}_x$$

$$\int_S \bar{J} \cdot d\bar{S} = \int_0^{\Delta z} -\sigma \frac{V_o}{b} i_x \cdot w\,dz\, i_x = -\frac{\sigma V_o w \Delta z}{b}$$

However, pick a new coordinate y' such that $y' = y - \frac{w}{2}$

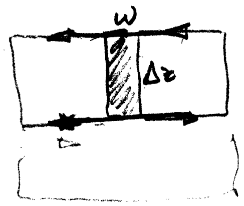
then $\int_S \bar{J} \cdot d\bar{S} = f(y') = -\frac{\sigma V_o k \Delta z}{b} \qquad |k| \leq \frac{w}{2}$

Because of the symmetry of the problem there can be no Y component of H. Furthermore $|H_z|$ must be the same on both sides of the resistor. (No!)

 this is an entirely closed loop of current

$$\therefore \quad \oint_c H \cdot dl = 2 H_z \Delta z \qquad \text{where} \quad \bar{H} = +H_z \bar{a}_z$$

(like a solenoid) and the field gets its interior only – outside $E = H = 0$

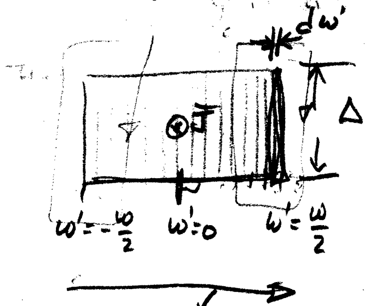


$$2 H_z \Delta z = -\frac{\sigma V_0 w \Delta z}{b}$$

$$H_z = -\frac{\sigma V_0 w}{2b}$$

good try

Now H_{tan} must be discontinuous by the amount of surface current. Pick the surface of length Δz.



for each surface of thickness dw' the discontinuity in H_z is:

$$|\bar{J}| \frac{dw' \Delta z}{\Delta z} = |\bar{J}| \, dw'$$

or $\quad dH = |\bar{J}| \, dw' = -\frac{\sigma V_0}{b} dw'$

dw'

$$\frac{dH}{dw'} = -\frac{\sigma V_0}{b}$$

$$H = -\frac{\sigma V_0}{b} w'$$

$\Rightarrow$ the magnetic field in the resistor is given by

$$\boxed{\bar{H} = -\frac{\sigma V_0}{b}\left(y - \frac{w}{2}\right) \bar{a}_z \qquad 0 \le y \le w}$$

(b) Show the solution for the potential within the space between the battery and the resistor is a solution of Laplaces equation.

$$\bar{E} \triangleq -\nabla\phi$$

$$\nabla\cdot\bar{E} = -\nabla\cdot\nabla\phi = -\nabla^2\phi$$

but $\nabla\cdot\epsilon\bar{E} = \rho$

or $\nabla\cdot\bar{E} = \frac{\rho}{\epsilon}$

$\therefore$ $\frac{\rho}{\epsilon} = -\nabla^2\phi$

$\nabla^2\phi = -\frac{\rho}{\epsilon}$ but there are no charges in this region of space, i.e. $\rho = 0$

$\Rightarrow \nabla^2\phi = 0$ (Laplace's equation) describes this region.

(c) Find and sketch the solution for potential and electric field within the space.

Because $\bar{E}$ is continuous across the boundary of the resistor

$\bar{E} = -\frac{v_0}{b}\bar{a}_x$ in the space

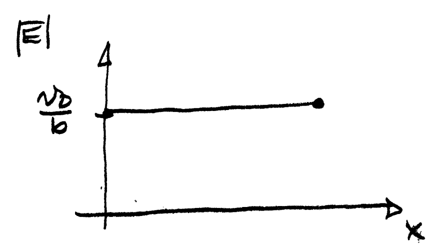
$$\phi = -\int_0^x E\cdot d\ell = -\int_0^x -\frac{v_0}{b}\bar{a}_x\cdot dx'\,\bar{a}_x = \int_0^x \frac{v_0}{b}dx'$$

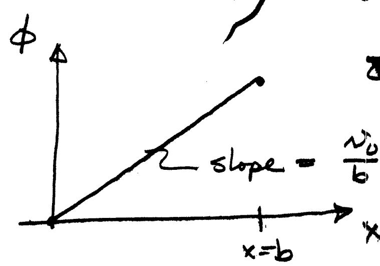
$\phi = \frac{v_0 x}{b}$; $\phi(0) = 0$

$\phi(b) = v_0$

$\therefore$ Laplaces equation is satisfied.

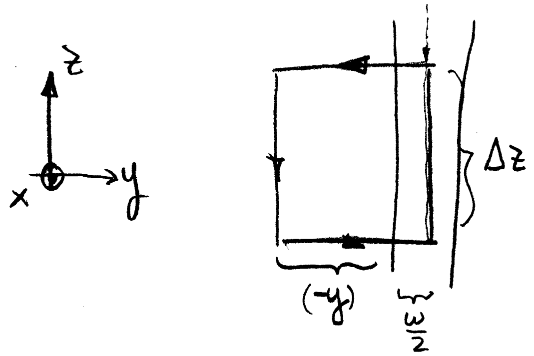
why is it satisfied within the free space region? It is with your solution but you only assumed the E-field. In a more complicated problem you could not have done this.





Note: E_y and $E_z \equiv 0$

(d) Find and sketch the magnetic field within the space.



$$\oint H \cdot dl = \int_S \bar{J} \cdot d\bar{S}$$

because of symmetry and the choise of the contours

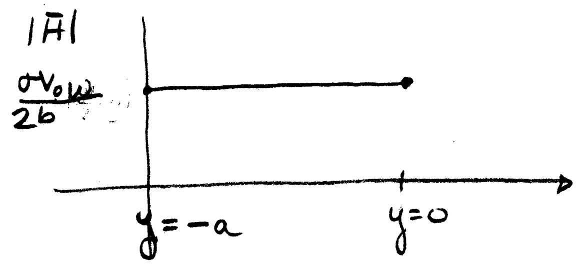
$$\oint H \cdot dl = -H_z \Delta z$$

$$\oint_S \bar{J} \cdot d\bar{S} = -\sigma \frac{V_0}{b} i_x \cdot \left(\frac{w}{2}\right) \Delta z \bar{a}_x$$

$$\Leftrightarrow \quad -H_z \cancel{\Delta z} = -\frac{\sigma V_0}{b}\left(\frac{w}{2}\right)\cancel{\Delta z}$$

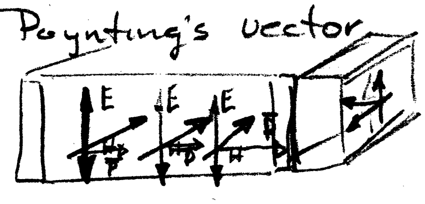
$$H_z = \frac{\sigma V_0}{b} \frac{w}{2}$$

$\Rightarrow$ The H field is constant throughout the resistive slab.



y
z x

(e) Sketch Poynting's vector



x
3
y

$$\bar{P} = \bar{E} \times \bar{H} = \left(-\frac{V_0}{b}\bar{a}_x\right) \times \left(\frac{\sigma V_0 w}{2b}\bar{a}_z\right)$$

$$= +\frac{\sigma V_0^2 w}{2b^2}\bar{a}_y$$

(f) Show that Poyntings Theorem is satisfied by your results by investigating it in integral form on the resistor.

$$\oint_\Sigma \bar{P}\cdot d\bar{S} \overset{?}{=} -\int_v \frac{\bar{J}\cdot\bar{E}}{2}\,dv - \int_v \left[E\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t}\right]dv$$

FAKE! The ½ factor only enters in time varying problems.

$$\int_v \frac{\bar{J}\cdot\bar{E}}{2}\,dv = \overset{NO}{\frac{1}{2}}\left(-\frac{\sigma V_0}{b}\bar{a}_x\right)\cdot -\frac{V_0}{b}\bar{a}_x\, wb\Delta z$$

$$= \frac{\sigma V_0^2\cdot w}{2b}\,\Delta z$$

If you had used the correct poyntings theorem you would have found a factor of ½ error which resulted from your improper solution for $\bar{H}$!

$$\oint_\Sigma \bar{P}\cdot dS = \frac{1}{2}\left(\frac{\sigma V_0^2 w}{2b^2}\bar{a}_y\right)\cdot\Delta z\, b\,(-\bar{a}_y) + \frac{1}{2}\left(\frac{\sigma V_0^2 w}{2b^2}\bar{a}_y\right)\cdot\Delta z\, b\,\bar{a}_y$$

2 terms because at the $y = w$ plane there is an E field

$$\oint_\Sigma \bar{P}\cdot d\bar{S} = -\frac{\sigma V_0^2 w}{4b}\Delta z - \frac{\sigma V_0^2 w}{4b}\Delta z$$

$$= -\frac{\sigma V_0^2 w}{2b}\Delta z$$

$$\Leftrightarrow \oint_\Sigma \bar{P}\cdot d\bar{S} = -\int_v \bar{J}\cdot\bar{E}\,dv$$

(g) From Ohm's law we would expect $\frac{\text{volt}}{\text{ohm-cm}}$

$$\langle P\rangle = \frac{V^2}{2R} \quad\text{or}\quad P = \frac{V_0^2}{2R} = \frac{V_0^2}{2\rho\frac{L}{A}} = \frac{V_0^2}{\frac{2L}{\sigma A}}$$

$$P = \frac{\sigma V_0^2}{2L}A$$

$$= \sigma V_0^2\,\frac{w\Delta z}{2b}$$

∴ the field concept does expain Power flow.

3. Find Poynting's vector, electric and magnetic energy density for a uniform plane wave.

Let $\hat{\underline{E}} = \hat{\underline{E}}_o e^{-jk_o \underline{n}\cdot\underline{r}}$ $k_o = \frac{\omega}{c}$: the wave number

where $\bar{n}\cdot\bar{r}$ defines a plane perpendicular to $\bar{n}$

$\bar{n}$ is the direction of propagation of the wave

$\hat{\underline{E}}_o$ is a constant vector.

Assume in free space. i.e. $B = \mu_o H$

Then $\nabla \times \hat{\underline{E}} = -j\omega\mu_o \hat{\underline{H}}$

$$\hat{\underline{H}} = \frac{j}{\omega\mu_o} \nabla \times \hat{\underline{E}} = \frac{j}{\omega\mu_o} \nabla \times \hat{\underline{E}}_o e^{-jk_o \bar{n}\cdot\underline{r}}$$

$$= \frac{j}{\omega\mu_o} \hat{\underline{E}}_o \times \nabla e^{jk_o \bar{n}\cdot\bar{r}} = \frac{j}{\omega\mu_o} \hat{\underline{E}}_o \times \underline{\bar{n}} -jk_o\, e^{-jk_o \underline{\bar{n}}\cdot\underline{r}}$$

$$\hat{\underline{H}} = +\frac{k_o}{\omega\mu_o} \underline{E}_o \times \underline{n}\; e^{-jk_o \underline{n}\cdot\underline{r}}$$

$$\hat{\underline{P}} = \frac{1}{2}\hat{\underline{E}} \times \hat{\underline{H}}^* = \frac{1}{2}\hat{\underline{E}}_o e^{-jk_o \underline{n}\cdot\underline{r}} \times +\frac{k_o}{\omega\mu_o} \hat{\underline{E}}_o^* \times \underline{n}\, e^{+jk_o \underline{n}\cdot\underline{r}} = \frac{-k_o}{2\omega\mu_o}\left[\hat{\underline{E}}_o \times \hat{\underline{E}}_o^* \times \underline{n}\right]$$

$$= \frac{-k_o}{2\omega\mu_o}\left[(\hat{\underline{E}}_o\cdot\underline{n})\,\hat{\underline{E}}_o^* - (\hat{\underline{E}}_o\cdot\hat{\underline{E}}_o^*)\,\underline{n}\right]$$

but $\hat{\underline{E}}_o\cdot\underline{n} = 0$ why $\nabla\cdot\hat{\underline{E}} = \nabla\cdot\hat{\underline{E}}_o e^{-jk_o\underline{n}\cdot\underline{r}} = \hat{\underline{E}}_o \cdot \nabla e^{-jk_o \underline{n}\cdot r}$

$$= \hat{\underline{E}}_o\cdot(-jk_o\underline{n})\, e^{-jk_o \underline{n}\cdot\underline{r}}$$

but $\nabla\cdot\hat{\underline{E}} = 0$ in free space $\Rightarrow \hat{\underline{E}}_o \cdot n = 0$

$$\hat{\underline{P}} = +\frac{k_o}{2\omega\mu_o}|\hat{\underline{E}}_o|^2\, \underline{n}$$

i.e. power is transported along the direction of travel of the wave.

$$\langle u_m \rangle = \frac{1}{4}\mu_0 |\hat{H}|^2 = \frac{1}{4}\mu_0\, \underline{\hat{H}} \cdot \underline{\hat{H}}^*$$

$$= \frac{1}{4}\cancel{\mu_0}\left(\frac{-k_0}{\omega \cancel{\mu_0}}\right)\left(\underline{\hat{E}}_0 \times \underline{n}\, e^{-jk_0 n\cdot r}\right) \cdot \left(\frac{-k_0}{\omega\mu_0}\right)\left(\underline{\hat{E}}_0^* \times \underline{n}\right) e^{+jk_0 \underline{n}\cdot\underline{r}}$$

$$= \frac{k_0^2}{4\omega^2\mu_0}\left(\underline{\hat{E}}_0 \times \underline{n}\right)\cdot\left(\underline{\hat{E}}_0^* \times n\right)$$

$$= \frac{k_0^2}{4\omega^2\mu_0}\left[(\underline{\hat{E}}_0 \cdot \underline{\hat{E}}_0^*)(\underline{n}\cdot\underline{n}) - \cancelto{0}{(\underline{\hat{E}}_0 \cdot \underline{n})}(\underline{n}\cdot\underline{\hat{E}}_0^*)\right]$$

$$= \frac{k_0^2}{4\omega^2\mu_0}\left(\underline{\hat{E}}_0 \cdot \underline{\hat{E}}_0^*\right)$$

$$\langle u_e \rangle = \frac{1}{4}\epsilon_0\, \underline{\hat{E}} \cdot \underline{\hat{E}}^* = \frac{1}{4}\epsilon_0\, \underline{\hat{E}}_0 e^{-jk_0\underline{n}\cdot\underline{r}} \cdot \underline{\hat{E}}_0^* e^{+jk_0\underline{n}\cdot\underline{r}}$$

$$\langle u_e \rangle = \frac{\epsilon_0}{4}\left(\underline{\hat{E}}_0 \cdot \underline{\hat{E}}_0^*\right)$$

4. assume a solution of the form $\underline{H} = -\nabla\psi$

$$\nabla\cdot H = -\nabla\cdot\nabla\psi$$

but $\nabla\cdot H = 0$

$$\therefore \quad \nabla^2\psi = 0$$

in spherical Coordinates $\Rightarrow$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

Assume $\psi \neq f(\phi)$ because of the symmetry of the problem. Furthermore assume $\psi = R(r)\,\Theta(\theta)$

then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial R\Theta}{\partial\theta}\right) = 0$$

when reduced

$$\frac{1}{R}\left[2r\frac{\partial R}{\partial r} + r^2\frac{\partial^2 R}{\partial r^2}\right] + \frac{1}{\Theta\sin\theta}\left[\cos\theta\,\frac{\partial\Theta}{\partial\theta} + \sin\theta\,\frac{\partial^2\Theta}{\partial\theta^2}\right] = 0$$

From Fano Chu & Adler solution is

$$\left.\begin{aligned} R(r) &= A_1 r^n + A_2 r^{-(n+1)} \\ \Theta(\theta) &= C P_n^m(\cos\theta) \end{aligned}\right\} \text{ with } n = m = 1$$

✓

$$R(r) = A_1 r + \frac{A_2}{r^2}$$

$$\Theta(\theta) = C\cos\theta$$

$$\psi = A_1C\cos\theta\, r + A_2C\frac{\cos\theta}{r^2}$$

$$H = -\nabla\psi = -a_r\frac{\partial\psi}{\partial r} - a_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$H_1 = -A_1\cos\theta\,\overline{a_r} + A_1\sin\theta\,\overline{a_\theta} \qquad r<R$$

$$H_2 = \frac{2A_3}{r^3}\cos\theta\,\overline{a_r} + \frac{A_3}{r^3}\sin\theta\,\overline{a_\theta} \qquad r>R$$

now with these two equations for H set $H_{1r}\big|_{r=R} = H_{2r}\big|_{r=R}$ and $H_{1\theta} - H_{2\theta} = K_\phi$ for 2 eqns & 2 unknowns A_1 & A_3

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_1\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_2\sin^2\theta\right) = 0$$

$$\frac{1}{r^{\cancel{2}}}2\cancel{r}A_1\cos\theta + \frac{1}{r\cancel{\sin\theta}}A_2\,2\cancel{\sin\theta}\cos\theta = 0$$

$$\therefore\ A_1 = -A_2$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{A_3}{r}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{A_4}{r^3}\sin^2\theta\right) = 0$$

$$\frac{1}{r^2}\ -\frac{A_3}{r^2}\cos\theta + \frac{A_4}{r^3}\frac{2\sin\theta\cos\theta}{r\sin\theta} = 0$$

All this work is unnecessary!

$$A_3 = 2A_4$$

NOTE: You must remember the coefficients A_1 & A_2 are the same and A_3 & A_4 are identical otherwise you cannot satisfy $\nabla\cdot H = 0$

$$\therefore\quad A_3 = -A_1R^3$$

$$A_2 = \frac{A_4}{R^3} - K_0$$

$$A_1 = -A_2$$

$$A_3 = 2A_4$$

$$-A_1 = 2\frac{A_3}{R^3}$$

and $\left(\frac{A_3}{R^3} - A_1\right)\sin\theta = K_0$

so $\frac{3A_3}{R^3} = K_0$

$$A_3 = \frac{K_0R^3}{3}$$

so $A_1 = -\frac{2K_0}{3}$

$$A_3 = A_2R^3 \qquad \therefore\ A_2 = \frac{A^3}{R^3}$$

$$A_2 = \frac{A_3}{2R_3} - K_0$$

$$\frac{A_3^3}{R^3} = \frac{A_3^3}{2R^3} - K_0$$

$$-K_0 = \frac{A_3}{2R^3} \qquad A_3 = -2R^3K_0$$

$$\underline{H} = -\nabla\psi = -\frac{\partial\psi}{\partial r}\bar{a}_r - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\bar{a}_\theta$$

$$\underline{H} = -\left(+A_1^c\cos\theta - A_2^c\frac{\cos\theta}{r^3}\right)\bar{a}_r + \frac{1}{r}\left(\frac{A_1^c r}{r} + \frac{A_2^c}{r^2}\right)\sin\theta\,\bar{a}_\theta$$

$$= -\left(A_1^c + \frac{A_2^c}{r^3}\right)\bar{a}_r\cos\theta + \left(A_1^c + \frac{A_2^c}{r^3}\right)\sin\theta\,\bar{a}_\theta$$

1) require $\underline{H}$ to be finite everywhere

says that $A_2 = 0$

i.e. $\underline{H}_1 = -A_1\cos\theta\,\bar{a}_r + A_2\sin\theta\,\bar{a}_\theta$ inside the sphere

1) require $\underline{H}$ to tend to 0 as $r \to \infty$

says that $A_1 = 0$

i.e. $\underline{H}_2 = +\frac{A_3}{r^3}\cos\theta\,\bar{a}_r + \frac{A_4}{r^3}\sin\theta\,\bar{a}_\theta$

require H_r to be continuous at $r=R$

$\Rightarrow \quad -A_1\cancel{\cos}\theta = +\frac{A_3}{R^3}\cancel{\cos}\theta \qquad A_3 = +A_1R^3$

Furthermore

$\bar{n} \times (\bar{H}_1 - \bar{H}_2) = K$

$\bar{a}_r \times (H_1 - H_2)_\theta = K_0\sin\theta\, a_\phi$

$\Rightarrow \quad H_{1\theta} - H_{2\theta} = K_0\sin\theta$

$+A_2\cancel{\sin\theta} + \frac{A_4}{R^3}\cancel{\sin\theta} = K_0\cancel{\sin\theta}$

$-A_1 \quad -A_2 + \frac{A_4}{R^3} = K_0$

$$\underline{H} = \begin{cases} +\frac{K_o}{2}\cos\theta\ \bar{a}_r - \frac{K_o}{2}\sin\theta\ \bar{a}_\theta \\ +\frac{K_o}{2}\left(\frac{R}{r}\right)^3\cos\theta\,\bar{a}_r + K_o\left(\frac{R}{r}\right)^3\sin\theta\ \bar{a}_\theta \end{cases}$$

$$\underline{H} = \frac{K_o}{2}\begin{cases} \cos\theta\ \bar{a}_r - \sin\theta\ \bar{a}_\theta & r < R \\ \left(\frac{R}{r}\right)^3 \cos\theta\ \bar{a}_r + \sin\theta\ \bar{a}_\theta & r > R \end{cases}$$

I know the result is incorrect but I can't find the error.

try again with the correction

FRANCIS L. MERAT
ENGR 336
OCTOBER 4, 1971

9/10

(1) in general

$$-\underline{a}_z \times \nabla_t \hat{e}_z - \gamma \underline{a}_z \times \hat{\underline{e}}_t = -j\omega\mu \hat{\underline{h}}_t$$
$$\nabla_t \times \hat{\underline{e}}_t = -j\omega\mu \hat{\underline{h}}_z$$

$$-\underline{a}_z \times \nabla_t \hat{\underline{h}}_z - \gamma \underline{a}_z \times \hat{\underline{h}}_t = j\omega\epsilon \hat{\underline{e}}_t$$
$$\nabla_t \times \hat{\underline{h}}_t = j\omega\epsilon \hat{\underline{e}}_z$$
$$\nabla_t \cdot \hat{\underline{e}}_t = \gamma \hat{e}_z$$
$$\nabla_t \cdot \hat{\underline{h}}_t = \gamma \hat{h}_z$$

for a TE wave $e_z = 0$

$$-\gamma \underline{a}_z \times \hat{\underline{e}}_t = -j\omega\mu \hat{\underline{h}}_t$$
$$\nabla_t \times \hat{\underline{e}}_t = -j\omega\mu \hat{\underline{h}}_z$$
$$-\underline{a}_z \times \nabla_t \hat{\underline{h}}_z - \gamma \underline{a}_z \times \hat{\underline{h}}_t = j\omega\epsilon \hat{\underline{e}}_t$$
$$\nabla_t \times \hat{\underline{h}}_t = 0$$
$$\nabla_t \cdot \hat{\underline{e}}_t = 0$$
$$\nabla_t \cdot \hat{\underline{h}}_t = \gamma \hat{h}_z$$

these waves must obey the Hemholtz equation

$$\nabla^2 H + \omega^2\mu\epsilon H = 0$$
$$(\nabla_t^2 + \gamma^2)\hat{\underline{h}}_t + \omega^2\mu\epsilon \hat{\underline{h}}_t = 0$$
$$(\nabla_t^2 + \gamma^2)\hat{\underline{h}}_z + \omega^2\mu\epsilon \hat{\underline{h}}_z = 0$$

$$\nabla_t^2 \hat{\underline{h}}_t + (\gamma^2 + \omega^2\mu\epsilon)\hat{\underline{h}}_t = 0$$
$$\nabla_t^2 \hat{\underline{h}}_z + (\gamma^2 + \omega^2\mu\epsilon)\hat{\underline{h}}_z = 0$$

$$\nabla_t^2 \underline{\hat{h}}_z + (\gamma^2 + \omega^2\mu\epsilon)\, \underline{\hat{h}}_z = 0$$

$$\nabla_t^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \underline{\hat{h}}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \underline{\hat{h}}_z}{\partial \phi^2} + (\gamma^2 + \omega^2\mu\epsilon)\, \underline{\hat{h}}_z = 0$$

let $\underline{\hat{h}}_z = \hat{h}_z a_z$

then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{h}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \hat{h}_z}{\partial \phi^2} + (r^2 + \omega^2\mu\epsilon)\, \hat{h}_z = 0 \quad \checkmark$$

(b) let $\hat{h}_z = f(r)\, g(\phi)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r g \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} f \frac{\partial^2 g}{\partial \phi^2} + (\gamma^2 + \omega^2\mu\epsilon) fg = 0$$

$$\frac{\cancel{g}}{\cancel{r}}\, \cancel{r}\, \frac{\partial^2 f}{\partial r^2} + \frac{g}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2} f \frac{\partial^2 g}{\partial \phi^2} + (\gamma^2 + \omega^2\mu\epsilon) fg = 0$$

$$\frac{1}{f(r)}\frac{\partial^2 f(r)}{\partial r^2} + \frac{1}{r f(r)}\frac{\partial f(r)}{\partial r} + \frac{1}{r^2 g(\phi)}\frac{\partial^2 g(\phi)}{\partial \phi^2} + (\gamma^2 + \omega^2\mu\epsilon) = 0$$

$$\frac{r^2}{f(r)}\frac{d^2 f(r)}{dr^2} + \frac{r}{f(r)}\frac{df(r)}{dr} + r^2(\gamma^2 + \omega^2\mu\epsilon) + \frac{1}{g(\phi)}\frac{d^2 g(\phi)}{d\phi^2} = 0$$

$$\frac{1}{g(\phi)}\frac{d^2 g(\phi)}{d\phi^2} = -n^2 \quad \checkmark$$

$$\frac{d^2 g(\phi)}{d\phi^2} = -n^2 g(\phi)$$

n must be an integer for ϕ to have periodicity of 2π

$$\Rightarrow g(\phi) = A_1 \cos n\phi + A_2 \sin n\phi$$

$$\Leftrightarrow \frac{r^2}{f(r)}\frac{d^2 f(r)}{dr^2} + \frac{r}{f(r)}\frac{df(r)}{dr} + r^2(\gamma^2 + \omega^2\mu\epsilon) = n^2$$

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + \left(\gamma^2 + \omega^2\mu\epsilon - \frac{n^2}{r^2}\right) f(r) = 0$$

$$k_c^2 \triangleq \gamma^2 + \omega^2\mu\epsilon$$

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + \left(k_c^2 - \frac{n^2}{r^2}\right) f(r) = 0$$

from p. 207-208 of Ramo, Whinnery, & Van Duzer

$$f(r) = B_1 J_n(k_c r) + B_2 Y_n(k_c r)$$

because $f(r)$ must be finite at the origin $B_2 = 0$

$$\therefore \quad f(r) = B_1 J_n(k_c r)$$

(4) and (5) $\quad h_z = J_n(k_c r)\left[C_1 \cos n\phi + C_2 \sin n\phi\right]$

the ~~because~~ there are no fields in the interior of a perfect conductor

$$n \cdot B = 0 \iff n \cdot H = 0$$

as $\quad n \times H = J_s$

$$n \cdot \hat{H} = 0 \qquad \text{but } H = \underline{h}_t + \underline{h}_z$$

$$n \cdot (\underline{\hat{h}}_t + \underline{\hat{h}}_z) = 0$$

$$n \cdot \underline{\hat{h}}_t = 0$$

but $\quad h_t = -\dfrac{\nabla_t(\gamma h_z)}{\gamma^2 + \omega^2\mu\epsilon}$

Proof: $\quad \nabla_t \times \underline{\hat{h}}_t = 0$

$$\nabla_t \times (\nabla_t \times \underline{\hat{h}}_t) = \nabla_t \nabla_t \underline{\hat{h}}_t - \nabla_t^2 \underline{\hat{h}}_t = 0$$

but $\quad \nabla_t^2 \underline{\hat{h}}_z = -(\gamma^2 + \omega^2\mu\epsilon)\,\underline{\hat{h}}_t$

$$\underline{\hat{h}}_t = -\frac{\gamma \nabla_t h_z}{\gamma^2 + \omega^2\mu\epsilon}$$

$$\Rightarrow \quad n \cdot \left(- \frac{\nabla_t (\gamma h_z)}{\gamma^2 + \omega^2 \mu \epsilon} \right) = 0$$

$$\underline{n} \cdot \nabla_t \hat{h}_z = 0$$

but $\nabla_t = \underline{a}_r \frac{\partial}{\partial r} + \underline{a}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}$

$$\hat{h}_z = J_n(k_c r) \left[C_1 \cos n\phi + C_2 \sin n\phi \right]$$

$$\underline{n} = \underline{a}_r$$

$$\underline{n} \cdot \nabla_t = \frac{\partial}{\partial r}$$

$$\Leftrightarrow \quad \frac{\partial}{\partial r} \hat{h}_z = \frac{\partial}{\partial r} \left[f(r) g(\phi) \right] = 0$$

$$\Leftrightarrow \quad \frac{\partial f(r)}{\partial r} = 0$$

but $f(r) = J_n(k_c r)$

$$\therefore \quad \left. \frac{d J_n(k_c r)}{dr} \right|_{r=a} = 0 \quad ✓$$

and given the boundary conditions of the system we can examine the dispersion relation of the system.

✓

Consider p_{mn} as that value of $k_c r$ which is the m-th zero of the derivative of the n-th order bessel function J_n

i.e. for all p_{mn} $\left. \frac{d J_n(k_c r)}{dr} \right|_{k_c r = p_{mn}} = 0$

in general $P_{mn} = k_c r$

or $k_c = \dfrac{P_{mn}}{r}$

but P_{mn} is only defined for $r = a$
i.e. the boundary condition

$$k_c = \frac{P_{mn}}{a}$$

but $k_c^2 = \omega^2 \mu \epsilon + \gamma^2$

$$\therefore \quad \gamma^2 + \omega^2 \mu \epsilon = \frac{P_{mn}^2}{a^2}$$

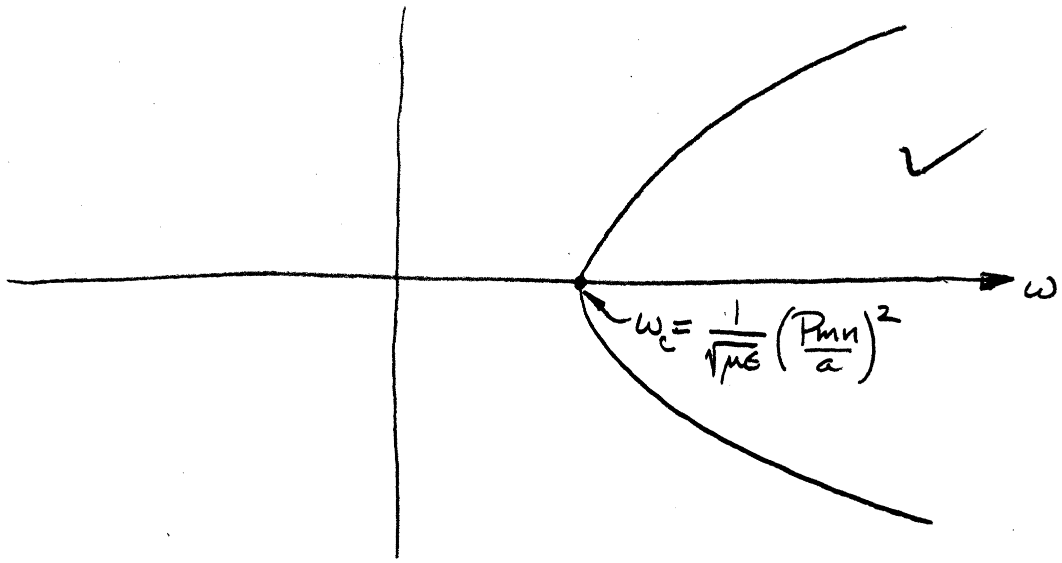
$$\gamma = \alpha + j\beta$$

assume $\alpha = 0$ in this case.

$$-\beta^2 + \omega^2 \mu \epsilon = \frac{P_{mn}^2}{a^2}$$

$$\beta^2 = \omega^2 \mu \epsilon - \frac{P_{mn}^2}{a^2}$$

$$\beta = \sqrt{\omega^2 \mu \epsilon - \left(\frac{P_{mn}}{a}\right)^2}$$



(6) TE$_{01}$ mode

1. no azimuthal variation
2. $\frac{1}{2}$ cycle variation of h_z in the radial direction from the axis to the wall

corresponds to the first zero of the zero-order Bessel function

i.e. $\hat{h}_z = J_0(k_c r)$

$$\nabla_t \times \underline{\hat{h}}_t = 0$$

$$\nabla_t \times (\nabla_t \times \underline{\hat{h}}_t) = \nabla_t \nabla_t \cdot \underline{\hat{h}}_t - \nabla_t^2 \underline{\hat{h}}_t = 0$$

$$= \nabla_t (\gamma \hat{h}_z) - \nabla_t^2 \underline{\hat{h}}_t = 0$$

$$\nabla_t \cdot \underline{\hat{h}}_t = \gamma \hat{h}_z$$

but these waves must obey the Hemholtz equation

$$\nabla^2 \underline{H} + \omega^2 \mu \epsilon \underline{H} = 0$$

$$(\nabla_t^2 + \gamma^2) \underline{h}_t + \omega^2 \mu \epsilon \, \underline{h}_t = 0$$

$$\nabla_t^2 \underline{\hat{h}}_t = -(\gamma^2 + \omega^2 \mu \epsilon) \underline{\hat{h}}_t$$

$$\therefore \nabla_t (\gamma \hat{h}_z) - \nabla_t^2 \underline{\hat{h}}_t = \nabla_t (\gamma \hat{h}_z) + (\gamma^2 + \omega^2 \mu \epsilon) \underline{\hat{h}}_t = 0$$

$$\therefore \underline{\hat{h}}_t = -\frac{\gamma}{\gamma^2 + \omega^2 \mu \epsilon} \nabla_t (\hat{h}_z)$$

$$\nabla_t = \underline{a_r}\frac{\partial}{\partial r} + \underline{a_\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\underline{a_\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$$

if $\hat{h}_z = c_1 J_0(k_c r)$

knowing $$\frac{d J_0(k_c r)}{d(k_c r)} = -J_1(k_c r)$$

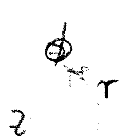
$$\nabla_t \hat{h}_z = \underline{a_r}\frac{d}{dr} J_0(k_c r) = k_c \underline{a_r}\frac{d J_0(k_c r)}{d(k_c r)}$$

$$\nabla_t \hat{h}_z = -\underline{a_r}\, k_c J_1(k_c r)$$

$$\underline{\hat{h}_t} = +\frac{\gamma}{\gamma^2 + \omega^2 \mu \epsilon}\,\underline{a_r}\, k_c J_1(k_c r)$$

$$= \frac{\gamma}{k_c^2}\,\underline{a_r}\,\cancel{k_c}\, J_1(k_c r)$$

$$= \frac{\gamma}{k_c} J_1(k_c r)\,\underline{a_r}$$



$$-\gamma\, \underline{a_z} \times \underline{\hat{e}_t} = -j\omega\mu\, \underline{\hat{h}_t}$$

$$-\gamma\left[\underline{a_z} \times \underline{a_z} \times \underline{\hat{e}_t}\right] = -j\omega\mu\left[\underline{a_z} \times \underline{\hat{h}_t}\right]$$

$$\gamma\,\underline{\hat{e}_t} = -j\omega\mu\left[\underline{a_z} \times \underline{\hat{h}_t}\right]$$

$$\underline{\hat{e}_t} = -\frac{j\omega\mu}{\gamma}\left[\underline{a_z} \times \underline{\hat{h}_t}\right]$$

$$= -\frac{j\omega\mu}{\gamma}\left[\frac{\gamma}{k_c} J_1(k_c r)\,\underline{a_\phi}\right] = -\frac{j\omega\mu}{k_c} J_1(k_c r)\,\underline{a_\phi}$$

in summary:

$$\hat{\underline{h}}_z = c_1 J_0(k_c r) \cancel{e^{-\gamma z}} \; \underline{a_z}$$

$$\hat{\underline{h}}_t = c_1 \frac{\gamma}{k_c} J_1(k_c r) \cancel{e^{-\gamma z}} \; \underline{a_r}$$

$$\hat{\underline{e}}_t = -\frac{j\omega\mu}{k_c} J_1(k_c r) \cancel{e^{-\gamma z}} \; \underline{a_\phi}$$

the lower case fields don't contain $e^{-\gamma z}$ factors

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{p_{mn}/a} = \frac{2\pi a}{p_{mn}}$$

$\underline{a} = 1\,cm$ given

$p_{mn} = 3.832$ Xerox notes p. 111

$$\omega_{co} = \left(\frac{p_{mn}}{a}\right) c$$

$$\lambda_c = \cancel{\frac{2\pi\,(1\,cm)}{3.832}} = \frac{2\pi}{3.832}\,cm$$

$$f_c = \frac{2\pi}{3.832\sqrt{\mu\epsilon}}$$

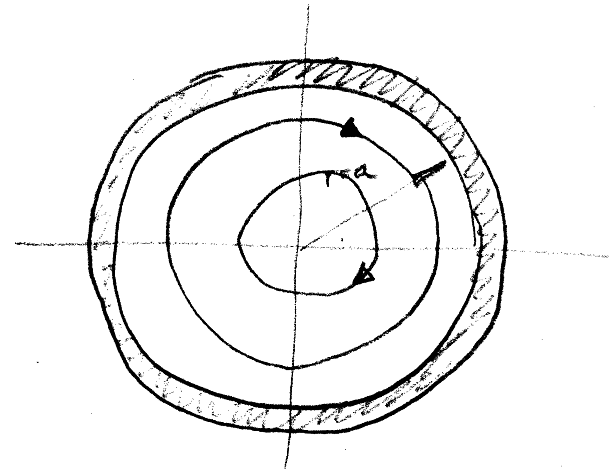
assume the center of the waveguide is free space.

$$c = \frac{1}{\sqrt{\mu\epsilon}} = 3\times10^{10}\,cm/sec.$$

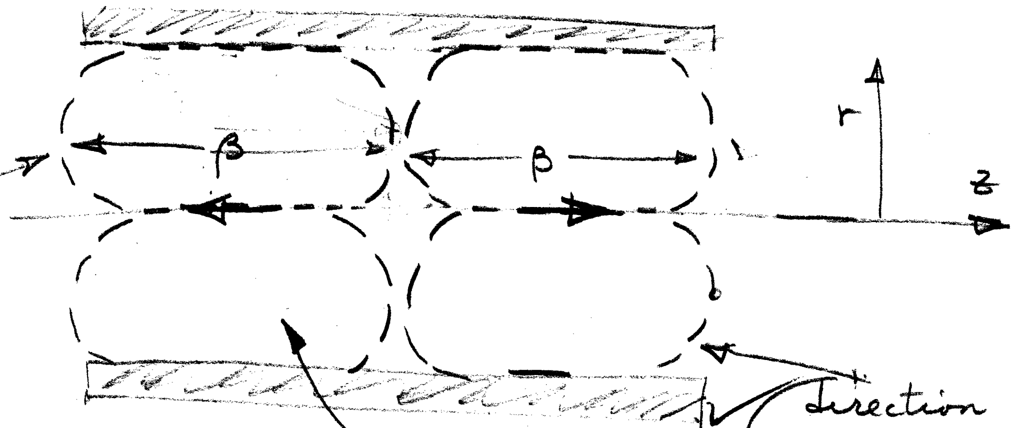
$$f_c = \frac{2\pi}{3.832}\left(3\times10^{10}/sec\right) = 4.92\times10^{10}\ Hz.$$

$$f_c = 49.2\ kMHz.$$

(7)



variation decreases as factor of $J_1(k_c)$



direction of rotation changes because of $e^{-\gamma z}$ factor

which becomes

$$e^{-\gamma z + j\omega t} = e^{-j\beta z + j\omega t} = e^{j(\omega t - \beta z)}$$

periodic in z

FRANCIS MERAT

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4.12 e

Eq. 4.12(2) $\nabla \times \underline{H} = \sigma \underline{E} + \frac{\partial \underline{D}}{\partial t}$

good work

$$\nabla \cdot (\nabla \times \bar{H}) = \sigma \nabla \cdot \underline{E} + \frac{\partial}{\partial t} \nabla \cdot \underline{D}$$

but $\nabla \cdot (\nabla \times \bar{H}) \equiv 0$

furthermore $\nabla \cdot \underline{D} = \rho$

$$0 = \sigma \nabla \cdot \underline{E} + \frac{\partial \rho}{\partial t}$$

but $\underline{D} = \epsilon \underline{E}$ or $\nabla \cdot E = \frac{1}{\epsilon} \nabla \cdot D = \frac{\rho}{\epsilon}$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0$$

solution is $\rho = c_1 e^{-\frac{\sigma}{\epsilon} t} + \cancel{c_2 e^{+\frac{\sigma}{\epsilon} t}}$ this couldn't possibly be a solution notice how it won't work by placing it in the equation

obviously $c_2 \equiv 0$

$$\rho = \rho(0) e^{-\frac{\sigma}{\epsilon} t}$$

where $\rho(0)$ is the initial charge density.

this implies that free charge cannot exist in the interior of a good conductor for a long period of time.

If we put excess charge q at the center of a solid conducting sphere of radius $r = 1\,cm$ This charge would rapidly move to the surface of The sphere.

this says that if we put total charge q at the center of the sphere $1/e$ of this charge would be on the surface of the sphere after $\frac{\epsilon}{\sigma}$ seconds.

$$\langle \text{velocity} \rangle = \frac{1 \text{ cm}}{(\frac{\epsilon}{\sigma}) \text{ sec}} = \frac{\sigma}{\epsilon} \frac{\text{cm}}{\text{sec}}$$

$$\frac{\sigma}{\epsilon} = \frac{5.8 \times 10^{7}}{\frac{1}{36\pi} \times 10^{-9}} = (36\pi)(5.8) \times 10^{16} = 656. \times 10^{16}$$

$$\frac{\sigma}{\epsilon} = 6.56 \times 10^{18} / \text{sec}$$

$$\langle \text{velocity} \rangle = 6.56 \times 10^{18} \frac{\text{cm}}{\text{sec}}$$

a speed vastly in excess of that of light.

Our fundamental assumptions were that the material be linear, homogeneous and isotropic. I believe that the material continues to be homogeneous and isotropic and that such large current flows (a few electrons at such a high speed constitude constitute a very short very large current, i.e. current = $\frac{\text{charge}}{\text{second}}$

$$= \frac{\#\text{ of electrons} \times \text{average velocity}}{\text{distance traveled}}$$

Because of this it would seem that $\underline{J} \neq \sigma \underline{E}$ and perhaps $\underline{D} \neq \epsilon \underline{E}$

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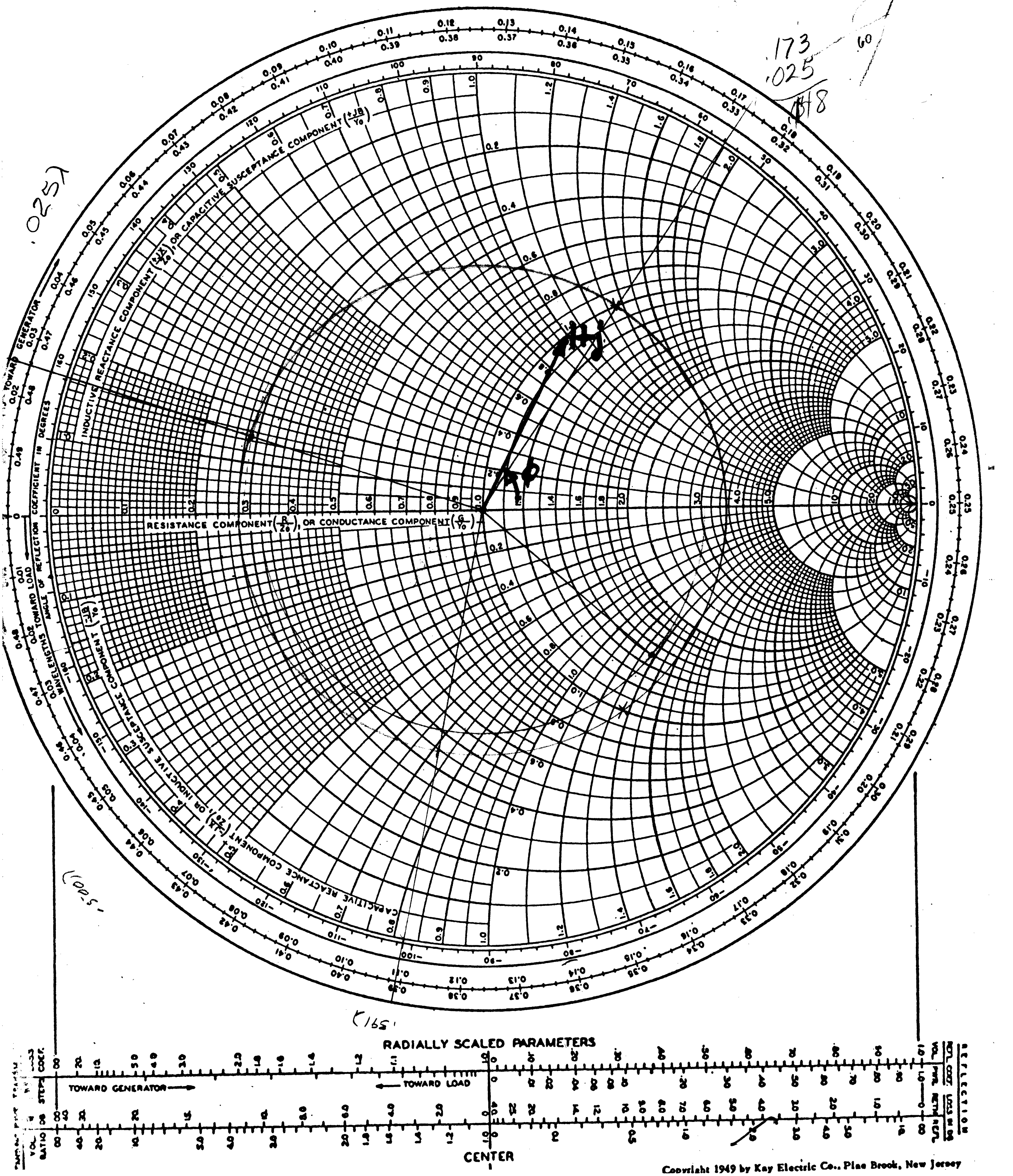

FRANCIS MERAT

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1.16a

$$\rho = \frac{R_L - Z_0}{R_L + Z_0} = \frac{V_-}{V_+} = \frac{\frac{V_-}{Z_0}}{\frac{V_+}{Z_0}} = \frac{I_-}{I_+}$$

$$\frac{P_{ref}}{P_{inc}} = \frac{I_- V_-}{I_+ V_+} = \left(\frac{I_-}{I_+}\right)\left(\frac{V_-}{V_+}\right) = \rho^2$$

$$\frac{P_{trans}}{P_{inc}} = \frac{V_L I_L}{V_+ I_+} = \tau \frac{I_L}{I_+} = \tau^2 \frac{Z_0}{R_L}$$

because $\frac{I_L}{I_+} = \frac{I_L Z_0 R_L}{I_+ Z_0 R_L} = \frac{Z_0 V_L}{V_+ R_L} = \frac{Z_0}{R_L}\tau$

furthermore $\frac{V_+ + V_-}{V_+} = \frac{V_L}{V_+}$

$$1 + \rho = \tau$$

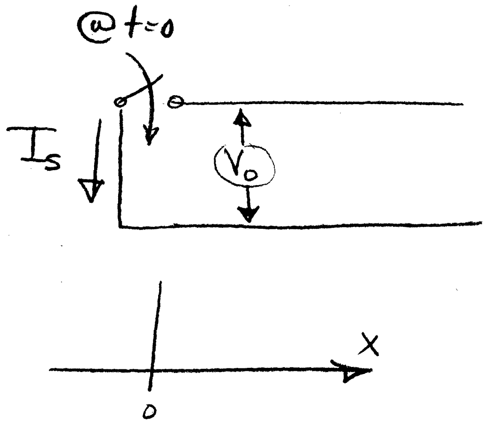
$$\frac{P_{trans}}{P_{inc}} = \frac{Z_0}{R_L}(1+\rho)^2$$

1.16b

		$\frac{P_{ref}}{P_{inc}} = \rho^2$	$\frac{P_{trans}}{P_{inc}} = \frac{Z_0}{R_L}\rho^2$
$R_L = 0$	$\rho = -1$	1	0
$R_L = \frac{1}{2}Z_0$	$\rho = -\frac{1}{3}$	$\frac{1}{9}$	$\frac{8}{9}$
$R_L = Z_0$	$\rho = 0$	0	1
$R_L = 2Z_0$	$\rho = \frac{1}{3}$	$\frac{1}{9}$	$\frac{8}{9}$
$R_L = \infty$	$\rho = 1$	1	0

$$\lim_{R_L \to 0} \frac{4R_L^2}{R_L(R_L+Z_0)^2} = \lim_{R_L \to 0} \frac{8R_L}{R_L 2(R_L+Z_0) + (R_L+Z_0)^2} = 0$$

1.17a



The current flowing into the short circuit is the negative of the current associated with the positively traveling wave, i.e. $I_s = -\frac{V_+}{Z_o}$.

Zero voltage at the short requires that $V_+ = -V_o$.
i.e. $I_s = \frac{V_o}{Z_o}$

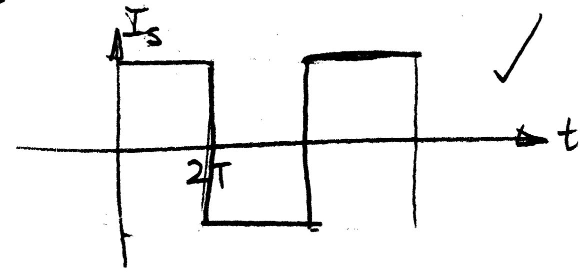
At the open end we require a reflected wave at the open end so that the total current is zero, so the current associated with the reflected wave is $-I_+$ or $\frac{V_o}{Z_o}$ $\left(I_+ = \frac{V_+}{Z_o} = -\frac{V_o}{Z_o} \right)$.

The voltage associated with this negatively traveling wave satisfies $I_- = -\frac{V_-}{Z_o}$

$$\therefore V_- = -I_- Z_o = -(-I_+) Z_o = I_+ Z_o = V_o.$$

This current reaches the short requiring a new voltage wave propagated in the +x direction. This new wave has associated with it a voltage of $V_+ = V_o$ volts, and an $I_s = -\frac{V_+}{Z_o} = -\frac{V_o}{Z_o}$ ✓

$T = \frac{l}{v}$



1.18

$$\beta = \frac{2\pi}{\lambda}$$

if $z = -\frac{\lambda}{3}$

$$\beta z = -\frac{2\pi}{3} \qquad Z_o = 100\,\Omega \qquad Z_L = 100 + j100$$

$$Z_L = \left.\frac{V}{I}\right|_{@z=0} = \frac{V_+ + V_-}{\frac{1}{Z_o}(V_+ - V_-)} = 100 + j100$$

$$\frac{V_+ + V_-}{V_+ - V_1} = \frac{100 + j100}{Z_o} = \frac{100 + j100}{100} = 1 + j$$

$$V_+ + V_- = (1+j)(V_+ - V_1)$$

$$\frac{V_-}{V_+} = \frac{j}{2+j} = \frac{1+j2}{5}$$

$$Z_i = \left.\frac{V}{I}\right|_{@z=-l} = \frac{V_+ e^{j\frac{2\pi}{3}} + V_- e^{-j\frac{2\pi}{3}}}{\frac{1}{Z_o}\left(V_+ e^{j\frac{2\pi}{3}} - V_- e^{-j\frac{2\pi}{3}}\right)}$$

$$= \frac{e^{j\frac{2\pi}{3}} + \frac{V_-}{V_+} e^{-j\frac{2\pi}{3}}}{e^{j\frac{2\pi}{3}} - \frac{V_-}{V_+} e^{-j\frac{2\pi}{3}}} Z_o$$

$$e^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$e^{-j\frac{2\pi}{3}} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$z_i = \frac{\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) + (.2+j.4)\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) - (.2+j.4)\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} z_0$$

$$z_i = \frac{-\frac{1}{2}+j\frac{\sqrt{3}}{2} + .2\left(-\frac{1}{2}\right) - j(.2)\frac{\sqrt{3}}{2} - (.4)\frac{1}{2}j + (.4)\frac{\sqrt{3}}{2}}{-\frac{1}{2}+j\frac{\sqrt{3}}{2} + .2\left(\frac{1}{2}\right) + (.4)\frac{1}{2}j + j(.2)\frac{\sqrt{3}}{2} - (.4)\frac{\sqrt{3}}{2}} z_0$$

$$z_i = \frac{(-1-.2+.4\sqrt{3}) + j(\sqrt{3}-.2\sqrt{3}-.4)}{(-1+.2-.4\sqrt{3}) + j(\sqrt{3}+.4+.2\sqrt{3})} z_0$$

$$z_i = \frac{(-1.2+.4\sqrt{3}) + j(.8\sqrt{3}-.4)}{(-.8-.4\sqrt{3}) + j(1.2\sqrt{3}+.4)} z_0$$

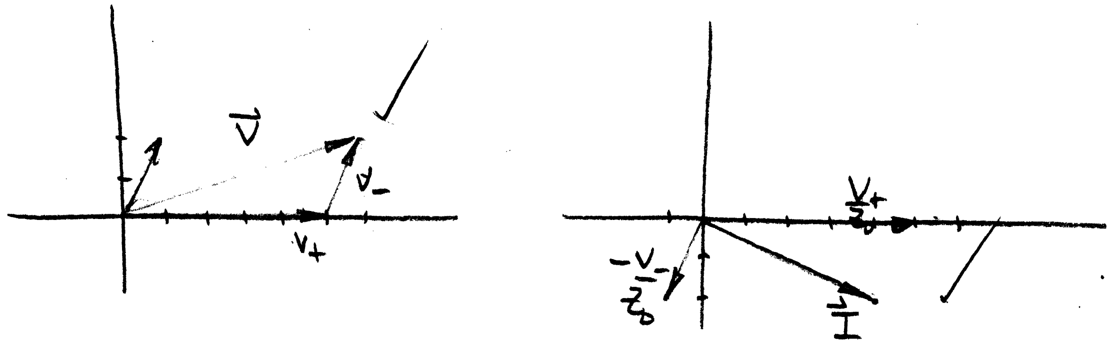
$$z_i = \frac{(-3+\sqrt{3}) + j(2\sqrt{3}-1)}{(-2-\sqrt{3}) + j(3\sqrt{3}+1)} z_0$$

$$z_i = \frac{20 + j(5\sqrt{3}-10)}{35+10\sqrt{3}} z_0$$

$$z_i = \frac{4 + j(\sqrt{3}-2)}{7+2\sqrt{3}} z_0$$

$$z_i = \frac{4 - j(2-\sqrt{3})}{7+2\sqrt{3}} (100)$$

phasor diagram for Z_L



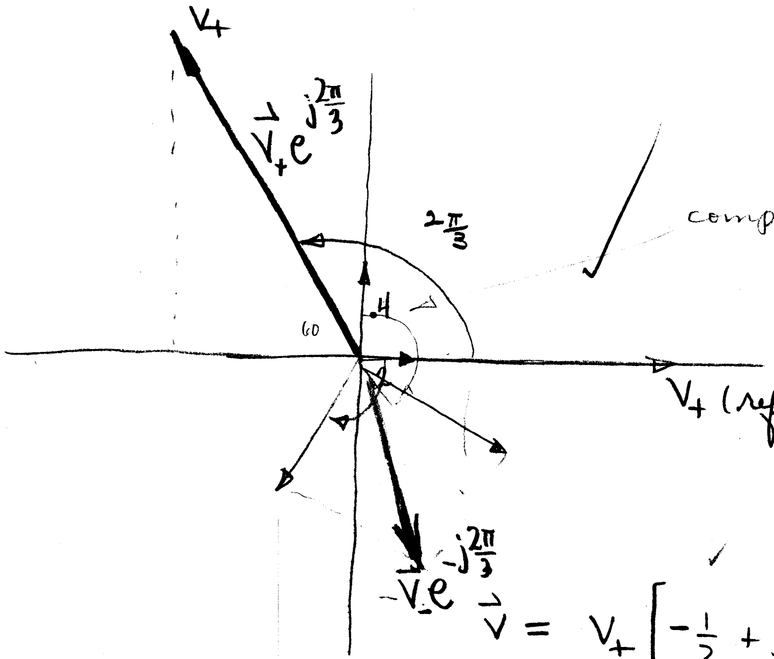
$$V_- = (.2 + j.4)\, V_+$$

$$\vec{V} = (1.2 + .4j)\, V_+ \qquad I = \frac{V_+}{Z_0}(.8 - .4j)$$

$$Z_L = \frac{1.2 + .4j}{.8 - .4j} Z_0 = \frac{3+j}{2-j} Z_0 = \frac{5+5j}{5} Z_0$$

$$Z_L = (1+j)\, Z_0 = (1+j)\, 100 = 100 + j100$$

phasor diagram for Z_i

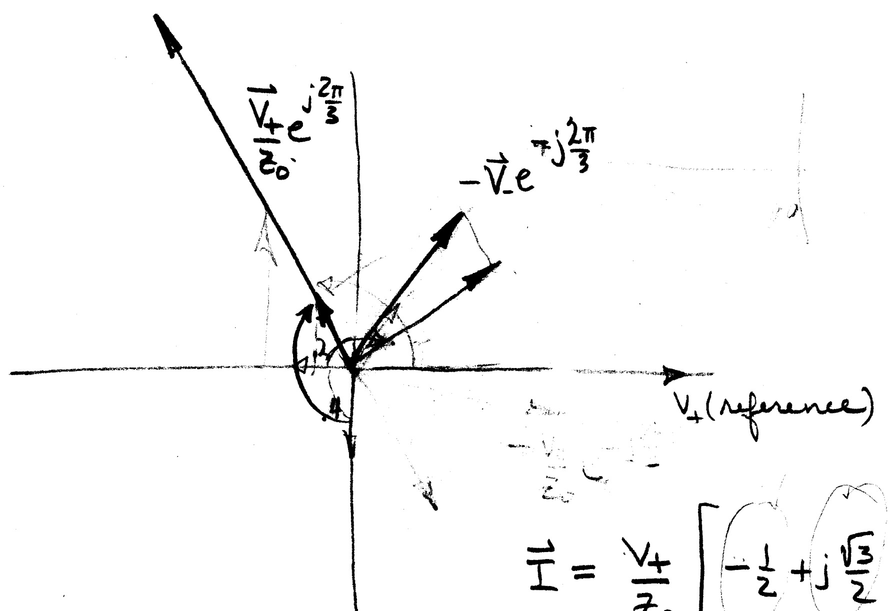


components of V_-

where's I_+, I_- ?

$$\vec{V} = V_+\left[-\frac{1}{2} + j\frac{\sqrt{3}}{2} + .4\frac{\sqrt{3}}{2} - j\frac{(.4)}{2} - .2\frac{1}{2} - j(.2)\frac{\sqrt{3}}{2}\right]$$

$$= \frac{V_+}{2}\left[(-1 - .2 + .4\sqrt{3}) + j(\sqrt{3} - .4 - .2\sqrt{3})\right)$$



$$\vec{I} = \frac{V_+}{Z_0}\left[-\frac{1}{2} + j\frac{\sqrt{3}}{2} + j(.4)\frac{1}{2} + \frac{\sqrt{3}}{2}(.4) + (.2)\frac{1}{2} - j(.2)\frac{\sqrt{3}}{2}\right]$$

$$= \frac{V_+}{2Z_0}\left[(-1 + .4\sqrt{3} + .2) + j(\sqrt{3} + .4 + .2\sqrt{3})\right]$$

and $$Z_i = Z_0\left[\frac{(-1.2 + .4\sqrt{3}) + j(.8\sqrt{3} - .4)}{(-.8 - .4\sqrt{3}) + j(1.2\sqrt{3} + .4)}\right]$$

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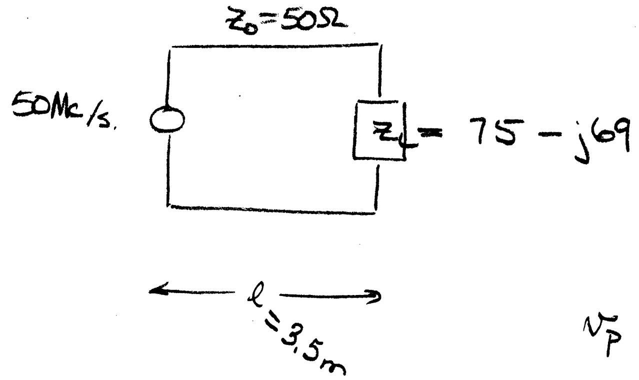
6.5/10

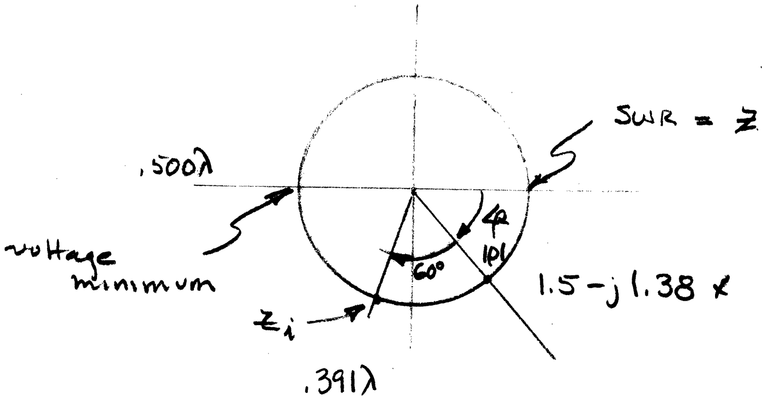
1.20 a:



$v_p = 3 \times 10^8$ m/sec.

$$Z_n = \frac{75 - j69}{50} = 1.5 - j1.38$$

to find Z_i



$$\beta\ell = \left(\frac{\omega}{v_p}\right)\ell = \left(\frac{2\pi f}{v_p}\right)\ell = \frac{2\pi\, 50 \times 10^6}{3 \times 10^8} = \frac{\pi}{3} = 30^\circ \text{ (toward generator)}$$

$$2\beta\ell = 60^\circ$$

$$Z_{i_n} = 0.5 - j0.77$$

$$Z_i = (0.5 - j(.77))\,50 = \boxed{25 + j38.5\ \Omega}$$

$$|\rho| = \sqrt{(1.5)^2 + (1.38)^2} = \sqrt{2.25 + 1.90} = \sqrt{4.15}$$

$\boxed{|\rho| = 1.72}$ read |ρ| from subsidiary scale or calculate it. but you've calculated

$\boxed{\angle\rho = -41^\circ}$

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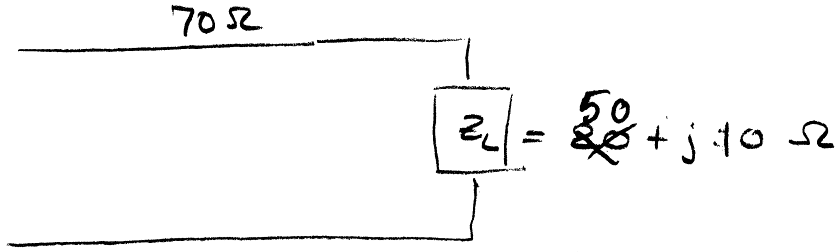
$\boxed{SWR = 3.05}$

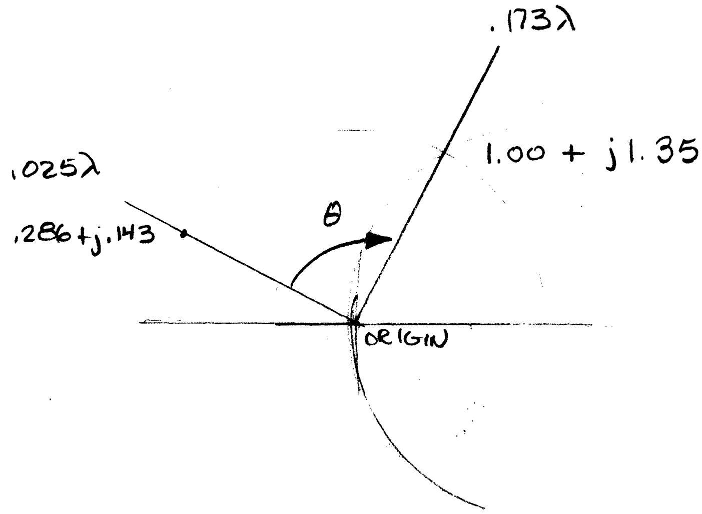
voltage minimum occurs at $(0.500 - 0.391)\lambda$ toward generator

$$.109\lambda = .109\frac{v_p}{f} = .109\,\frac{3 \times 10^8 \text{ m/sec}}{.50 \times 10^8 \text{ m/sec}} = \boxed{.654\,m \text{ toward generator}}$$

1.20 d.



$$Z_{Ln} = \frac{20 + j\,10}{70} = .286 + j\,.143$$



$$\theta = .173\lambda - .025\lambda = .148\lambda$$

matching can be done by putting a capacitive reactance (X_c = 1.35) 0.148λ from the load.

right idea ✓

consider $(A+jB)^{\frac{1}{2}} = A^{\frac{1}{2}} + \frac{1}{2}A^{-\frac{1}{2}}(jB) + \frac{1}{2}\left(-\frac{1}{2}\right)A^{-\frac{3}{2}}(jB)^2 + \dots$

$$= \sqrt{A} + \frac{j}{2}\frac{B}{\sqrt{A}} + \frac{1}{4}\frac{B^2}{(\sqrt{A})^3} - \dots$$

if $Z = R + j\omega L \quad Y = G + j\omega C$

$$\gamma^2 = ZY = RG + j\omega LG + j\omega RC - \omega^2 LC$$

$$\gamma^2 = (RG - \omega^2 LC) + j(\omega LG + \omega RC)$$

$$\therefore \gamma = \left[(RG - \omega^2 LC) + j(\omega LG + \omega RC)\right]^{\frac{1}{2}}$$

identify $A = (RG - \omega^2 LC)$

$B = (\omega LG + \omega RC)$

retaining upto 2nd order terms.

$$\sqrt{A} = \sqrt{RG - \omega^2 LG} \approx \sqrt{-\omega^2 LC} = j\omega\sqrt{LC}$$

$$\frac{j}{2}\frac{B}{\sqrt{A}} = \frac{\cancel{j}}{2}\frac{\cancel{\omega}(LG+RC)}{\cancel{j\omega}\sqrt{LC}} = \frac{1}{2}\frac{LG}{\sqrt{LC}} + \frac{1}{2}\frac{RC}{\sqrt{LC}}$$

$$= \frac{1}{2}G\sqrt{\frac{L}{C}} + \frac{1}{2}R\sqrt{\frac{C}{L}}$$

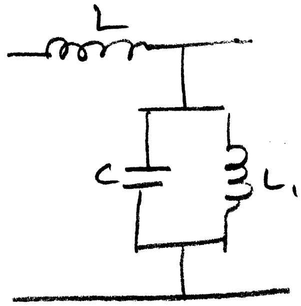
$$= \frac{1}{2}GZ_0 + \frac{1}{2}\frac{R}{Z_0} \qquad \text{where } Z_0 = \sqrt{\frac{L}{C}}$$

$$\gamma \approx \frac{1}{2}GZ_0 + \frac{1}{2}\frac{R}{Z_0} + j\omega\sqrt{LC}$$

if $\gamma = \alpha + j\beta$

identify $\alpha = \frac{1}{2}GZ_0 + \frac{1}{2}\frac{R}{Z_0}$; $\beta = \omega\sqrt{LC}$

4.



$$Z = j\omega L$$

$$Y = j\omega C + \frac{1}{j\omega L_1}$$

$$\gamma^2 = ZY = j\omega L\left(j\omega C + \frac{1}{j\omega L_1}\right)$$

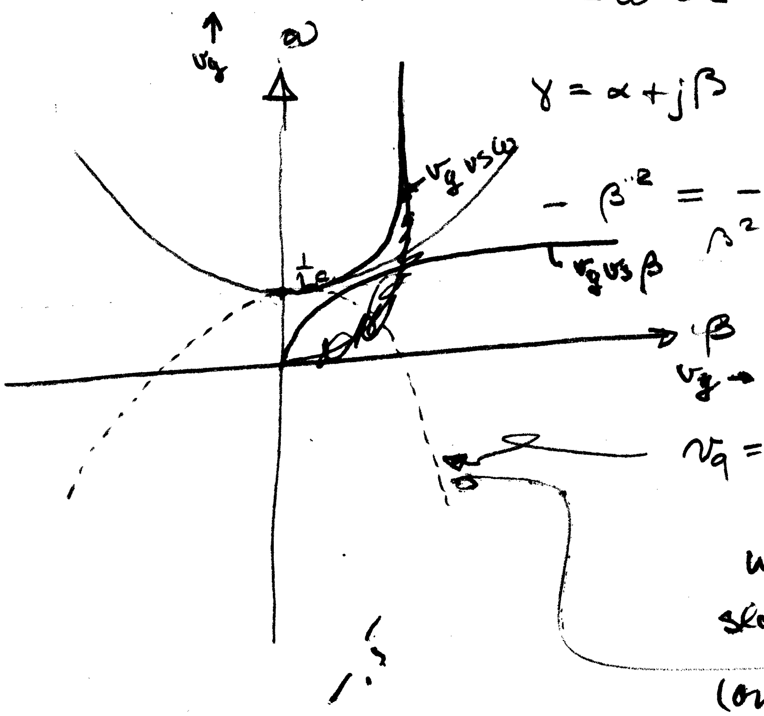
$$= -\omega^2 LC + \frac{L}{L_1}$$

$\gamma = \alpha + j\beta$ assume $\alpha = 0$

$$-\beta^2 = -\omega^2 LC + \frac{L}{L_1}$$

$$\beta^2 = \omega^2 LC - \frac{L}{L_1}$$

$$\omega^2 = \frac{\beta^2}{LC} + \frac{1}{L_1 C}$$



$$v_g = \frac{d\omega}{d\beta}$$

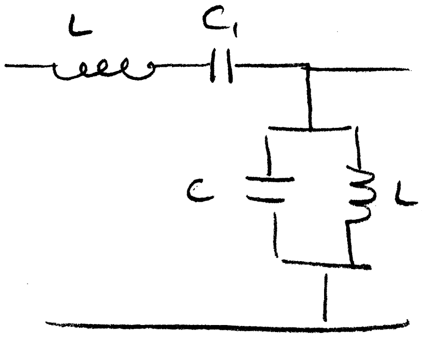
why do you think the slope becomes negative here?

(or are you plotting v_g vs β)

It's important to label your plot.

In any case its wrong.

(b)



$$LC_1 > L_1C$$

$$Z = j\omega L + \frac{1}{j\omega C_1}$$

$$Y = j\omega C + \frac{1}{j\omega L_1}$$

$$\gamma^2 = ZY = -\omega^2 LC + \frac{C}{C_1} + \frac{L}{L_1} + \frac{1}{-\omega^2 L_1 C_1}$$

$$-\beta^2 = -\omega^2 LC - \frac{1}{\omega^2 L_1 C_1} + \left(\frac{C}{C_1} + \frac{L}{L_1}\right)$$

$$\beta^2 = \omega^2 LC + \frac{1}{\omega^2 L_1 C_1} - \left(\frac{C}{C_1} + \frac{L}{L_1}\right)$$

$$\beta^2 = \left(\omega^2 LC - \frac{C}{C_1}\right) + \left(\frac{1}{\omega^2 L_1 C_1} - \frac{L}{L_1}\right)$$

1.5

$$= \frac{C\left(\omega^2 LC_1 - 1\right)}{C_1} + \frac{1 - \omega^2 C}{\omega^2 L_1 C_1}$$

$$\beta^2 = \omega^2 LC + \frac{1}{\omega^2 L_1 C_1} - \left(\frac{C}{C_1} + \frac{L}{L_1}\right)$$

$$\beta^2\omega^2 L_1 C_1 = \omega^4 LCL_1 C_1 - \left(\frac{C}{C_1} + \frac{L}{L_1}\right)\omega^2 L_1 C_1 + 1$$

$$\omega^4 LCL_1C_1 - \left(\frac{C}{C_1} + \frac{L}{L_1}\right)\omega^2 L_1 C_1 - \beta^2\omega^2 L_1 C_1 + 1 = 0$$

$$\omega^4 LCL_1C_1 - \omega^2 L_1 C - \omega^2 LC_1 - \beta^2\omega^2 L_1 C_1 + 1 = 0$$

$$\omega^4 (LCL_1C_1) + \omega^2 (L_1C + LC_1 + L_1C_1\beta^2) + 1 = 0$$

$$\omega^2 = \frac{L_1C + LC_1 + L_1C_1\beta^2 \pm \sqrt{(L_1C + LC_1 + L_1C_1\beta^2)^2 - 4LCL_1C_1}}{2LCL_1C_1}$$

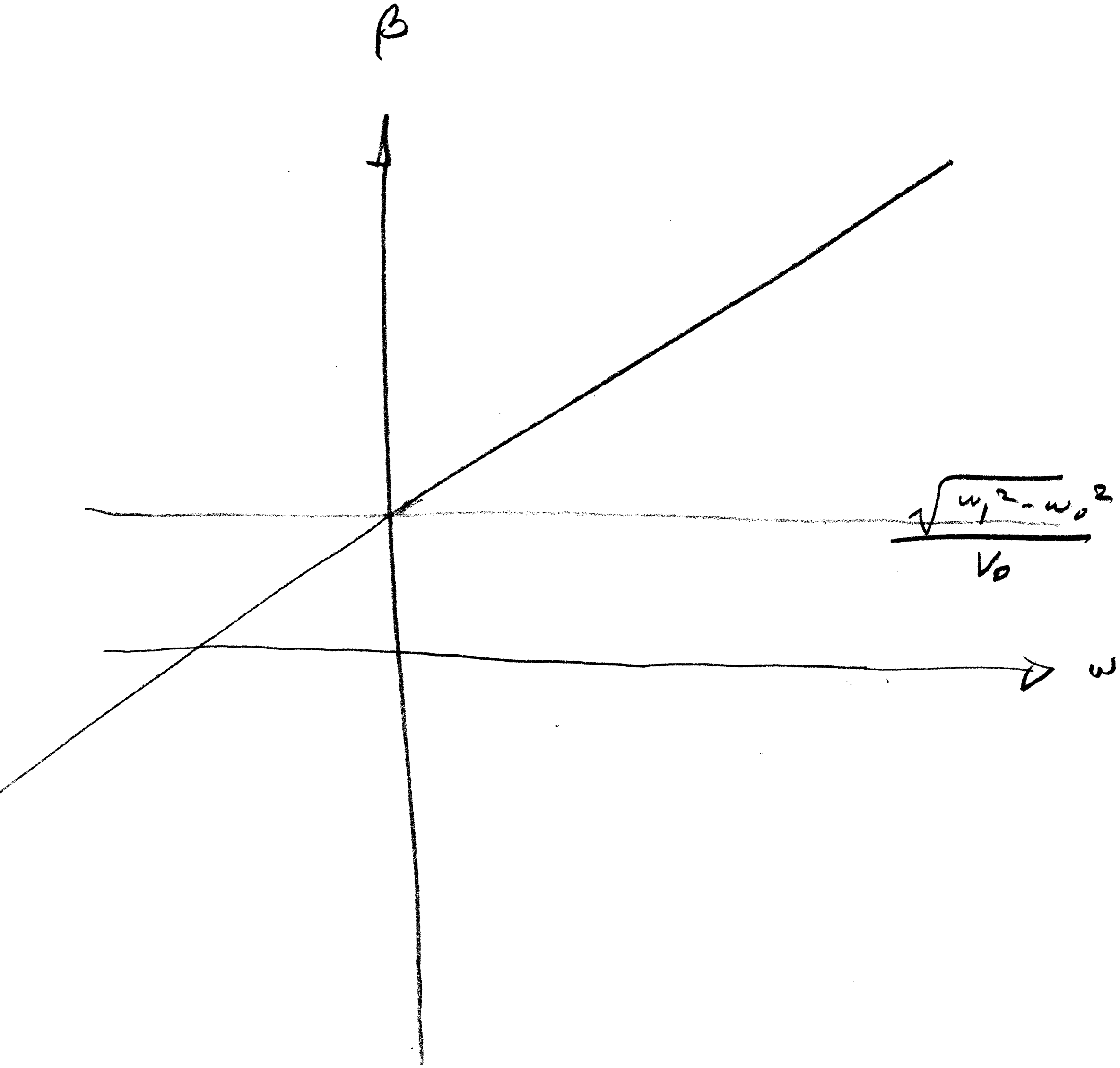
$$\omega^2 LC + \frac{1}{\omega^2 L_1 C_1} = \beta^2 + \left(\frac{C}{C_1} + \frac{L}{L_1}\right)$$

$$\frac{\omega^4 LCL_1C_1 + 1}{\omega^2 L_1 C_1} = \beta^2 + \left(\frac{C}{C_1} + \frac{L}{L_1}\right)$$

$$\omega^4 LCL_1C_1 + 1 = \beta^2\omega^2 L_1 C_1 + \omega^2 L_1 C + \omega^2 LC_1$$

I cannot come up with a reasonable equation to plot.

$$\approx \beta_0 \sqrt{\frac{\omega_p^2}{(\omega - \beta V_0)^2 - \omega_c^2} - 1}$$



$$\omega_p \geq (\omega - \beta V_0)^2 - \omega_c^2$$

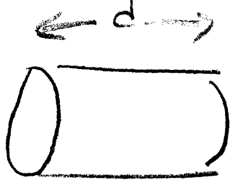
$$\beta' \quad \omega_p^2 - \omega_c^2 \geq (\omega - \beta V_0)^2$$

$$\omega - \beta V_0 \geq \sqrt{\omega_p^2 - \omega_c^2}$$

$$\omega \geq \sqrt{\omega_p^2 + \omega_c^2} + \beta V_0$$

$$\frac{\omega + \sqrt{\omega_p^2 - \omega_c^2}}{V_0} \geq \beta$$

6.



$$\hat{h}_z = H_0 J_0 \left(\frac{3.832}{a} r \right) \quad \underline{a_z}$$

$$\hat{h}_t = \frac{H_0 \gamma}{k_c^2} \left(\frac{3.832}{a} \right) J_1 \left(\frac{3.832}{a} r \right) \underline{a_r}$$

$$\hat{e}_t = \frac{H_0}{j\omega\epsilon_0} \left(\frac{3.832}{a} \right) J_1 \left(\frac{3.822}{a} r \right) \underline{a_\phi}$$

boundary condition is that

$$a_z \cdot \nabla H_z = 0 \quad @ \quad z=0 \quad z=L$$

$$\frac{\partial}{\partial z} \left(H_0 J_0 \left(\frac{3.822}{a} r \right) e^{-j\beta z} \right) = 0$$

$$-j\beta H_0 J_0 \left(\frac{3.832}{a} r \right) e^{-j\beta z} = 0$$

$\therefore$ require $Re \; -je^{-j\beta z} = 0 \quad @ \quad z=0 \quad z=L$

$$-j(\cos\beta z - j\sin\beta z) = 0$$

$$-j\cos\beta z - \sin\beta z = 0$$

$$\therefore \beta d = n\pi$$

$$\beta = \frac{n\pi}{d}$$

$$U = \frac{\epsilon_0}{2}\int_0^d\int_0^{2\pi}\int_0^a \frac{H_0^2}{\omega^2\epsilon_0^2}\left(\frac{3.832}{a}\right)^2 J_1^2\left(\frac{3.832}{a}r\right) r\,dr\,d\phi\,dz$$

$$= \frac{\epsilon_0}{2}\int_0^d\int_0^{2\pi} \frac{H_0^2}{\omega^2\epsilon_0^2}\left(\frac{3.832}{\cancel{a}}\right)^2 \frac{\cancel{a^2}}{2} J_0^2\left(3.832\right) d\phi\,dz$$

$$= \frac{d\pi H_0^2}{2\omega^2\epsilon_0}(3.832)^2 J_0^2(3.822)$$

loss in walls from homework #1

$$W_L = \frac{1}{2}R_s\int |H_{tan}|^2\,dS$$

$$= \frac{R_s}{2}\left[H_0^2\,\frac{1}{2}J_0^2(3.832)\,2\pi a d\right] + \frac{R_s}{2}\cdot 2\int_0^a \frac{H_0^2\gamma^2}{k_c^2}\left(\frac{3.832}{a}\right) J_1^2\left(\frac{3.832}{a}r\right) 2\pi r\,dr$$

$$= \frac{R_s}{2}\left[\frac{1}{2}J_0^2(3.832)\,2\pi a d + R_s\left[\frac{H_0^2\gamma^2}{k_c^2}\;\frac{3.832}{a}\;\pi a^2 J_0^2(3.832)\right]\right]$$

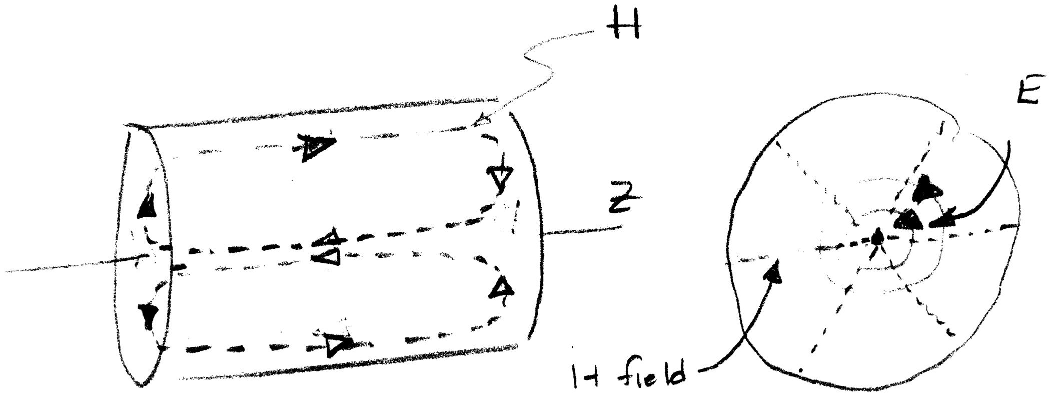
$$Q = \frac{\omega U}{W_L} = \frac{\cancel{\omega}\,\frac{d\cancel{\pi}H_0^2}{2\cancel{\omega^2}\epsilon_0^2}(3.832)^2\,\cancel{J_0^2(3.832)}}{\frac{R_s}{4}\cancel{J_0^2(3.832)}\,2\cancel{\pi}ad + \frac{R_s H_0^2\gamma^2}{k_c^2}\left(\frac{3.832}{\cdot}\right)a\cancel{\pi}\cancel{J_0^2(3.832)}}$$

$$= \frac{\frac{dH_0^2}{2\omega\epsilon_0^2}(3.832)^2}{\frac{R_s}{2}ad + \frac{R_s H_0^2\gamma^2}{k_c^2}(3.832)a}$$

✓

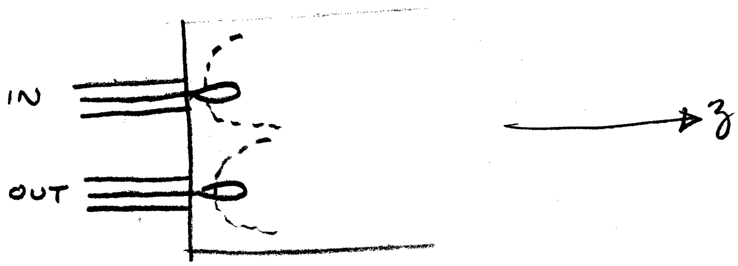
I dont like the "3.832"'s in the expression

why not?!



cross section ⊥ to z

filter



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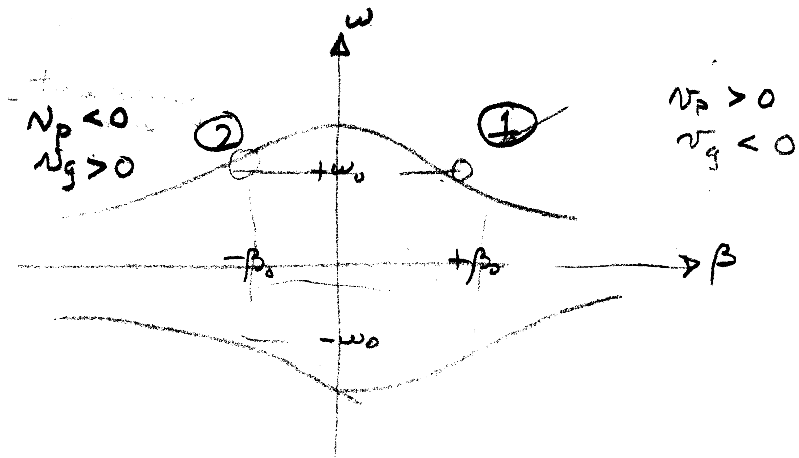
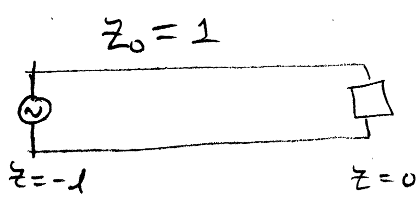
$e^{j(\omega t - \beta z)}$

1.



obviously $v_g > 0$ for real waves
Therefore if a signal of frequency ω_0 appeared at the load the point on the ω-β diagram being considered is point ②.



$Z_L = j\omega_0 \cancel{X} = j$ wow! $j\omega L = j$

$$\rho(z=0) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{j\omega_0 - 1}{j\omega_0 + 1} \qquad \tau = \frac{2Z_L}{Z_L + Z_0} = \frac{j2\omega_0}{j\omega_0 + 1}$$

let the voltage measured across the load be denoted by V_L

then $$\frac{V_L}{V_+} = \tau = \frac{j2\omega_0}{j\omega_0 + 1}$$

$$\boxed{V_+ = \frac{j\omega_0 + 1}{j2\omega_0}}$$

$$\frac{V_-}{V_+} = \rho = \frac{j\omega_0 - 1}{j\omega_0 + 1}$$

$$V_- = \frac{j\omega_0 - 1}{j\omega_0 + 1} V_+$$

$$V_- = \frac{j\omega_0 - 1}{\cancel{j\omega_0 + 1}} \cdot \frac{\cancel{j\omega_0 + 1}}{j2\omega_0} V_L = \frac{j\omega_0 - 1}{j2\omega_0} V_L$$

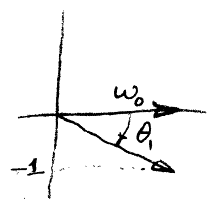
$$\therefore \boxed{V_- = \frac{j\omega_0 - 1}{j2\omega_0}}$$

$$I_+ = \frac{V_+}{Z_0} \quad ; \quad I_- = -\frac{V_-}{Z_0}$$

$$I_+ = \frac{j\omega_0 + 1}{j2\omega_0} \qquad I_- = \frac{1 - j\omega_0}{j2\omega_0}$$

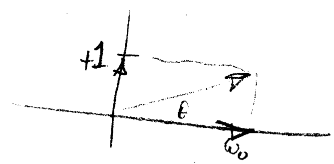
$$V_+ = \frac{j\omega_0 + 1}{j2\omega_0} \qquad V_- = \frac{j\omega_0 - 1}{j2\omega_0}$$

$$V_+ = \frac{j\omega_0 + 1}{j2\omega_0} = \frac{\omega_0 - j}{2\omega_0} = \frac{1}{2\omega_0}\sqrt{\omega_0^2 + 1}\; e^{j\theta_1} \; : \; \theta_1 = -\mathrm{Tan}^{-1}\frac{1}{\omega_0}$$



$$I_+ = \frac{1}{2\omega_0}\sqrt{\omega_0^2 + 1}\; e^{j\theta_1}$$

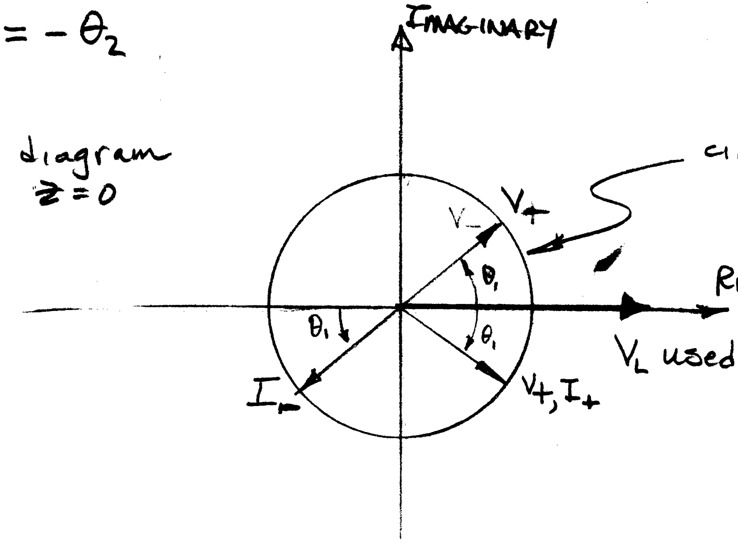
$$V_- = \frac{j\omega_0 - 1}{j2\omega_0} = \frac{\omega_0 + j}{2\omega_0} = \frac{1}{2\omega_0}\sqrt{\omega_0^2 + 1}\; e^{j\theta_2} \; ; \; \theta_2 = \mathrm{Tan}^{-1}\frac{1}{\omega_0}$$



$$I_- = \frac{1 - j\omega_0}{j2\omega_0} = -\frac{\omega_0 + j}{2\omega_0} = -\frac{1}{2\omega_0}\sqrt{\omega_0^2 + 1}\; e^{j\theta_2}$$

Note: $\theta_1 = -\theta_2$

Phasor diagram at $z = 0$



circle of magnitude $\frac{1}{2\omega_0}\sqrt{\omega_0^2 + 1}$

V_L used as phase reference.

(c) at the source

$$\hat{V}_+(z) = \hat{V}_+ e^{-j\beta z} \quad ; \quad \hat{V}_-(z) = \hat{V}_- e^{+j\beta z}$$

if $z = -\frac{\lambda}{4}$ and $\beta < 0 \Rightarrow \beta = -\frac{2\pi}{\lambda}$ $\lambda > 0$

$$\beta z = -\beta \frac{\lambda}{4} = +\frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$$

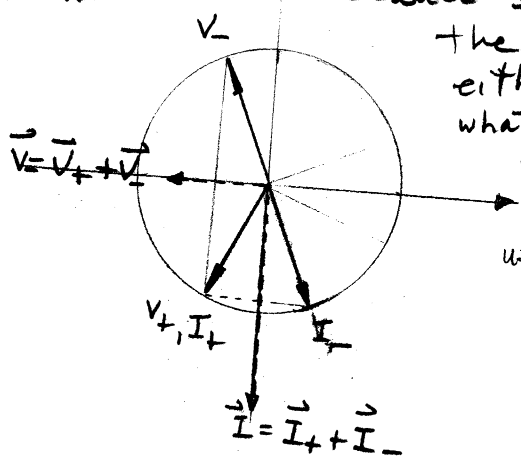
$$\therefore \hat{V}_+(-l) = \hat{V}_+ e^{-j\frac{\pi}{2}}$$

$$\hat{V}_-(-l) = \hat{V}_+ e^{+j\frac{\pi}{2}}$$

$$\hat{I}_+(-l) = \hat{V}_+ e^{-j\frac{\pi}{2}}$$

$$\hat{I}_-(-l) = -\hat{V}_+ e^{+j\frac{\pi}{2}}$$

I cannot say what the source sees in terms of reflected and incident waves because I do not know enough about the source, i.e. I need to know either its internal impedance or what voltage it is putting out (phase & magnitude)



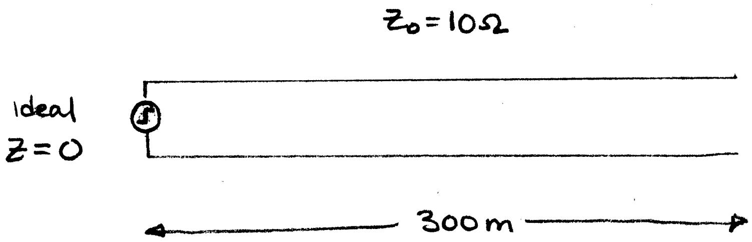
use as phase reference V_L

"the source sees" does not mean to include its own impedance

i.e.

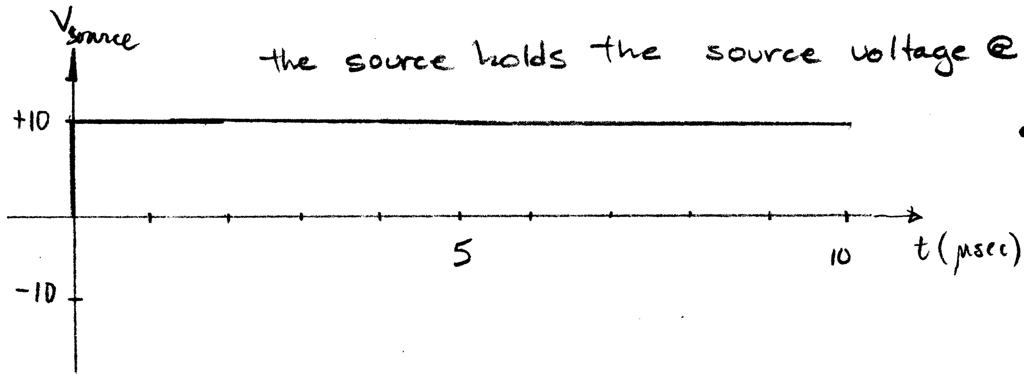
(d) by inspection I leads v by 90° and the line appears to be a capacitive load to the voltage source.

2.

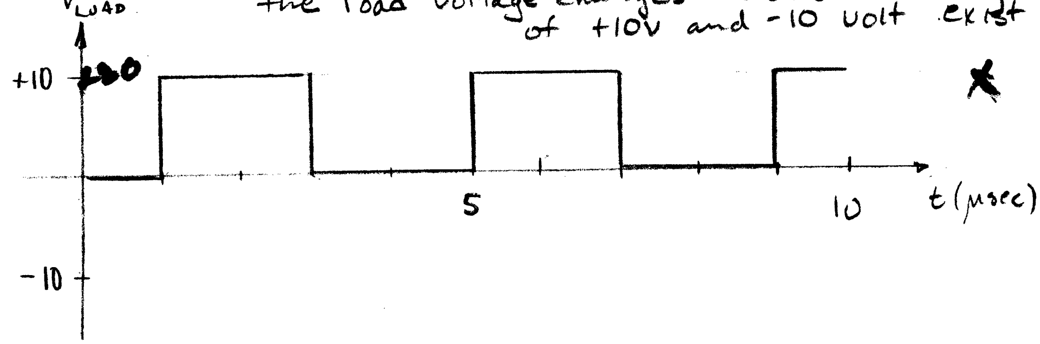


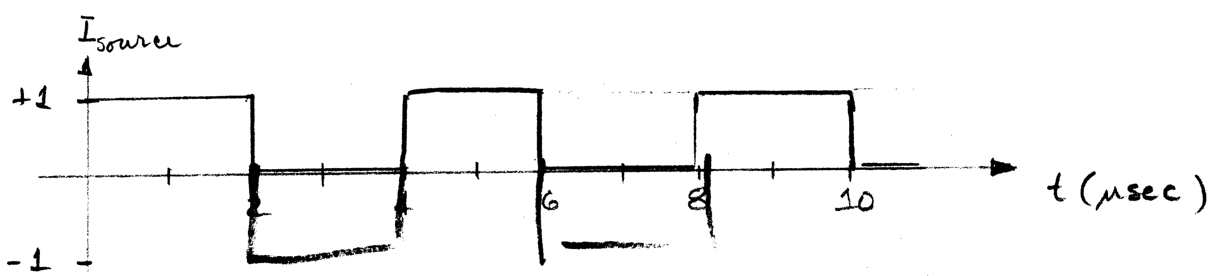
$$t = \frac{3\times10^{2}\,m}{3\times10^{8}\,m/sec} = 10^{-6}\,sec$$
$$= 1\ \mu sec.$$

the source holds the source voltage @ 10 V.

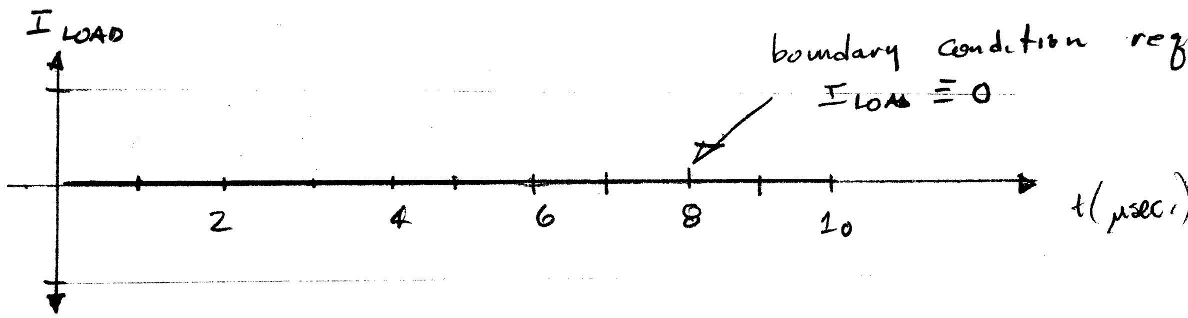


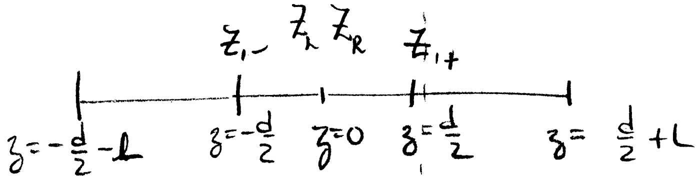
the load voltage changes because traveling waves of +10V and −10 volt exist on the line.





boundary condition requires $I_{LOAD} = 0$





$$Z_{1+} = Z_0\left[\frac{j Z_0 \sin k'L}{Z_0 \cos k'L}\right] \quad \text{because} \quad Z_L = 0$$

$$Z_{1+} = Z_0\left[j \tan k'L\right] \checkmark$$

$$Z_{1-} = Z_0\left[\frac{j Z_0 \sin(-k'L)}{Z_0 \cos(-k'L)}\right] \quad \text{because} \quad Z_L = 0$$

$$= Z_0\left[-j \tan k'L\right]$$

$$Z_R = Z_1\left[\frac{Z_{1+} \cos k\frac{d}{2} + j Z_1 \sin \frac{kd}{2}}{Z_1 \cos \frac{kd}{2} + j Z_{1+} \sin \frac{kd}{2}}\right] \checkmark$$

$$Z_L = Z_1\left[\frac{Z_{1-} \cos \frac{kd}{2} - j Z_1 \sin \frac{kd}{2}}{Z_1 \cos \frac{kd}{2} - j Z_{1-} \sin \frac{kd}{2}}\right]$$

for resonance $Z_R = Z_L$ ✓

$$\frac{Z_{1+} \cos \frac{kd}{2} + j Z_1 \sin \frac{kd}{2}}{Z_1 \cos \frac{kd}{2} + j Z_{1+} \sin \frac{kd}{2}} = \frac{Z_{1-} \cos \frac{kd}{2} - j Z_1 \sin \frac{kd}{2}}{Z_1 \cos \frac{kd}{2} - j Z_{1-} \sin \frac{kd}{2}}$$

$$Z_1 Z_{1+}\cos^2\frac{kd}{2} - jZ_{1+}Z_{1-}\cos\frac{kd}{2}\sin\frac{kd}{2} + jZ_1Z_1\sin\frac{kd}{2}\cos\frac{kd}{2} + Z_1Z_{1-}\sin^2\frac{kd}{2}$$

$$= Z_1Z_{1-}\cos^2\frac{kd}{2} - jZ_1Z_1\sin\frac{kd}{2}\cos\frac{kd}{2} + jZ_{1-}Z_{1+}\cos\frac{kd}{2}\sin\frac{kd}{2} + Z_1Z_{1+}\sin^2\frac{kd}{2}$$

$\therefore$ for equality

$$Z_1Z_{1+}\cos^2\frac{kd}{2} + Z_1Z_{1-}\sin^2\frac{kd}{2} = Z_1Z_{1-}\cos^2\frac{kd}{2} + Z_1Z_{1+}\sin^2\frac{kd}{2}$$

$$Z_{1+}\cos^2\frac{kd}{2} + Z_{1-}\sin^2\frac{kd}{2} = Z_{1-}\cos^2\frac{kd}{2} + Z_{1+}\sin^2\frac{kd}{2}$$

$$-Z_{1+}Z_{1-}\cancel{\cos\frac{kd}{2}}\cancel{\sin\frac{kd}{2}} + Z_1Z_1\cancel{\sin\frac{kd}{2}}\cancel{\cos\frac{kd}{2}} = -Z_1Z_1\cancel{\sin\frac{kd}{2}}\cancel{\cos\frac{kd}{2}} + Z_{1-}Z_{1+}\cancel{\cos\frac{kd}{2}}\cancel{\sin\frac{kd}{2}}$$

$$-Z_{1+}Z_{1-} + Z_1^2 = -Z_1^2 + Z_{1-}Z_{1+}$$

$$\left(-Z_{1+}Z_{1-} + Z_1^2\right) = -\left(Z_1^2 - Z_{1-}Z_{1+}\right)$$

$$\therefore \quad Z_1^2 = Z_{1+}Z_{1-}$$

$$\cancel{Z_0}\left[+\cancel{j}\tan k'L\right]\cos^2\frac{kd}{2} + \cancel{Z_0}\left[-\cancel{j\tan k'L}\right]\sin^2\frac{kd}{2} = \cancel{Z_0}\left[-\cancel{j\tan k'L}\right]\cos^2\frac{kd}{2} + \cancel{Z_0}\left[\cancel{j}\tan kL\right]\sin^2\frac{kd}{2}$$

$$\cos^2\frac{kd}{2} + \sin^2\frac{kd}{2} = -\cos^2\frac{kd}{2} + \sin^2\frac{kd}{2}$$

$$\therefore \quad \cos^2\frac{kd}{2} - \sin^2\frac{kd}{2} = 0$$

$$\therefore \quad \frac{kd}{2} = \frac{\pi}{4}$$

$$\boxed{k} = \frac{\pi}{2d}$$

k is measured in the dielectric region of the wave structure

i.e. $k = \omega\sqrt{\mu_0 \epsilon_0}$

$$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_1}}$$

$$\therefore \quad \omega\sqrt{\mu_0 \epsilon_0} = \frac{\pi}{2d}$$

$$\omega = \frac{\pi}{2d\sqrt{\mu_0 \epsilon_0}}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$k' \triangleq$ propagation constant in the free space region

$$\sqrt{\frac{\mu_0}{\epsilon_1}}\sqrt{\frac{\mu_0}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_0}}(j\tan k'L)\sqrt{\frac{\mu_0}{\epsilon_0}}(-j\tan k'L)$$

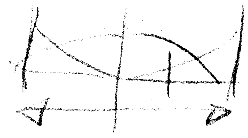
$$\frac{\epsilon_0 \epsilon_1}{\cancel{\mu_0}} \frac{\cancel{\mu_0}}{\epsilon_1} = \frac{\cancel{\mu_0}}{\cancel{\epsilon_0}} \tan^2 k'L \quad \frac{\epsilon_0 \epsilon_1}{\mu_0}$$

$$\frac{\epsilon_0}{\epsilon_1} = \tan^2 k'L$$

$$\sqrt{\frac{\epsilon_0}{\epsilon_1}} = \tan k'L$$

$$k'L = \mathrm{Tan}^{-1}\sqrt{\frac{\epsilon_0}{\epsilon_1}}$$

$$k' = \frac{\mathrm{Tan}^{-1}\sqrt{\frac{\epsilon_0}{\epsilon_1}}}{L}$$



for resonance :

The propagation constants β & β' must be equal at the interface

$$\Rightarrow k = k'$$

$$\omega \sqrt{\mu_0 \epsilon_0'} = \frac{\mathrm{Tan}^{-1} \sqrt{\frac{\epsilon_0}{\epsilon_1}}}{L}$$

$$\omega_{resonant} = \frac{\mathrm{Tan}^{-1} \sqrt{\frac{\epsilon_0}{\epsilon_1}}}{L \sqrt{\mu_0 \epsilon_0'}}$$

I was trying to do this using an impedance concept

I used as a reference

"Electromagnetic Theory For Engineering Applications"

by W.L Weeks

which said that if one breaks open a cavity at a point, measures the impedance looking to the right, and then measures the impedance looking to the left, call these values Z_R & Z_L.

If $Z_R = Z_L$ the circuit is resonant.

But using this technique I only obtained one resonant frequency. $\Rightarrow$ something is wrong.