3

Transmission lines and waveguides

Transmission lines and waveguides are used to transport electromagnetic energy at microwave frequencies from one point in a system to another without radiation of energy taking place. The two main characteristics desired in a transmission line or waveguide are single-mode propagation over a wide band of frequencies and small attenuation. A great variety of transmission lines and waveguides having these two essential features have been investigated. Most of the structures considered fall into one of the following three categories: (1) transmission lines on which the dominant mode of propagation is a transverse electromagnetic wave, (2) closed cylindrical conducting tubes, and (3) open-boundary structures that support a surface-wave mode of propagation. It will not be possible to examine in detail all the different structures that have been introduced for waveguiding. We shall restrict the discussion to examining the basic theory of transmission lines and empty cylindrical waveguides, with specific reference to the commonly used coaxial transmission line and the rectangular and circular waveguides, and to presenting an introduction to surface waveguides. The extensions and modifications of the theory, necessary for analyzing other specific structures, are not difficult. The student should find little difficulty in reading the literature devoted to various types of waveguides, once familiarity with the theory given here is acquired.

Transmission lines consist of two or more parallel conductors and will guide a transverse electromagnetic (TEM) wave. The common forms of transmission lines are the two-conductor line, the shielded two-conductor line, and the coaxial line. Another form of transmission line that has come into prominence in the last few years is the microwave strip line, which consists of a thin conducting ribbon separated from a wider ground plane by a dielectric sheet or placed between two ground planes to form a shielded structure. The two main advantages obtained with strip lines are the reduction in size and weight and the ability to use printed-

circuit techniques for the construction of the strip lines and associated components such as bends, junctions, filters, etc. A good introduction to strip lines and associated components may be found in a special issue of *IRE Transactions* devoted entirely to this subject.†

The common forms of waveguides are the rectangular and circular guides, which will be analyzed in detail in this chapter. A number of other structures have also been proposed that offer some distinct advantages for certain applications, but the theory of many of them is not sufficiently different from that of the common rectangular and circular guides to warrant detailed treatment.‡

3.1 Classification of wave solutions

The transmission lines and waveguides to be analyzed in this chapter are all characterized by having axial uniformity. Their cross-sectional shape and electrical properties do not vary along the axis, which is chosen as the z axis. Since sources are not considered, the electric and magnetic fields are solutions of the homogeneous vector Helmholtz equation, i.e.,

$$\nabla^2\mathbf{E} + k_0^2\mathbf{E} = 0 \qquad \nabla^2\mathbf{H} + k_0^2\mathbf{H} = 0$$

The type of solution sought is that corresponding to a wave that propagates along the z axis. Since the Helmholtz equation is separable, it is possible to find solutions of the form $f(z)g(x, y)$, where f is a function of z only and g is a function of x and y or other suitable transverse coordinates only. The second derivative with respect to z enters into the wave equation in a manner similar to the second derivative with respect to time. By analogy with $e^{j\omega t}$ as the time dependence, the z dependence can be assumed to be $e^{\pm j\beta z}$. This assumption will lead to wave solutions of the

† Special Issue on Microwave Strip Circuits, *IRE Trans.*, vol. MTT-3, March, 1955.

‡ As an introduction to some of these other structures, the following references may be consulted. For a complete survey with an extensive bibliography the book by Harvey should be consulted.

S. B. Cohn, Properties of Ridged Waveguides, *Proc. IRE*, vol. 35, pp. 783–788, August, 1947.

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F. J. Tischer, Properties of the H-guide at Microwave and Millimetre Wave Regions, *Proc. IEE*, vol. 106B, suppl. 13, 1959.

F. J. Tischer, The Groove Guide: A Low Loss Waveguide for Millimetre Waves, *IEEE Trans.*, vol. MTT-11, pp. 291–296, September, 1963.

form $\cos(\omega t \pm \beta z)$ and $\sin(\omega t \pm \beta z)$, which are appropriate for describin wave propagation along the z axis. A wave propagating in the positiv z direction is represented by $e^{-j\beta z}$, and $e^{j\beta z}$ corresponds to a wave propa gating in the negative z direction. With an assumed z dependence $e^{-j\beta}$ the del operator becomes $\nabla = \nabla_t + \nabla_z = \nabla_t - j\beta \mathbf{a}_z$, since $\nabla_z = \mathbf{a}_z\, \partial/\partial$ Note that ∇_t is the transverse part and equals $\mathbf{a}_x\, \partial/\partial x + \mathbf{a}_y\, \partial/\partial y$ i rectangular coordinates. The propagation phase constant β will tur out to depend on the waveguide configuration.

Considerable simplification of Maxwell's equations is obtained b decomposing all fields into transverse and axial components and sepa rating out the z dependence. Thus, let (the time dependence $e^{j\omega t}$ i suppressed)

$$\mathbf{E}(x,y,z) = \mathbf{E}_t(x,y,z) + \mathbf{E}_z(x,y,z) = \mathbf{e}(x,y)e^{-j\beta z} + \mathbf{e}_z(x,y)e^{-j\beta z} \tag{3.1}$$

$$\mathbf{H}(x,y,z) = \mathbf{H}_t(x,y,z) + \mathbf{H}_z(x,y,z) = \mathbf{h}(x,y)e^{-j\beta z} + \mathbf{h}_z(x,y)e^{-j\beta z} \tag{3.2}$$

where $\mathbf{E}_t$, $\mathbf{H}_t$ are the transverse (x and y) components, and $\mathbf{E}_z$, $\mathbf{H}_z$ are th axial components. Note also that $\mathbf{e}(x,y)$, $\mathbf{h}(x,y)$ are transverse vecto functions of the transverse coordinates only, and $\mathbf{e}_z(x,y)$, $\mathbf{h}_z(x,y)$ ar axial vector functions of the transverse coordinates.

Consider the $\nabla \times \mathbf{E}$ equation, which may be expanded to give

$$\nabla \times \mathbf{E} = (\nabla_t - j\beta\mathbf{a}_z) \times (\mathbf{e} + \mathbf{e}_z)e^{-j\beta z} = -j\omega\mu_0(\mathbf{h} + \mathbf{h}_z)e^{-j\beta z}$$

or

$$\nabla_t \times \mathbf{e} - j\beta\mathbf{a}_z \times \mathbf{e} + \nabla_t \times \mathbf{e}_z - j\beta\mathbf{a}_z \times \mathbf{e}_z = -j\omega\mu_0\mathbf{h} - j\omega\mu_0\mathbf{h}_z$$

The term $\mathbf{a}_z \times \mathbf{e}_z = 0$, and $\nabla_t \times \mathbf{e}_z = \nabla_t \times \mathbf{a}_z e_z = -\mathbf{a}_z \times \nabla_t e_z$. Note als that $\nabla_t \times \mathbf{e}$ is directed along the z axis only, since it involves factors suc as $\mathbf{a}_x \times \mathbf{a}_y$, $\mathbf{a}_x \times \mathbf{a}_x$, $\mathbf{a}_y \times \mathbf{a}_x$, and $\mathbf{a}_y \times \mathbf{a}_y$, whereas $\mathbf{a}_z \times \mathbf{e}$ and $\nabla_t \times \mathbf{e}_z$ hav transverse components only. Consequently, when the transverse an axial components of the above equation are equated, there results

$$\nabla_t \times \mathbf{e} = -j\omega\mu_0\mathbf{h}_z \tag{3.3a}$$

$$\nabla_t \times \mathbf{e}_z - j\beta\mathbf{a}_z \times \mathbf{e} = -\mathbf{a}_z \times \nabla_t e_z - j\beta\mathbf{a}_z \times \mathbf{e} = -j\omega\mu_0\mathbf{h} \tag{3.3b}$$

In a similar manner the $\nabla \times \mathbf{H}$ equation yields

$$\nabla_t \times \mathbf{h} = j\omega\epsilon_0\mathbf{e}_z \tag{3.3c}$$

$$\mathbf{a}_z \times \nabla_t h_z + j\beta\mathbf{a}_z \times \mathbf{h} = -j\omega\epsilon_0\mathbf{e} \tag{3.3d}$$

The divergence equation $\nabla \cdot \mathbf{B} = 0$ becomes

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mu_0\mathbf{H} = (\nabla_t - j\beta\mathbf{a}_z) \cdot (\mathbf{h} + \mathbf{h}_z)\mu_0 e^{-j\beta z} = (\nabla_t \cdot \mathbf{h} - j\beta\mathbf{a}_z \cdot \mathbf{h}_z)\mu_0 e^{-j\beta z} = 0$$

or

$$\nabla_t \cdot \mathbf{h} = j\beta h_z \tag{3.3e}$$

Similarly, $\nabla \cdot \mathbf{D} = 0$ gives

$$\nabla_t \cdot \mathbf{e} = j\beta e_z \tag{3.3f}$$

This reduced form of Maxwell's equations will prove to be very useful in formulating the solutions for waveguiding systems.

For a large variety of waveguides of practical interest it turns out that all the boundary conditions can be satisfied by fields that do not have all components present. Specifically, for transmission lines, the solution of interest is a transverse electromagnetic wave with transverse components only, that is, $E_z = H_z = 0$, whereas for waveguides, solutions with $E_z = 0$ or $H_z = 0$ are possible. Because of the widespread occurrence of such field solutions, the following classification of solutions is of particular interest.

1. Transverse electromagnetic (TEM) waves. For TEM waves, $E_z = H_z = 0$. The electric field may be found from the transverse gradient of a scalar function $\Phi(x, y)$, which is a function of the transverse coordinates only and is a solution of the two-dimensional Laplace's equation.
2. Transverse electric (TE), or H, modes. These solutions have $E_z = 0$, but $H_z \neq 0$. All the field components may be derived from the axial component H_z of magnetic field.
3. Transverse magnetic (TM), or E, modes. These solutions have $H_z = 0$, but $E_z \neq 0$. The field components may be derived from E_z.

In some cases it will be found that a TE or TM mode by itself will not satisfy all the boundary conditions. However, in such cases linear combinations of TE and TM modes may be used, since such linear combinations always provide a complete and general solution. Although other possible types of wave solutions may be constructed, the above three types are the most useful in practice and by far the most commonly used ones.

The appropriate equations to be solved to obtain TEM, TE, or TM modes will be derived below by placing E_z and H_z, E_z, and H_z, respectively, equal to zero in Maxwell's equations.

TEM waves

For TEM waves $e_z = h_z = 0$; so (3.3) reduces to

$$\nabla_t \times \mathbf{e} = 0 \tag{3.4a}$$

$$\beta\mathbf{a}_z \times \mathbf{e} = \omega\mu_0\mathbf{h} \tag{3.4b}$$

$$\nabla_t \times \mathbf{h} = 0 \tag{3.4c}$$

$$\beta \mathbf{a}_z \times \mathbf{h} = -\omega\epsilon_0 \mathbf{e} \tag{3.4d}$$

$$\nabla_t \cdot \mathbf{h} = 0 \tag{3.4e}$$

$$\nabla_t \cdot \mathbf{e} = 0 \tag{3.4f}$$

The vanishing of the transverse curl of **e** means that the line integral of **e** around any closed path in the xy plane is zero. This must clearly be so since there is no axial magnetic flux passing through such a contour. Although $\nabla_t \times \mathbf{h} = 0$ when there are no volume currents present, the line integral of **h** will not vanish for a transmission line with conductors on which axial currents may exist. This point will be considered again later when transmission lines are analyzed. Equation (3.4a) is just the condition that permits **e** to be expressed as the gradient of a scalar potential. Hence, let

$$\mathbf{e}(x, y) = -\nabla_t \Phi(x, y) \tag{3.5}$$

Using (3.4f) shows that Φ is a solution of the two-dimensional Laplace equation

$$\nabla_t^2 \Phi(x, y) = 0 \tag{3.6}$$

The electric field is thus given by

$$\mathbf{E}_t(x, y, z) = -\nabla_t \Phi(x, y) e^{-j\beta z}$$

But this field must also satisfy the Helmholtz equation

$$\nabla^2 \mathbf{E}_t + k_0^2 \mathbf{E}_t = 0$$

Since $\nabla = \nabla_t - j\beta \mathbf{a}_z$, $\nabla^2 = \nabla_t^2 - \beta^2$, that is, the second derivative with respect to z gives a factor $-\beta^2$, this reduces to

$$\nabla_t^2 \mathbf{E}_t + (k_0^2 - \beta^2)\mathbf{E}_t = 0$$

or

$$\nabla_t[\nabla_t^2 \Phi + (k_0^2 - \beta^2)\Phi] = 0$$

This shows that $\beta = \pm k_0$ for TEM waves, a result to be anticipated from the wave solutions discussed in Chap. 2. The magnetic field may be found from the $\nabla \times \mathbf{E}$ equation, i.e., from (3.4b); thus

$$\frac{\omega\mu_0}{k_0} \mathbf{h} = \mathbf{a}_z \times \mathbf{e} = Z_0 \mathbf{h} \tag{3.7}$$

In summary, for TEM waves, first find a scalar potential Φ which is a solution of

$$\nabla_t^2 \Phi(x, y) = 0 \tag{3.8a}$$

and satisfies the proper boundary conditions. The fields are then given by

$$\mathbf{E} = \mathbf{E}_t = \mathbf{e} e^{\mp jk_0 z} = -\nabla_t \Phi e^{\mp jk_0 z} \tag{3.8b}$$

$$\mathbf{H} = \mathbf{H}_t = \pm \mathbf{h} e^{\mp jk_0 z} = \pm Y_0 \mathbf{a}_z \times \mathbf{e} e^{\mp jk_0 z} \tag{3.8c}$$

where $k_0 = \omega(\mu_0\epsilon_0)^{\frac{1}{2}}$, $Y_0 = (\epsilon_0/\mu_0)^{\frac{1}{2}}$, and $e^{-jk_0 z}$ represents a wave propagating in the $+z$ direction and $e^{jk_0 z}$ corresponds to wave propagation in the $-z$ direction. For TEM waves, Z_0 is the wave impedance, and from (3.8c) it is seen that, for wave propagation in the $+z$ direction,

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = Z_0 \tag{3.9a}$$

whereas for propagation in the $-z$ direction,

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = -Z_0 \tag{3.9b}$$

TE waves

For transverse electric (TE) waves, h_z plays the role of a potential function from which the rest of the field components may be obtained. The magnetic field **H** is a solution of

$$\nabla^2 \mathbf{H} + k_0^2 \mathbf{H} = 0$$

Separating the above equation into transverse and axial parts and replacing ∇^2 by $\nabla_t^2 - \beta^2$ yield

$$\nabla_t^2 h_z(x, y) + k_c^2 h_z(x, y) = 0 \tag{3.10a}$$

$$\nabla_t^2 \mathbf{h} + k_c^2 \mathbf{h} = 0 \tag{3.10b}$$

where $k_c^2 = k_0^2 - \beta^2$ and a z dependence $e^{-j\beta z}$ is assumed. Unlike the case of TEM waves, β^2 will not equal k_0^2 for TE waves. Instead, β is determined by the parameter k_c^2 in (3.10a). When this equation is solved, subject to appropriate boundary conditions, the eigenvalue k_c^2 will be found to be a function of the waveguide configuration.

The Maxwell equations (3.3) with $\mathbf{e}_z = 0$ become

$$\nabla_t \times \mathbf{e} = -j\omega\mu_0 \mathbf{h}_z \tag{3.11a}$$

$$\beta \mathbf{a}_z \times \mathbf{e} = \omega\mu_0 \mathbf{h} \tag{3.11b}$$

$$\nabla_t \times \mathbf{h} = 0 \tag{3.11c}$$

$$\mathbf{a}_z \times \nabla_t h_z + j\beta \mathbf{a}_z \times \mathbf{h} = -j\omega\epsilon_0 \mathbf{e} \tag{3.11d}$$

$$\nabla_t \cdot \mathbf{h} = j\beta h_z \tag{3.11e}$$

$$\nabla_t \cdot \mathbf{e} = 0 \tag{3.11f}$$

The transverse curl of (3.11c) gives

$$\nabla_t \times (\nabla_t \times \mathbf{h}) = \nabla_t \nabla_t \cdot \mathbf{h} - \nabla_t^2 \mathbf{h} = 0$$

Replacing $\nabla_t \cdot \mathbf{h}$ by $j\beta h_z$ from (3.3c) and $\nabla_t^2 \mathbf{h}$ by $-k_c^2 \mathbf{h}$ from (3.10b) leads to the solution for $\mathbf{h}$ in terms of h_z; namely,

$$\mathbf{h} = -\frac{j\beta}{k_c^2} \nabla_t h_z \qquad (3.12)$$

To find $\mathbf{e}$ in terms of $\mathbf{h}$, take the vector product of (3.11b) with $\mathbf{a}_z$ to obtain

$$\beta \mathbf{a}_z \times (\mathbf{a}_z \times \mathbf{e}) = \beta[(\mathbf{a}_z \cdot \mathbf{e})\mathbf{a}_z - (\mathbf{a}_z \cdot \mathbf{a}_z)\mathbf{e}] = -\beta \mathbf{e} = \omega\mu_0 \mathbf{a}_z \times \mathbf{h}$$

or

$$\mathbf{e} = -\frac{\omega\mu_0}{\beta} \mathbf{a}_z \times \mathbf{h} = -\frac{k_0}{\beta} Z_0 \mathbf{a}_z \times \mathbf{h} \qquad (3.13)$$

The factor $k_0 Z_0/\beta$ has the dimensions of an impedance, and is called the wave impedance of TE, or H, modes. It will be designated by the symbol Z_h, so that

$$Z_h = \frac{k_0}{\beta} Z_0 \qquad (3.14)$$

Thus, in component form, (3.13) gives

$$\frac{e_x}{h_y} = -\frac{e_y}{h_x} = Z_h \qquad (3.15)$$

for a wave with z dependence $e^{-j\beta z}$.

The remaining equations in the set (3.11) do not yield any new results; so the solution for TE waves may be summarized as follows: First find a solution for h_z, where

$$\nabla_t^2 h_z + k_c^2 h_z = 0 \qquad (3.16a)$$

Then

$$\mathbf{h} = -\frac{j\beta}{k_c^2} \nabla_t h_z \qquad (3.16b)$$

and

$$\mathbf{e} = -Z_h \mathbf{a}_z \times \mathbf{h} \qquad (3.16c)$$

where

$$\beta = (k_0^2 - k_c^2)^{\frac{1}{2}} \qquad \text{and} \qquad Z_h = \frac{k_0 Z_0}{\beta}$$

Complete expressions for the fields are

$$\mathbf{H} = \pm \mathbf{h} e^{\mp j\beta z} + \mathbf{h}_z e^{\mp j\beta z} \qquad (3.16d)$$

$$\mathbf{E} = \mathbf{E}_t = \mathbf{e} e^{\mp j\beta z} \qquad (3.16e)$$

Note that in (3.16d) the sign in front of $\mathbf{h}$ is reversed for a wave propagating in the $-z$ direction since $\mathbf{h}$ will be defined by (3.16b), with β positive

regardless of whether propagation is in the $+z$ or $-z$ direction. The sign in front of $\mathbf{e}$ does not change since it involves the factor β twice, once in the expression for $\mathbf{h}$ and again in Z_h. Only the sign of one of $\mathbf{e}$ or $\mathbf{h}$ can change if a reversal in the direction of energy flow is to occur. That is, the solution for a wave propagating in the $-z$ direction can be chosen as $\mathbf{E} = -\mathbf{e}e^{j\beta z}$, $\mathbf{H} = (\mathbf{h} - \mathbf{h}_z)e^{j\beta z}$ or as $\mathbf{E} = \mathbf{e}e^{j\beta z}$, $\mathbf{H} = (-\mathbf{h} + \mathbf{h}_z)e^{j\beta z}$. One solution is the negative of the other. The latter solution is arbitrarily chosen as the standard in this text.

TM waves

The TM, or E, waves have $\mathbf{h}_z = 0$, but the axial electric field $\mathbf{e}_z$ is not zero. These modes may be considered the dual of the TE modes in that the roles of electric and magnetic fields are interchanged. The derivation of the equations to be solved parallels that for TE waves, and hence only the final results will be given.

First obtain a solution for e_z, where

$$\nabla_t^2 e_z + k_c^2 e_z = 0 \qquad (3.17a)$$

subject to the boundary conditions imposed. This will serve to determine the eigenvalue k_c^2. The transverse fields are then given by

$$\mathbf{E}_t = \mathbf{e}e^{\mp j\beta z} = -\frac{j\beta}{k_c^2} \nabla_t e_z e^{\mp j\beta z} \qquad (3.17b)$$

$$\mathbf{H}_t = \pm \mathbf{h}e^{\mp j\beta z} = \pm Y_e \mathbf{a}_z \times \mathbf{e}e^{\mp j\beta z} \qquad (3.17c)$$

where $\beta = (k_0^2 - k_c^2)^{\frac{1}{2}}$ and the wave admittance Y_e for TM waves is given by

$$Y_e = Z_e^{-1} = \frac{k_0}{\beta} Y_0 \qquad (3.17d)$$

The dual nature of TE and TM waves is exhibited by the relation

$$Z_e Z_h = Z_0^2 \qquad (3.18)$$

which holds when both types of waves have the same value of β and is derivable from (3.14) and (3.17d). The complete expression for the electric field is

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_z = \mathbf{e}e^{\mp j\beta z} \pm \mathbf{e}_z e^{\mp j\beta z}$$

$$= \left(-\frac{j\beta}{k_c^2} \nabla_t e_z \pm e_z\right) e^{\mp j\beta z} \qquad (3.19)$$

It is convenient to keep the sign of $\mathbf{e}$ the same for propagation in either the $+z$ or $-z$ direction. Since $\nabla \cdot \mathbf{E} = 0$, that is, $\nabla_t \cdot \mathbf{E}_t + \partial E_z/\partial z = 0$, this requires that the z component of electric field be $-\mathbf{e}_z e^{j\beta z}$ for a wave propagating in the $-z$ direction, because $\nabla_t \cdot \mathbf{E}_t$ does not change sign, whereas $\partial E_z/\partial z$ does, in view of the change in sign in front of β. The transverse magnetic field must also change sign upon reversal of the direction of propagation in order to obtain a change in the direction of energy flow. For reference, this sign convention is summarized below. The transverse variations of the fields are represented by the functions $\mathbf{e}$, $\mathbf{h}$, $\mathbf{e}_z$, and $\mathbf{h}_z$, independent of the direction of propagation. Waves propagating in the $+z$ direction are then given by

$$\mathbf{E} = \mathbf{E}^+ = (\mathbf{e} + \mathbf{e}_z)e^{-j\beta z} \tag{3.20a}$$

$$\mathbf{H} = \mathbf{H}^+ = (\mathbf{h} + \mathbf{h}_z)e^{-j\beta z} \tag{3.20b}$$

For propagation in the $-z$ directions the fields are

$$\mathbf{E} = \mathbf{E}^- = (\mathbf{e} - \mathbf{e}_z)e^{j\beta z} \tag{3.21a}$$

$$\mathbf{H} = \mathbf{H}^- = (-\mathbf{h} + \mathbf{h}_z)e^{j\beta z} \tag{3.21b}$$

Additional superscripts (+) or (−) will be used when it is necessary to indicate the direction of propagation. The previously derived equations for TEM, TE, and TM modes are valid in a medium with electrical constants ϵ, μ, provided these are used to replace ϵ_0, μ_0. A finite conductivity can also be taken into account by making ϵ complex, i.e., replacing ϵ by $\epsilon - j\sigma/\omega$.

The wave impedance introduced in the solutions is an extremely useful concept in practice. The wave impedance is always chosen to relate the transverse components of the electric and magnetic fields. The sign is always such that if i, j, k is a cyclic labeling of the coordinates and propagation is along the positive direction of coordinate k, the ratio $E_i/H_j = (Z_w)_k$ is positive. Here $(Z_w)_k$ is the wave impedance referred to the k axis as the direction of propagation. If i, j, k form an odd permutation of the coordinates, then E_i/H_j is negative. The usefulness of the wave-impedance concept stems from the fact that the power is given in terms of the transverse fields alone. For example, for TE waves

$$\begin{aligned} P &= \tfrac{1}{2}\,\mathrm{Re}\int_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_z\,dx\,dy \\ &= \tfrac{1}{2}\,\mathrm{Re}\int_S \mathbf{e}\times\mathbf{h}^*\cdot\mathbf{a}_z\,dx\,dy \\ &= -\tfrac{1}{2}\,\mathrm{Re}\int_S Z_h(\mathbf{a}_z\times\mathbf{h})\times\mathbf{h}^*\cdot\mathbf{a}_z\,dx\,dy \\ &= \frac{Z_h}{2}\int_S \mathbf{h}\cdot\mathbf{h}^*\,dx\,dy = \frac{Y_h}{2}\int_S \mathbf{e}\cdot\mathbf{e}^*\,dx\,dy \end{aligned}$$

upon expanding the integrand. Thus the wave impedance enables the power transmitted to be expressed in terms of one of the transverse fields alone. A further property of the wave impedance, which will be dealt with later, is that it provides a basis for an analogy between conventional multiconductor transmission lines and waveguides.

3.2 Transmission lines (field analysis)

Lossless transmission line

A transmission line consists of two or more parallel conductors. Typical examples are the two-conductor line, shielded two-conductor line, and coaxial line with cross sections, as illustrated in Fig. 3.1. Initially, it will be assumed that the conductors are perfectly conducting and that the medium surrounding the conductors is air, with $\epsilon \approx \epsilon_0$, $\mu \approx \mu_0$. The effect of small losses will be considered later.

With reference to Fig. 3.2, let the one conductor be at a potential $V_0/2$ and the other conductor at $-V_0/2$. To determine the field of a TEM wave, a suitable potential $\Phi(x, y)$ must be found first. It is necessary that Φ be a solution of

$$\nabla_t^2\Phi = 0$$

and satisfy the boundary conditions

$$\Phi = \begin{cases} \dfrac{V_0}{2} & \text{on } S_2 \\ -\dfrac{V_0}{2} & \text{on } S_1 \end{cases}$$

Since Φ is unique only to within an additive constant, we could equally

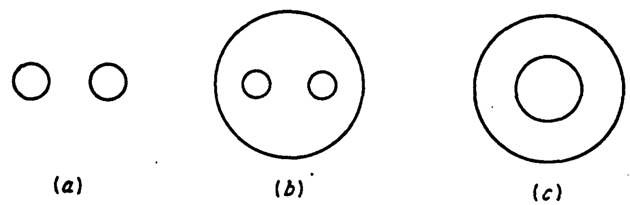

Fig. 3.1 Cross sections of typical transmission lines. (*a*) Two-conductor line; (*b*) shielded two-conductor line; (*c*) coaxial line.

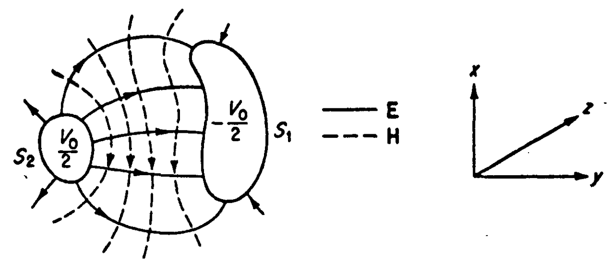

Fig. 3.2 Cross section of a general two-conductor line showing transverse field patterns.

well choose $\Phi = V_0$ on S_2 and $\Phi = 0$ on S_1. If a solution for Φ is possible, a TEM mode or field solution is also possible. When two or more conductors are present, this is always the case. The solution for Φ is an electrostatic problem that can be solved when the line configuration is simple enough, as exemplified in Fig. 3.1.

The fields are given by (3.8), and for propagation in the $+z$ direction are

$$\mathbf{E} = \mathbf{E}_t = \mathbf{e}e^{-jk_0z} = -\nabla_t\Phi e^{-jk_0z} \tag{3.22a}$$

$$\mathbf{H} = \mathbf{H}_t = Y_0\mathbf{a}_z \times \mathbf{e}e^{-jk_0z} \tag{3.22b}$$

The line integral of $\mathbf{e}$ between the two conductors is

$$\int_{S_1}^{S_2} \mathbf{e}\cdot d\mathbf{l} = \int_{S_1}^{S_2} -\nabla_t\Phi\cdot d\mathbf{l}$$

$$= -\int_{S_1}^{S_2}\frac{d\Phi}{dl}\,dl = -[\Phi(S_2) - \Phi(S_1)] = -V_0$$

Associated with the electric field is a unique voltage wave

$$V = V_0e^{-jk_0z} \tag{3.23}$$

since the line integral of $\mathbf{e}$ between S_1 and S_2 is independent of the path chosen because $\mathbf{e}$ is the gradient of a scalar potential.

The line integral of $\mathbf{h}$ around one conductor, say S_2, gives

$$\oint_{S_2}\mathbf{h}\cdot d\mathbf{l} = \oint_{S_2} J_z\,dl = I_0$$

by application of Ampère's law, $\nabla\times\mathbf{H} = j\omega\mathbf{D} + \mathbf{J}$, and noting that there is no axial displacement flux D_z for a TEM mode. On the conductors the boundary conditions require $\mathbf{n}\times\mathbf{e} = 0$ and $\mathbf{n}\times\mathbf{h} = \mathbf{J}_s$, where $\mathbf{n}$ is a unit outward normal and $\mathbf{J}_s$ is the surface current density. Since $\mathbf{n}$ and $\mathbf{h}$ lie in a transverse plane, the current $\mathbf{J}_s$ is in the axial direction. In the region remote from the conductors, $\nabla_t\times\mathbf{h} = 0$, but the line integral around a conductor is not zero because of the current that exists. The current on the two conductors is oppositely directed, as may be verified from the expression $\mathbf{n}\times\mathbf{h} = \mathbf{J}_s$. Associated with the magnetic field there is a unique current wave

$$I = I_0e^{-jk_0z} \tag{3.24}$$

Since the potential Φ is independent of frequency, it follows that the transverse fields $\mathbf{e}$ and $\mathbf{h}$ are also independent of frequency and are, in actual fact, the static field distributions which exist between the conductors if the potential difference is V_0 and currents I_0, $-I_0$ exist on S_2, S_1, respectively. The magnetic lines of flux coincide with the equipotential lines, since $\mathbf{e}$ and $\mathbf{h}$ are orthogonal, as seen from (3.22b).

Example 3.1 Coaxial line Figure 3.3 illustrates a coaxial transmission line for which the solution for a TEM mode will be constructed. In

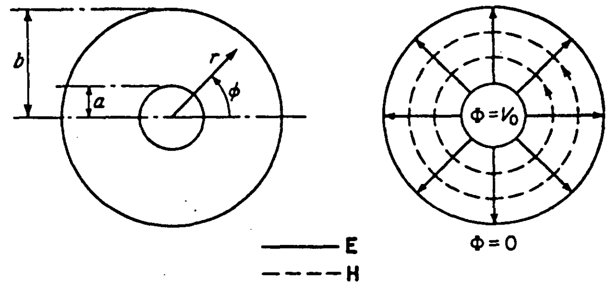

Fig. 3.3 Coaxial transmission line.

cylindrical coordinates r, ϕ, z, the two-dimensional Laplace equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

or for a potential function independent of the angular coordinate ϕ,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = 0$$

Integrating this equation twice gives

$$\Phi = C_1\ln r + C_2$$

Imposing the boundary conditions $\Phi = V_0$ at $r = a$, $\Phi = 0$ at $r = b$, gives

$$V_0 = C_1\ln a + C_2 \qquad 0 = C_1\ln b + C_2$$

and hence $C_2 = -C_1\ln b$, $C_1 = V_0/[\ln(a/b)]$,

$$\Phi = V_0\frac{\ln(r/b)}{\ln(a/b)} \tag{3.25}$$

The electric and magnetic fields of a TEM mode propagating in the $+z$ direction are given by (3.22) and are

$$\mathbf{E} = -\mathbf{a}_r\frac{\partial\Phi}{\partial r}e^{-jk_0z} = -\frac{V_0}{\ln(a/b)}\frac{\mathbf{a}_r}{r}e^{-jk_0z}$$

$$= \frac{V_0}{\ln(b/a)}\frac{\mathbf{a}_r}{r}e^{-jk_0z} \tag{3.26a}$$

$$\mathbf{H} = Y_0\mathbf{a}_z\times\mathbf{e}e^{-jk_0z} = \frac{Y_0V_0}{\ln(b/a)}\frac{\mathbf{a}_\phi}{r}e^{-jk_0z} \tag{3.26b}$$

The potential difference between the two conductors is obviously V_0; so the voltage wave associated with the electric field is

$$V = V_0e^{-jk_0z} \tag{3.27}$$

The current density on the inner conductor is

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H} = \mathbf{a}_r \times \mathbf{H} = \frac{Y_0 V_0}{\ln (b/a)} \frac{\mathbf{a}_z}{a} e^{-jk_0 z}$$

The total current, apart from the factor e^{-jk_0z}, is

$$I_0 = \frac{Y_0 V_0}{a \ln (b/a)} \int_0^{2\pi} a\, d\phi = \frac{Y_0 V_0 2\pi}{\ln (b/a)} \tag{3.28}$$

The current on the inner surface of the outer conductor is readily shown to be equal to I_0 also, but directed in the $-z$ direction. The current wave associated with the magnetic field is therefore

$$I = I_0 e^{-jk_0 z} \tag{3.29}$$

The power, or rate of energy flow, along the line is given by

$$P = \frac{1}{2} \operatorname{Re} \int_a^b \int_0^{2\pi} \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{a}_z r\, dr\, d\phi = \frac{1}{2} \frac{Y_0 V_0^2}{[\ln (b/a)]^2} \int_a^b \int_0^{2\pi} \frac{d\phi\, dr}{r} = \frac{\pi Y_0 V_0^2}{\ln (b/a)} \tag{3.30}$$

The power transmitted is seen to be also given, as anticipated, by the expression

$$\tfrac{1}{2} \operatorname{Re} (VI^*) = \tfrac{1}{2} V_0 I_0 = \tfrac{1}{2} V_0^2 \frac{2\pi Y_0}{\ln (b/a)}$$

The characteristic impedance of the line is defined by the ratio

$$\frac{V_0}{I_0} = Z_c \tag{3.31}$$

in terms of which the power may be expressed as $P = \frac{1}{2} Z_c I_0^2 = \frac{1}{2} Y_c V_0^2$, where Y_c is the characteristic admittance of the line and equal to Z_c^{-1}. The characteristic impedance is a function of the cross-sectional shape of the transmission line.

Transmission line with small losses

Practical transmission lines always have some loss caused by the finite conductivity of the conductors and also loss that may be present in the dielectric material surrounding the conductors. Consider first the case when the conductors are surrounded by a dielectric with permittivity $\epsilon = \epsilon' - j\epsilon''$ but the conductors are still considered to be perfect. The presence of a lossy dielectric does not affect the solution for the scalar potential Φ. Consequently, the field solution is formally the same as for the ideal line, except that k_0 and Y_0 are replaced by $k = k_0(\kappa' - j\kappa'')^{\frac{1}{2}}$ and $Y = Y_0(\kappa' - j\kappa'')^{\frac{1}{2}}$, where the dielectric constant $\kappa = \kappa' - j\kappa'' = \epsilon/\epsilon_0$.

For small losses such that $\kappa'' \ll \kappa'$, the propagation constant is

$$jk = \alpha + j\beta = j(\kappa')^{\frac{1}{2}} k_0 \left(1 - j\frac{\kappa''}{\kappa'}\right)^{\frac{1}{2}} \approx j(\kappa')^{\frac{1}{2}} k_0 + \frac{\kappa'' k_0}{2(\kappa')^{\frac{1}{2}}}$$

Thus

$$\alpha = \frac{\kappa'' k_0}{2(\kappa')^{\frac{1}{2}}} \tag{3.32a}$$

$$\beta = (\kappa')^{\frac{1}{2}} k_0 \tag{3.32b}$$

where α is the attenuation constant and β is the phase constant. The wave consequently attenuates according to $e^{-\alpha z}$ as it propagates in the $+z$ direction.

It will be instructive to derive the above expression for α by means of a perturbation method that is widely used in the evaluation of the attenuation, or damping, factor for a low-loss physical system. This method is based on the assumption that the introduction of a small loss does not substantially perturb the field from its loss-free value. The known field distribution for the loss-free case is then used to evaluate the loss in the system, and from this the attenuation constant can be calculated. In the present case, if $\kappa'' = 0$, the loss-free solution is

$$\mathbf{E} = -\nabla_t \Phi e^{-jkz} \qquad \mathbf{H} = Y \mathbf{a}_z \times \mathbf{E}$$

where $k = (\kappa')^{\frac{1}{2}} k_0$ and $Y = (\kappa')^{\frac{1}{2}} Y_0$. When κ'' is small but not zero, the imaginary part of ϵ, that is, ϵ'', is equivalent to a conductivity

$$\sigma = \omega\epsilon'' = \omega\epsilon_0\kappa''$$

A conductivity σ results in a shunt current $\mathbf{J} = \sigma\mathbf{E}$ between the two conductors. The power loss per unit length of line is

$$P_l = \frac{1}{2\sigma} \int_S \mathbf{J} \cdot \mathbf{J}^*\, dS = \frac{\omega\epsilon''}{2} \int_S \mathbf{E} \cdot \mathbf{E}^*\, dS \tag{3.33}$$

where the integration is over the cross section of the line, and the loss-free solution for $\mathbf{E}$ is used to carry out the evaluation of P_l. Since loss is present, the power propagated along the line must decrease according to a factor $e^{-2\alpha z}$. The rate of decrease of power propagated along the line equals the power loss. If the power at $z = 0$ is P_0, then at z it is $P = P_0 e^{-2\alpha z}$. Consequently,

$$-\frac{\partial P}{\partial z} = P_l = 2\alpha P_0 e^{-2\alpha z} = 2\alpha P \tag{3.34}$$

which states that the power loss at any plane z is directly proportional to the total power P present at this plane. The power propagated along the line is given by

$$P = \tfrac{1}{2}\,\mathrm{Re}\int_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_z\,dS$$
$$= \frac{Y}{2}\,\mathrm{Re}\int_S \mathbf{E}\times(\mathbf{a}_z\times\mathbf{E}^*)\cdot\mathbf{a}_z\,dS = \frac{Y}{2}\int_S \mathbf{E}\cdot\mathbf{E}^*\,dS$$

Hence the attenuation α is given by

$$\alpha = \frac{P_l}{2P} = \frac{\sigma}{2Y} = \frac{\omega\epsilon''}{2Y_0(\kappa')^{\frac{1}{2}}} = k_0\frac{\kappa''}{2(\kappa')^{\frac{1}{2}}}$$

which is the same as the expression (3.32a). For this example the perturbation method does not offer any advantage. However, often the field solution for the lossy case is very difficult to find, in which case the perturbation method is extremely useful and simple to carry out by comparison with other methods. The case of transmission lines with conductors having finite conductivity is an important example of this, and is discussed below.

If the conductors of a transmission line have a finite conductivity, they exhibit a surface impedance

$$Z_m = \frac{1+j}{\sigma\delta_s} \tag{3.35}$$

where $\delta_s = (2/\omega\mu\sigma)^{\frac{1}{2}}$ is the skin depth (Sec. 2.9). At the surface the electric field must have a tangential component equal to $Z_m\mathbf{J}_s$, where $\mathbf{J}_s$ is the surface current density. Therefore it is apparent that an axial component of electric field must be present, and consequently the field is no longer that of a TEM wave. The axial component of electric field gives rise to a component of the Poynting vector directed into the conductor, and this accounts for the power loss in the conductor. Generally, it is very difficult to find the exact solution for the fields when the conductors have finite conductivity. However, since $|Z_m|$ is very small compared with Z_0, the axial component of electric field is also very small relative to the transverse components. Thus the field is very nearly that of the TEM mode in the loss-free case. The perturbation method outlined earlier may be used to evaluate the attenuation caused by finite conductivity.

The current density $\mathbf{J}_s$ is taken equal to $\mathbf{n}\times\mathbf{H}$, where $\mathbf{n}$ is the unit outward normal to the conductor surface and $\mathbf{H}$ is the *loss-free* magnetic field. The power loss in the surface impedance per unit length of line is

$$P_l = \tfrac{1}{2}\,\mathrm{Re}\,Z_m\oint_{S_1+S_2}\mathbf{J}_s\cdot\mathbf{J}_s^*\,dl$$
$$= \frac{R_m}{2}\oint_{S_1+S_2}(\mathbf{n}\times\mathbf{H})\cdot(\mathbf{n}\times\mathbf{H}^*)\,dl$$
$$= \frac{R_m}{2}\oint_{S_1+S_2}\mathbf{H}\cdot\mathbf{H}^*\,dl \tag{3.36}$$

where $R_m = 1/\sigma\delta_s$ is the high-frequency surface resistance, and

$$(\mathbf{n}\times\mathbf{H})\cdot(\mathbf{n}\times\mathbf{H}^*) = \mathbf{n}\cdot\mathbf{H}\times(\mathbf{n}\times\mathbf{H}^*)$$
$$= \mathbf{n}\cdot[(\mathbf{H}\cdot\mathbf{H}^*)\mathbf{n} - (\mathbf{H}\cdot\mathbf{n})\mathbf{H}^*] = \mathbf{H}\cdot\mathbf{H}^*$$

since $\mathbf{n}\cdot\mathbf{H} = 0$ for the infinite-conductivity case. The integration is taken around the periphery $S_1 + S_2$ of the two conductors. The attenuation constant arising from conductor loss is thus

$$\alpha = \frac{P_l}{2P} = \frac{R_m\oint_{S_1+S_2}\mathbf{H}\cdot\mathbf{H}^*\,dl}{2Z\int\mathbf{H}\cdot\mathbf{H}^*\,dS} \tag{3.37}$$

where the power propagated along the line is given by

$$\mathrm{Re}\,\tfrac{1}{2}\int\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_z\,dS = \tfrac{1}{2}Z\int\mathbf{H}\cdot\mathbf{H}^*\,dS$$

and Z is the intrinsic impedance of the medium; that is, $Z = (\mu/\epsilon)^{\frac{1}{2}}$.

When both dielectric and conductor losses are present, the attenuation constant is the sum of the attenuation constants arising from each cause, provided both attenuation constants are small.

Example 3.2 Lossy coaxial line Let the coaxial line in Fig. 3.3 be filled with a lossy dielectric ($\epsilon = \epsilon' - j\epsilon''$), and let the conductors have finite conductivity σ. For the loss-free case ($\epsilon'' = 0, \sigma = \infty$) the fields are given by (3.26), with k_0 and Y_0 replaced by $k = (\epsilon'/\epsilon_0)^{\frac{1}{2}}k_0$,

$$Y = \left(\frac{\epsilon'}{\epsilon_0}\right)^{\frac{1}{2}}Y_0$$

Thus

$$\mathbf{E} = \frac{V_0}{\ln(b/a)}\frac{\mathbf{a}_r}{r}e^{-jkz} \tag{3.38a}$$

$$\mathbf{H} = \frac{YV_0}{\ln(b/a)}\frac{\mathbf{a}_\phi}{r}e^{-jkz} \tag{3.38b}$$

The power propagated along the line is

$$P = \tfrac{1}{2}\,\mathrm{Re}\int_0^{2\pi}\int_a^b\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_z r\,d\phi\,dr = \frac{\pi YV_0^2}{\ln(b/a)} \tag{3.39}$$

The power loss P_{l1} from the lossy dielectric is, from (3.33),

$$P_{l1} = \frac{\omega\epsilon''}{2}\int_0^{2\pi}\int_a^b\mathbf{E}\cdot\mathbf{E}^* r\,d\phi\,dr = \frac{\omega\epsilon''V_0^2\pi}{\ln(b/a)} \tag{3.40a}$$

The power loss from finite conductivity is given by (3.36), and is

$$P_{l2} = \frac{R_m}{2}\frac{Y^2V_0^2}{[\ln(b/a)]^2}\int_0^{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)d\phi$$
$$= \frac{R_m\pi Y^2V_0^2}{[\ln(b/a)]^2}\frac{b+a}{ab} \tag{3.40b}$$

Hence the attenuation constant α for the coaxial line is given by

$$\begin{aligned}\alpha &= \frac{P_{l1} + P_{l2}}{2P} = \frac{\omega\epsilon''}{2Y} + \frac{R_m Y}{2\ln(b/a)}\frac{b+a}{ab}\\ &= \frac{k_0\kappa''}{2(\kappa')^{\frac{1}{2}}} + \frac{R_m}{2Z\ln(b/a)}\frac{b+a}{ab}\end{aligned} \tag{3.41}$$

For the lossy case the propagation constant is consequently taken as

$$\alpha + j\beta = \alpha + jk$$

with α given by (3.41).

3.3 Transmission lines (distributed-circuit analysis)

In the previous section it was shown that a unique voltage and current wave was associated with the electric and magnetic fields of a TEM mode on a transmission line. Also, the transverse fields of a TEM mode have a transverse variation with the coordinates that is the same as for static fields. For these reasons the transmission line can be described in a unique manner as a distributed-parameter electric network. Energy storage in the magnetic field is accounted for by the series inductance L per unit length, whereas energy storage in the electric field is accounted for by the distributed shunt capacitance C per unit length. Power loss in the conductors is taken into account by a series resistance R per unit length. Finally, the power loss in the dielectric may be included by introducing a shunt conductance G per unit length. Suitable definitions for the parameters L, C, R, and G based on the above concepts are

$$L = \frac{\mu}{I_0 I_0^*}\int_S \mathbf{H}\cdot\mathbf{H}^*\,dS \tag{3.42a}$$

$$C = \frac{\epsilon'}{V_0 V_0^*}\int_S \mathbf{E}\cdot\mathbf{E}^*\,dS \tag{3.42b}$$

$$R = \frac{R_m}{I_0 I_0^*}\oint_{S_1+S_2} \mathbf{H}\cdot\mathbf{H}^*\,dl \tag{3.42c}$$

$$G = \frac{\omega\epsilon''}{V_0 V_0^*}\int \mathbf{E}\cdot\mathbf{E}^*\,dS \tag{3.42d}$$

where I_0 is the total current on the line, and V_0 the potential difference. These expressions are obtained, for example, by equating the magnetic energy $\frac{1}{4}I_0 I_0^* L = W_m$ stored in the equivalent series inductance L to the expression for W_m in terms of the field. The above definitions are readily

Table 3.1 Parameters of common transmission lines†

Z_c	R
$\frac{1}{\pi}\left(\frac{\mu_0}{\epsilon'}\right)^{\frac{1}{2}}\cosh^{-1}\frac{D}{d}$	$\frac{2R_m}{\pi d}\frac{D/d}{[(D/d)^2-1]^{\frac{1}{2}}}$
$\frac{1}{2\pi}\left(\frac{\mu_0}{\epsilon'}\right)^{\frac{1}{2}}\ln\frac{b}{a}$	$\frac{R_m}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$
$\frac{1}{\pi}\left(\frac{\mu_0}{\epsilon'}\right)^{\frac{1}{2}}\left[\ln\left(2p\frac{1-q^2}{1+q^2}\right) - \frac{1+4p^2}{16p^4}(1-4q^2)\right]$	$\frac{2R_m}{\pi d}\left[1+\frac{1+2p^2}{4p^4}(1-4q^2)\right] + \frac{8R_m}{\pi a}q^2\left[(1+q^2)-\frac{1+4p^2}{8p^4}\right]$

† For all TEM transmission lines

$$C = \frac{(\mu_0\epsilon')^{\frac{1}{2}}}{Z_c} \qquad L = (\mu_0\epsilon')^{\frac{1}{2}}Z_c \qquad G = \frac{\omega\epsilon''C}{\epsilon'}$$

$$\alpha_d = \frac{GZ_c}{2} \qquad \alpha_c = \frac{RY_c}{2} \qquad R_m = \frac{1}{\sigma\delta_s} = \left(\frac{\omega\mu}{2\sigma}\right)^{\frac{1}{2}}$$

shown to be equivalent to the other commonly used definitions such as†

$$L = \frac{\text{magnetic flux linkage}}{\text{total current}} \tag{3.43a}$$

$$C = \frac{\text{total charge per unit length}}{\text{voltage difference between conductors}} \tag{3.43b}$$

$$G = \frac{\text{total shunt current}}{\text{voltage difference between conductors}} \tag{3.43c}$$

The series resistance R is most conveniently defined on an energy basis as in (3.42c) since the current density $\mathbf{J}_s = \mathbf{n}\times\mathbf{H}$ is not always uniformly distributed around the conductor periphery. Parameters of some common transmission lines are given in Table 3.1.

The equivalent circuit of a section of transmission line of differential length dz is illustrated in Fig. 3.4. If the voltage and current at the input are $\mathcal{V}(z,t)$, $\mathcal{I}(z,t)$ and at the output are

$$\mathcal{V} + \frac{\partial\mathcal{V}}{\partial z}dz \qquad \mathcal{I} + \frac{\partial\mathcal{I}}{\partial z}dz$$

† See, for example, R. Plonsey and R. E. Collin, "Principles and Applications of Electromagnetic Fields," sec. 10.5, McGraw-Hill Book Company, New York, 1961.

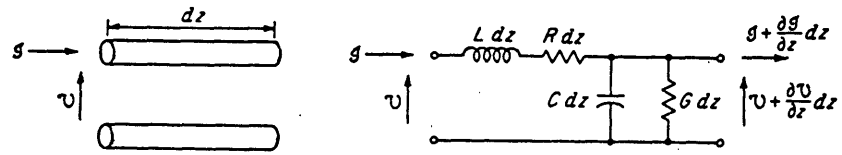

Fig. 3.4 Equivalent circuit of a differential length of transmission line.

then Kirchhoff's laws give

$$\mathcal{V} - \left(\mathcal{V} + \frac{\partial \mathcal{V}}{\partial z}\, dz\right) = \mathcal{I} R\, dz + L\, dz\, \frac{\partial \mathcal{I}}{\partial t}$$

or

$$\frac{\partial \mathcal{V}}{\partial z} = -\mathcal{I} R - L\, \frac{\partial \mathcal{I}}{\partial t} \tag{3.44a}$$

Similarly,

$$\mathcal{I} - \left(\mathcal{I} + \frac{\partial \mathcal{I}}{\partial z}\, dz\right) = \mathcal{V} G\, dz + C\, dz\, \frac{\partial \mathcal{V}}{\partial t}$$

or

$$\frac{\partial \mathcal{I}}{\partial z} = -\mathcal{V} G - C\, \frac{\partial \mathcal{V}}{\partial t} \tag{3.44b}$$

The first equation states that the potential difference between the input and output is equal to the potential drop across R and L. The second equation states that the output current is less than the input current by an amount equal to the shunt current flowing through C and G. Differentiating (3.44*a*) with respect to z and (3.44*b*) with respect to time t gives

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = -R\, \frac{\partial \mathcal{I}}{\partial z} - L\, \frac{\partial^2 \mathcal{I}}{\partial t\, \partial z} \tag{3.45a}$$

$$\frac{\partial^2 \mathcal{I}}{\partial t\, \partial z} = -G\, \frac{\partial \mathcal{V}}{\partial t} - C\, \frac{\partial^2 \mathcal{V}}{\partial t^2} \tag{3.45b}$$

Using (3.44*b*) and (3.45*b*) in (3.45*a*) now gives the following equation for the line voltage $\mathcal{V}$:

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = R\left(G\mathcal{V} + C\, \frac{\partial \mathcal{V}}{\partial t}\right) + L\left(G\, \frac{\partial \mathcal{V}}{\partial t} + C\, \frac{\partial^2 \mathcal{V}}{\partial t^2}\right)$$

or

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} - (RC + LG)\, \frac{\partial \mathcal{V}}{\partial t} - LC\, \frac{\partial^2 \mathcal{V}}{\partial t^2} - RG\mathcal{V} = 0 \tag{3.46}$$

The current $\mathcal{I}$ satisfies this one-dimensional wave equation also. If a

solution in the form of a propagating wave

$$\mathcal{V} = \text{Re}\,(Ve^{-\gamma z + j\omega t})$$

is assumed, substitution into (3.46) shows that the propagation constant must be a solution of

$$\gamma^2 - j\omega(RC + LG) + \omega^2 LC - RG = 0 \tag{3.47}$$

If only the steady-state sinusoidally time-varying solution is desired phasor notation may be used. If we let V and I represent the voltage and current without the time dependence $e^{j\omega t}$, the basic equations (3.44) may be written as

$$\frac{\partial V}{\partial z} = -(R + j\omega L)I \tag{3.48a}$$

$$\frac{\partial I}{\partial z} = -(G + j\omega C)V \tag{3.48b}$$

The wave equation (3.46) becomes

$$\frac{\partial^2 V}{\partial z^2} - (RG - \omega^2 LC)V - j\omega(RC + LG)V = 0 \tag{3.49}$$

The general solution to (3.49) is

$$V = V^+e^{-\gamma z} + V^-e^{\gamma z} \tag{3.50}$$

where $\gamma = \alpha + j\beta$ is given by

$$\gamma = [-\omega^2 LC + RG + j\omega(RC + LG)]^{\frac{1}{2}} \tag{3.51}$$

from (3.47). The constants V^+ and V^- are arbitrary amplitude constants for waves propagating in the $+z$ and $-z$ directions, respectively. The solution for the current I may be found from (3.48*a*), and is

$$I = I^+e^{-\gamma z} - I^-e^{+\gamma z} = \frac{\gamma}{R + j\omega L}\,(V^+e^{-\gamma z} - V^-e^{\gamma z}) \tag{3.52}$$

The parameter

$$Z_c = \frac{R + j\omega L}{\gamma} = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{\frac{1}{2}} \tag{3.53}$$

is called the characteristic impedance of the line since it is equal to the ratio V^+/I^+ and V^-/I^-. Note that $\gamma = [(R + j\omega L)(G + j\omega C)]^{\frac{1}{2}}$.

Loss-free transmission line

For a line without loss, i.e., for which $R = G = 0$, the propagation constant is

$$\gamma = j\beta = j\omega\sqrt{LC} \tag{3.54}$$

and the characteristic impedance is pure real and given by

$$Z_c = \sqrt{\frac{L}{C}} \tag{3.55}$$

According to the field analysis, β is also equal to $\omega(\mu\epsilon)^{\frac{1}{2}}$, and hence

$$LC = \mu\epsilon \tag{3.56}$$

for a transmission line. This result may also be verified from the solutions for L and C, as shown below in the section on transmission-line parameters. Using (3.56) in (3.55) shows that the characteristic impedance is also given by

$$Z_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu\epsilon}{C^2}} = \frac{\epsilon}{C}\sqrt{\frac{\mu}{\epsilon}} = Z\frac{\epsilon}{C} \tag{3.57}$$

where Z is the intrinsic impedance of the medium. The characteristic impedance differs from the intrinsic impedance Z by a factor ϵ/C, which is a function of the line configuration only.

Low-loss transmission line

For most microwave transmission lines the losses are very small; that is, $R \ll \omega L$ and $G \ll \omega C$. When this is the case, the term RG in the expression (3.51) for γ may be neglected. A binomial expansion then gives

$$\gamma \approx j\omega\sqrt{LC} + \tfrac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right) = \alpha + j\beta \tag{3.58}$$

To first order the characteristic impedance is still given by (3.55) or (3.57). Thus the phase constant for a low-loss line is

$$\beta = \omega\sqrt{LC} \tag{3.59a}$$

and the attenuation constant α is

$$\alpha = \tfrac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right) = \tfrac{1}{2}(RY_c + GZ_c) \tag{3.59b}$$

where $Y_c = Z_c^{-1} = \sqrt{C/L}$ is the characteristic admittance of the transmission line.

★3.4 Transmission-line parameters

In this section the field analysis to determine the circuit parameters L, R, C, and G for a transmission line is examined in greater detail. This will serve further to correlate the field analysis and circuit analysis of transmission lines.

Consider first the case of a loss-free line such as that illustrated in Fig. 3.2. When the scalar potential Φ has been determined, the charge density on the conductors may be found from the normal component of

electric field at the surface; that is, $\rho_s = \epsilon\mathbf{n}\cdot\mathbf{e} = -\epsilon\mathbf{n}\cdot\nabla\Phi = -\epsilon\,\partial\Phi/\partial n$, where ϵ is the permittivity of the medium surrounding the conductors. The total charge Q per unit length on conductor S_2 is

$$Q = \oint_{S_2} \epsilon\mathbf{n}\cdot\mathbf{e}\,dl$$

The total charge on the conductor S_1 is $-Q$ per meter. The potential of S_2 is V_0, and hence the capacitance C per unit length is

$$C = \frac{Q}{V_0} = \frac{\epsilon\int_{S_2}\mathbf{n}\cdot\mathbf{e}\,dl}{\int_{S_2}^{S_1}\mathbf{e}\cdot d\mathbf{l}} \tag{3.60}$$

The total current on S_2 is

$$I_0 = \oint_{S_2}\mathbf{h}\cdot d\mathbf{l} = \oint_{S_2} Y\mathbf{n}\cdot\mathbf{e}\,dl = \frac{YQ}{\epsilon}$$

since $|\mathbf{h}| = Y|\mathbf{e}| = Y\mathbf{n}\cdot\mathbf{e}$ at the surface of S_2 because the normal component of $\mathbf{h}$ and the tangential component of $\mathbf{e}$ are zero at the perfectly conducting surface S_2. The characteristic impedance of the line is given by

$$Z_c = \frac{V_0}{I_0} = \frac{V_0\epsilon}{YQ} = \frac{\epsilon Z}{C} \tag{3.61}$$

A knowledge of the capacitance per unit length suffices to determine the characteristic impedance.

To determine the inductance L per unit length, refer to Fig. 3.5, which illustrates the magnetic flux lines around the conductors. Since $\mathbf{h}$ is orthogonal to $\mathbf{e}$, these coincide with the equipotential lines. All the flux lines from the $\Phi = 0$ to the $\Phi = V_0/2$ line link the current on S_2. The flux linkage is the total flux cutting any path joining the $\Phi = 0$ line to the surface S_2. If a path such as P_1S_2 or P_2S_2 is chosen, which is orthogonal to the flux lines, this path coincides with a line of electric force. The

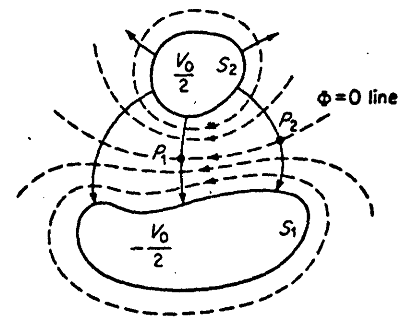

Fig. 3.5 **Magnetic flux lines in a transmission line.**

flux cutting such a path is

$$\psi = \int_{P_1}^{S_2} \mu h \, dl = \mu Y \int_{P_1}^{S_2} -\mathbf{e} \cdot d\mathbf{l} = \mu Y \frac{V_0}{2}$$

since $|\mathbf{h}| = Y|\mathbf{e}|$ for a TEM wave. The inductance of one conductor of the line is

$$L_1 = \frac{\psi}{I_0} = \mu Y \frac{V_0}{2I_0}$$

The inductance of both conductors per unit length is twice this value; so

$$L = \mu Y \frac{V_0}{I_0} = \mu Y Z_c \tag{3.62}$$

From this relation and (3.61) it is seen that $Z = \mu Z_c/L = CZ_c/\epsilon$, and hence

$$Z^2 = \frac{\mu}{\epsilon} = \frac{\mu Z_c}{L} \frac{CZ_c}{\epsilon}$$

which gives

$$Z_c = \sqrt{\frac{L}{C}} \tag{3.63}$$

Equations (3.61) and (3.62) also show that

$$\mu\epsilon = LC \tag{3.64}$$

for a transmission line. The above expressions for C and L can also be obtained from the definitions based on stored energy. The derivation is left as a problem.

If the dielectric has a complex permittivity $\epsilon = \epsilon' - j\epsilon''$, where ϵ'' includes the conductivity of the dielectric if it is not zero, the total shunt current consists of a displacement current I_D and a conduction current I_S. The current leaving conductor S_2 per unit length is

$$I = I_D + I_S = j\omega\epsilon \oint_{S_2} \mathbf{e} \cdot \mathbf{n} \, dl = j\omega\epsilon' \oint_{S_2} \mathbf{e} \cdot \mathbf{n} \, dl + \omega\epsilon'' \oint_{S_2} \mathbf{e} \cdot \mathbf{n} \, dl$$

where the first integral on the right gives the displacement current and the second integral gives the conduction current. The total shunt admittance is given by $Y = j\omega C + G = (I_S + I_D)/V_0$, and hence it is seen that

$$G = \frac{I_S}{V_0} = \frac{I_S}{I_D} \frac{I_D}{V_0} = \frac{\omega\epsilon''}{\epsilon'} C \tag{3.65}$$

since $j\omega C = I_D/V_0$ and $j\omega C/j\omega\epsilon' = C/\epsilon'$. This relation shows that G differs from C by the factor $\omega\epsilon''/\epsilon'$ only.

The transmission-line loss from finite conductivity may be accounted

for by a series resistance R per unit length provided R is chosen so that

$$\tfrac{1}{2} R I_0^2 = \frac{R_m}{2} \oint_{S_1+S_2} |\mathbf{h}|^2 \, dl \tag{3.66}$$

The right-hand side gives the total power loss per unit length arising from the high-frequency resistance of the conductors. In terms of this quantity, the resistance R is thus defined as

$$R = R_m \frac{\oint_{S_1+S_2} |\mathbf{h}|^2 \, dl}{\left(\oint_{S_2} |\mathbf{h}| \, dl\right)^2} \tag{3.67}$$

where $R_m = 1/\sigma\delta_s$ and δ_s is the skin depth.

A further effect of the finite conductivity is to increase the series inductance of the line by a small amount because of the penetration of the magnetic field into the conductor. This skin-effect inductance L_s is readily evaluated on an energy basis. The surface impedance Z_m has an inductive part $jX_m = j/\sigma\delta_s$ equal in magnitude to R_m. The magnetic energy stored in X_m is (note that X_m is equivalent to a surface inductance $X_m/\omega = L_m$)

$$W_m = \frac{X_m}{4\omega} \oint_{S_1+S_2} |\mathbf{J}_s|^2 \, dl$$

$$= \frac{X_m}{4\omega} \oint_{S_1+S_2} |\mathbf{h}|^2 \, dl = \frac{X_m}{4\omega} \frac{RI_0^2}{R_m} = \frac{RI_0^2}{4\omega}$$

by using (3.66) to replace the integral. Defining L_s by the relation

$$\tfrac{1}{4} L_s I_0^2 = W_m$$

gives

$$\omega L_s = R \tag{3.68}$$

The series inductive reactance of the line is increased by an amount equal to the series resistance. However, for low-loss lines, $R \ll \omega L$, so that $L_s \ll L$, and the correction is not significant for most practical lines. The inductance L_s is called the internal inductance since it arises from flux linkage internal to the conductor surfaces.

It should not come as a surprise to find that $\omega L_s = R$ since both the inductive reactance and resistance arise from the penetration of the current and fields into the conductor. The effect of this penetration into the conductor by an effective distance equal to the skin depth δ_s is correctly accounted for in a simplified manner by introduction of the surface impedance $Z_m = (1 + j)/\sigma\delta_s$.

Example 3.3 Coaxial-line parameters For the coaxial line of Fig. 3.3 the potential Φ is given by

$$\Phi = V_0 \frac{\ln (r/b)}{\ln (a/b)}$$

The charge on the inner conductor is

$$Q = \epsilon \int_0^{2\pi} \mathbf{a}_r \cdot \mathbf{e} a\, d\phi = \epsilon \int_0^{2\pi} - \frac{\partial \Phi}{\partial r} a\, d\phi$$

$$= \frac{-\epsilon V_0}{\ln (a/b)} \int_0^{2\pi} d\phi = \frac{2\pi\epsilon V_0}{\ln (b/a)}$$

Hence the capacitance per unit length is

$$C = \frac{\epsilon'}{\epsilon} \frac{Q}{V_0} = \frac{2\pi\epsilon'}{\ln (b/a)} \tag{3.69}$$

since the capacitance arises only from that part of the charge associated with ϵ' whereas ϵ'' gives rise to the shunt conductance.

The magnetic field is given by (3.38*b*) as

$$\mathbf{H} = \mathbf{h} e^{-jk_0 z} = \frac{YV_0}{\ln (b/a)} \frac{\mathbf{a}_\phi}{r} e^{-jk_0 z}$$

The current I_0 is

$$I_0 = \int_0^{2\pi} \mathbf{h} \cdot \mathbf{a}_\phi a\, d\phi = \frac{2\pi Y V_0}{\ln (b/a)}$$

Thus the characteristic impedance is

$$Z_c = \frac{V_0}{I_0} = \frac{Z}{2\pi \ln (b/a)} \tag{3.70}$$

The flux linking the center conductor is

$$\psi = \mu \int_a^b \mathbf{h} \cdot \mathbf{a}_\phi\, dr = \frac{\mu Y V_0}{\ln (b/a)} \int_a^b \frac{dr}{r} = \mu Y V_0$$

Consequently, the inductance per unit length is

$$L = \frac{\psi}{I_0} = \frac{\mu Y V_0}{2\pi Y V_0} \ln \frac{b}{a} = \frac{\mu}{2\pi} \ln \frac{b}{a} \tag{3.71}$$

from which it is seen that $LC = \mu\epsilon'$ and $Z_c = (L/C)^{\frac{1}{2}}$.

The shunt conductance G is given by $\omega\epsilon'' C/\epsilon'$, and is

$$G = \frac{\omega\epsilon''}{\epsilon'} \frac{2\pi\epsilon'}{\ln (b/a)} = \frac{2\pi\omega\epsilon''}{\ln (b/a)} \tag{3.72}$$

To find the series resistance the power loss in the inner and outer conductors must be evaluated. This was done in Example 3.2, with the result [Eq. (3.40*b*)]

$$\tfrac{1}{2} R I_0^2 = P_{l2} = \frac{R_m \pi Y^2 V_0^2}{[\ln (b/a)]^2} \frac{b + a}{ab}$$

Solving for R gives

$$R = \frac{R_m}{2\pi} \frac{b + a}{ab} \tag{3.73}$$

The internal inductance L_i is equal to R/ω; so the total series line inductance per unit length is

$$L + L_i = \frac{\mu}{2\pi} \ln \frac{b}{a} + \frac{b + a}{2\pi\omega a b \delta_s \sigma} \tag{3.74}$$

3.5 Terminated transmission line

In this section the properties of a transmission line terminated in an arbitrary load impedance Z_L are examined. This will serve to illustrate how the forward and backward propagating waves can be combined to satisfy the boundary conditions at a termination. Figure 3.6 illustrates schematically a transmission line terminated in a load impedance Z_L. The line is assumed lossless and with a characteristic impedance Z_c and a propagation constant $\gamma = j\beta$. It should be noted that at microwave frequencies conventional low-frequency resistors, inductors, or capacitors, when connected across the two conductors of a transmission line, may behave as impedance elements with quite different characteristics from the low-frequency behavior.

If a voltage wave $V^+e^{-j\beta z}$ with an associated current $I^+e^{-j\beta z}$ is incident on the termination, a reflected voltage wave $V^-e^{j\beta z}$ with a current $-I^-e^{j\beta z}$ will, in general, be created. The ratio of the reflected and incident wave amplitudes is determined by the load impedance only. At the load the total line voltage must equal the impressed voltage across the load and the line current must be continuous through the load. Hence, if Z_L is located at $z = 0$,

$$V = V^+ + V^- = V_L \tag{3.75a}$$

$$I = I^+ - I^- = I_L \tag{3.75b}$$

But $I^+ = Y_c V^+$, $I^- = Y_c V^-$, and $V_L/I_L = Z_L$ by definition of load impedance. Therefore

$$V^+ + V^- = V_L \tag{3.76a}$$

$$V^+ - V^- = \frac{Z_c}{Z_L} V_L \tag{3.76b}$$

The ratio of V^- to V^+ is usually described by a voltage reflection coeffi-

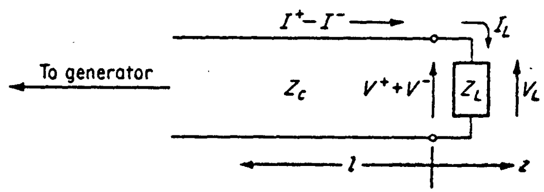

Fig. 3.6 Terminated transmission line.

cient Γ defined as

$$\Gamma = \frac{V^-}{V^+} \tag{3.77}$$

In place of (3.76) we may write

$$V^+(1 + \Gamma) = V_L$$

$$V^+(1 - \Gamma) = \frac{Z_c}{Z_L} V_L$$

Dividing one equation by the other yields

$$\frac{1+\Gamma}{1-\Gamma} = \frac{Z_L}{Z_c} \tag{3.78}$$

The quantity Z_L/Z_c is called the normalized load impedance (load impedance measured in units of Z_c), and $(1 + \Gamma)/(1 - \Gamma)$ is then the normalized input impedance seen looking toward the load at $z = 0$. The normalized load impedance will be expressed as $\bar{Z}_L$, with the bar on top signifying a normalized impedance in general. Solving for the voltage reflection coefficient Γ gives

$$\Gamma = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{Z_L/Z_c - 1}{Z_L/Z_c + 1} = \frac{\bar{Z}_L - 1}{\bar{Z}_L + 1} \tag{3.79}$$

Analogous to a voltage reflection coefficient, a current reflection coefficient Γ_I could also be introduced. In the present case

$$\Gamma_I = \frac{-I^-}{I^+} = -\frac{Y_c V^-}{Y_c V^+} = -\Gamma$$

In this text, however, only the voltage reflection coefficient will be used; so the adjective "voltage" can be dropped without confusion.

The incident voltage wave can be considered as transmitting a voltage V_L across the load, and a voltage transmission coefficient T can be defined as giving V_L in terms of V^+; thus

$$V_L = TV^+ = (1 + \Gamma)V^+$$

So

$$T = 1 + \Gamma \tag{3.80}$$

A corresponding current transmission coefficient is not used in this book.

Returning to (3.79), it is seen that if $Z_L = Z_c$, the reflection coefficient is zero. In this case all the power in the incident wave is transmitted to the load and none of it is reflected back toward the generator. The power delivered to the load in this case is

$$P = \tfrac{1}{2}\operatorname{Re}(VI^*) = \tfrac{1}{2}|V^+|^2 Y_c = \tfrac{1}{2}|V_L|^2 Y_L \tag{3.81}$$

The load is said to be matched to the transmission line when $\Gamma = 0$.

If Z_L does not equal Z_c, the load is mismatched to the line and a reflected wave is produced. The power delivered to the load is now given by

$$\begin{aligned} P &= \tfrac{1}{2}\operatorname{Re}(V_L I_L^*) = \tfrac{1}{2}\operatorname{Re}[(V^+ + V^-)(I^+ - I^-)^*] \\ &= \tfrac{1}{2}\operatorname{Re}[Y_c(V^+ + V^-)(V^+ - V^-)^*] = \tfrac{1}{2}\operatorname{Re}[Y_c|V^+|^2(1+\Gamma)(1-\Gamma)^*] \\ &= \tfrac{1}{2}Y_c|V^+|^2(1 - |\Gamma|^2) \end{aligned} \tag{3.82}$$

The final result states the physically obvious result that the power delivered to the load is the incident power minus that reflected from the load.

In the absence of reflection, the magnitude of the voltage along the line is a constant equal to $|V^+|$. When a reflected wave also exists, the incident and reflected waves interfere to produce a standing-wave pattern along the line. The voltage at any point on the line ($z < 0$) is given by

$$V = V^+ e^{-j\beta z} + \Gamma V^+ e^{j\beta z}$$

and has a magnitude given by

$$|V| = |V^+|\,|1 + \Gamma e^{2j\beta z}| = |V^+|\,|1 + \Gamma e^{-2j\beta l}|$$

where $l = -z$ is the positive distance measured from the load toward the generator, as in Fig. 3.6. Let Γ be equal to $\rho e^{j\theta}$, where $\rho = |\Gamma|$; then†

$$\begin{aligned} |V| &= |V^+|\,|1 + \rho e^{j(\theta - 2\beta l)}| = |V^+|\{[1 + \rho\cos(\theta - 2\beta l)]^2 \\ &\qquad + \rho^2 \sin^2(\theta - 2\beta l)\}^{1/2} \\ &= |V^+|\{(1+\rho)^2 - 2\rho[1 - \cos(\theta - 2\beta l)]\}^{1/2} \\ &= |V^+|\left[(1+\rho)^2 - 4\rho\sin^2\left(\beta l - \frac{\theta}{2}\right)\right]^{1/2} \end{aligned} \tag{3.83}$$

This result shows that $|V|$ oscillates back and forth between maximum values of $|V^+|(1 + \rho)$ when $\beta l - \theta/2 = n\pi$ and minimum values $|V^+|(1 - \rho)$ when $\beta l - \theta/2 = n\pi + \pi/2$, where n is an integer. These results also agree with physical intuition since they state that voltage maxima occur when the incident and reflected waves add in phase and that voltage minima occur when they add 180° out of phase. Successive maxima and minima are spaced a distance $d = \pi/\beta = \lambda\pi/2\pi = \lambda/2$ apart, where λ is the wavelength for TEM waves in the medium surrounding the conductors. The distance between a maximum and the nearest minimum is $\lambda/4$.

Since the current reflection coefficient is equal to $-\Gamma$, the current waves subtract whenever the voltage waves add up in phase. Hence

† The symbol ρ denotes both charge density and the modulus of the reflection coefficient. The context makes it clear which quantity is under discussion; so confusion should not occur.

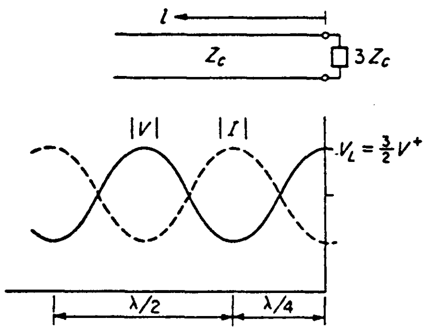

Fig. 3.7 Voltage and current standing-wave patterns on a line terminated in a load impedance equal to $3Z_c$.

current maxima and minima are displaced $\lambda/4$ from the corresponding voltage maxima and minima. Figure 3.7 illustrates the voltage and current standing-wave patterns that result when Z_L is a pure resistance equal to $3Z_c$.

The ratio of the maximum line voltage to the minimum line voltage is called the voltage standing-wave ratio S; thus

$$S = \frac{|V^+|(1+\rho)}{|V^+|(1-\rho)} = \frac{1+\rho}{1-\rho} \tag{3.84}$$

This is a parameter of considerable importance in practice for the following reasons: At microwave frequencies instruments for the direct absolute measurement of voltage or current are difficult to construct and use. On the other hand, devices to measure relative voltage or current (or electric or magnetic field) amplitudes are easy to construct. A typical device is a small probe inserted into the region of the electric field around a line. The output of the probe is connected to a crystal rectifier, and produces an output current which is a measure of the relative electric field or voltage at the probe position. By moving the probe along the line, the standing-wave ratio can be measured directly in terms of the maximum and minimum probe currents. The location of a voltage minimum can also be measured, and this permits the phase angle θ of Γ to be calculated. Since ρ is known from the measured value of S, Γ is specified, and the normalized load impedance may be calculated from (3.78).

Although the reflection coefficient was introduced as a measure of the ratio of reflected- to incident-wave amplitudes at the load, the definition may be extended to give the corresponding voltage ratio at any point on the line. Thus, at $z = -l$, the reflection coefficient is

$$\Gamma(l) = \frac{V^- e^{-j\beta l}}{V^+ e^{j\beta l}} = \frac{V^-}{V^+} e^{-2j\beta l} = \Gamma_L e^{-2j\beta l} \tag{3.85}$$

where $\Gamma_L = V^-/V^+$ now denotes the reflection coefficient of the load.

The normalized impedance, seen looking toward the load, at $z = -l$, is

$$\bar{Z}_{in} = \frac{Z_{in}}{Z_c} = \frac{V}{IZ_c} = \frac{V^+ e^{j\beta l} + V^- e^{-j\beta l}}{V^+ e^{j\beta l} - V^- e^{-j\beta l}} = \frac{1+\Gamma(l)}{1-\Gamma(l)} = \frac{1+\Gamma_L e^{-2j\beta l}}{1-\Gamma_L e^{-2j\beta l}} \tag{3.86}$$

By replacing Γ_L by $(Z_L - Z_c)/(Z_L + Z_c)$ and $e^{\pm j\beta l}$ by $\cos \beta l \pm j \sin \beta l$, this result may be expressed as

$$\bar{Z}_{in} = \frac{Z_{in}}{Z_c} = \frac{Z_L + jZ_c \tan \beta l}{Z_c + jZ_L \tan \beta l} \tag{3.87}$$

A similar result holds for the normalized input admittance; so

$$\bar{Y}_{in} = \frac{Y_{in}}{Y_c} = \frac{Y_L + jY_c \tan \beta l}{Y_c + jY_L \tan \beta l} = \frac{\bar{Y}_L + j \tan \beta l}{1 + j\bar{Y}_L \tan \beta l} \tag{3.88}$$

Of particular interest are two special cases, namely, $\beta l = \pi$ or $l = \lambda/2$ and $\beta l = \pi/2$ or $l = \lambda/4$, for which

$$Z_{in}\left(l = \frac{\lambda}{2}\right) = Z_L \tag{3.89a}$$

$$Z_{in}\left(l = \frac{\lambda}{4}\right) = \frac{Z_c^2}{Z_L} \tag{3.89b}$$

The first is equivalent to an ideal one-to-one impedance transformer, whereas in the second case the impedance has been inverted with respect to Z_c^2.

Terminated lossy line

In the case of a lossy line with propagation constant $\gamma = j\beta + \alpha$, the previous equations hold except that $j\beta$ must be replaced by $j\beta + \alpha$, where α is usually so small that, for the short lengths of line used in most experimental setups, the neglect of α is justified. Nevertheless, it is of some interest to examine the behavior of a lossy transmission line terminated in a load Z_L. One simplifying assumption will be made, and this is that the characteristic impedance Z_c can still be considered real. This assumption is certainly valid for low-loss lines of the type used at microwave frequencies. A detailed calculation justifying this assumption for a typical case is called for in Prob. 3.6.

Clearly, the presence of an attenuation constant α does not affect the definition of the voltage reflection coefficient Γ_L for the load. However, at any other point a distance l toward the generator, the reflection coefficient is now given by

$$\Gamma(l) = \Gamma_L e^{-2j\beta l - 2\alpha l} \tag{3.90}$$

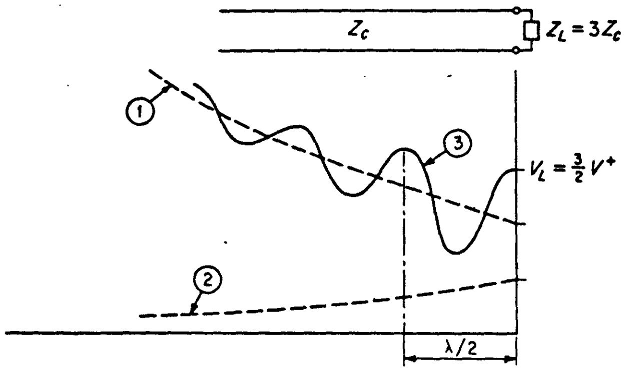

Fig. 3.8 Voltage-standing-wave pattern on a lossy transmission line. (1) Envelope of incident-wave amplitude; (2) envelope of reflected-wave amplitude; (3) standing-wave pattern.

As l is increased, Γ decreases exponentially until, for large l, it essentially vanishes. Thus, whenever a load Z_L is viewed through a long section of lossy line, it appears to be matched to the line since Γ is negligible at the point considered. This effect may also be seen from the expression for the input impedance, namely,

$$Z_{in} = Z_c \frac{1 + \Gamma_L e^{-2j\beta l - 2\alpha l}}{1 - \Gamma_L e^{-2j\beta l - 2\alpha l}} = Z_c \frac{Z_L + Z_c \tanh (j\beta l + \alpha l)}{Z_c + Z_L \tanh (j\beta l + \alpha l)} \tag{3.91}$$

which approaches Z_c for l large since $\tanh x$ approaches 1 for x large and not a pure imaginary quantity.

The losses also have the effect of reducing the standing-wave ratio S toward unity as the point of observation is moved away from the load toward the generator. As the generator is approached, the incident-wave amplitude increases exponentially whereas the reflected-wave amplitude decreases exponentially. The result is a standing-wave pattern of the type illustrated in Fig. 3.8. For illustrative purposes a relatively large value of α has been assumed here.

The power delivered to the load is given by

$$P_L = \tfrac{1}{2} \operatorname{Re} (V_L I_L^*) = \tfrac{1}{2}|V_L|^2 Y_L = \frac{Y_c}{2} |V^+|^2 (1 - |\Gamma_L|^2) \tag{3.92}$$

as before. At some point $z = -l$, the power directed toward the load is

$$P(l) = \tfrac{1}{2} \operatorname{Re} (VI^*) = \tfrac{1}{2}|V|^2 Y_c = \frac{Y_c}{2} |V^+ e^{\alpha l}|^2 [1 - |\Gamma(l)|^2]$$

$$= \frac{Y_c}{2} |V^+|^2 (e^{2\alpha l} - |\Gamma_L|^2) \tag{3.93}$$

where $|\Gamma|e^{\alpha l}$ has been replaced by $|\Gamma_L|$. Of the power given by (3.93), only that portion corresponding to P_L as given by (3.92) is delivered to the load. The remainder is dissipated in the lossy line, this remainder being given by

$$P(l) - P_L = \frac{Y_c}{2} |V^+|^2 (e^{2\alpha l} - 1)$$

3.6 Rectangular waveguide

The rectangular waveguide with a cross section as illustrated in Fig. 3.9 is an example of a waveguiding device that will not support a TEM wave. Consequently, it turns out that unique voltage and current waves do not exist, and the analysis of the waveguide properties has to be carried out as a field problem rather than as a distributed-parameter-circuit problem.

In a hollow cylindrical waveguide a transverse electric field can exist only if a time-varying axial magnetic field is present. Similarly, a transverse magnetic field can exist only if either an axial displacement current or an axial conduction current is present, as Maxwell's equations show. Since a TEM wave does not have any axial field components and there is no center conductor on which a conduction current can exist, a TEM wave cannot be propagated in a cylindrical waveguide.

The types of waves that can be supported (propagated) in a hollow empty waveguide are the TE and TM modes discussed in Sec. 3.1. The essential properties of all hollow cylindrical waveguides are the same, so that an understanding of the rectangular guide provides insight into the behavior of other types as well. As for the case of the transmission line, the effect of losses is initially neglected. The attenuation is computed later by using the perturbation method given earlier, together with the loss-free solution for the currents on the walls.

The essential properties of empty loss-free waveguides, which the detailed analysis to follow will establish, are that there is a double infinity of possible solutions for both TE and TM waves. These waves, or modes, may be labeled by two identifying integer subscripts n and m, for example, TE_{nm}. The integers n and m pertain to the number of standing-wave interference maxima occurring in the field solutions that describe

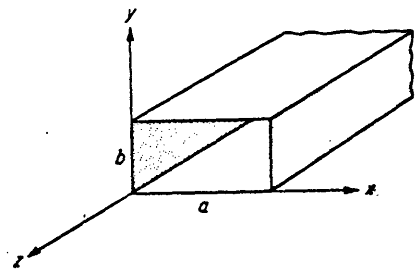

Fig. 3.9 Rectangular waveguide.

the variation of the fields along the two transverse coordinates. It will be found that each mode has associated with it a characteristic cutoff frequency $f_{c,nm}$ below which the mode does not propagate and above which the mode does propagate. The cutoff frequency is a geometrical parameter dependent on the waveguide cross-sectional configuration. When f_c has been determined, it is found that the propagation factor β is given by

$$\beta = (k_0^2 - k_c^2)^{\frac{1}{2}} \tag{3.94}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and $k_c = 2\pi f_c\sqrt{\mu_0\epsilon_0}$. The guide wavelength is readily seen to be given by

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda_0}{(1 - \lambda_0^2/\lambda_c^2)^{\frac{1}{2}}} = \frac{\lambda_0}{\sqrt{1 - f_c^2/f^2}} \tag{3.95}$$

where λ_0 is the free-space wavelength of plane waves at the frequency $f = \omega/2\pi$. Since k_c differs for different modes, there is always a lower band of frequencies for which only one mode propagates (except when k_c may be the same for two or more modes). In practice, waveguides are almost universally restricted to operation over this lower-frequency band for which only the dominant mode propagates, because of the difficulties associated with coupling signal energy into and out of a waveguide when more than one mode propagates. This latter difficulty arises because of the different values of the propagation phase constant β for different modes, since this means that the signal carried by the two or more modes does not remain in phase as the modes propagate along the guide. This necessitates the use of separate coupling probes for each mode at both the input and output and thus leads to increased system complexity and cost.

Another feature common to all empty uniform waveguides is that the phase velocity v_p is greater than the velocity of light c by the factor λ_g/λ_0. On the other hand, the velocity at which energy and a signal are propagated is the group velocity v_g and is smaller than c by the factor λ_0/λ_g. Also, since β, and hence λ_g, v_p, and v_g, are functions of frequency, any signal consisting of several frequencies is dispersed, or spread out, in both time and space as it propagates along the guide. This dispersion results from the different velocities at which the different frequency components propagate. If the guide is very long, considerable signal distortion may take place. Group and signal velocities are discussed in detail in Sec. 3.11.

With some of the general properties of waveguides considered, it is now necessary to consider the detailed analysis that will establish the above properties and that, in addition, will provide the relation between k_c and the guide configuration, the expressions for power and attenuation, etc. The case of TE modes in a loss-free empty rectangular guide is considered first.

TE waves

For TE, or H, modes, $e_z = 0$ and all the remaining field components can be determined from the axial magnetic field h_z by means of (3.16). The axial field h_z is a solution of

$$\nabla_t^2 h_z + k_c^2 h_z = 0$$

or

$$\frac{\partial^2 h_z}{\partial x^2} + \frac{\partial^2 h_z}{\partial y^2} + k_c^2 h_z = 0 \tag{3.96}$$

If a product solution $h_z = f(x)g(y)$ is assumed, (3.96) becomes

$$\frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} + k_c^2 = 0$$

after substituting fg for h_z and dividing the equation by fg. The term $\frac{1}{f}\frac{d^2f}{dx^2}$ is a function of x only, $\frac{1}{g}\frac{d^2g}{dy^2}$ is a function of y only, and k_c^2 is a constant, and hence this equation can hold for all values of x and y only if each term is constant. Thus we may write

$$\frac{1}{f}\frac{d^2f}{dx^2} = -k_x^2 \qquad \text{or} \qquad \frac{d^2f}{dx^2} + k_x^2 f = 0$$

$$\frac{1}{g}\frac{d^2g}{dy^2} = -k_y^2 \qquad \text{or} \qquad \frac{d^2g}{dy^2} + k_y^2 g = 0$$

where $k_x^2 + k_y^2 = k_c^2$ in order that the sum of the three terms may vanish. The use of the *separation-of-variables* technique has reduced the partial differential equation (3.96) to two ordinary simple-harmonic second-order equations. The solutions for f and g are easily found to be

$$f = A_1 \cos k_x x + A_2 \sin k_x x$$

$$g = B_1 \cos k_y y + B_2 \sin k_y y$$

where A_1, A_2, B_1, B_2 are arbitrary constants. These constants, as well as the separation constants k_x, k_y, can be further specified by considering the boundary conditions that h_z must satisfy. Since the normal component of the transverse magnetic field $\mathbf{h}$ must vanish at the perfectly conducting waveguide wall, (3.16*b*) shows that $\mathbf{n} \cdot \nabla_t h_z = 0$ at the walls, where $\mathbf{n}$ is a unit normal vector at the walls. When this condition holds, tangential $\mathbf{e}$ will also vanish on the guide walls, as (3.16*c*) shows. The requirements on h_z are thus

$$\frac{\partial h_z}{\partial x} = 0 \qquad \text{at } x = 0, a$$

$$\frac{\partial h_z}{\partial y} = 0 \qquad \text{at } y = 0, b$$

where the guide cross section is taken to be that in Fig. 3.9. In the solution for f, the boundary conditions give

$$-k_x A_1 \sin k_x x + k_x A_2 \cos k_x x = 0 \qquad \text{at } x = 0, a$$

Hence, from the condition at $x = 0$, it is found that $A_2 = 0$. At $x = a$, it is necessary for $\sin k_x a = 0$, and this specifies k_x to have the values

$$k_x = \frac{n\pi}{a} \qquad n = 0, 1, 2, \ldots$$

In a similar manner it is found that $B_2 = 0$ and

$$k_y = \frac{m\pi}{b} \qquad m = 0, 1, 2, \ldots$$

Both n and m equal to zero yields a constant for the solution for h_z and no other field components; so this trivial solution is of no interest.

If we use the above relations and put $A_1B_1 = A_{nm}$, the solutions for h_z are seen to be

$$h_z = A_{nm} \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \tag{3.97}$$

for $n = 0, 1, 2, \ldots$; $m = 0, 1, 2, \ldots$; $n = m \neq 0$. The constant A_{nm} is an arbitrary amplitude constant associated with the nmth mode. For the nmth mode the cutoff wave number is designated $k_{c,nm}$, given by

$$k_{c,nm} = \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}} \tag{3.98}$$

and is clearly a function of the guide dimensions only. The propagation constant for the nmth mode is given by

$$\gamma_{nm} = j\beta_{nm} = j(k_0^2 - k_{c,nm}^2)^{\frac{1}{2}}$$
$$= j\left[\left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}} \tag{3.99}$$

When $k_0 > k_{c,nm}$, β_{nm} is pure real and the mode propagates; when $k_0 < k_{c,nm}$, then γ_{nm} is real but β_{nm} is imaginary and the propagation factor is $e^{-\gamma_{nm}|z|}$, which shows that the mode decays rapidly with distance $|z|$ from the point at which it is excited. This decay is not associated with energy loss, but is a characteristic feature of the solution. Such decaying, or evanescent, modes may be used to represent the local diffraction, or fringing, fields that exist in the vicinity of coupling probes and obstacles in waveguides. The frequency separating the propagation and no-propagation bands is designated the cutoff frequency $f_{c,nm}$. This is given by the solution of $k_0 = k_{c,nm}$; that is,

$$f_{c,nm} = \frac{c}{\lambda_{c,nm}} = \frac{c}{2\pi} k_{c,nm} = \frac{c}{2\pi}\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}} \tag{3.100}$$

where c is the velocity of light. The cutoff wavelength is given by

$$\lambda_{c,nm} = \frac{2ab}{(n^2b^2 + m^2a^2)^{\frac{1}{2}}} \tag{3.101}$$

A typical guide may have $a = 2b$, in which case

$$\lambda_{c,nm} = \frac{2a}{(n^2 + 4m^2)^{\frac{1}{2}}}$$

and $\lambda_{c,10} = 2a$, $\lambda_{c,01} = a$, $\lambda_{c,11} = 2a/\sqrt{5}$, etc. In this example there is a band of wavelengths from a to $2a$, that is, a frequency band

$$\frac{c}{2a} < f < \frac{c}{a}$$

for which only the H_{10} mode propagates. This is the dominant mode in a rectangular guide and the one most commonly used in practice. Above the frequency c/a, other modes may propagate; so the useful frequency band in the present case is a one-octave band from $c/2a$ to c/a.

The remainder of the field components for the TE_{nm}, or H_{nm}, mode are readily found from (3.97) by using (3.16). The results for the complete nmth solution are

$$H_z = A_{nm} \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} e^{\mp j\beta_{nm} z} \tag{3.102a}$$

$$H_x = \pm j \frac{\beta_{nm}}{k_{c,nm}^2} A_{nm} \frac{n\pi}{a} \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} e^{\mp j\beta_{nm} z} \tag{3.102b}$$

$$H_y = \pm j \frac{\beta_{nm}}{k_{c,nm}^2} A_{nm} \frac{m\pi}{b} \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{\mp j\beta_{nm} z} \tag{3.102c}$$

$$E_x = Z_{h,nm} A_{nm} j \frac{\beta_{nm}}{k_{c,nm}^2} \frac{m\pi}{b} \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{\mp j\beta_{nm} z} \tag{3.102d}$$

$$E_y = -Z_{h,nm} A_{nm} j \frac{\beta_{nm}}{k_{c,nm}^2} \frac{n\pi}{a} \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} e^{\mp j\beta_{nm} z} \tag{3.102e}$$

where the wave impedance for the nmth H mode is given by

$$Z_{h,nm} = \frac{k_0}{\beta_{nm}} Z_0 \tag{3.103}$$

When the mode does not propagate, $Z_{h,nm}$ is imaginary, indicating that there is no net energy flow associated with the evanescent mode. A general field with $E_z = 0$ can be described in a complete manner by a linear superposition of all the H_{nm} modes.

Power

For a propagating H_{nm} mode the power, or rate of energy flow, in the positive z direction is given by

$$\begin{aligned} P_{nm} &= \tfrac{1}{2}\,\mathrm{Re}\int_0^a\int_0^b \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_z\,dx\,dy \\ &= \tfrac{1}{2}\,\mathrm{Re}\int_0^a\int_0^a (E_xH_y^* - E_yH_x^*)\,dx\,dy \\ &= \tfrac{1}{2}\,\mathrm{Re}\,Z_{h,nm}\int_0^a\int_0^b (H_yH_y^* + H_xH_x^*)\,dx\,dy \end{aligned} \tag{3.104}$$

If we substitute from (3.102*b*) and (3.102*c*) and note that

$$\int_0^a\int_0^b \sin^2\frac{n\pi x}{a}\cos^2\frac{m\pi y}{b}\,dx\,dy = \int_0^a\int_0^b \cos^2\frac{n\pi x}{a}\sin^2\frac{m\pi y}{b}\,dx\,dy = \begin{cases} \dfrac{ab}{4} & n \neq 0,\ m\neq 0 \\ \dfrac{ab}{2} & n \text{ or } m = 0 \end{cases}$$

we find that

$$P_{nm} = |A_{nm}|^2\frac{ab}{\epsilon_{0n}\epsilon_{0m}}\frac{\beta_{nm}{}^2}{k_{c,nm}^4}Z_{h,nm}\left[\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right] = \frac{|A_{nm}|^2ab}{\epsilon_{0n}\epsilon_{0m}}\left(\frac{\beta_{nm}}{k_{c,nm}}\right)^2 Z_{h,nm} \tag{3.105}$$

where ϵ_{0m} is the Neumann factor and equal to 1 for $m = 0$ and equal to 2 for $m > 0$.

If two modes, say the H_{nm} and H_{rs} modes, were present simultaneously, it would be found that the power is the sum of that contributed by each individual mode, that is, $P_{nm} + P_{rs}$. This is a general property of loss-free waveguides, and is discussed in detail in a later section. This power orthogonality arises because of the orthogonality of the functions (eigenfunctions) that describe the transverse variation of the fields when integrated over the guide cross section; e.g.,

$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{r\pi x}{a}\,dx = 0 \qquad n\neq r$$

Even when small losses are present the energy flow may be taken to be that contributed by each individual mode, with negligible error in all cases except when two or more degenerate modes are present. Degenerate modes are modes which have the same propagation constant γ, and for these the presence of even small losses may result in strong coupling between the modes. This phenomenon is explained more fully in Sec. 3.10.

Attenuation

If the waveguide walls have finite conductivity, there will be a continuous loss of power to the walls as the modes propagate along the guide. Consequently, the phase constant $j\beta$ is perturbed and becomes $\gamma = \alpha + j\beta$, where α is an attenuation constant that gives the rate at which the mode amplitude must decay as the mode progresses along the guide. For practical waveguides the losses caused by finite conductivity are so small that the attenuation constant may be calculated using the perturbation method outlined in Sec. 3.2 in connection with lossy transmission lines.† The method will be illustrated for the dominant H_{10} mode only. For the H_{nm} and also the E_{nm} modes, the calculation differs only in that somewhat greater algebraic manipulation is required.

For the H_{10} mode, the fields are given by (apart from the factor $e^{-j\beta_{10}z}$)

$$h_z = A_{10}\cos\frac{\pi x}{a}$$

$$h_x = j\frac{\beta_{10}}{k_{c,10}^2}A_{10}\frac{\pi}{a}\sin\frac{\pi x}{a}$$

$$e_y = -Z_{h,10}A_{10}\frac{j\beta_{10}}{k_{c,10}^2}\frac{\pi}{a}\sin\frac{\pi x}{a}$$

as reference to (3.102) shows. From (3.105) the rate of energy flow along the guide is

$$P_{10} = |A_{10}|^2\frac{ab}{2}\left(\frac{\beta_{10}}{k_{c,10}}\right)^2 Z_{h,10}$$

The currents on the lossy walls are assumed to be the same as the loss-free currents, and hence are given by

$$\mathbf{J}_s = \mathbf{n}\times\mathbf{H}$$

where $\mathbf{n}$ is a unit inward directed normal at the guide wall. Thus, on the walls at $x = 0, a$, the surface currents are

$$\mathbf{J}_s = \begin{cases} \mathbf{a}_x\times\mathbf{H} = -\mathbf{a}_yA_{10} & x = 0 \\ -\mathbf{a}_x\times\mathbf{H} = -\mathbf{a}_yA_{10} & x = a \end{cases}$$

whereas on the upper and lower walls the currents are

$$\mathbf{J}_s = \begin{cases} \mathbf{a}_y\times\mathbf{H} = -\mathbf{a}_z\dfrac{j\beta_{10}}{k_{c,10}^2}A_{10}\dfrac{\pi}{a}\sin\dfrac{\pi x}{a} + \mathbf{a}_xA_{10}\cos\dfrac{\pi x}{a} & y = 0 \\ -\mathbf{a}_y\times\mathbf{H} = \mathbf{a}_z\dfrac{j\beta_{10}}{k_{c,10}^2}A_{10}\dfrac{\pi}{a}\sin\dfrac{\pi x}{a} - \mathbf{a}_xA_{10}\cos\dfrac{\pi x}{a} & y = b \end{cases}$$

With a finite conductivity σ, the waveguide walls may be characterized

† The case for degenerate modes may require a modified analysis, and this is covered in Sec. 3.10.

Table 3.2 Properties of modes in a rectangular guide†

	TE modes	TM modes
H_z	$\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\,e^{-j\beta_{nm}z}$	0
E_z	0	$\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,e^{-j\beta_{nm}z}$
E_x	$Z_{h,nm}H_y$	$\frac{-j\beta_{nm}n\pi}{ak_{c,nm}^2}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,e^{-j\beta_{nm}z}$
E_y	$-Z_{h,nm}H_x$	$\frac{-j\beta_{nm}m\pi}{bk_{c,nm}^2}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\,e^{-j\beta_{nm}z}$
H_x	$\frac{j\beta_{nm}n\pi}{ak_{c,nm}^2}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\,e^{-j\beta_{nm}z}$	$-\frac{E_y}{Z_{e,nm}}$
H_y	$\frac{j\beta_{nm}m\pi}{bk_{c,nm}^2}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,e^{-j\beta_{nm}z}$	$\frac{E_x}{Z_{e,nm}}$
$Z_{h,nm}$	$\frac{k_0}{\beta_{nm}}Z_0$	
$Z_{e,nm}$		$\frac{\beta_{nm}}{k_0}Z_0$
$k_{c,nm}$	$\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}$	
β_{nm}	$(k_0^2-k_{c,nm}^2)^{\frac{1}{2}}$	
$\lambda_{c,nm}$	$\frac{2ab}{(n^2b^2+m^2a^2)^{\frac{1}{2}}}$	
α	$\frac{2R_m}{bZ_0(1-k_{c,nm}^2/k_0^2)^{\frac{1}{2}}}\left[\left(1+\frac{b}{a}\right)\frac{k_{c,nm}^2}{k_0^2}+\frac{b}{a}\left(\frac{\epsilon_{0m}}{2}-\frac{k_{c,nm}^2}{k_0^2}\right)\frac{n^2ab+m^2a^2}{n^2b^2+m^2a^2}\right]$	$\frac{2R_m}{bZ_0(1-k_{c,nm}^2/k_0^2)^{\frac{1}{2}}}\frac{n^2b^3+m^2a^3}{n^2b^2a+m^2a^3}$

† $R_m = (\omega\mu_0/2\sigma)^{\frac{1}{2}}$, $\epsilon_{0m} = 1$ for $m = 0$ and 2 for $m > 0$. The expression for α is not valid for degenerate modes (Sec. 3.10).

as exhibiting a surface impedance given by

$$Z_m = \frac{1+j}{\sigma\delta_s} = (1+j)R_m$$

where δ_s is the skin depth. The power loss in the resistive part R_m of Z_m per unit length of guide is

$$P_l = \frac{R_m}{2}\oint_{\substack{\text{guide}\\\text{walls}}} \mathbf{J}_s\cdot\mathbf{J}_s^*\,dl$$

$$= \frac{R_m|A_{10}|^2}{2}\left(2\int_0^b dy + 2\int_0^a \frac{\beta_{10}{}^2}{k_{c,10}^4}\frac{\pi^2}{a^2}\sin^2\frac{\pi x}{a}\,dx + 2\int_0^a \cos^2\frac{\pi x}{a}\,dx\right)$$

Since $k_{c,10} = \pi/a$, the above gives

$$P_l = R_m|A_{10}|^2\left[b + \frac{a}{2}\left(\frac{\beta_{10}}{k_{c,10}}\right)^2 + \frac{a}{2}\right]$$

If P_0 is the power at $z = 0$, then $P_{10} = P_0e^{-2\alpha z}$ is the power in the guide at any z. The rate of decrease of power propagated is

$$-\frac{dP_{10}}{dz} = 2\alpha P_{10} = P_l$$

and equals the power loss, as indicated in the above equation. The attenuation constant α for the H_{10} mode is thus seen to be

$$\alpha = \frac{P_l}{2P_{10}} = \frac{R_m\left[b + \frac{a}{2}\left(\frac{\beta_{10}}{k_{c,10}}\right)^2 + \frac{a}{2}\right]}{\frac{ab}{2}\left(\frac{\beta_{10}}{k_{c,10}}\right)^2 Z_{h,10}}$$

$$= \frac{R_m}{ab\beta_{10}k_0Z_0}(2bk_{c,10}^2 + ak_0^2) \qquad \text{nepers/m} \tag{3.106}$$

The attenuation for other TE_{nm} modes is given by the formula in Table 3.2, which summarizes the solutions for TE_{nm} and also TM_{nm} modes. In Fig. 3.10 the attenuation for the H_{10} mode in a copper rectangular guide is given as a function of frequency. To convert attenuation given in nepers to decibels, multiply by 8.7.

The theoretical formulas for attenuation give results in good agreement with experimental values for frequencies below about 5,000 Mc. For higher frequencies, measured values of α may be considerably higher, depending on the smoothness of the waveguide surface. If surface imperfections of the order of magnitude of the skin depth δ_s are present, it is readily appreciated that the effective surface area is much greater, resulting in greater loss. By suitably polishing the surface, the experi-

Fig. 3.10 Attenuation of H_{10} mode in a copper rectangular waveguide, $a = 2b$, f in units of 10^{10} cps.

mental values of attenuation are found to be in substantial agreement with the theoretical values.†

Dominant TE_{10} mode

Since the TE_{10} mode is the dominant mode in a rectangular guide, and also the most commonly used mode, it seems appropriate to examine this mode in more detail. From the results given earlier, the field components for this mode are described by the following (propagation in the $+z$ direction assumed):

$$H_z = A \cos \frac{\pi x}{a} e^{-j\beta z} \tag{3.107a}$$

$$H_x = \frac{j\beta}{k_c} A \sin \frac{\pi x}{a} e^{-j\beta z} \tag{3.107b}$$

$$E_y = -jZ_h \frac{\beta}{k_c} \sin \frac{\pi x}{a} e^{-j\beta z} \tag{3.107c}$$

where the subscript 10 has been dropped for convenience since this discussion pertains only to the TE_{10} mode. The parameters β, k_c, and Z_h are given by

$$k_c = \frac{\pi}{a} \tag{3.108a}$$

$$\beta = \left[k_0^2 - \left(\frac{\pi}{a}\right)^2\right]^{\frac{1}{2}} \tag{3.108b}$$

$$Z_h = -\frac{E_y}{H_x} = \frac{k_0}{\beta} Z_0 \tag{3.108c}$$

The guide wavelength λ_g is

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda_0}{[1 - (\lambda_0/2a)^2]^{\frac{1}{2}}} \tag{3.108d}$$

since the cutoff wavelength $\lambda_c = 2a$. The phase and group velocities are

$$v_p = \frac{\lambda_g}{\lambda_0} c \tag{3.108e}$$

$$v_g = \frac{\lambda_0}{\lambda_g} c \tag{3.108f}$$

and are discussed in detail in Sec. 3.11.

In Fig. 3.11 the magnetic and electric field lines associated with the TE_{10} mode are illustrated. Note that the magnetic flux lines encircle the electric field lines; so these can be considered to be the source (displacement current) for the magnetic field. On the other hand, the

† See J. Allison and F. A. Benson, Surface Roughness and Attenuation of Precision Drawn, Chemically Polished, Electropolished, Electroplated and Electroformed Waveguides, *Proc. IEE (London)*, vol. 102, pt. B, pp. 251–259, 1955.

Fig. 3.11 Magnetic and electric field lines for the TE_{10} mode. (*a*) Transverse plane; (*b*) top view; (*c*) mutual total current and magnetic field linkages.

Fig. 3.12 Decomposition of TE_{10} mode into two plane waves.

electric field lines terminate in an electric charge distribution on the inner surface of the upper and lower waveguide walls. This charge oscillates back and forth in the axial and transverse directions and thus constitutes an axial and transverse conduction current that forms the continuation of the displacement current. The total current, displacement plus conduction, forms a closed linkage of the magnetic field lines, and as such may be regarded as being generated by the changing magnetic flux they enclose. This completes the required mutual-support action between the electric and magnetic fields which is required for wave propagation.

The fields for a TE_{10} mode may be decomposed into the sum of two plane TEM waves propagating along zigzag paths between the two waveguide walls at $x = 0$ and $x = a$, as in Fig. 3.12. For the electric field we have

$$E_y = -\frac{Z_h}{2}\frac{\beta}{k_c}\left(e^{j\pi x/a - j\beta z} - e^{-j\pi x/a - j\beta z}\right)$$

If π/a and β are expressed as

$$\frac{\pi}{a} = k_0 \sin\theta \qquad \beta = k_0 \cos\theta$$

the relation $(\pi/a)^2 + \beta^2 = k_0^2$ still holds. The electric field is now given by

$$E_y = \frac{Z_h}{2}\frac{\beta}{k_c}\left(e^{-jk_0(x\sin\theta + z\cos\theta)} - e^{-jk_0(-x\sin\theta + z\cos\theta)}\right)$$

which is clearly two plane waves propagating at angles $\pm\theta$ with respect to the z axis, as illustrated. Alternatively, the field may be pictured as a plane wave reflecting back and forth between the two guide walls. As shown in Sec. 2.7, the constant phase planes associated with these obliquely propagating plane waves move in the z direction at the phase velocity $c/\cos\theta = \beta c/k_0$, and this is the reason why the phase velocity of the TE_{10} mode exceeds the velocity of light. Since the energy in a TEM wave propagates with the velocity c in the direction in which the plane wave propagates, this energy will propagate down the guide at a velocity equal to the component of c along the z axis. This component is $v_g = c\cos\theta = (k_0/\beta)c$ and is the group velocity for the TE_{10} mode. When $\theta = \pi/2$, the plane waves reflect back and forth, but do not progress down the guide; so the mode is cutoff.

The above decomposition of the TE_{10} mode into two plane waves may be extended to the TE_{nm} modes also. When n and m are both different from zero, four plane waves result. Although such superpositions of plane waves may be used to construct the field solutions for rectangular guides, this is a rather cumbersome approach. However, it does lend insight into why the phase velocity exceeds that of light, as well as other properties of the modes.

TM modes

For TM modes, h_z equals zero and e_z plays the role of a potential function from which the remaining field components may be derived. This axial electric field satisfies the reduced Helmholtz equation

$$\nabla_t^2 e_z + k_c^2 e_z = 0 \tag{3.109}$$

of the same type encountered earlier for h_z, that is, (3.96). The solution may be found by using the separation-of-variables method. In the present case the boundary conditions require that e_z vanish at $x = 0$, a and $y = 0$, b. This condition requires that the solution for e_z be

$$e_z = A_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \tag{3.110}$$

instead of a product of cosine functions which was suitable for describing h_z. Again, there are a doubly infinite number of solutions corresponding

to various integers n and m. However, unlike the situation for TE modes, $n = 0$ and $m = 0$ are not solutions. The cutoff wave number is given by the same expression as for TE modes; that is,

$$k_{c,nm} = \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2} \tag{3.111}$$

and the propagation factor β_{nm} by

$$\beta_{nm} = (k_0^2 - k_{c,nm}^2)^{1/2} \tag{3.112}$$

The lowest-order propagating mode is the $n = m = 1$ mode, and this has a cutoff wavelength equal to $2ab/(a^2 + b^2)^{1/2}$. Note that the TE_{10} mode can propagate at a lower frequency (longer wavelength), thus verifying that this is the dominant mode.† It should also be noted that for the same values of n and m, the TE_{nm} and TM_{nm} modes are degenerate since they have the same propagation factor. Another degeneracy occurs when $a = b$, for in this case the four modes TE_{nm}, TE_{mn}, TM_{nm}, and TM_{mn} all have the same propagation constant. Still further degeneracies exist if a is an integer multiple of b, or vice versa.

The rest of the solution for TM modes is readily constructed using the general equations (3.17) given in Sec. 3.1. A summary of this solution is given in Table 3.2. The TM modes are the dual of the TE modes and apart from minor differences have essentially the same properties. For this reason it does not seem necessary to repeat the preceding discussion.

3.7 Circular waveguides

Figure 3.13 illustrates a cylindrical waveguide with a circular cross section of radius a. In view of the cylindrical geometry involved, cylindrical coordinates are most appropriate for the analysis to be carried out. Since the general properties of the modes that may exist are similar to those for the rectangular guide, this section is not as detailed.

† In any hollow waveguide the dominant mode is a TE mode because the boundary conditions $e_z = 0$ for TM modes always require e_z to have a greater spatial variation in the transverse plane than that for h_z for the lowest-order TE mode, and hence the smallest value of k_c occurs for TE modes. Hence a TE mode has the lowest cutoff frequency, i.e., is the dominant mode.

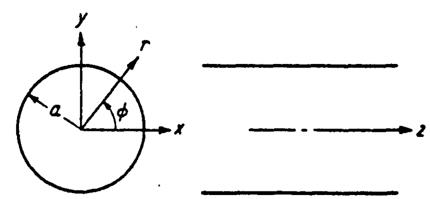

Fig. 3.13 The circular cylindrical waveguide.

TM modes

For the TM modes a solution of

$$\nabla_t^2 e_z + k_c^2 e_z = 0$$

is required such that e_z will vanish at $r = a$. When we express the transverse laplacian ∇_t^2 in cylindrical coordinates (Appendix I), this equation becomes

$$\frac{\partial^2 e_z}{\partial r^2} + \frac{1}{r}\frac{\partial e_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 e_z}{\partial \phi^2} + k_c^2 e_z = 0 \tag{3.113}$$

The separation-of-variables method may be used to reduce the above to two ordinary differential equations. Consequently, it is assumed that a product solution $f(r)g(\phi)$ exists for e_z. Substituting for e_z into (3.113) and dividing the equation by fg yield

$$\frac{1}{f}\frac{d^2f}{dr^2} + \frac{1}{rf}\frac{df}{dr} + \frac{1}{r^2g}\frac{d^2g}{d\phi^2} + k_c^2 = 0$$

Multiplying this result by r^2 gives

$$\frac{r^2}{f}\frac{d^2f}{dr^2} + \frac{r}{f}\frac{df}{dr} + r^2k_c^2 = -\frac{1}{g}\frac{d^2g}{d\phi^2}$$

The left-hand side is a function of r only, whereas the right-hand side depends on ϕ only. Therefore this equation can hold for all values of the variables only if both sides are equal to some constant ν^2. As a result, (3.113) is seen to separate into the following two equations:

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(k_c^2 - \frac{\nu^2}{r^2}\right)f = 0 \tag{3.114a}$$

$$\frac{d^2g}{d\phi^2} + \nu^2 g = 0 \tag{3.114b}$$

In this case the field inside the waveguide must be periodic in ϕ with period 2π, that is, single-valued. It is therefore necessary to choose ν equal to an integer n, in which case the general solution to (3.114b) is

$$g(\phi) = A_1 \cos n\phi + A_2 \sin n\phi$$

where A_1 and A_2 are arbitrary constants.

Equation (3.114a) is Bessel's differential equation and has two solutions (a second-order differential equation always has two independent solutions) $J_\nu(k_cr)$ and $Y_\nu(k_cr)$, called Bessel functions of the first and second kind, respectively, and of order ν.† For the problem under investigation here, only $J_n(k_cr)$ is a physically acceptable solution since $Y_n(k_cr)$ becomes infinite at $r = 0$. The final solution for e_z may thus

† Y_ν is also called a Neumann function.

be expressed as

$$e_z(r, \phi) = (A_1 \cos n\phi + A_2 \sin n\phi)J_n(k_cr) \tag{3.115}$$

Reference to Appendix II shows that $J_n(x)$ behaves like a damped sinusoidal function and passes through zero in a quasi-periodic fashion. Since e_z must vanish when $r = a$, it is necessary to choose k_ca in such a manner that $J_n(k_ca) = 0$. If the mth root of the equation $J_n(x) = 0$ is designated p_{nm}, the allowed values (eigenvalues) of k_c are

$$k_{c,nm} = \frac{p_{nm}}{a} \tag{3.116}$$

The values of p_{nm} for the first three modes for $n = 0, 1, 2$ are given in Table 3.3. As in the case of the rectangular guide, there are a doubly infinite number of solutions.

Each choice of n and m specifies a particular TM_{nm} mode (eigenfunction). The integer n is related to the number of circumferential variations in the field, whereas m relates to the number of radial variations. The propagation constant for the nmth mode is given by

$$\beta_{nm} = \left(k_0^2 - \frac{p_{nm}^2}{a^2}\right)^{\frac{1}{2}} \tag{3.117}$$

the cutoff wavelength by

$$\lambda_{c,nm} = \frac{2\pi a}{p_{nm}} \tag{3.118}$$

and the wave impedance by

$$Z_{e,nm} = \frac{\beta_{nm}}{k_0} Z_0 \tag{3.119}$$

A cutoff phenomenon similar to that for the rectangular guide exists. For the dominant TM mode, $\lambda_c = 2\pi a/p_{01} = 2.61a$, a value 30 percent greater than the waveguide diameter.

Expressions for the remaining field components may be derived by using the general equations (3.17). Energy flow and attenuation may

Table 3.3 Values of p_{nm} for TM modes

n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

Table 3.4 Properties of modes in circular waveguides

	TE modes	TM modes
H_z	$J_n\left(\frac{p'_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	0
E_z	0	$J_n\left(\frac{p_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$
H_r	$-\frac{j\beta_{nm}p'_{nm}}{ak^2_{c,nm}}J'_n\left(\frac{p'_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$	$-\frac{E_\phi}{Z_{e,nm}}$
H_ϕ	$-\frac{jn\beta_{nm}}{rk^2_{c,nm}}J_n\left(\frac{p'_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{Bmatrix}-\sin n\phi\\ \cos n\phi\end{Bmatrix}$	$\frac{E_r}{Z_{e,nm}}$
E_r	$Z_{h,nm}H_\phi$	$-\frac{j\beta_{nm}p_{nm}}{ak^2_{c,nm}}J'_n\left(\frac{p_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{Bmatrix}\cos n\phi\\ \sin n\phi\end{Bmatrix}$
E_ϕ	$-Z_{h,nm}H_r$	$-\frac{jn\beta_{nm}}{rk^2_{c,nm}}J_n\left(\frac{p_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{Bmatrix}-\sin n\phi\\ \cos n\phi\end{Bmatrix}$
β_{nm}	$\left[k_0^2-\left(\frac{p'_{nm}}{a}\right)^2\right]^{\frac{1}{2}}$	$\left[k_0^2-\left(\frac{p_{nm}}{a}\right)^2\right]^{\frac{1}{2}}$
$Z_{h,nm}$	$\frac{k_0}{\beta_{nm}}Z_0$	
$Z_{e,nm}$		$\frac{\beta_{nm}}{k_0}Z_0$
$k_{c,nm}$	$\frac{p'_{nm}}{a}$	$\frac{p_{nm}}{a}$
$\lambda_{c,nm}$	$\frac{2\pi a}{p'_{nm}}$	$\frac{2\pi a}{p_{nm}}$
Power	$\frac{Z_0k_0\beta_{nm}\pi}{4k^4_{c,nm}}(p'^2_{nm}-n^2)J_n^2(p'_{nm})$	$\frac{Y_0k_0\beta_{nm}\pi}{4k^4_{c,nm}}p_{nm}^2[J'_n(k_{c,nm}a)]^2$
α	$\frac{R_m}{aZ_0}\left(1-\frac{k^2_{c,nm}}{k_0^2}\right)^{-\frac{1}{2}}\times\left[\frac{k^2_{c,nm}}{k_0^2}+\frac{n^2}{(p'_{nm})^2-n^2}\right]$	$\frac{R_m}{aZ_0}\left(1-\frac{k^2_{c,nm}}{k_0^2}\right)^{-\frac{1}{2}}$

be determined by methods similar to those used for the rectangular guide. A summary of the results is given in Table 3.4.

TE modes.

The solution for TE modes parallels that for the TM modes with the exception that the boundary conditions require that $\partial h_z/\partial r$ vanish at

$r = a$. An appropriate solution for h_z is

$$h_z(r, \phi) = (B_1 \cos n\phi + B_2 \sin n\phi)J_n(k_c r) \tag{3.120}$$

with the requirement that

$$\frac{dJ_n(k_c r)}{dr} = 0 \qquad \text{at } r = a \tag{3.121}$$

The roots of (3.121) will be designated by p'_{nm}; so the eigenvalues $k_{c,nm}$ are given by

$$k_{c,nm} = \frac{p'_{nm}}{a} \tag{3.122}$$

Table 3.5 lists the values of the roots for the first few modes. Note that $p'_{0m} = p_{1m}$ since $dJ_0(x)/dx = -J_1(x)$, and hence the TE_{0m} and TM_{1m} modes are degenerate.

Table 3.5 Values of p'_{nm} for TE modes

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

The first TE mode to propagate is the TE_{11} mode, having a cutoff wavelength $\lambda_{c,11} = 3.41a$. This mode is seen to be the dominant mode for the circular waveguide, and is normally the one used. A sketch of the field lines in the transverse plane for this mode is given in Fig. 3.14.

If the expression for the attenuation constant for TE modes is examined, it will be seen that, for the TE_{0m} modes, the attenuation is

$$\alpha = \frac{R_m}{aZ_0}\frac{f^2_{c,0m}}{f(f^2 - f^2_{c,0m})^{\frac{1}{2}}} \tag{3.123}$$

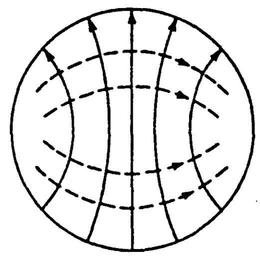

Fig. 3.14 **Field lines for TE_{11} mode in a circular guide.**

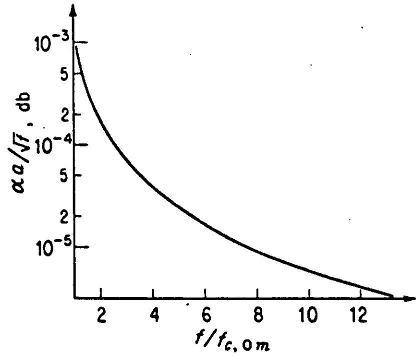

Fig. 3.15 Attenuation of low-loss TE_{0m} modes in a circular copper waveguide, f in units of 10^{10} cps, $f_{c,10} = 1.83 \times 10^{10}/a$ cps, where a = radius in centimeters.

and falls off as $f^{-\frac{3}{2}}$ for high frequencies since R_m increases as $f^{\frac{1}{2}}$. The rapid decrease in attenuation with increasing frequency is a unique property of the TE_{0m} modes in circular waveguides and makes possible the construction of very long low-loss waveguide communication links.† In Fig. 3.15 the attenuation in decibels for the TE_{0m} modes as a function of frequency for a copper waveguide is plotted. Although very low attenuations are achieved for frequencies well above the cutoff frequency $f_{c,01}$, certain practical difficulties are encountered which limit the overall performance of such guides to less than the theoretical predictions. These practical difficulties stem from operating the guide at a frequency well above the dominant-TE_{11}-mode cutoff frequency, i.e., in a frequency region where many modes can propagate. Any small irregularity in the guide causes some of the power in the TE_{01} mode to be converted into power in other modes (mode conversion). Two serious effects arise from mode conversion. The most obvious effect is the loss of power in the desired TE_{01} mode when some of this power is converted into other more rapidly attenuating modes. The more serious effect arises when the power in the TE_{01} mode is converted into power in other modes, with different propagation phase constants and at a position farther along the guide converted back into a TE_{01} mode, since this leads to signal distortion. To avoid signal distortion arising from this mode conversion and reconversion, it is desirable that the waveguide have a high attenuation for the undesired modes, so that these will be rapidly attenuated and not converted back into a TE_{01} mode. The currents associated with the TE_{0m} modes are in the circumferential direction only. This property may be utilized to construct mode filters that will suppress modes having

† S. E. Miller, Waveguide as a Communication Medium, *Bell System Tech. J.*, vol. 33, pp. 1209–1265, November, 1954; also Millimeter Waves in Communications, *Proc. Symp. Millimeter Waves*, Polytechnic Institute of Brooklyn, New York, 1959, pp.