Complex Poynting's Theorem

Maxwell's equations in Complex form for a linear medium (isotropic)

(1) $\nabla \times \underline{\hat{E}} = -j\omega\mu \underline{\hat{H}}$

(2) $\nabla \times \underline{\hat{H}} = \underline{\hat{J}} + j\omega\epsilon \underline{\hat{E}}$

(3) $\nabla \cdot \mu \underline{\hat{H}} = 0$

(4) $\nabla \cdot \epsilon \underline{\hat{E}} = \hat{\rho}$

Let us mathematically manipulate (1) and (2) by taking (1) and dotting it into $\underline{\hat{H}}$ conjugate $\equiv \underline{\hat{H}}^*$ and taking (2) conjugated and dot multiply it by $\underline{\hat{E}}^*$ then subtracting one resulting equation from the other so that we have

$$\underline{\hat{E}} \cdot \nabla \times \underline{\hat{H}}^* - \underline{\hat{H}}^* \cdot \nabla \times \underline{\hat{E}} = \underline{\hat{E}} \cdot \underline{\hat{J}}^* + j\omega(\mu \underline{\hat{H}} \cdot \underline{\hat{H}}^* - \epsilon \underline{\hat{E}} \cdot \underline{\hat{E}}^*)$$

or

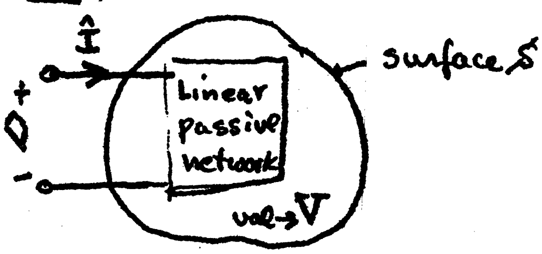
$$-\nabla \cdot (\underline{\hat{E}} \times \underline{\hat{H}}^*/2) = \frac{\underline{\hat{E}} \cdot \underline{\hat{J}}^*}{2} + 2j\omega\left[\frac{1}{4}\mu|\hat{H}|^2 - \frac{1}{4}\epsilon|\hat{E}|^2\right]$$

now $\frac{1}{4}\mu|\hat{H}|^2 = \langle u_m \rangle$, the time average magnetic energy density stored

and $\frac{1}{4}\epsilon|\hat{E}|^2 = \langle u_e \rangle$ " " " electric energy density

we define $\underline{\hat{P}} = \frac{1}{2}(\hat{E} \times \hat{H}^*)$ as the complex Poynting vector

Example



within some surface we have a linear passive network (which can be a lumped element representation of a distributed system at the frequency ω)

we can ~~b~~ define the impedance of this network at some position in space by determining the complex voltage $\hat{V}$ and current $\hat{I}$ at this position

$$Z = \hat{V}/\hat{I} = \frac{\hat{V}\,\hat{I}^*}{|I|^2}$$

but $\frac{1}{2}\hat{V}\hat{I}^* = -\oint_S \vec{P}\cdot d\vec{s}$ since it is the complex power flowing into the volume! hence

$$Z = \frac{2}{|I|^2}\left[\frac{1}{2}\int_{vol}(\hat{J}^*\cdot\hat{E})\,dv + 2j\omega\int_{vol}(\langle u_m\rangle - \langle u_e\rangle)\,dv\right]$$

$$= \frac{2}{|I|^2}\left[\langle P_d\rangle_{\text{(average power dissipated)}} + 2j\omega(\langle U_m\rangle - \langle U_e\rangle)\right]$$

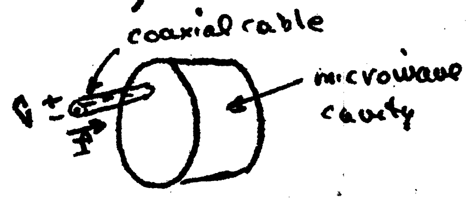
NOTE SIGN

$\langle U_m\rangle$: time ave stored mag. energy

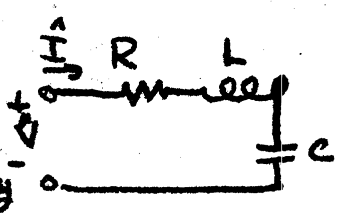
$\langle U_e\rangle$: time average elect. energy stored

resistance, $R = \mathrm{Re}\,Z = \dfrac{2\langle P_d\rangle}{|I|^2}$

reactance, $X = \mathrm{Im}\,Z = \dfrac{4\omega}{|I|^2}(\langle U_m\rangle - \langle U_e\rangle)$



≡ equivalent to ⇒ at some frequency near resonance.



$$\langle U_m\rangle = \frac{1}{4}L|I|^2$$

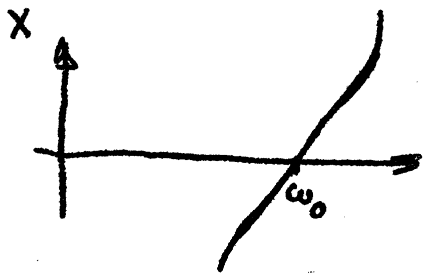
$$\langle U_e\rangle = \frac{1}{4}\frac{|I|^2}{\omega^2 C}$$

$$\langle P_d\rangle = \frac{1}{2}R|I|^2$$

$$Z = R + jX = \frac{2}{|I|^2}\left[\frac{|I|^2 R}{2} + 2\left[\frac{\omega L|I|^2}{4} - \frac{|I|^2}{4\omega C}\right]\right]$$

resonance occurs for $X = 0$

or $\omega_0^2 = \dfrac{1}{LC}$



let us expand the reactance near resonance in a Taylor series

$$X \approx X(\omega_0) + (\omega-\omega_0)\left.\frac{\partial X}{\partial \omega}\right|_{\omega=\omega_0}$$

but $\frac{\partial X}{\partial \omega} = \frac{4}{|I|^2}\left[\frac{L|I|^2}{4} + \frac{|I|^2}{4\omega^2 C}\right] = \frac{4}{|I|^2}\left[\langle U_m\rangle + \langle U_e\rangle\right]$

by direct substitution.

this is a general result in spite of our specific use of a series RLC net.

hence near resonance

$$Z = \frac{2}{|I|^2}\left[\langle P_d\rangle + 2j(\omega-\omega_0)\left(\langle U_m\rangle_{\omega=\omega_0} + \langle U_e\rangle_{\omega=0}\right)\right]$$

(note $\langle U_m\rangle = \langle U_e\rangle$ at resonance)

the Q is found by setting $X = R$ at some $\omega - \omega_0$ and is given by $Q = \frac{2\langle U_e\rangle_{\omega=\omega_0}}{\langle P_d\rangle}$ $\quad \frac{\text{resonant frequency}}{\text{bandwidth}}$

now let us perturb the system in the volume by introducing a small modification of its dielectric constant say which gives a $\Delta\langle U_e\rangle_{\omega=\omega_0} \ll \langle U_e\rangle_{\omega=\omega_0}$ then the change in reactance is

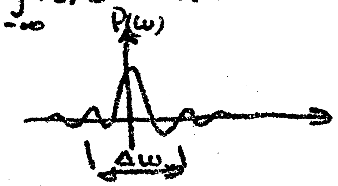
$$\Delta X = \left.\frac{\partial X}{\partial \omega}\right|_{\omega_0}(\omega-\omega_0) + \left.\frac{\partial X}{\partial \langle U_e\rangle}\right|_{\omega=\omega_0}\langle \Delta U_e\rangle = \frac{4}{|I|^2}\left(\langle U_m\rangle + \langle U_e\rangle\right)(\omega-\omega_0) - \frac{4\omega_0}{|I|^2}\langle \Delta U_e\rangle$$

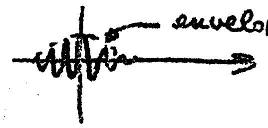
so the frequency shifts $\omega - \omega_0 = \Delta\omega$ to return to resonance $(\Delta X = 0)$ by an amount $\frac{\Delta\omega}{\omega_0} = \langle \Delta U_e\rangle / 2\langle U_e\rangle$ (we have used $\langle U_m\rangle = \langle U_e\rangle$)

GROUP VELOCITY (See Sec 1.27)

we consider propagation of a pulse modulated r.f. carrier ($f(t)$)

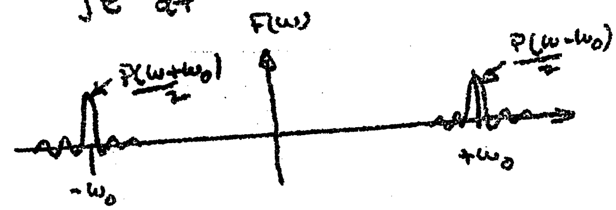
pulse $= p(t)$  $P(\omega) = \int_{-\infty}^{\infty} p(t)\, e^{-j\omega t}\, dt$

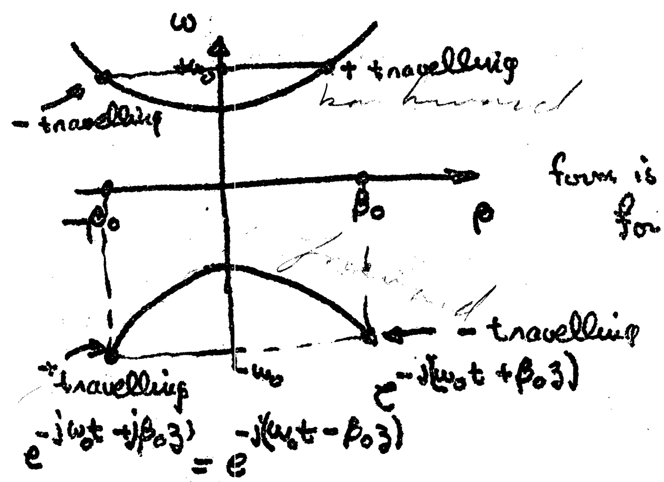


$f(t) = p(t)\cos\omega_0 t$ 

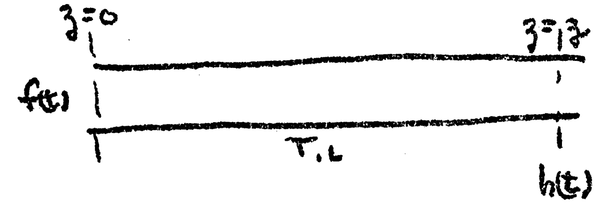
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \frac{1}{2}\int_{-\infty}^{\infty} p(t)\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] e^{-j\omega t}\, dt \quad \text{where } \Delta\omega \ll \omega_0$$

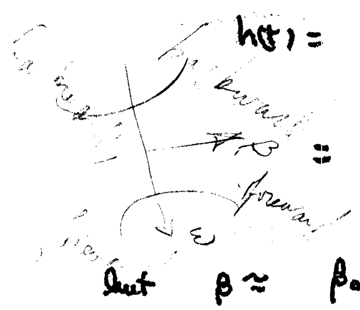
$$= \frac{1}{2}\left[P(\omega+\omega_0) + P(\omega-\omega_0)\right]$$





form is $e^{j(\omega t - \beta z)}$ for + wave





$$h(t) = \frac{1}{2\pi}\int_0^{\infty} e^{j(\omega t - \beta z)} F(\omega)\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{0} e^{j(\omega t + \beta z)} F(\omega)\, d\omega$$

$$= 2\,\mathrm{Re}\,\frac{1}{2\pi}\int_0^{\infty} F(\omega)\, e^{j(\omega t - \beta z)}\, d\omega = 2\,\mathrm{Re}\,\frac{1}{2\pi}\int_0^{\infty} \frac{P(\omega-\omega_0)}{2}\, e^{j(\omega t - \beta z)}\, d\omega$$

but $\beta \simeq \beta_0 + \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega=\omega_0}(\omega-\omega_0)$ so $h(t) = 2\,\mathrm{Re}\, e^{j(\omega_0 t - \beta_0 z)}\frac{1}{2\pi}\int_0^{\infty} \frac{P(\omega-\omega_0)}{2}\, e^{j(\omega-\omega_0)\left[t - \frac{\partial\beta}{\partial\omega}\big|_{\omega_0} z\right]}\, d\omega$

$$\text{or } h(t) = \cos(\omega_0 t - \beta_0 z)\; p\left(t - \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} z\right)$$

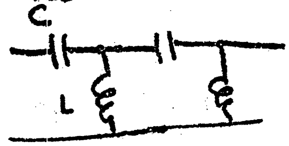
so the envelope propagates with the velocity, $v_{g\,roup}$ (called the group velocity) which is $v_g = \dfrac{1}{\left.\partial\beta/\partial\omega\right|_{\omega=\omega_0}}$.

The carrier propagates with the velocity $v_{phase} = \omega_0/\beta_0$.

Backward Waves

the Group velocity is $\frac{1}{\partial\beta/\partial\omega}$ which we saw is the velocity of the propagation of the envelope of a pulsed r.f. signal.

Let's consider a simple periodic transmission line whose structure is



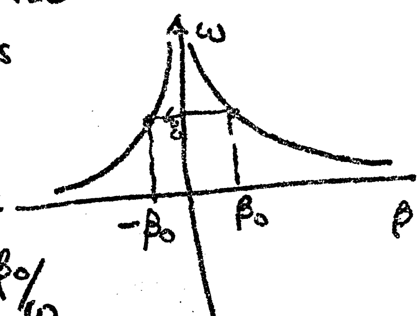
the propagation constant

$\gamma = \sqrt{ZY} = \pm \frac{j}{\omega\sqrt{LC}} = j\beta$ so the

$Z = \frac{1}{j\omega C}$

dispersion diagram is

$Y = \frac{1}{j\omega L}$



$-\beta^2 = \gamma^2 = \frac{1}{j\omega C} \cdot \frac{1}{j\omega L} = -\frac{1}{\omega^2 LC}$ $\quad \beta = \frac{1}{\omega\sqrt{LC}}, \ \beta_0 = \frac{1}{\omega_0\sqrt{LC}}$

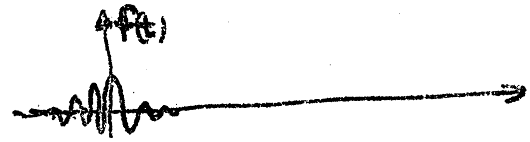
$\beta^2 = \frac{1}{\omega^2 LC} \qquad \beta = \frac{1}{\omega\sqrt{LC}}$ $\qquad \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} = -\frac{1}{\omega_0^2\sqrt{LC}} = -\beta_0/\omega_0$

hence β_0 and $\left.\partial\beta/\partial\omega\right|_{\omega_0}$ have opposite signs!

for a wave travelling in the + z-direction could we use ~~A(t)cos~~ $p(t + \beta_0/\omega_0\, z)\cos\omega_0(t - \beta_0 z/\omega_0)$?

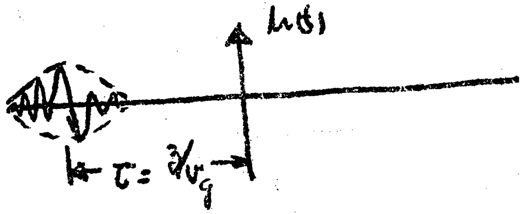
what does this form imply? envelope arrives before the carrier...

at $z=0$



actually for a backward wave the direction is the same sign as appears in the equation!

at $z=l$



that is the wave has arrived before it has been transmitted! (called ESP)

actually the + travelling wave is $p(t - \beta_0/\omega_0\, z)\cos\omega_0(t + \frac{\beta_0}{\omega_0}z)$

These are called BACKWARD WAVES