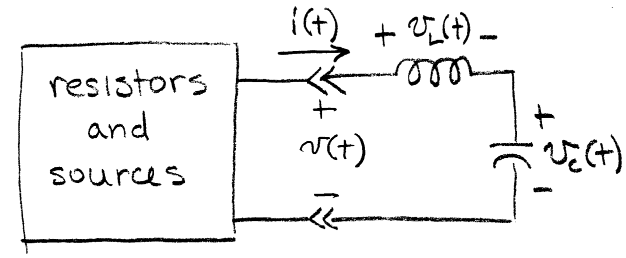
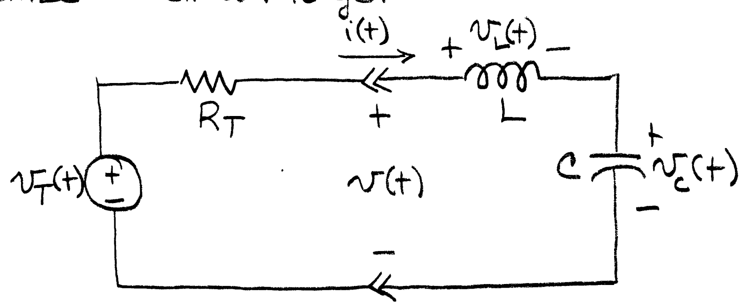
# 7-5 The Series RLC circuit

Second order circuits contain two energy storage elements that cannot be replaced by a single equivalent element.




Thevenize the circuit to get




This is known as a series RLC circuit.

On the left hand side use KVL to get $-v_T(t) + i(t) R_T + v(t) = 0$

On the right hand side use KVL to get $-v(t) + v_L(t) + v_c(t) = 0$

The i-v characteristics of the inductor and capacitor give

$$v_L = L \frac{di}{dt}$$

$$i = C \frac{dv_c}{dt}$$

These are four equations in four unknowns $(i, v, v_c, v_L)$

To get a single equation in $v_c(t)$

$$v_L(t) + v_c(t) - v_T(t) + i(t) R_T = 0$$

$$v_L(t) + v_c(t) + R_T C \frac{dv_c}{dt} = v_T(t)$$

substituting $L \frac{di}{dt} = L \frac{d}{dt}\left(C \frac{dv_c}{dt}\right) = LC \frac{d^2 v_c}{dt^2}$

$$LC \frac{d^2 v_c}{dt^2} + R_T C \frac{dv_c}{dt} + v_c = v_T(t)$$

This is a second-order linear differential equation with constant coefficients.

The initial conditions are

$$v_c(0) = V_0$$

$$\frac{dv_c(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C}$$

Zero-input response

With $v_T = 0$ the differential equation for $v_c$ becomes

$$LC\frac{d^2 v_c}{dt^2} + R_T C \frac{dv_c}{dt} + v_c = 0 \qquad (1)$$

We could also solve the circuit in terms of the inductor current $i$.

$$\underbrace{v_L(t)}_{v_L = L\frac{di}{dt}} + \underbrace{v_c(t)}_{\frac{1}{C}\int_0^t i(x)\,dx + v_c(0)} - \cancelto{0}{v_T(t)} + i(t)R_T = 0$$

$$L\frac{di}{dt} + \frac{1}{C}\int_0^t i(x)\,dx + v_c(0) + iR_T = 0$$

differentiate once more to get

$$LC\frac{d^2 i(t)}{dt^2} + R_T C\frac{di(t)}{dt} + i(t) = 0 \qquad (2)$$

Equations (1) and (2) are identical in form so we will analyze just (1)

Example 7-14

A series RLC circuit has $C = 0.25\,\mu f$ and $L = 1H$. Find the roots of the characteristic equation for $R_T = 8.5k$, $4k$ and $1k\Omega$.

The characteristic equation for a series RLC is

$$LCs^2 + R_TCs + 1 = 0$$

For $R_T = 8.5k$

$$LC = (1)(.25\times10^{-6}) = .25\times10^{-6}$$

$$R_TC = (8.5\times10^{3})(.25\times10^{-6}) = 2.125\times10^{-3}$$

$$s_1, s_2 = \frac{-2.125\times10^{-3} \pm \sqrt{(2.125\times10^{-3})^2 - 4(.25\times10^{-6})(1)}}{2(.25\times10^{-6})}$$

$$s_1, s_2 = \frac{-2.125\times10^{-3} \pm 1.875\times10^{-3}}{0.5\times10^{-6}} = -4250 \pm 3750$$

$$s_1 = -500 \qquad s_2 = -8000$$

For $R_T = 4k$

$$R_TC = (4\times10^{3})(.25\times10^{-6}) = 10^{-3}$$

$$s_1, s_2 = \frac{-10^{-3} \pm \sqrt{(10^{-3})^2 - 4(.25\times10^{-6})(1)}}{2(.25\times10^{-6})} = \frac{-10^{-3} \pm \sqrt{10^{-6} - 10^{-6}}}{2(.25\times10^{-6})}$$

$$s_1, s_2 = -2000$$

For $R_T = 1k$

$$R_TC = (1\times10^{3})(.25\times10^{-6}) = .25\times10^{-3}$$

$$s_1, s_2 = \frac{-.25\times10^{-3} \pm \sqrt{(.25\times10^{-3})^2 - 4(.25\times10^{-6})(1)}}{2(.25\times10^{-6})}$$

$$s_1, s_2 = \frac{-.25\times10^{-3} \pm \sqrt{6.25\times10^{-8} - 1\times10^{-6}}}{0.5\times10^{-6}}$$

$$s_1, s_2 = \frac{-.25\times10^{-3} \pm \sqrt{-9.375\times10^{-7}}}{.5\times10^{-6}} = \frac{-.25\times10^{-3} \pm j9.682\times10^{-4}}{.5\times10^{-6}}$$

$$s_1, s_2 = -500 \pm j1936.5$$

Based upon previous experience we try a solution of the form $v_C = Ke^{st}$ to get the characteristic equation

$$LC\,Ks^2e^{st} + R_T C\,Kse^{st} + Kse^{st} = 0$$

$$Ke^{st}\left(LCs^2 + R_T Cs + 1\right) = 0$$

This equation has two roots. Using the quadratic formula

$$s_1, s_2 = \frac{-R_T C \pm \sqrt{(R_T C)^2 - 4LC}}{2LC}$$

This has three cases depending upon the square root.

CASE A: $(R_T C)^2 - 4LC > 0$ two real, unequal roots

$s_1 = -\alpha_1,\ s_2 = -\alpha_2$

CASE B: $(R_T C)^2 - 4LC = 0$ two equal, real roots

$s_1 = s_2 = -\alpha$

CASE C: $(R_T C)^2 - 4LC < 0$ two complex conjugate roots

$s_1 = -\alpha - j\beta$ and $s_2 = -\alpha + j\beta$

CASE A: $\quad s_1 = -\alpha_1 \;,\; s_2 = -\alpha_2$

$$v_c(t) = \frac{-\alpha_2 V_0 - I_0/C}{-\alpha_2 + \alpha_1}\, e^{-\alpha_1 t} + \frac{+\alpha_1 V_0 + I_0/C}{-\alpha_2 + \alpha_1}\, e^{-\alpha_2 t}$$

$$v_c(t) = \frac{\alpha_2 V_0 + I_0/C}{\alpha_2 - \alpha_1}\, e^{-\alpha_1 t} + \frac{-\alpha_1 V_0 - I_0/C}{\alpha_2 - \alpha_1}\, e^{-\alpha_2 t}$$

This is a sum of two exponential functions.
See Example 5-14.

CASE B: $\quad s_1 = s_2 = -\alpha$

The solution becomes

$$v_c(t) = \frac{-\alpha V_0 - I_0/C}{-\alpha + \alpha}\, e^{-\alpha t} + \frac{-\alpha V_0 + I_0/C}{-\alpha + \alpha}\, e^{-\alpha t}$$

The denominators go to zero but the numerator also goes to zero.
Let $s_1 = -\alpha$ and $s_2 = -\alpha + x$ and take limit as $x \to 0$

$$v_c(t) = \frac{(-\alpha + x) V_0 - I_0/C}{-\alpha + x + \alpha}\, e^{-\alpha t} + \frac{+\alpha V_0 + I_0/C}{-\alpha + x + \alpha}\, e^{(-\alpha + x)t}$$

$$v_c(t) = e^{-\alpha t}\left[\frac{-\alpha V_0 + x V_0 - I_0/C + \alpha V_0 e^{xt} + I_0/C\, e^{xt}}{x}\right]$$

$$v_c(t) = e^{-\alpha t}\left[V_0 - \frac{\alpha V_0}{x} - \frac{I_0/C}{x} + \frac{\alpha V_0}{x} e^{xt} + \frac{I_0/C}{x} e^{xt}\right]$$

$$v_c(t) = e^{-\alpha t}\left[V_0 - (\alpha V_0 + I_0/C)\frac{1}{x} + (\alpha V_0 + I_0/C)\frac{e^{xt}}{x}\right]$$

$$v_c(t) = e^{-\alpha t}\left[V_0 - (\alpha V_0 + I_0/C)\frac{1 - e^{xt}}{x}\right].$$

## Zero-input response

There are two roots to the characteristic equation which correspond to two solutions.

The general solution for zero-input is the sum of these solutions

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

The initial conditions are

$$v_c(0) = V_0$$

$$\frac{dv_c(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C}$$

For $t=0$ $\quad v_c(0) = K_1 e^0 + K_2 e^0 = K_1 + K_2 = V_0$

To use the second initial condition we need to differentiate our expression for $v_c(t)$

$$\frac{dv_c(t)}{dt} = K_1 s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t}$$

$$\frac{dv_c(0)}{dt} = K_1 s_1 + K_2 s_2 = \frac{I_0}{C}$$

$$\begin{aligned} s_1 K_1 + s_2 K_2 &= \frac{I_0}{C} \\ K_1 + K_2 &= V_0 \end{aligned}$$

$$\begin{aligned} s_1 K_1 + s_2 K_2 &= \frac{I_0}{C} \\ s_1 K_1 + s_1 K_2 &= s_1 V_0 \end{aligned}$$

$$(s_2 - s_1) K_2 = \frac{I_0}{C} - s_1 V_0$$

$$K_2 = \frac{-s_1 V_0 + I_0/C}{s_2 - s_1}$$

$$K_1 = V_0 - K_2 = V_0 - \frac{-s_1 V_0 + I_0/C}{s_2 - s_1}$$

$$K_1 = \frac{s_2 V_0 - s_1 V_0 + s_1 V_0 - I_0/C}{s_2 - s_1} = \frac{s_2 V_0 - I_0/C}{s_2 - s_1}$$

This response is different depending upon cases A, B, or C.

The indeterminacy in this equation comes from the second term. We can use L'Hopital's rule to evaluate it

$$\lim_{x \to 0} \underbrace{\frac{1-e^{xt}}{x}}_{\text{take derivatives with respect to } x} = \lim_{x \to 0} \left( \frac{-te^{xt}}{1} \right) = -t$$

Substituting this result gives

$$v_c(t) = e^{-\alpha t}\left[V_0 - \left(\alpha V_0 + \frac{I_0}{C}\right)(-t)\right]$$

$$v_c(t) = V_0 e^{-\alpha t} + \left(\alpha V_0 + I_0/C\right) t e^{-\alpha t} \qquad t \geq 0$$

This solution is an exponential and a damped ramp.

CASE C is perhaps the most interesting.

$$s_1 = -\alpha - j\beta \qquad s_2 = -\alpha + j\beta$$

Substituting gives

$$v_c(t) = \frac{(-\alpha + j\beta)V_0 - I_0/C}{(-\alpha + j\beta) - (-\alpha - j\beta)} e^{-\alpha t - j\beta t} + \frac{-(-\alpha - j\beta)V_0 + I_0/C}{(-\alpha + j\beta) - (-\alpha - j\beta)} e^{-\alpha t + j\beta t}$$

$$v_c(t) = \frac{-\alpha V_0 + j\beta V_0 - I_0/C}{j2\beta} e^{-\alpha t} e^{-j\beta t} + \frac{\alpha V_0 + j\beta V_0 + I_0/C}{j2\beta} e^{-\alpha t} e^{j\beta t}$$

$$v_c(t) = \frac{j\beta V_0 e^{-\alpha t} e^{-j\beta} + j\beta V_0 e^{-\alpha t} e^{j\beta t}}{j2\beta} + \frac{-\alpha V_0 e^{-j\beta t} - I_0/C\, e^{-j\beta t} + \alpha V_0 e^{j\beta t} + I_0/C\, e^{j\beta t}}{j2\beta} e^{-\alpha t}$$

$$v_c(t) = V_0 e^{-\alpha t}\left[\frac{e^{-j\beta t} + e^{+j\beta t}}{2}\right] + \frac{\alpha V_0 + I_0/C}{\beta} e^{-\alpha t}\left[\frac{e^{j\beta t} - e^{-j\beta t}}{j2}\right]$$

We can use Euler's identity to re-write the expressions in the square brackets.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\therefore \quad \left.\begin{aligned} \cos\theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin\theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \end{aligned}\right\}$$

adding and subtracting the above equations gives these useful expressions for the sine and cosine.

We can use these expressions to simplify our solution

$$v_c(t) = V_0 e^{-\alpha t} \cos\beta t + \frac{\alpha V_0 + I_0/C}{\beta} e^{-\alpha t} \sin\beta t$$

These are two damped sinusoids. The real part of the roots describes the exponential decay, and the imaginary part of the roots defines the frequency of the sinusoidal oscillations.

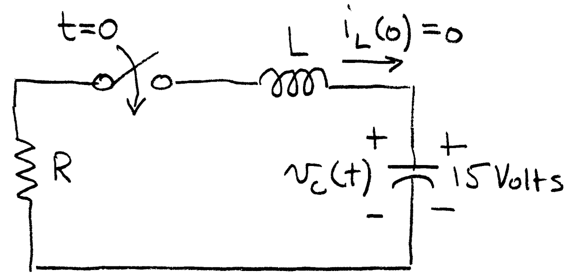
$s$ – has units of complex frequency

$\alpha$ – has units of neper frequency

$\beta$ – has units of radian frequency

Example 7-15

The circuit shown below has $C = 0.25\mu f$ and $L = 1H$. The switch has been open for a long time and is closed at $t=0$. Find the capacitor voltage for $t \geq 0$ for (a) $R_T = 8.5k$, (b) $R = 4k$ and (c) $R = 1k$. The initial conditions are $I_0 = 0$ and $V_0 = 15$.




SOLUTION:

The roots of the characteristic equation were found in Example 7-14.

(a) $R_T = 8.5k$ $\quad s_1 = -500 \quad s_2 = -8000$

$$v_c(t) = K_1 e^{-500t} + K_2 e^{-8000t}$$

$$v_c(0) = K_1 + K_2 = V_0 = 15$$

$$\frac{d v_c(t)}{dt} = \frac{I_0}{C} = K_1(-500)e^{-500t} + K_2(-8000)e^{-8000t}$$

$$\frac{d v_c(0)}{dt} = 0 = -500K_1 - 8000K_2$$

$$K_1 + K_2 = 15$$
$$K_1 + 16K_2 = 0$$

$$K_2 - 16K_2 = 15$$

$$K_2 = \frac{15}{-15} = -1$$

$$K_1 + 16(-1) = 0$$
$$K_1 = 16$$

$$v_c(t) = 16e^{-500t} - e^{-8000t} \qquad t \geq 0$$

(b) $R_T = 4k\Omega$ $\quad s_1 = s_2 = -2000.$

The zero-input solution is an exponential plus an exponentially damped ramp.

$$v_c(t) = K_1 e^{-2000t} + K_2 t e^{-2000t}$$

We now solve for the initial conditions in the same manner as (a).

$$v_c(0) = K_1 = V_0 = 15$$

$$\frac{dv_c(t)}{dt} = K_1(-2000)e^{-2000t} + K_2(-2000)te^{-2000t} + K_2 e^{-2000t} = \frac{I_0}{C}$$

$$\frac{dv_c(0)}{dt} = -2000K_1 e^{0} + 0 + K_2 e^{0} = \frac{0}{C}$$

$$-2000K_1 + K_2 = 0 \quad \Rightarrow K_2 = 2000K_1 = 30000$$

$$v_c(t) = 15e^{-2000t} + 30000te^{-2000t}, \quad t \geq 0$$

(c) $R_T = 1k\Omega$ $\quad s_1 = -500 - j\,1936.5 \quad s_2 = -500 + j\,1936.5$

$$v_c(t) = K_1 e^{-500t}\cos(1936.5t) + K_2 e^{-500t}\sin(1936.5t)$$

$$v_c(0) = K_1 e^{0} + K_2 \cdot 0 = 15 \quad \therefore K_1 = 15$$

$$\frac{dv_c(t)}{dt} = K_1(-500)e^{-500t}\cos(1936.5t) + K_1 e^{-500t}(-1936.5)\sin(1936.5t)$$
$$+ K_2(-500)e^{-500t}\sin(1936.5t) + K_2 e^{-500t}(1936.5)\cos(1936.5t)$$

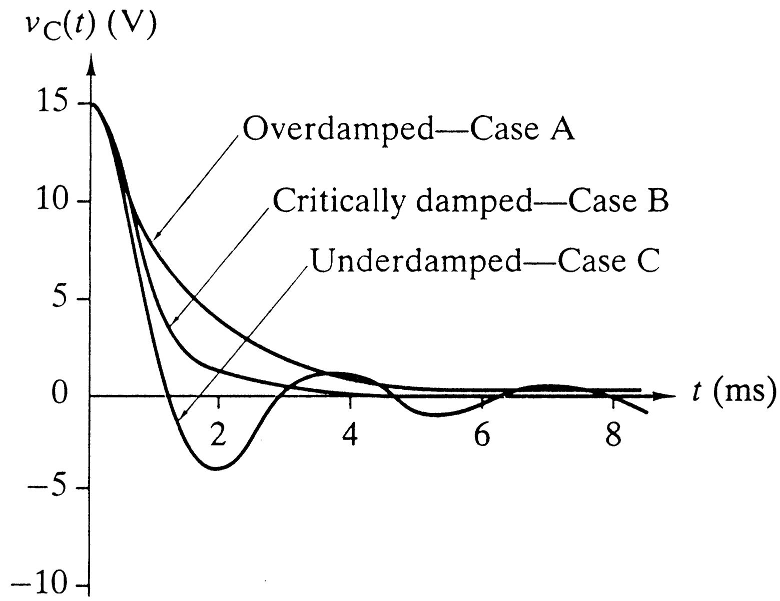
$$\frac{dv_c(0)}{dt} = K_1(-500)e^{0}\cdot 1 + K_1 e^{0}(-1936.5)(0) + K_2(-500)e^{0}(0) + K_2 e^{0}(1936.5)(1) = 0$$

$$-500K_1 + 1936.5K_2 = 0$$

$$K_2 = \frac{500}{1936.5}K_1 = \frac{500}{1936.5}(15) = 3.873$$

$$v_c(t) = 15e^{-500t}\cos(1936.5t) + 3.873e^{-500t}\sin(1936.5t) \quad t \geq 0$$

The plots of these are very interesting.




All three cases start at $V_c(0) = 15$ and all damp out but in different ways.

Example 7-16

In a series RLC circuit the zero-input voltage across the 1μf capacitor is

$$v_C(t) = 10e^{-1000t} \sin 2000t \qquad V \geq 0$$

(a) Find the circuit characteristic equation.

The circuit is underdamped because it has a sine
By inspection $\alpha = 1000$, $\beta = 2000$
the solutions are then

$$(s+\alpha+j\beta)(s+\alpha-j\beta) = (s+1000+j2000)(s+1000-j2000)$$

$$= s^2 + (1000+j2000+1000-j2000)s + (1000^2+2000^2)$$

$$= s^2 + 2000s + 5\times10^6 = 0$$

(b) Find the R and L.

The characteristic equation can also be written as

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Since $C = 1\mu f$ we can solve for L and R.

$$\frac{1}{LC} = 5\times10^6 \Rightarrow L = \frac{1}{(5\times10^6)(1\times10^{-6})} = 0.2\ H$$

$$\frac{R}{L} = 2000 \Rightarrow R = 2000L = 2000(0.2) = 400\,\Omega$$

(c) Find $i_L(t)$ for $t \geq 0$

In a series RLC circuit $i_L(t) = i_C(t)$ and we know $i_C(t) = C\frac{dv_C(t)}{dt}$

$$i_L(t) = (1\times10^{-6})\left[(10)(-1000)e^{-1000t}\sin 2000t + 10e^{-1000t}(+2000)\cos 2000t\right]$$

$$i_L(t) = 0.01e^{-1000t}\sin 2000t + 0.02e^{-1000t}\cos 2000t$$

(d) Find the initial values of the state variables.

By inspection $\quad v_C(0) = 0$

$\qquad i_L(0) = 0.02$ Amps.