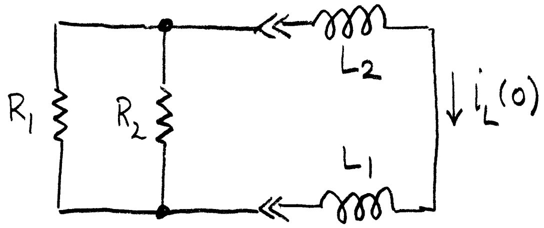
Example 7-2

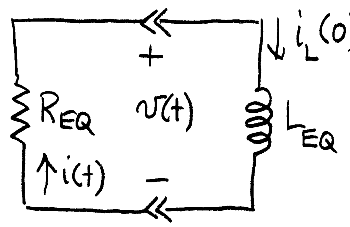
Find the response of the state variable of the RL circuit in the figure below using $L_1 = 10mH$, $L_2 = 30mH$, $R_1 = 2k\Omega$, $R_2 = 6k\Omega$, and $i_L(0) = 100mA$.



The inductors are in series and can be replaced by an equivalent L_{EQ}.

$$L_{EQ} = L_1 + L_2 = 10mH + 30mH = 40mH$$

The resistors are in parallel and can be replaced by $R_{EQ} = \frac{R_1 R_2}{R_1 + R_2} = \frac{(2k)(6k)}{2k+6k} = 1.5k$



$i_L(0)$ remains the same since L_1 & L_2 were in series

Using KVL $\quad +i(t)R_{EQ} + v(t) = 0$

The inductor constraint is $\quad v(t) = L_{EQ}\frac{di(t)}{dt}$

$$i(t)R_{EQ} + L_{EQ}\frac{di(t)}{dt} = 0$$

$$\frac{L_{EQ}}{R_{EQ}}\frac{di(t)}{dt} + i(t) = 0$$

Assume $i(t) = Ke^{st}$, then

$$\frac{L_{EQ}}{R_{EQ}}Kse^{st} + Ke^{st} = 0$$

$$\frac{L_{EQ}}{R_{EQ}}s + 1 = 0 \quad \Rightarrow s = -\frac{R_{EQ}}{L_{EQ}} = -\frac{1500}{40\times10^{-3}}$$

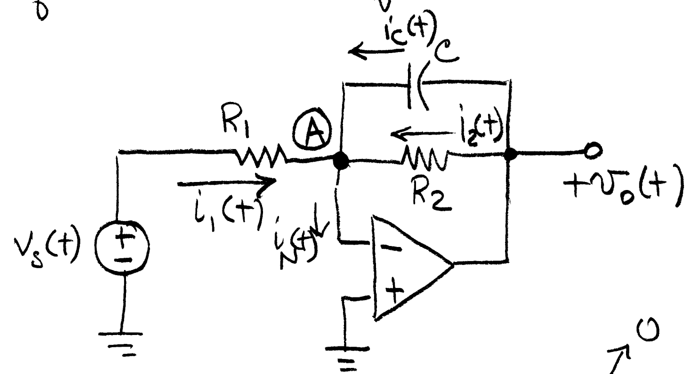
$$s = -37500$$

$$\therefore \; i(t) = Ke^{-37500t} = Ke^{-\frac{t}{26.7\mu s}}$$

$$i(t=0) = Ke^{0} = 100mA$$

$$\therefore \; i(t) = 0.1e^{-\frac{t}{26.7\mu s}}\ A.$$

Consider a circuit which is difficult to Thevenize.
Derive equations in terms of a more convenient variable.



Using KCL at node A $\quad \sum i = 0 \quad$ +in $\qquad +i_1(t) - \cancelto{0}{i_N}(t) + i_c(t) + i_2(t) = 0$

$$\frac{v_s - 0}{R_1} + 0 + C\frac{d}{dt}(v_o - 0) + \frac{v_o - 0}{R_2} = 0$$

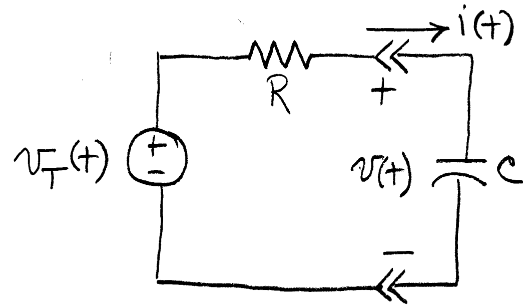
$$C\frac{dv_o}{dt} + \frac{v_o}{R_2} = -\frac{v_s}{R_1}$$

$$\underbrace{R_2 C\frac{dv_o}{dt} + v_o} = -\frac{R_2}{R_1}v_s(t)$$

differential equation in v_o
rather that the
capacitor voltage v_c

7-2 First Order Circuit Step Response

Consider the following circuit



We solved this circuit previously for $v_T(t) = 0, \quad t \geq 0$

Consider the case where $v_T(t) = V_A u(t)$

The circuit differential equation is

$$R_T C \frac{dv}{dt} + v = V_A u(t)$$

or $$R_T C \frac{dv}{dt} + v = V_A \quad \text{for } t \geq 0$$

While there are many methods to solve this equation we will use superposition.

$$v(t) = \underbrace{v_N(t)}_{\substack{\text{natural response} \\ \text{when input is} \\ \text{set to zero.}}} + \underbrace{v_F(t)}_{\substack{\text{forced response} \\ \text{to the input} \\ \text{step function}}}$$

<u>Natural response</u>:

$$R_T C \frac{dv_N}{dt} + v_N = 0$$

Solution is $v_N(t) = K e^{-\frac{t}{R_T C}} \quad t \geq 0$

which we have seen previously.

Forced response

$$R_T C \frac{dv_F(t)}{dt} + v_F(t) = V_A \qquad t \geq 0$$

A solution is $\quad v_F(t) = V_A \quad t \geq 0$

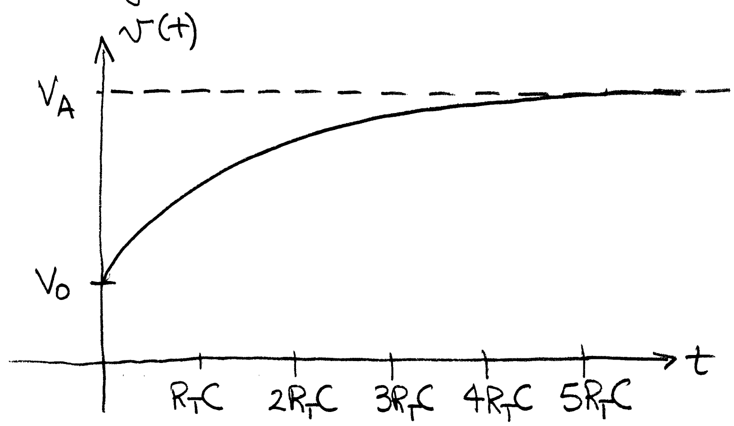
since $\dfrac{dv_F}{dt} = 0$ for $t > 0$

The Total response is the sum of the natural and forced response.

$$v(t) = v_N(t) + v_F(t)$$

$$v(t) = ke^{-\frac{t}{R_T C}} + V_A$$

This is the general solution and is plotted below.

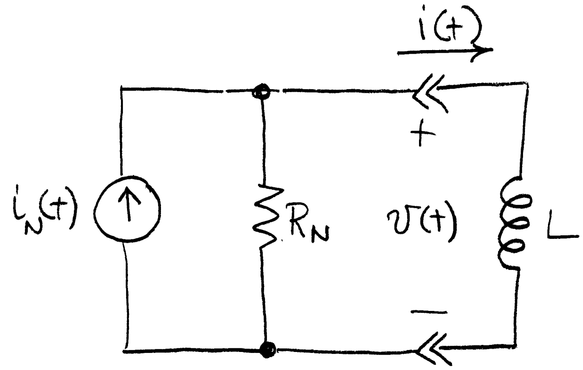


From this plot we can easily see that

$$\lim_{t \to 0^+} v(t) = V_0 \qquad \text{initial value}$$

$$\lim_{t \to \infty} v(t) = V_A \qquad \text{final value}$$

For a RL circuit with a $I_A u(t)$ input,



The differential equation is $\dfrac{L}{R_N}\dfrac{di(t)}{dt} + i(t) = I_A$ for $t \geq 0$

The natural response $\dfrac{L}{R_N}\dfrac{di_N}{dt} + i_N = 0$

has the solution $i_N(t) = ke^{-\frac{R_N}{L}t} = ke^{-\frac{t}{L/R_N}}$ for $t \geq 0$

The forced response is from $\dfrac{L}{R_N}\dfrac{di_F}{dt} + i_F = I_A$ for $t \geq 0$

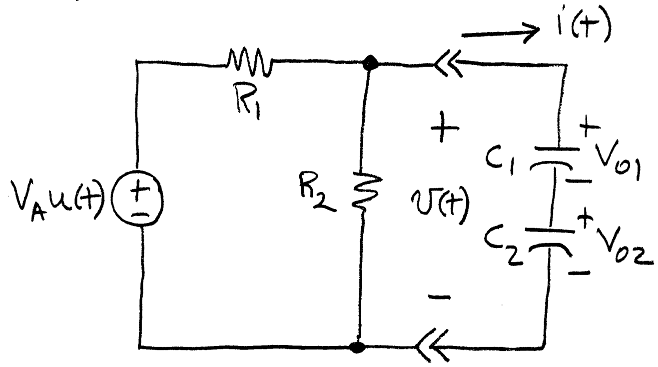
which has solution $i_F(t) = I_A$ for $t \geq 0$

The general solution is then

$$i(t) = i_N(t) + i_F(t) = ke^{-\frac{t}{L/R_N}} + I_A \qquad t \geq 0$$

A step function drives the state variable from an initial value (determined by what happened for $t<0$) to a final value (determined by the magnitude of the step at $t=0$)

Example 7-4 Find the response of the given RC circuit.



The two capacitors can be replaced by a single equivalent capacitor

$$C_{EQ} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{C_1 C_2}{C_1+C_2}$$

$V_A = 100\text{ V}$

at $t=0$ $V_{01} = 5\text{V}$

$V_{02} = 10\text{V}$

$C_1 = 0.1\,\mu f$

$C_2 = 0.5\,\mu f$

$R_1 = 30k$

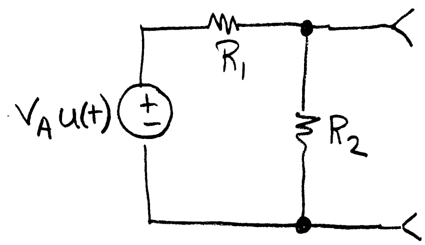
$R_2 = 10k$

$$C_{EQ} = \frac{(0.1)(0.5)}{0.1+0.5} = 0.0833\,\mu f$$

The initial voltage on C_{EQ} is the sum of the initial voltages on C_1 and C_2

$$V_0 = V_{01} + V_{02} = 5+10 = 15 \text{ volts}.$$

We next find the Thevenin equivalent seen by C_{EQ}.



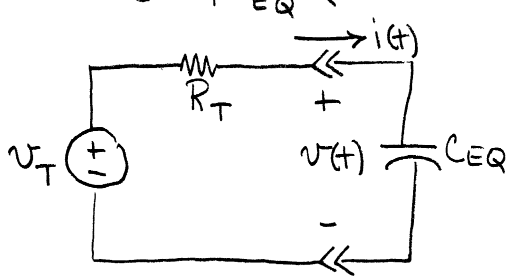
$$v_T = v_{oc} = \frac{R_2}{R_1+R_2} V_A u(t) = \frac{10k}{30k+10k} 100\,u(t)$$

$$v_T(t) = 25\,u(t)$$

Replacing the voltage source by a short we see that

$$R_T = R_1 \| R_2 = \frac{(30k)(10k)}{30k+10k} = 7.5k$$

$$T_c = R_T C_{EQ} = (7.5\times10^{3})(0.0833\times10^{-6}) = 0.625\text{ ms}$$



The circuit equation is

$$-v_T(t) + i(t)R_T + v(t) = 0$$

but $i(t) = C_{EQ}\dfrac{dv(t)}{dt}$

$$-v_T(t) + R_T C_{EQ}\frac{dv}{dt} + v = 0$$

$$R_T C_{EQ}\frac{dv}{dt} + v = v_T(t)$$