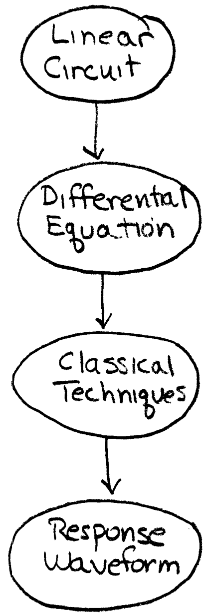
Chapter 7 - First- and Second-Order Circuits

7-1 RC and RL Circuits

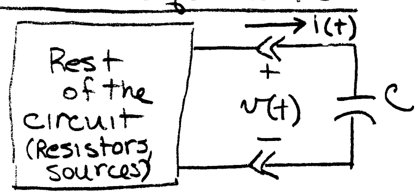


use device and connection equations to write differential equation describing the circuit

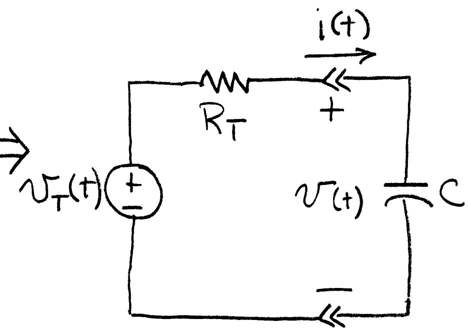
solve differential equation

use classical techniques for solving D.E.
(other techniques include phasors & Laplace transform)

RC and RL Equations



$\Rightarrow$



Doing KVL gives

$$-v_T(t) + i(t)R_T + v(t) = 0$$

From the previous chapter the capacitor is described by

$$i(t) = C\frac{dv(t)}{dt}$$

Substituting

$$-v_T(t) + C\frac{dv(t)}{dt}R_T + v(t) = 0$$

Re-arranging

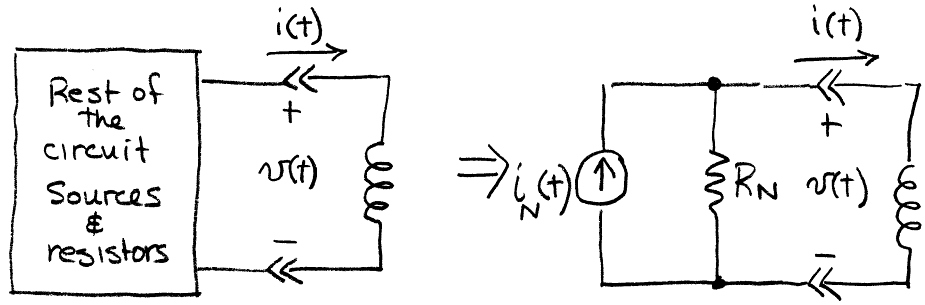
first order linear differential equation with constant coefficients

$$R_TC\frac{dv(t)}{dt} + v(t) = v_T(t)$$

$v_T(t)$ is the input and $v(t)$ is the response

$v(t)$ is called the <u>state variable</u> and determines the amount of energy stored in the RC circuit.

We can do the same with an inductor.



Using KCL at the node gives $\sum_{+in} i = 0$

$$+i_N(t) - \frac{v(t)}{R_N} - i(t) = 0$$

The element constraint is $v(t) = L\frac{di(t)}{dt}$

Substituting

$$i_N(t) - \frac{L}{R_N}\frac{di(t)}{dt} - i(t) = 0$$

Re-arranging

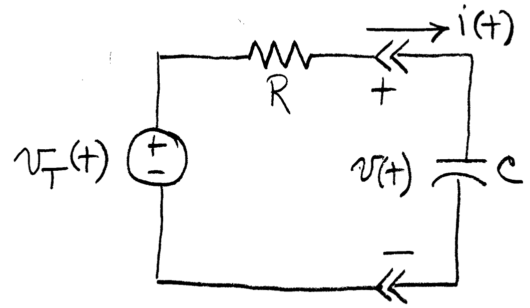
$$\frac{L}{R_N}\frac{di(t)}{dt} + i(t) = i_N(t)$$

- first-order linear differential equation with constant coefficients
- $i_N(t)$ is the forcing function
- $i(t)$ is the state variable as it defines the amount of energy stored in the RL circuit

Any circuit containing a single capacitor or inductor and resistors is a first-order circuit described by a first-order differential equation.

7-2 First Order Circuit Step Response

Consider the following circuit



We solved this circuit previously for $v_T(t) = 0, \quad t \geq 0$

Consider the case where $v_T(t) = V_A u(t)$

The circuit differential equation is

$$R_T C \frac{dv}{dt} + v = V_A u(t)$$

or

$$R_T C \frac{dv}{dt} + v = V_A \quad \text{for } t \geq 0$$

While there are many methods to solve this equation we will use superposition.

$$v(t) = \underbrace{v_N(t)}_{\substack{\text{natural response} \\ \text{when input is} \\ \text{set to zero.}}} + \underbrace{v_F(t)}_{\substack{\text{forced response} \\ \text{to the input} \\ \text{step function}}}$$

Natural response:

$$R_T C \frac{dv_N}{dt} + v_N = 0$$

Solution is $v_N(t) = K e^{-\frac{t}{R_T C}} \quad t \geq 0$

which we have seen previously.

7-4 First-order Circuit Sinusoidal Response

In previous sections we determined that the solution of a linear first-order differential equation consisted of a forced and natural response. We determined the response to a step-input. In this section we consider the response to a causal sinusoid. $V_A \cos\omega t\, u(t)$.

The complete differential equation for a first order RC circuit is

$$R_T C \frac{dv(t)}{dt} + v(t) = V_A \cos\omega t\, u(t)$$

The natural response does not change and is given by

$$v_N(t) = Ke^{-\frac{t}{R_T C}} \qquad t \geq 0$$

The forced response is a solution of

$$R_T C \frac{dv_F(t)}{dt} + v_F(t) = V_A \cos\omega t \qquad t \geq 0$$

The only solution of this equation is another sinusoid.
Consider $\quad v_F(t) = a\cos\omega t + b\sin\omega t$

This is called the method of undetermined coefficients.

$$R_T C \frac{d}{dt}(a\cos\omega t + b\sin\omega t) + (a\cos\omega t + b\sin\omega t) = V_A \cos\omega t$$

$$R_T C(-a\omega\sin\omega t + b\omega\cos\omega t) + (a\cos\omega t + b\sin\omega t) = V_A\cos\omega t$$

$$[-a\omega R_T C + b]\sin\omega t + [+b\omega R_T C + a - V_A]\cos\omega t = 0$$

This is only true when the sine and cosine coefficients are identically zero.

$$a + (R_T C\omega)\, b = V_A$$
$$(-\omega R_T C)\, a + \quad b = 0$$

$$a + \omega R_T C\, b = V_A$$

$$-(\omega R_T C)^2 a + (\omega R_T C) b = 0$$

subtracting $1 + (\omega R_T C)^2 a = V_A$

$$a = \frac{V_A}{1+(\omega R_T C)^2}$$

Substituting $\quad b = (\omega R_T C)\, a = \dfrac{\omega R_T C\, V_A}{1+(\omega R_T C)^2}$

The forced response is then

$$v_F(t) = \frac{V_A}{1+(\omega R_T C)^2}\left[\cos\omega t + \omega R_T C \sin\omega t\right] \qquad t \geq 0$$

The total solution is then

$$v(t) = K e^{-\frac{t}{R_T C}} + \frac{V_A}{1+(\omega R_T C)^2}\left[\cos\omega t + \omega R_T C \sin\omega t\right] \qquad t \geq 0$$

The initial condition $v_0(t=0) = V_0$ requires

$$v(0) = K e^{0} + \frac{V_A}{1+(\omega R_T C)^2}\left[1 + 0\right] = V_0$$

$$\therefore \quad K = V_0 - \frac{V_A}{1+(\omega R_T C)^2}$$

The total solution is then

$$v(t) = \underbrace{\left[V_0 - \frac{V_A}{1+(\omega R_T C)^2}\right] e^{-\frac{t}{R_T C}}}_{\text{natural response}} + \underbrace{\frac{V_A}{1+(\omega R_T C)^2}\left[\cos\omega t + \omega R_T C \sin\omega t\right]}_{\text{forced response}} \qquad t \geq 0$$

The more commonly used form of the solution requires converting the second term to magnitude and phase format.

$$v(t) = \underbrace{\left[V_0 - \frac{V_A}{1+(\omega R_T C)^2}\right] e^{-\frac{t}{R_T C}}}_{\text{natural response}} + \underbrace{\frac{V_A}{\sqrt{1+(\omega R_T C)^2}} \cos(\omega t + \theta)}_{\text{forced response}^*} \qquad t \geq 0$$

where we used

$$\cos \omega t + \omega R_T C \sin \omega t = \sqrt{1+(\omega R_T C)^2}\, \cos(\omega t + \theta)$$

$$\theta = \tan^{-1}\left(\frac{-\omega R_T C}{1}\right) = \tan^{-1}(-\omega R_T C)$$

Observations:

1. The forced sinusoidal response lasts whereas the natural response decays to zero.
2. The forced sinusoidal response is of the same frequency (ω) as the input but with a different magnitude and phase
3. The forced response is proportional to V_A.

* the forced response is called
  the sinusoidal steady-state response
  the ac steady-state response
  the ac response

Technically we have found the solution to the step-function

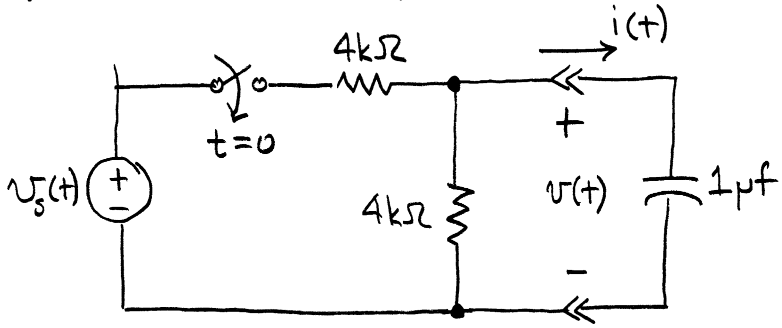
$$V_A[\cos \omega t]u(t)$$

If $\omega = 0$ this reduces to the previous solution for

$$V_A u(t)$$

Example 7-12

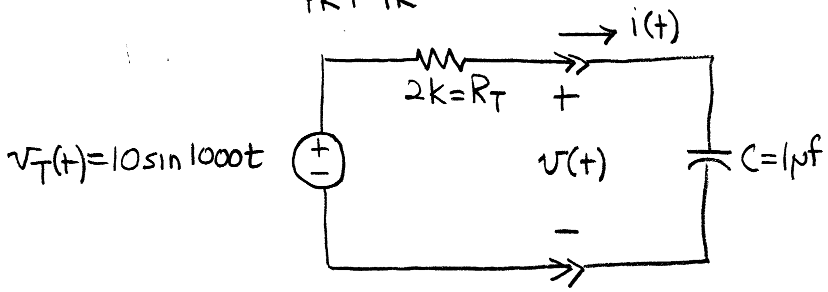
The switch in the figure below has been open for a long time and is closed at $t=0$. Find the voltage $v(t)$ for $t \geq 0$ when
$v_s(t) = 20 \sin 1000t \; u(t)$



We have to derive the circuit differential equation by Thevenizing the circuit.

Shorting $v_s(t)$, $R_T = 4k \| 4k = 2k\Omega$

$$v_T(t) = \frac{4k}{4k+4k} v_s(t) = 10 \sin 1000t \; u(t)$$



$$R_T C = (2\times10^3)(1\times10^{-6}) = 2\times10^{-3}$$

$$-v_T(t) + i(t) R_T + v(t) = 0$$

$$i(t) = C\frac{dv}{dt}$$

$$R_T C \frac{dv}{dt} + v(t) = v_T(t)$$

$$(2\times10^{-3}) \frac{dv(t)}{dt} + v(t) = 10 \sin 1000t$$

Now we can compute the natural and forced responses.

$$v_N(t) = Ke^{-\frac{t}{R_T C}} \qquad t \geq 0$$

Using undetermined Fourier coefficients we write

$$v_F(t) = a \cos 1000t + b \sin 1000t$$

We substitute $v_F(t)$ into the differential equation

$$(2\times10^{-3})\frac{d}{dt}\left(a\cos1000t+b\sin1000t\right)+\left(a\cos1000t+b\sin1000t\right)$$
$$=10\sin1000t$$

$$\frac{1}{500}\left[-a1000\sin1000t+b1000\cos1000t\right]+\left[a\cos1000t+b\sin1000t\right]=10\sin1000t$$

$$\left[-a2+b-10\right]\sin\omega t+\left[b2+a\right]\cos\omega t=0$$

$$a+2b=0$$
$$-2a+b=10$$

$$2a+4b=0$$
$$-2a+b=10$$

$$5b=10\Rightarrow b=2$$
$$a=-2b=-2(2)=-4$$

$$\therefore v(t)=v_N(t)+v_F(t)$$
$$v(t)=Ke^{-\frac{t}{.002}}-4\cos1000t+2\sin1000t$$

K is found from the initial condition that $v(0)=0$ since the switch was open for a long time the capacitor is uncharged.

$$\therefore v(0)=0=Ke^{0}-4+0\quad\Rightarrow K=4$$

The complete response is

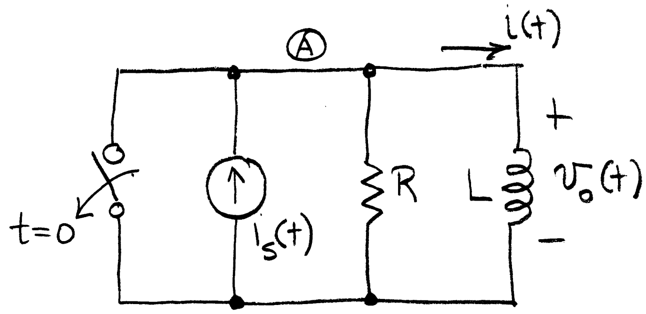
$$v(t)=4e^{-\frac{t}{.002}}-4\cos1000t+2\sin1000t$$

In magnitude and phase

$$v(t)=4e^{-\frac{t}{.002}}+\sqrt{(4)^2+(2)^2}\cos\left(1000t+\tan^{-1}\left(\frac{2}{4}\right)\right)$$
$$v(t)=4e^{-\frac{t}{.002}}+4.472\cos\left(1000t+26.5^\circ\right)$$

Example 7-13

Find the sinusoidal steady-state response of the output voltage $v_o(t)$ in the circuit shown below. when the input current is
$i_s(t) = [I_A \cos \omega t]\,u(t)$



The switch guarantees that $i(t<0) = 0$

For $t>0$ using KCL @ A $\sum_{+in} i = 0$

$$+i_s(t) - \frac{v_o(t)}{R} - i(t) = 0$$

Using the inductor constraint $v_o(t) = L\frac{di}{dt}$

$$-\frac{L}{R}\frac{di(t)}{dt} - i(t) + i_s(t) = 0$$

$$\frac{L}{R}\frac{di(t)}{dt} + i(t) = i_s(t) = I_A \cos \omega t \qquad t \geq 0$$

We are only asked to find the forced component
Using the method of undetermined coefficients we write

$$i_F(t) = a \cos \omega t + b \sin \omega t$$

Substituting this into the differential equation gives

$$\frac{L}{R}\frac{d}{dt}\left[a\cos\omega t + b\sin\omega t\right] + \left[a\cos\omega t + b\sin\omega t\right] = I_A \cos \omega t$$

$$-\frac{L}{R}a\omega\sin\omega t + \frac{L}{R}b\omega\cos\omega t + a\cos\omega t + b\sin\omega t = I_A\cos\omega t$$

Collecting like terms

$$\left[-\frac{L}{R}a\omega + b\right]\sin\omega t + \left[\frac{L}{R}b\omega + a - I_A\right]\cos\omega t = 0$$

We require

$$-\frac{L}{R}\omega\, a + b = 0$$

$$a + \frac{L}{R}\omega\, b = I_A$$

$$+\frac{L}{R}\omega a + \left(\frac{L}{R}\omega\right)^2 b = \left(\frac{L}{R}\omega\right) I_A$$

$$\left[1+\left(\frac{L}{R}\omega\right)^2\right] b = \left(\frac{L}{R}\omega\right) I_A$$

$$b = \frac{\left(\frac{L}{R}\omega\right) I_A}{1+\left(\frac{L}{R}\omega\right)^2}$$

$$a = \frac{b}{\left(\frac{L}{R}\omega\right)} = \frac{I_A}{1+\left(\frac{L}{R}\omega\right)^2}$$

The forced component is then

$$i_F(t) = \frac{I_A}{1+\left(\frac{L}{R}\omega\right)^2}\cos\omega t + \frac{\left(\frac{L}{R}\omega\right) I_A}{1+\left(\frac{L}{R}\omega\right)^2}\sin\omega t$$

The output voltage $v_o(t) = L\,\frac{di_F(t)}{dt}$

$$v_o(t) = L\,\frac{d}{dt}\left[\frac{I_A}{1+\left(\frac{L}{R}\omega\right)^2}\cos\omega t + \frac{\left(\frac{L}{R}\omega\right) I_A \sin\omega t}{1+\left(\frac{L}{R}\omega\right)^2}\right]$$

$$v_o(t) = \frac{-L I_A\omega}{1+\left(\frac{L}{R}\omega\right)^2}\sin\omega t + \frac{\frac{L^2}{R}\omega^2 I_A\cos\omega t}{1+\left(\frac{L}{R}\omega\right)^2}$$

$$v_o(t) = \frac{I_A\omega L}{1+\left(\frac{L}{R}\omega\right)^2}\left[-\sin\omega t + \omega\frac{L}{R}\cos\omega t\right]$$

$$v_o(t) = \frac{I_A\omega L}{1+\left(\frac{L}{R}\omega\right)^2}\sqrt{1+\left(\omega\frac{L}{R}\right)^2}\cos\left(\omega t + \mathrm{Tan}^{-1}\left(+\frac{1}{\omega\frac{L}{R}}\right)\right)$$

$$v_o(t) = \underbrace{\frac{I_A \omega L}{\sqrt{1+\left(\frac{L}{R}\omega\right)^2}}}_{\substack{\text{amplitude}\\ \text{changes}\\ \text{with the}\\ \text{frequency of}\\ \text{the sinusoidal}\\ \text{input}}} \cos\left(\omega t + \tan^{-1}\left(\frac{1}{\omega \frac{L}{R}}\right)\right)$$

At dc ($\omega = 0$) the amplitude goes to zero since the inductor acts like a short.

As $\omega \to \infty$ the amplitude goes to $\dfrac{I_A \cancel{\omega} L}{\frac{L}{R}\cancel{\omega}} = I_A R$

which makes sense since the inductor acts like an open forcing all current through the resistor

This is a frequency dependent response. ■

Chapter 8 - Sinusoidal Steady-state response

The forced sinusoid remaining <u>after</u> the natural component disappears is called the sinusoidal steady-state response.

8-1 Sinusoids and Phasors

The fundamental relationship between sinewaves and complex numbers comes from Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Define $\cos\theta = Re\left\{e^{j\theta}\right\}$ ⟵ we use the cosine to describe the eternal sinewave

$$\sin\theta = \mathcal{I}m\left\{e^{j\theta}\right\}$$

Expanding upon the general sinusoid

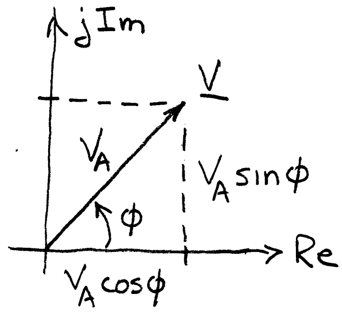
$$v(t) = V_A\cos(\omega t + \phi)$$

$$v(t) = V_A\,Re\left\{e^{j(\omega t+\phi)}\right\} = Re\left\{\underline{\underline{(V_A e^{j\phi})}}\, e^{j\omega t}\right\}$$

this is defined to be the phasor representation of the sinusoid $v(t)$

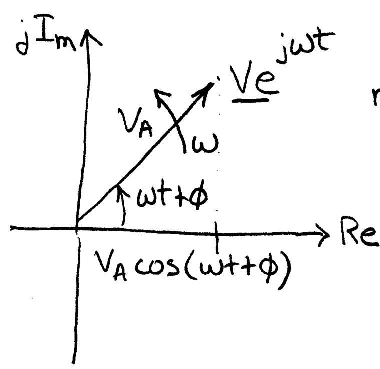
$$\underline{V} \triangleq V_A e^{j\phi} = V_A\cos\phi + jV_A\sin\phi$$

↑ the phasor $\underline{V}$ is a complex number



1. Phasors will be written with an underline ($\underline{V}$) to distinguish them from signal waveforms such as $v(t)$.

2. A phasor is determined by amplitude and phase angle and does not contain any information about the frequency.



this is a rotating phasor