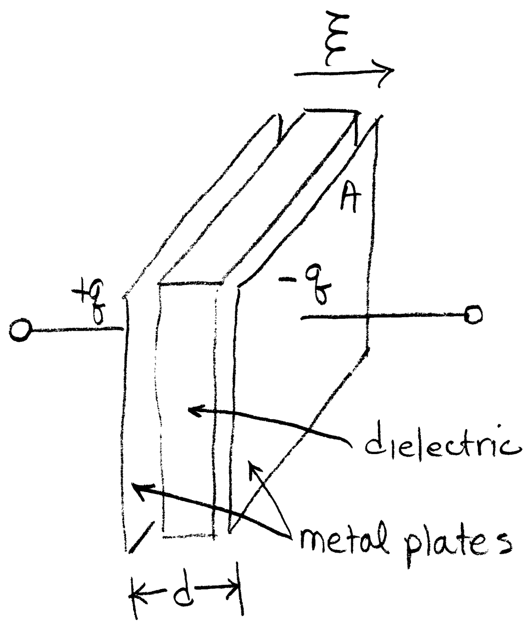
Chapter 6 Capacitance and Inductance



For small d the electric field is given by

$$\xi(t) = \frac{q(t)}{\epsilon A}$$

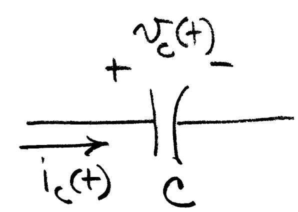
but $$\xi(t) = \frac{v_c(t)}{d}$$

Solving for q $$q(t) = \epsilon A\,\xi(t) = \underbrace{\frac{\epsilon A}{d}}\, v_c(t)$$

This is a constant dependent upon the physical construction called the capacitance. C

$$q(t) = C\, v_c(t)$$

Differentiating $$i_c(t) = \frac{dq(t)}{dt} = C\,\frac{dv_c}{dt}$$



This the the i-v characteristic of the capacitor.

Consider what this means

1. when v_c = constant (dc) the current is zero
2. discontinuous voltage like $\delta(t)$ would require and infinite current, Because of this voltage across a capacitor must be continuous.

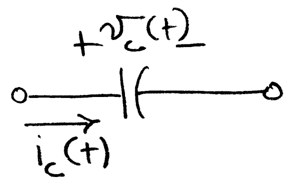
Integrate $i_c(t) = C\frac{dv_c}{dt}$

$$\frac{1}{C} i_c(t)dt = dv_c$$

$$\frac{1}{C}\int_{t_0}^{t} i_c(x)dx = \int_{v_c(t_0)}^{v_c(t)} dv_c = v_c(t) - v_c(t_0)$$

$$\therefore\ v_c(t) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(x)dx$$

In most of our problems we will
1. define $t_0 = 0$
2. recognize $v_c(t_0)$ as the initial voltage on the capacitor.



Consider the power $p_c(t) = i_c(t)\, v_c(t)$

$$= \left[C\frac{dv_c}{dt}\right] v_c(t)$$

$$p_c(t) = \frac{d}{dt}\left[\frac{1}{2} C v_c^2(t)\right]$$

This is VERY interesting.

$p_c(t) > 0$ capacitor absorbing power

$p_c(t) < 0$ capacitor releasing stored power

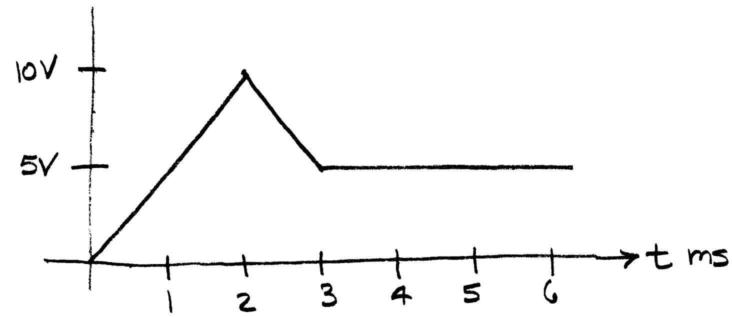
Since $p_c(t) = \frac{d}{dt}\left[w_c(t)\right]$ energy in the capacitor

$\therefore$ energy in capacitor $w_c(t) = \frac{1}{2} C v_c^2(t) + \cancel{\text{constant}}$ 0

since $\xi = 0$ when $v_c = 0$

Example 6-1

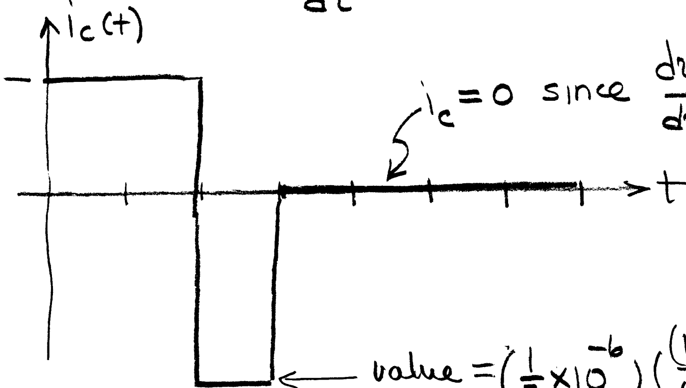
If the capacitor voltage is as shown across a $\frac{1}{2}\mu F$ capacitor. What is the corresponding current?



Solution: since $i_c = C\frac{dv_c}{dt}$

$\left(\frac{1}{2}\times10^{-6}\right)\left(\frac{10-0}{2-0\times10^{-3}}\right)$

$+2.5\times10^{-3}$



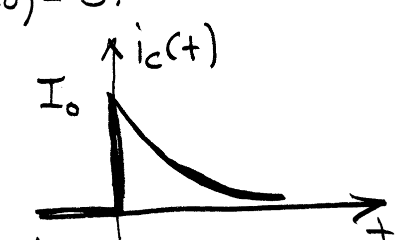
$i_c = 0$ since $\frac{dv_c}{dt} = 0$

value $=\left(\frac{1}{2}\times10^{-6}\right)\left(\frac{(10-5)}{(2-3)\times10^{-3}}\right) = \frac{10^{-6}\ (5)}{2(-1)\ 10^{-3}}$

$i_c = -2.5\times10^{-3}A = -2.5mA$

Example 6-2

If $i_c(t) = I_o e^{-\frac{t}{T_c}} u(t)$ find $v_c(t)$ if $v_c(t=0) = 0$.



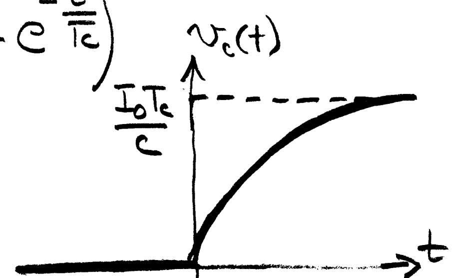
dis continuous

$$i_c = C\frac{dv_c}{dt}$$

$$\frac{1}{C}i_c dt = dv_c$$

$$v_c(t) = v_c(t=0) + \frac{1}{C}\int_0^t I_o e^{-\frac{x}{T_c}}dx$$

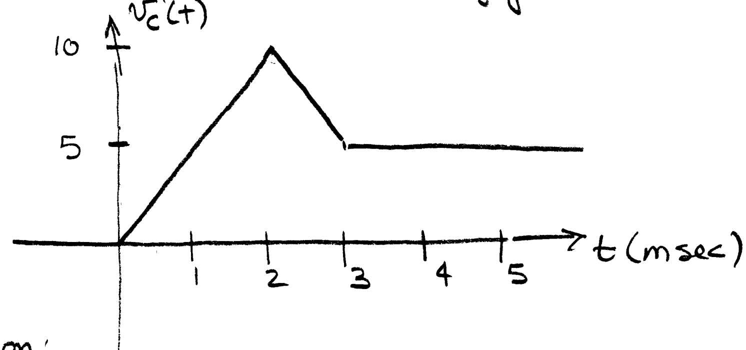
$$v_c(t) = 0 + \frac{I_o}{C}\left.\frac{e^{-\frac{x}{T_c}}}{-\frac{1}{T_c}}\right|_0^t = \frac{I_o T_c}{C}\left(1 - e^{-\frac{t}{T_c}}\right)$$



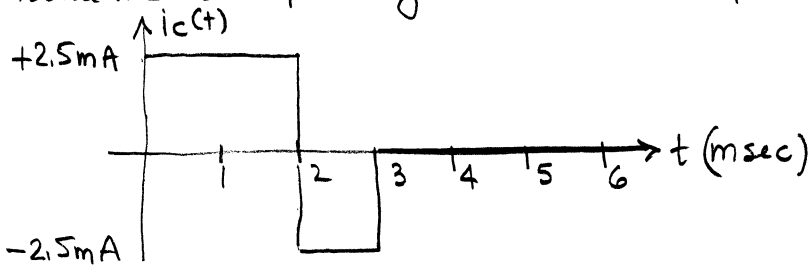
this is continuous

Example 6-3

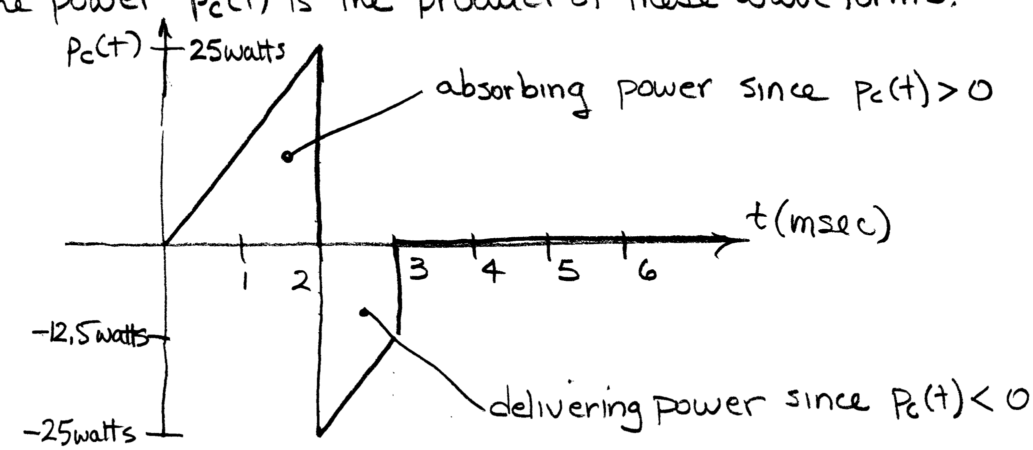
The voltage across a capacitor is given below. Find the capacitor's power and energy.



Solution:
We found the corresponding current in Example 6-1.



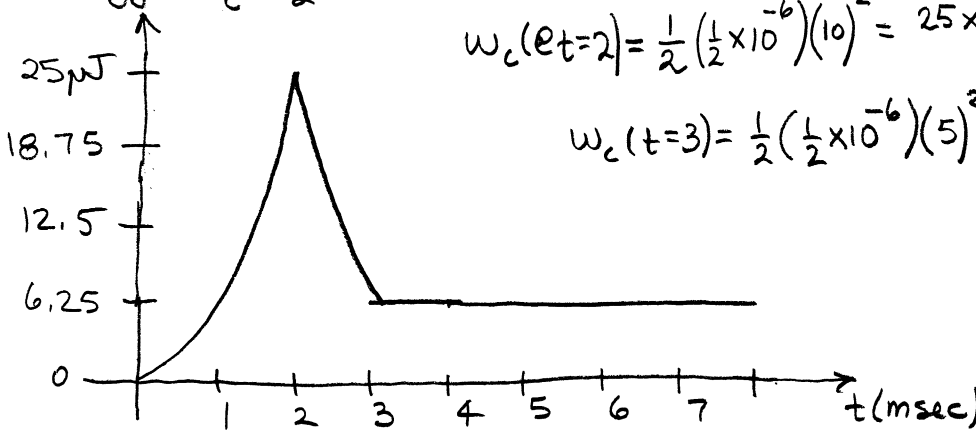
The power $p_c(t)$ is the product of these waveforms.



The energy $w_c = \frac{1}{2} C V^2$

$$w_c(@t=2) = \frac{1}{2}\left(\frac{1}{2}\times10^{-6}\right)(10)^2 = 25\times10^{-6}\,J$$

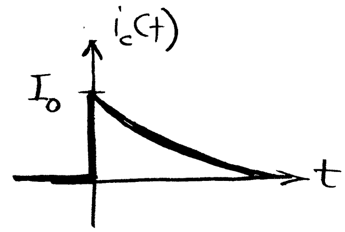
$$w_c(t=3) = \frac{1}{2}\left(\frac{1}{2}\times10^{-6}\right)(5)^2 = 6.25\times10^{-6}\,J$$



Example 6-4

The current through a capacitor is given by

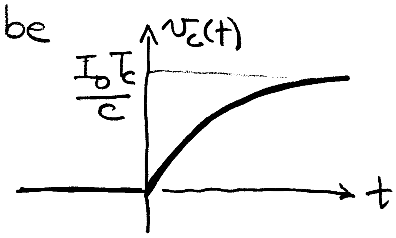
$$i_c(t) = I_o e^{-\frac{t}{T_c}} u(t)$$



Find the capacitor's energy and power.

The voltage was found in Example 6-2 to be

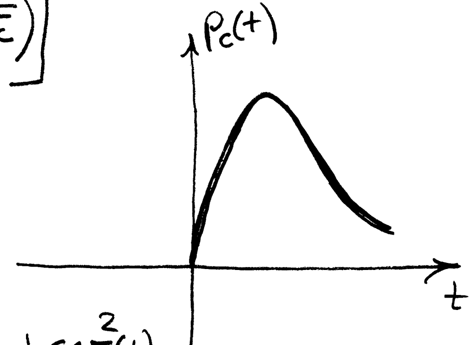
$$v_c(t) = \frac{I_o T_c}{C}\left[1 - e^{-\frac{t}{T_c}}\right]$$



The power is given by $p_c(t) = i_c(t)\, v_c(t)$

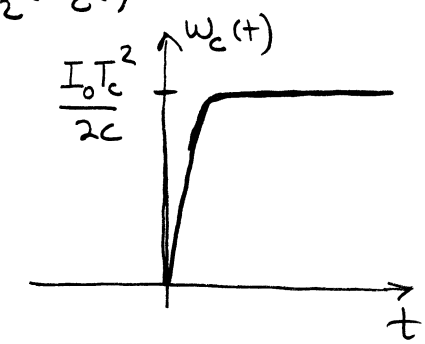
$$p_c(t) = \left[I_o e^{-\frac{t}{T_c}}\right]\left[\frac{I_o T_c}{C}\left(1 - e^{-\frac{t}{T_c}}\right)\right]$$

$$p_c(t) = \frac{I_o^2 T_c}{C}\left[e^{-\frac{t}{T_c}} - e^{-\frac{2t}{T_c}}\right]$$



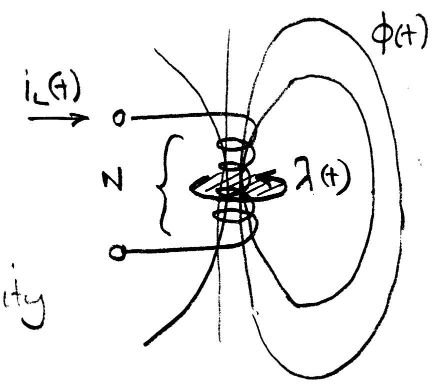
The energy is most easily calculated as $w_c(t) = \frac{1}{2} C v_c^2(t)$

$$w_c(t) = \frac{1}{2} C\left[\frac{I_o T_c}{C}\left(1 - e^{-\frac{t}{T_c}}\right)\right]^2$$



6-2 The Inductor

From magneto statics



$$\phi(t) = k_1 N i_L(t)$$

N — # of turns

k_1 — constant of proportionality

$\phi(t)$ — magnetic flux (webers)

$$\lambda(t) = N\phi(t)$$

$\phi(t)$ — magnetic flux

N — # of turns

$\lambda(t)$ — flux linkage, flux/unit area.

Substituting

$$\lambda(t) = N\phi(t) = N k_1 N i_L(t) = \underbrace{\left[N^2 k_1\right]}_{\text{this is the inductance } L} i_L(t)$$

$$\lambda(t) = L\, i_L(t)$$

Differentiate this to get the i-v relationship

$$\frac{d\lambda(t)}{dt} = L\frac{di_L(t)}{dt}$$

by Faraday's Law $v_L(t) = \dfrac{d\lambda(t)}{dt}$

$$\therefore \quad v_L(t) = L\frac{di_L(t)}{dt}$$

Observations

1. If $i_L(t)$ is constant, $v_L = 0$ and the inductor looks like a short
2. A discontinuity $u(t)$ or $\delta(t)$ in i_L would create an infinite voltage so $i_L(t)$ must be continuous.

We often want i_L in terms of voltage

$$v_L(t) = L\,\frac{di_L(t)}{dt}$$

$$di_L(t) = \frac{1}{L}\,v_L(t)\,dt$$

Assuming $i_L(t_0)$ is known we can integrate this to get

$$\int_{i_L(t_0)}^{i_L(t)} di_L = \frac{1}{L}\int_{t_0}^{t} v_L(x)\,dx$$

$$i_L(t) - i_L(t_0) = \frac{1}{L}\int_{0}^{t} v_L(x)\,dx$$

$$i_L(t) = i_L(t_0) + \frac{1}{L}\int_{0}^{t} v_L(x)\,dx$$

<u>Power & energy</u>

$$p_L(t) = i_L(t)\,v_L(t)$$

$$p_L(t) = i_L(t)\,L\,\frac{di_L(t)}{dt}$$

$$p_L(t) = \frac{d}{dt}\Big[\underbrace{\tfrac{1}{2}L\,i_L^2(t)}\Big]$$

recognize this as the energy w_L stored in the inductor

$$w_L(t) = \frac{1}{2}L\,i_L^2(t) + \cancel{\text{constant}}^{\,0}$$

since it is the energy stored when $i_L = 0$

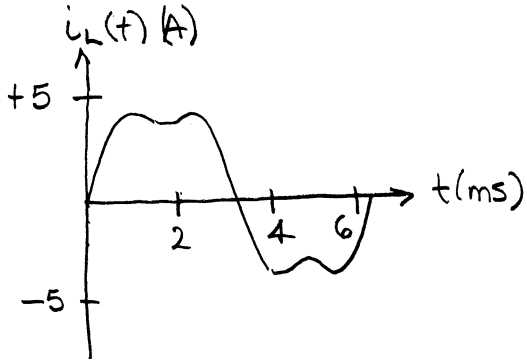
Example 6-6

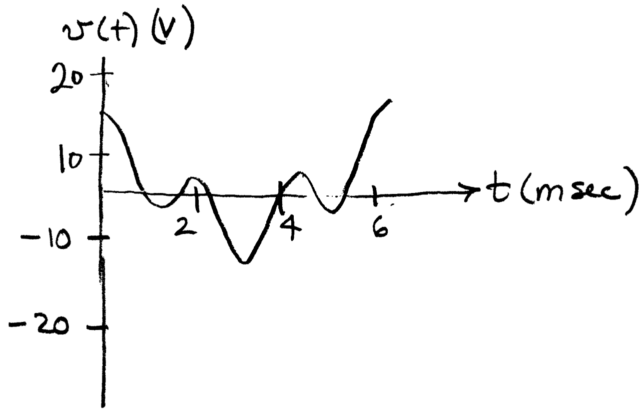
The current through a 2-mH inductor is $i_L(t) = 4\sin 1000t + \sin 3000t$
Find the corresponding $v_L(t)$.

$$v_L(t) = L\frac{di_L}{dt} = L\frac{d}{dt}\left(4\sin 1000t + \sin 3000t\right)$$

$$= (.002)\left[4(1000)\cos 1000t + 3000\cos 3000t\right]$$

$$v_L(t) = 8\cos 1000t + 6\cos 3000t.$$

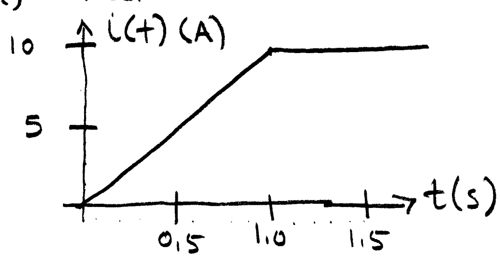




Example 6-7

The following figures show the current and voltage across an unknown energy storage element.

(a) What is the element and its numerical value?

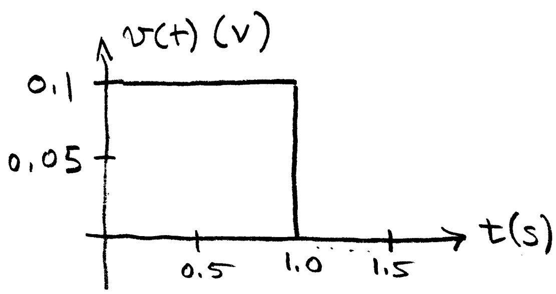


This is an inductor since $v = 0$ when i = constant.

Since $v = L\frac{di}{dt}$ for $0 < t < 1$

$$0.1 = L\left(\frac{10-0}{1-0}\right) = 10L$$

$$L = .01\text{ H} = 10\text{mH}$$



(b) What is the energy stored at t=1 sec?

$$w_L = \frac{1}{2}Li^2 = \frac{1}{2}(.01)(10)^2 = \frac{1}{2}\text{ J}.$$