7-3 Initial and Final Conditions

For $t \geq 0$ the state variable step responses can be written as

RC circuit $$v_C(t) = [v_C(0) - v_C(\infty)]\, e^{-\frac{t}{T_C}} + v_C(\infty) \qquad t \geq 0$$

RL circuit $$i_L(t) = [i_L(0) - i_L(\infty)]\, e^{-\frac{t}{T_C}} + i_L(\infty) \qquad t \geq 0$$

The general form is

$$\begin{matrix}\text{state}\\ \text{variable}\\ \text{response}\end{matrix} = \left[\begin{matrix}\text{initial value}\\ \text{of}\\ \text{state variable}\end{matrix} - \begin{matrix}\text{final value}\\ \text{of}\\ \text{state variable}\end{matrix}\right] e^{-\frac{t}{T_C}} + \begin{matrix}\text{final value}\\ \text{of}\\ \text{state variable}\end{matrix}$$

This argues that all we need are

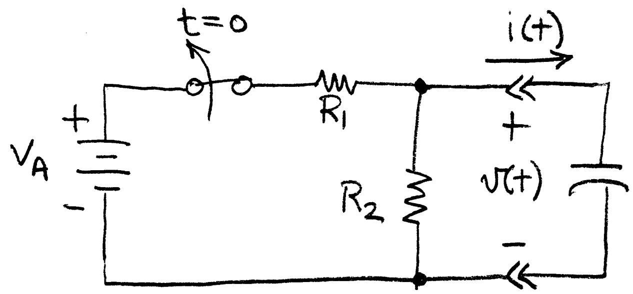
- the initial value
- the final value
- the time constant

Use dc analysis to find final values for

- capacitor $\longrightarrow$ open
- inductor $\longrightarrow$ short

Final value must be greater than $5\,T_C$ from initial conditions

Use dc analysis to determine initial values based upon previous final values



STEP 1: Find the initial value by applying dc analysis to the circuit configuration for $t<0$. when the switch is closed,

For $t<0$ the capacitor becomes an open circuit.

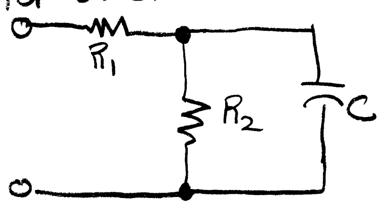
$$v(0) = \frac{R_2}{R_1+R_2} V_A$$

STEP 2: The final condition is found by dc analysis of the circuit configuration when the switch is open.

Since there is no source connected to the capacitor when the switch is open

$$v(\infty) = 0$$

STEP3 The time constant can now be found from the circuit for $t>0$.



Since R_1 is not connected

$$T_c = R_2 C$$

The capacitor voltage for $t \geq 0$ is then

$$v(t) = [v(0) - v(\infty)] e^{-\frac{t}{T_c}} + v(\infty)$$

Substituting values

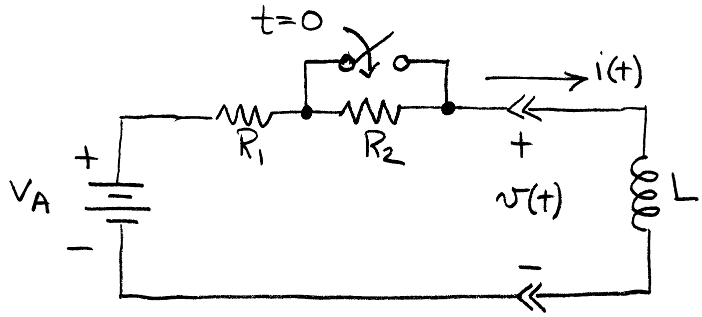
$$v(t) = \frac{R_2}{R_1+R_2} V_A e^{-\frac{t}{R_2 C}} \quad t \geq 0$$

We can find i using the element constraint

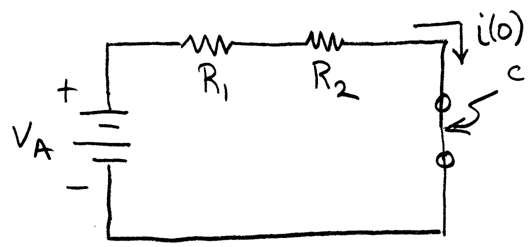
$$i(t) = C\frac{dv}{dt} = \cancel{C}\frac{\cancel{R_2}}{R_1+R_2} V_A \left(-\frac{1}{\cancel{R_2 C}}\right) e^{-\frac{t}{R_2 C}} = \frac{-V_A}{R_1+R_2} e^{-\frac{t}{R_2 C}} \quad t \geq 0$$

Example 7-8

The switch in the figure below has been open for a long time and is closed a $t=0$. Find $i(t)$ for $t \geq 0$,



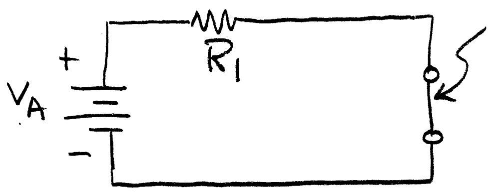
STEP 1: Find the initial value applying dc analysis for $t<0$,



circuit is in steady state so replace L by a short

$$i(0) = \frac{V_A}{R_1+R_2}$$

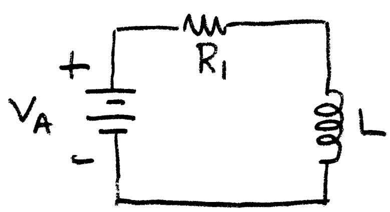
STEP 2: The final condition is found by dc analysis of the circuit configuration when the switch is closed and the circuit is in steady state.



circuit is in steady state so replace L by a short

$$i(\infty) = \frac{V_A}{R_1}$$

STEP 3: The time constant for $t>0$ is given by



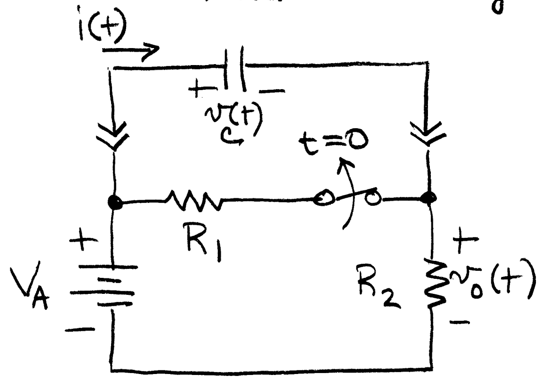
$$T_c = L/R_1$$

The inductor current is then given by

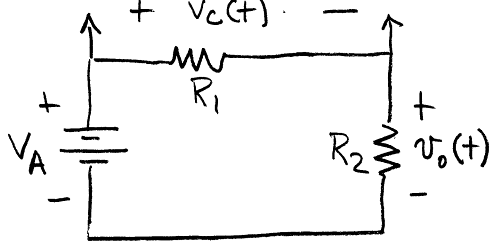
$$i_L(t) = \left[i_L(0) - i_L(\infty)\right] e^{-\frac{t}{T_c}} + i(\infty)$$

$$i_L(t) = \left[\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1}\right] e^{-\frac{t}{L/R_1}} + \frac{V_A}{R_1} \qquad t \geq 0$$

Example 7-9 The switch in the circuit shown below has been closed for a long time and is opened at $t=0$. Find the voltage $v_o(t)$.

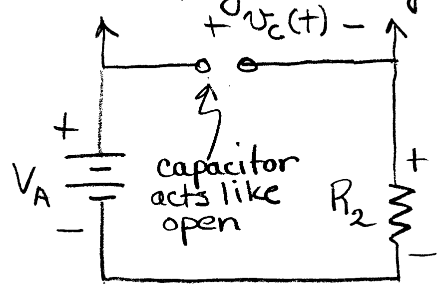


STEP 1: Apply dc analysis for $t<0$.



$$v_c(0) = \frac{R_1}{R_1+R_2} V_A$$

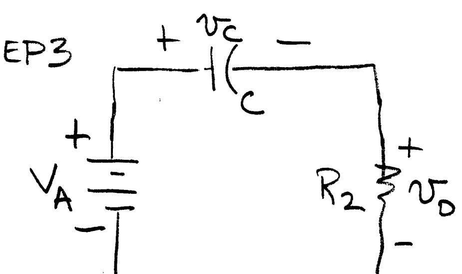
STEP 2: Apply dc analysis to final circuit.



This is a tough circuit.
When switch opens, capacitor looks like an open.

$$v_c(\infty) = V_A$$

STEP3



$$T_c = R_2 C$$

The capacitor voltage is given by

$$v_c(t) = \left[v_c(0) - v_c(\infty)\right] e^{-t/T_c} + v_c(\infty) = \left[\frac{R_1}{R_1+R_2} V_A - V_A\right] e^{-\frac{t}{R_2 C}} + V_A$$

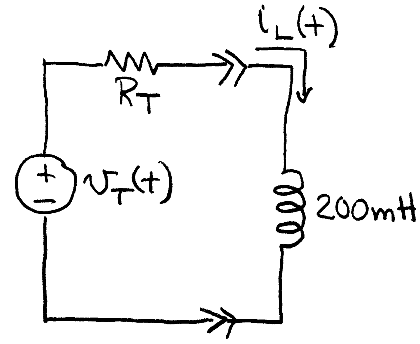
$$v_c(t) = \frac{R_2}{R_1+R_2} V_A e^{-\frac{t}{R_2 C}} + V_A$$

Using KVL $\quad -V_A + v_c + v_o = 0 \Rightarrow v_o(t) = V_A - v_c(t) = -\frac{R_2 V_A}{R_1+R_2} e^{-\frac{t}{R_2 C}}$

Example 7-10

For $t \geq 0$ the state-variable response of the RL circuit is observed to be

$$i_L(t) = 50 + 100e^{-5000t} \text{ mA}$$



(a) Identify the forced and natural components of the response.

The forced component is what is left after $t \to \infty$

$$i_F(t) = 50 \text{mA}$$

The natural component is $i_N(t) = 100e^{-5000t}$

(b) Find the circuit time constant.

$$\frac{1}{T_c} = 5000 \qquad T_c = 0.2 \text{ mS}$$

(c) Find the Thevenin equivalent circuit seen by the inductor.

$$T_c = \frac{L}{R_T} = 0.2 \times 10^{-3} \qquad \therefore R_T = \frac{200 \times 10^{-3}}{0.2 \times 10^{-3}} = 1000\,\Omega$$

As $t \to \infty$ $L \to$ short so $i_L(\infty) = \frac{v_T(\infty)}{R_T}$

$$i_L(\infty) = 50 \text{ mA}$$

$$\therefore v_T(\infty) = i_L(\infty) R_T = (50 \times 10^{-3})(1000)$$

$$v_T(\infty) = 50 \text{ volts.}$$