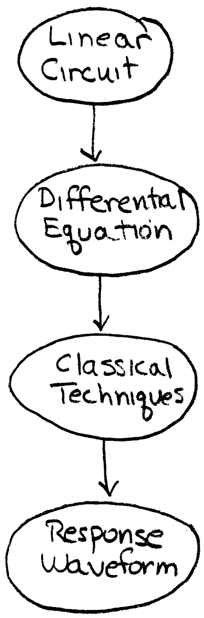
Chapter 7 - First- and Second-Order Circuits

7-1 RC and RL Circuits

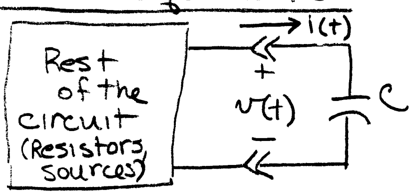


use device and connection equations to write differential equation describing the circuit

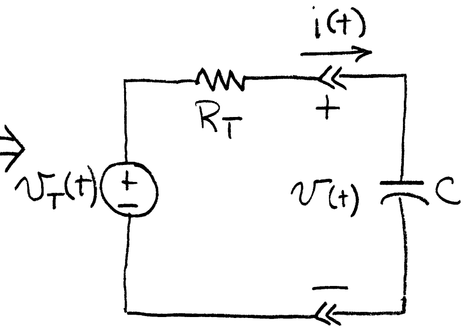
solve differential equation

use classical techniques for solving D.E.
(other techniques include phasors & Laplace transform)

RC and RL Equations



$\Rightarrow$



Doing KVL gives

$$-v_T(t) + i(t)R_T + v(t) = 0$$

From the previous chapter the capacitor is described by

$$i(t) = C\frac{dv(t)}{dt}$$

Substituting

$$-v_T(t) + C\frac{dv(t)}{dt}R_T + v(t) = 0$$

Re-arranging

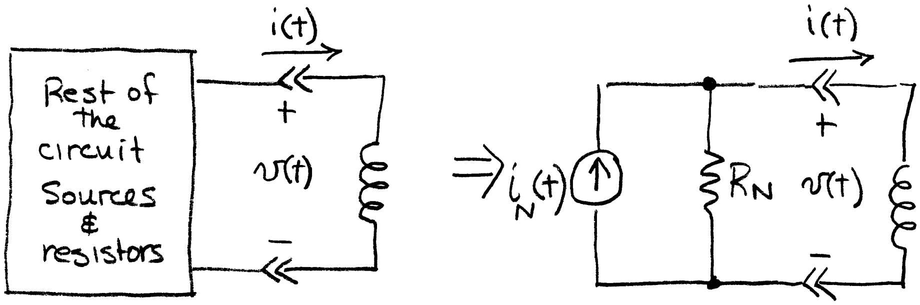
first order linear differential equation with constant coefficients

$$R_TC\frac{dv(t)}{dt} + v(t) = v_T(t)$$

$v_T(t)$ is the input and $v(t)$ is the response

$v(t)$ is called the <u>state variable</u> and determines the amount of energy stored in the RC circuit.

We can do the same with an inductor.



Using KCL at the node gives $\sum_{+in} i = 0$

$$+i_N(t) - \frac{v(t)}{R_N} - i(t) = 0$$

The element constraint is $v(t) = L\frac{di(t)}{dt}$

Substituting

$$i_N(t) - \frac{L}{R_N}\frac{di(t)}{dt} - i(t) = 0$$

Re-arranging

$$\frac{L}{R_N}\frac{di(t)}{dt} + i(t) = i_N(t)$$

- first-order linear differential equation with constant coefficients
- $i_N(t)$ is the forcing function
- $i(t)$ is the state variable as it defines the amount of energy stored in the RL circuit

Any circuit containing a single capacitor or inductor and resistors is a first-order circuit described by a first-order differential equation.

Zero-input response of First-order circuits

The response $v(t)$ for an RC circuit depends upon

1. The input $v_T(t)$
2. The circuit values R_T and C
3. The initial condition, i.e., $v(t=0)$

   This can use a response even when $v_T(t) = 0$.

Consider the zero-input response when $v_T(t) = 0$. for $t \geq 0$

$$R_T C \frac{dv(t)}{dt} + v(t) = 0$$

This is a homogeneous differential equation with a solution of the form

$$v(t) = Ke^{st}$$

Substituting gives

$$R_T C\left(Kse^{st}\right) + Ke^{st} = 0$$

$$Ke^{st}\left(R_T C s + 1\right) = 0$$

This can only be zero if

$$R_T C s + 1 = 0$$

This is called the characteristic equation of the differential equation.

Solving gives $\quad s = -\frac{1}{R_T C}$

The solution $v(t)$ is then

$$v(t) = Ke^{-\frac{t}{R_T C}} \qquad t \geq 0$$

The constant K comes from the initial condition $v(t=0) = V_0$

$$v(t=0) = Ke^{0} = K = V_0$$

The final zero-input response is then

$$v(t) = V_0 e^{-\frac{t}{R_T C}} \qquad t \geq 0$$

We can do the same for the RL circuit.

$$\frac{L}{R_N}\frac{di(t)}{dt} + i(t) = 0 \qquad \text{where we set } i_N(t)=0,\ t\geq 0$$

This is also a homogeneous linear differential equation with a solution of the form

$$i(t) = Ke^{st}$$

Substituting gives

$$\frac{L}{R_N}Kse^{st} + Ke^{st} = 0$$

$$Ke^{st}\left(\frac{L}{R_N}s+1\right) = 0$$

This requires the characteristic equation

$$\frac{L}{R_N}s+1 = 0$$

Solving gives $\quad s = -\dfrac{1}{\frac{L}{R_N}} = -\dfrac{R_N}{L}$

The solution is

$$i(t) = Ke^{-\frac{R_N}{L}t} \qquad t \geq 0$$

The constant K comes from the initial condition $i(t=0) = I_0$

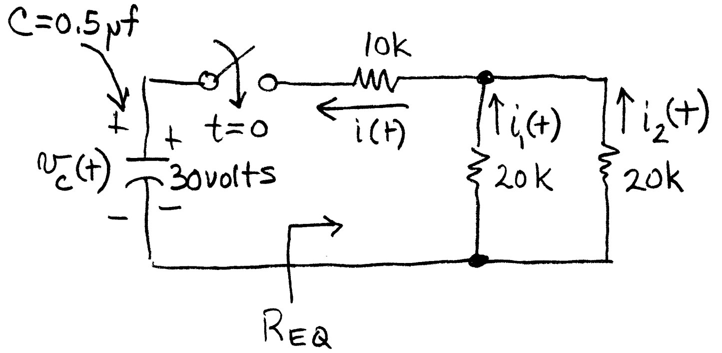
$$i(t=0) = Ke^{0} = K = I_0$$

The final zero-input response is then

$$i(t) = I_0 e^{-\frac{R_N}{L}t} \qquad t \geq 0$$

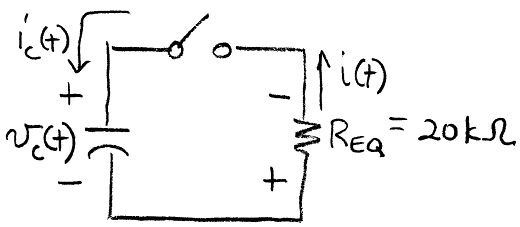
Example 7-1

The switch in the circuit shown below is closed at $t=0$, connecting a capacitor with an initial voltage of 30V to the resistances shown. Find the responses $v_C(t)$, $i(t)$, $i_1(t)$, and $i_2(t)$ for $t \geq 0$.



We first determine the equivalent resistance R_{EQ} as seen by the capacitor.

$$R_{EQ} = 10k + 20k \parallel 20k = 10k + 10k = 20k\Omega$$



Doing KVL gives

$$-v_C(t) - i(t)R_{EQ} = 0$$

The element constraint for C is $i_C = C\dfrac{dv_C}{dt}$

Since we chose $i_C(t)$ to be in the same direction as $i(t)$, $i_C(t) = i(t)$

Substituting,

$$-v_C(t) - C\frac{dv_C(t)}{dt}R_{EQ} = 0$$

$$R_{EQ}C\frac{dv_C(t)}{dt} + v_C(t) = 0$$

The solution is $v_C(t) = Ke^{st}$. Substituting gives

$$R_{EQ}C\,Ks\,e^{st} + Ke^{st} = 0$$

$$R_{EQ}Cs + 1 = 0$$

$$s = -\frac{1}{R_{EQ}C} = -\frac{1}{(20\times10^3)(0.5\times10^{-6})} = -100.$$

Using the initial condition $v_C(t=0) = 30$ volts

$$v_C(t=0) = Ke^0 = 30 \text{ volts} \Rightarrow K = 30$$

The solution is $v_C(t) = 30e^{-100t}, \quad t \geq 0$

We can now calculate the other circuit values using $v_c(t)$

$$i(t) = i_c(t) = C\frac{dv_c}{dt} = (0.5\times10^{-6})\frac{d}{dt}\left(30e^{-100t}\right)$$

$$= (0.5\times10^{-6})(30)(-100)\,e^{-100t} = -1.5\times10^{-3}e^{-100t}, \quad t\geq 0$$

i_1 and i_2 are given by a current divider.

$$i_1 = \frac{20k}{20k+20k}\,i(t) = \frac{1}{2}i(t) = -0.75\times10^{-3}e^{-100t}, \quad t\geq 0$$

$$i_2 = \frac{20k}{20k+20k}\,i(t) = \frac{1}{2}i(t) = -0.75\times10^{-3}e^{-100t}, \quad t\geq 0$$