Properties of sinusoids

periodicity $\quad v(t+T_0) = v(t) \qquad T_0 \equiv \text{period}$

Signals that are not periodic are called aperiodic.

Consider $\quad v(t) = V_A \cos\left(2\pi \frac{t}{T_0}\right)$

$$v(t+T_0) = V_A \cos\left(2\pi\left(\frac{t+T_0}{T_0}\right)\right) = V_A \cos\left(2\pi \frac{t}{T_0} + 2\pi\right)$$

$\therefore \; v(t+T_0) = v(t)$ for the cosine.

additive property

Sums of sinusoids cannot be done by adding amplitudes and phase representations,
Must use Fourier representation to add sinusoids.

$$v_1(t) = a_1 \cos(2\pi f_0 t) + b_1 \sin(2\pi f_0 t)$$

$$v_2(t) = a_2 \cos(2\pi f_0 t) + b_2 \sin(2\pi f_0 t)$$

$$v_{TOTAL}(t) = v_1(t) + v_2(t)$$

$$= (a_1+a_2)\cos(2\pi f_0 t) + (b_1+b_2)\sin(2\pi f_0 t)$$

derivative

$$\frac{d}{dt}(V_A \cos\omega t) = -\omega V_A \sin\omega t = \omega V_A \cos\left(\omega t + \frac{\pi}{2}\right)$$

use trig identity

integral

$$\int V_A \cos\omega t \, dt = V_A \frac{\sin\omega t}{\omega} = \frac{V_A}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right)$$

trig identity

Example 5-8

(a) Find the period, cyclic and radian frequency of

$$v_1(t) = 17\cos(2000t - 30^\circ) \qquad v_2(t) = 12\cos(2000t + 30^\circ)$$

(b) Find the sum $v_3(t) = v_1(t) + v_2(t)$

SOLUTION

(a) Always first check the frequency

$v_1(t)$ and $v_2(t)$ are at the same frequency.

By inspection $\omega_0 = 2000$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{2000}{2\pi} = 318.3\text{ Hz}$$

$$T_0 = \frac{1}{f_0} = \frac{1}{318.3} = 3.14 \times 10^{-3} \text{ seconds.}$$

(b) To compute the sum we have to find the fourier representation of v_1 and v_2, and add those representations.

$$v_1(t) = 17\cos(2000t - 30^\circ)$$

watch sign $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$$v_1(t) = 17\left[\cos 2000t \cos(-30^\circ) - \sin 2000t \sin(-30^\circ)\right]$$

$$v_1(t) = 14.7\cos 2000t + 8.5\sin 2000t$$

$$v_2(t) = 12\cos(2000t + 30^\circ)$$

$$v_2(t) = 12\left[\cos 2000t \cos 30^\circ - \sin 2000t \sin 30^\circ\right]$$

$$v_2(t) = \left[12\cos 30^\circ\right]\cos 2000t + \left[-12\sin 30^\circ\right]\sin 2000t$$

$$v_2(t) = 10.4\cos 2000t - 6\sin 2000t$$

$$\therefore\ v_3(t) = (14.7 + 10.4)\cos 2000t + (8.5 - 6)\sin 2000t$$

$$v_3(t) = 25.1\cos 2000t + 2.5\sin 2000t$$

This can be converted into a single expression.

$$V_A = \sqrt{(25.1)^2 + (2.5)^2} = 25.2$$

$$\phi = \tan^{-1}\left(\frac{-2.5}{25.1}\right) = -5.69^\circ$$

$$v_3(t) = 25.2\cos(2000t - 5.69^\circ)$$

5.5 Composite Waveforms produced by combining waveforms

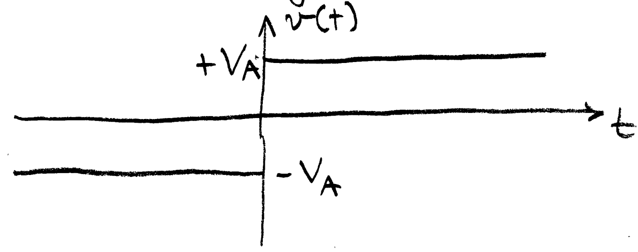
Example 5-10

$$v(t) = V_A u(t) - V_A u(-t)$$

What is new here is the time reflected $u(-t)$

$$u(-t) = \begin{cases} 1 & t<0 \\ 0 & t>0 \end{cases}$$

The waveform has a jump discontinuity of $2V_A$ at $t=0$ and is called the signum function

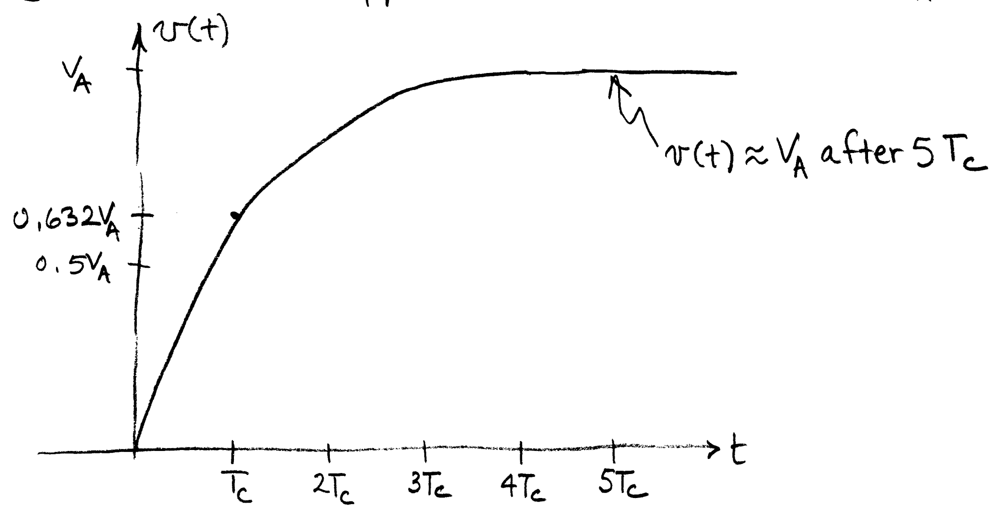


Example 5-11

$$v(t) = V_A u(t) - V_A e^{-\frac{t}{T_C}} u(t)$$

For $t \leq 0$ $v(t)=0$

For $t \gg T_C$ the waveform approaches the constant value V_A



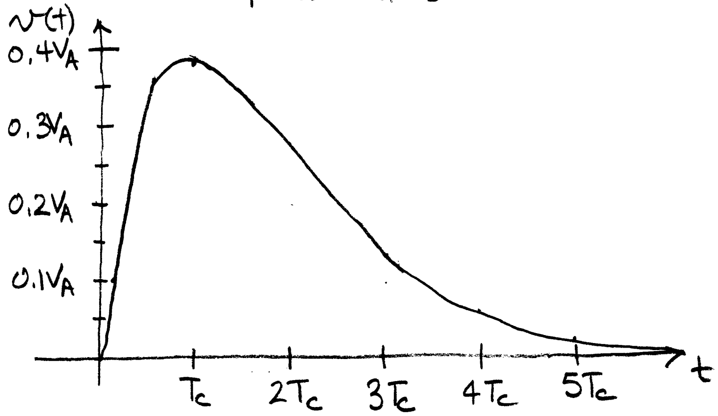
This waveform is called a "charging exponential" or a "exponential rise."

Example 5-12

$$v(t) = \frac{r(t)}{T_c} V_A e^{-\frac{t}{T_c}} u(t)$$

for small t the ramp dominates

for large t the exponential dominates

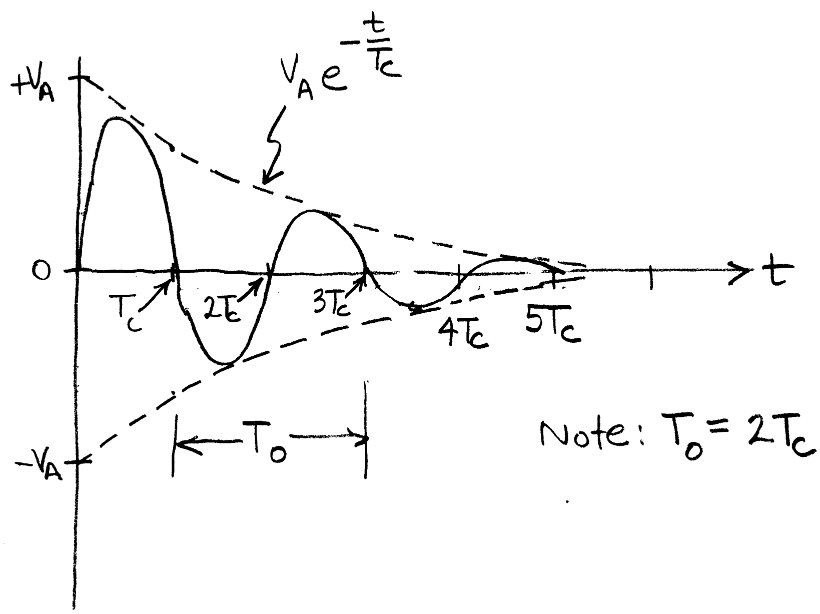


This is called a "damped ramp"

Example 5-13

$$v(t) = \sin\omega_0 t \, V_A e^{-\frac{t}{T_c}} u(t) = V_A \left[e^{-\frac{t}{T_c}} \sin\omega_0 t \right] u(t)$$

This waveform is NOT periodic because the decaying exponential changes the amplitude.

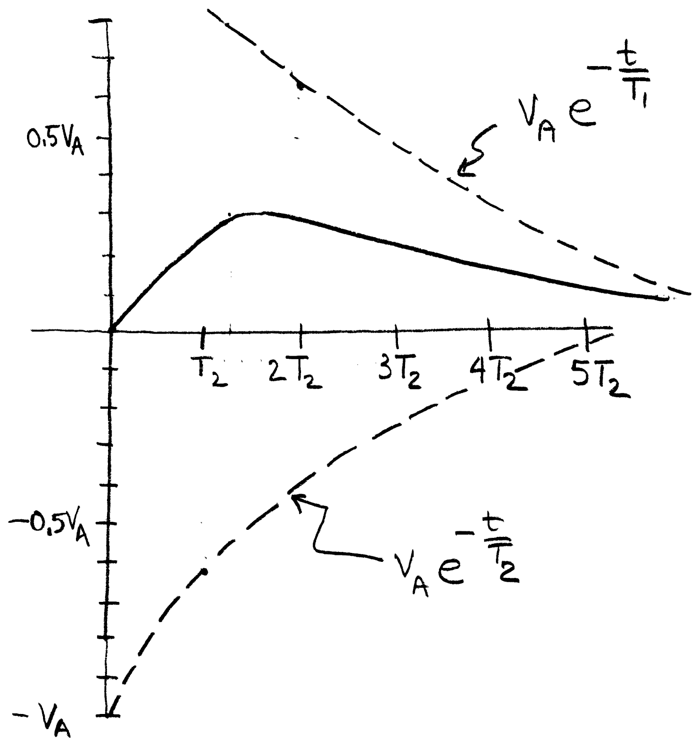


This waveform is called a "damped sine"

Example 5-14

$$v(t) = V_A e^{-\frac{t}{T_1}} u(t) - V_A e^{-\frac{t}{T_2}} u(t)$$

Assume $T_1 = 2T_2$



This is called a "double" exponential.

Fourier description of wave forms

Example - 5-15

$$v(t) = 5 - \frac{10}{\pi}\sin(2\pi 500t) - \frac{10}{2\pi}\sin(2\pi 1000t) - \frac{10}{3\pi}\sin(2\pi 1500t)$$

(this is a ramp approximation)

Second example

$$v(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin n\omega_0 t}{n}$$

(this is a square wave approximation)

The constant is called the dc term.

The term at ω_0 is called the fundamental.
This is $\omega_0 = 2\pi 500$ in Example 5-15

The term at $2\omega_0$ is called the 2nd harmonic.
This is $2\omega_0 = 2\pi 1000$ in Example 5-15

The term at $3\omega_0$ is called the 3rd harmonic,
This is $3\omega_0 = 2\pi 1500$ in Example 5-15

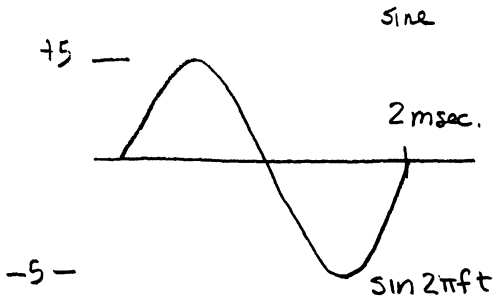
5-6 Waveform Partial Descriptors

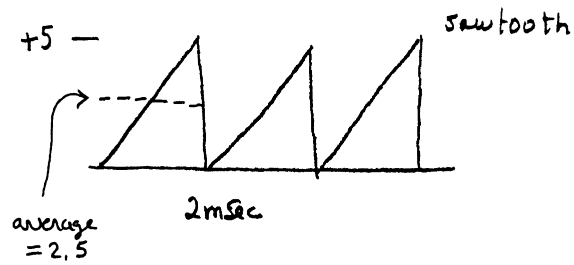
peak V_p

peak-to-peak V_{p-p}

average $V_{avg} = \frac{1}{T}\int_t^{t+T} v(x)\,dx$ mechanical meters measure this

rms $V_{rms} = \sqrt{\frac{1}{T}\int_t^{t+T} v^2(x)\,dx}$ use for ~~power~~ average power calculations





$f = \frac{1}{2\times10^{-3}} = 500\,Hz$

$V_{peak} = 5$

$V_{peak-peak} = 10$

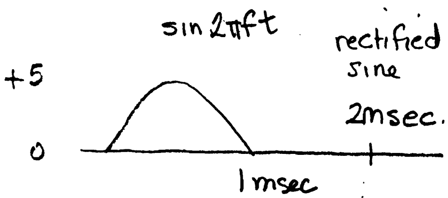
$V_{average} = 0$

$V_{peak} = 5$

$V_{peak-peak} = 5$

$V_{average} = \frac{1}{T_0}\int_0^{T_0} \frac{5t}{T_0}\,dt = \frac{1}{T_0^2}\left.\frac{5t^2}{2}\right|_0^{T_0}$

$= \frac{1}{T_0^2}\frac{5T_0^2}{2} = \frac{5}{2}.$



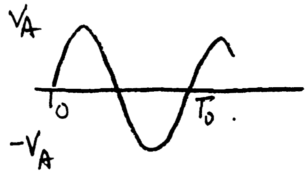
$V_{peak} = 5$

$V_{peak-peak} = 5$

One msec $V_{average} = \frac{1}{10^{-3}}\int_0^{10^{-3}} 5\sin(2\pi 500t)\,dt$

$V_{average} = \frac{5}{10^{-3}} \left.\frac{-\cos(2\pi 500t)}{2\pi 500}\right|_0^{10^{-3}} = \frac{5}{10^{-3}} \frac{-\cancel{\cos}(\pi)^{-1} + \cancel{\cos}(0)^{1}}{1000\pi} = \frac{5\,(1+1)}{\pi} = 3.18$

2 msec $V_{average} = \frac{5}{2\times10^{-3}} \left.\frac{-\cos(2\pi 500t)}{2\pi 500}\right|_0^{10^{-3}} = \frac{5}{2\times10^{-3}}\left(\frac{1+1}{1000\pi}\right) = \frac{10}{2\pi} = 1.59 \text{ volts}.$



$$p(t) = \frac{v^2(t)}{R}$$ instantaneous

$$P_{AVG} = \frac{1}{T_0}\int_t^{t+T_0} p(t)\,dt$$ We are usually more interested in average power.

$$P_{AVG} = \frac{1}{T_0}\int_t^{t+T_0} \frac{v^2(t)}{R}\,dt = \frac{1}{R}\underbrace{\left[\frac{1}{T_0}\int_t^{t+T_0} v^2(t)\,dt\right]}_{\text{average value of } v^2(t)}$$

$$P_{AVG} = \frac{V_{rms}^2}{R}$$ we call this $V_{rms} = \sqrt{\frac{1}{T}\int_t^{t+T_0} v^2(t)\,dt}$

average power to resistor

utilize RMS quantity for power measurements

<u>Simple example</u>; measure $V_{Rms} = 120\text{ V}$

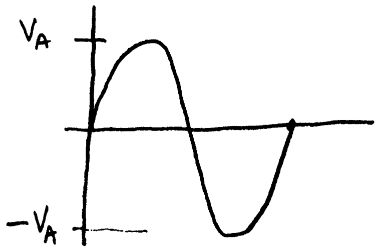
electric heater with $R = 10\Omega$.

$$P_{heater} = \frac{V_{Rms}^2}{R} = \frac{14400}{10} = 1440 \text{ watts}$$

typical home/dorm space heater

how would you measure R? hot 10 vs cold 1

RMS value of a sinusoid – very important and useful.



$V_A \sin\left(\frac{2\pi}{T_0} t\right)$

What is RMS? in this case

$$V_{rms} = \sqrt{\frac{1}{T_0}\int_0^{T_0} V_A^2 \sin^2\left(\frac{2\pi t}{T_0}\right) dt}$$

$$= \sqrt{\frac{V_A^2}{T_0}\int_0^{T_0} \sin^2\left(\frac{2\pi t}{T_0}\right) dt}$$

use $\sin^2 x = \frac{1}{2}(1 - \sin 2x)$

$$\int_0^{T_0} \frac{1}{2}\left(1 - \sin \frac{4\pi t}{T_0}\right) dt$$

$$\frac{1}{2}\left(t + \frac{\cos \frac{4\pi t}{T_0}}{\frac{4\pi}{T_0}}\right)\Bigg|_0^{T_0}$$

$$\frac{1}{2}(T_0 - 0) + \frac{T_0}{8\pi}\underbrace{\left[\cos(\overset{1}{\cancel{4\pi}}) - \overset{1}{\cancel{\cos}}(0)\right]}_{0}$$

$$\therefore V_{rms} = \sqrt{\frac{V_A^2}{\cancel{T_0}}\,\frac{\cancel{T_0}}{2}} = \frac{V_A}{\sqrt{2}} \quad \text{for a sine or cosine}$$

Simple example: If $V_{RMS} = 120\,V$ what is V_{peak} for household voltage

$V_{peak} = 120\sqrt{2} = 169.7$ volts.

$V_{peak\text{-}peak} = 339.4$ volts

MEASURING RMS VOLTAGES

Mechanical meter measures V_{AVG}

- calibrated for sinusoidal V_{RMS}

Rectifier DMM (cheap)

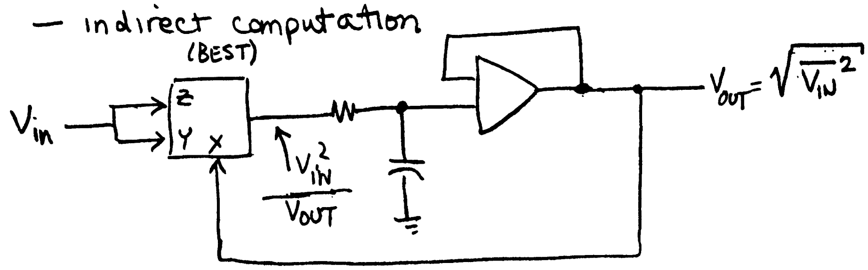
- measures average of rectified sine wave and multiplies by 1.11 to get V_{RMS} (accurate only for sine waves)

Averaging DMM

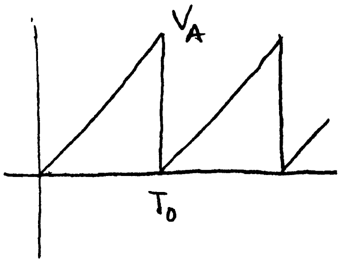
- measure V_{AVG} of actual waveform and multiply by conversion for sine wave (accurate only for sine waves)

True RMS DMM

- uses RMS converter
  - actual thermal measurement compare heat with a calibrated heater
  - direct computation square, average, square root (analog or digital)
  - indirect computation (BEST)



Example for sawtooth.



$$v(t) = \frac{V_A t}{T_0} \qquad 0 \le t \le T_0$$

$$V_{Rms} = \sqrt{\frac{1}{T_0}\int_0^{T_0} \left(\frac{V_A t}{T_0}\right)^2 dt}$$

$$= \sqrt{\frac{1}{T_0}\frac{V_A^2}{T_0^2}\int_0^{T_0} t^2\,dt}$$

$$= \sqrt{\frac{V_A^2}{T_0^3}\left.\frac{t^3}{3}\right|_0^{T_0}}$$

$$= \sqrt{\frac{V_A^2}{T_0^3}\frac{T_0^3}{3}}$$

$$V_{Rms} = \frac{V_A}{\sqrt{3}}.$$