Chapter 3 – Circuit Analysis Techniques.

As circuits increase in complexity we need a systematic method to analyze circuits. There are two basic methods:

1. node voltage analysis
2. mesh current analysis

3-1 Node Voltage Analysis

rather than use element voltages and currents as our variables we can reduce the number of equations needed to analyze a circuit by using node voltages

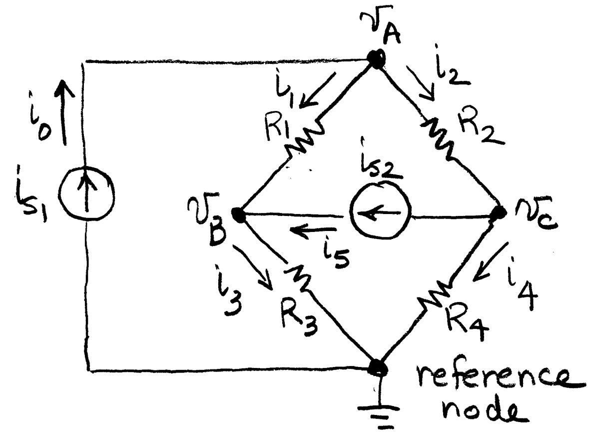
$$\underbrace{v_K}_{\text{element voltage}} = \underbrace{v_x}_{\text{voltage at node x}} - \underbrace{v_y}_{\text{voltage at node y}}$$

To develop the node voltage equations

1. Select a reference node – this is node ϕ.
   Identify a node voltage at each of the remaining nodes.
   Identify a current with every element in the circuit.
2. Write KCL connection equations for all non-reference nodes,
3. Write the element currents in terms of node voltages,
4. Substitute the equations from step 3 into the KCL equations from step 2 and arrange the resulting N–1 equations into standard form,

Example 3-1

Formulate node voltage equations for the bridge circuit shown below.



① The reference node and all remaining voltages and currents are identified.

② Write the KCL connection constraints for all the non-reference nodes.

$$\sum_{+in} i \text{ @ A}: \quad i_0 - i_1 - i_2 = 0$$

$$\sum_{+in} i \text{ @ B}: \quad i_1 - i_3 + i_5 = 0$$

$$\sum_{+in} i \text{ @ C}: \quad i_2 - i_4 - i_5 = 0$$

③ Use the element constraints to write the currents in terms of node voltages.

$$i_0 = i_{s1} \qquad i_1 = \frac{v_A - v_B}{R_1} \qquad i_3 = \frac{v_B - 0}{R_3}$$

$$i_5 = i_{s2} \qquad i_2 = \frac{v_A - v_C}{R_2} \qquad i_4 = \frac{v_C - 0}{R_4}$$

④ Substitute ③ into ② and put into standard form

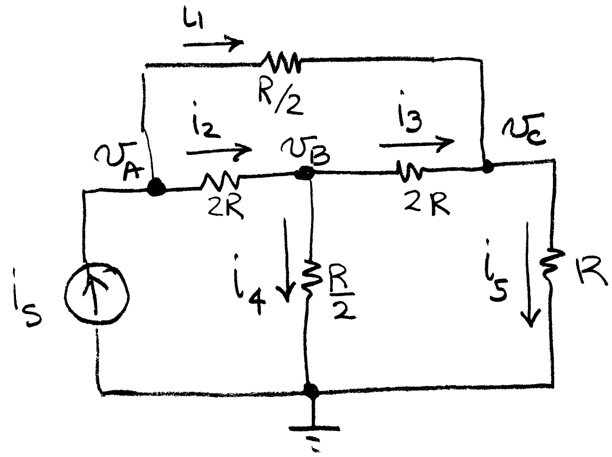
$$(i_{s1}) - \left(\frac{v_A - v_B}{R_1}\right) - \left(\frac{v_A - v_C}{R_2}\right) = 0 \qquad \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_A - \left(\frac{1}{R_1}\right)v_B - \left(\frac{1}{R_2}\right)v_C = i_{s1}$$

$$\left(\frac{v_A - v_B}{R_1}\right) - \frac{v_B}{R_3} + i_{s2} = 0 \qquad \left(-\frac{1}{R_1}\right)v_A + \left(\frac{1}{R_1} + \frac{1}{R_3}\right)v_B = i_{s2}$$

$$\left(\frac{v_A - v_C}{R_2}\right) - \left(\frac{v_C}{R_4}\right) - i_{s2} = 0 \qquad \left(-\frac{1}{R_2}\right)v_A + \left(\frac{1}{R_2} + \frac{1}{R_4}\right)v_C = i_{s2}$$

Example 3-2

Formulate node-voltage equations for the given bridged-T circuit.



① label all voltages and currents

currents given by Ohm's Law

$$i = \frac{\Delta v}{R}$$

② Write the node equations

$$\sum_{+in} i \text{ @A} \quad +i_s - i_1 - i_2 = 0$$

$$\sum_{+in} i \text{ @ B} \quad +i_2 - i_3 - i_4 = 0$$

$$\sum_{+in} i \text{ @ C} \quad +i_1 + i_3 - i_5 = 0$$

③ write the element equations to write currents in terms of node voltages

$$i_1 = \frac{v_A - v_C}{R/2} \qquad i_3 = \frac{v_B - v_C}{2R} \qquad i_5 = \frac{v_C - 0}{R}$$

$$i_2 = \frac{v_A - v_B}{2R} \qquad i_4 = \frac{v_B - 0}{R/2}$$

④ Substitute and write equations in standard form

$$i_s - \left(\frac{v_A - v_C}{R/2}\right) - \left(\frac{v_A - v_B}{2R}\right) = 0 \qquad \left(\frac{2}{R} + \frac{1}{2R}\right)v_A - \left(\frac{1}{2R}\right)v_B - \left(\frac{1}{R/2}\right)v_C = i_s$$

$$\left(\frac{v_A - v_B}{2R}\right) - \left(\frac{v_B - v_C}{2R}\right) - \left(\frac{v_B}{R/2}\right) = 0 \qquad \left(\frac{1}{2R}\right)v_A - \left(\frac{1}{2R} + \frac{1}{2R} + \frac{1}{R/2}\right)v_B + \frac{v_C}{2R} = 0$$

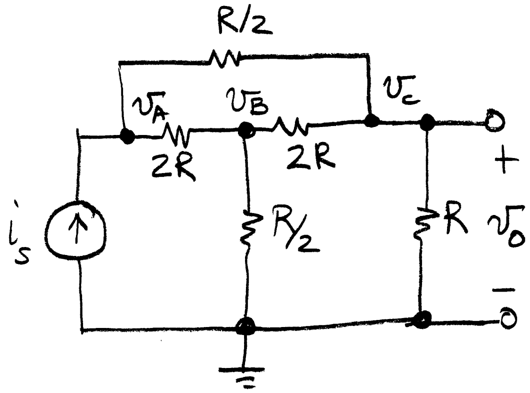
$$\left(\frac{v_A - v_C}{R/2}\right) + \left(\frac{v_B - v_C}{2R}\right) - \left(\frac{v_C}{R}\right) = 0 \qquad \left(\frac{1}{\frac{R}{2}}\right)v_A + \left(\frac{1}{2R}\right)v_B - \left(\frac{1}{R/2} + \frac{1}{2R} - \frac{1}{R}\right)v_C = 0$$

Solving linear systems of equations
(Use a calculator or a computer, MATLAB, mathcad)

Suppose you have a system of equations

$$2.5\,v_A - 0.5\,v_B - 2\,v_C = i_s R$$
$$-0.5\,v_A + 3\,v_B - 0.5\,v_C = 0$$
$$-2\,v_A - 0.5\,v_B + 3.5\,v_C = 0$$

for the T circuit of Fig 3-9



Cramer's Rule

$$v_A = \frac{\Delta_A}{\Delta} = \frac{\begin{bmatrix} i_s R & -0.5 & -2 \\ 0 & +3 & +.5 \\ 0 & -0.5 & +3.5 \end{bmatrix}}{\begin{bmatrix} 2.5 & -0.5 & -2 \\ -0.5 & +3 & -0.5 \\ -2 & -0.5 & +3.5 \end{bmatrix}}$$

Now expand by cofactors

$$v_A = \frac{i_s R \begin{bmatrix} +3 & +0.5 \\ -0.5 & +3.5 \end{bmatrix}}{2.5 \begin{bmatrix} +3 & -0.5 \\ -0.5 & +3.5 \end{bmatrix} - (-0.5) \begin{bmatrix} -0.5 & -2 \\ -0.5 & +3.5 \end{bmatrix} - 2 \begin{bmatrix} -0.5 & -2 \\ +3 & -0.5 \end{bmatrix}}$$

the − sign comes about because cofactors do alternate sign.

$$v_A = \frac{10.25\, i_s R}{11.75} = 0.872\, i_s R$$

$$R_{in} \equiv \frac{v_A}{i_s} = 0.872\, R$$

You could also solve for the output voltage.

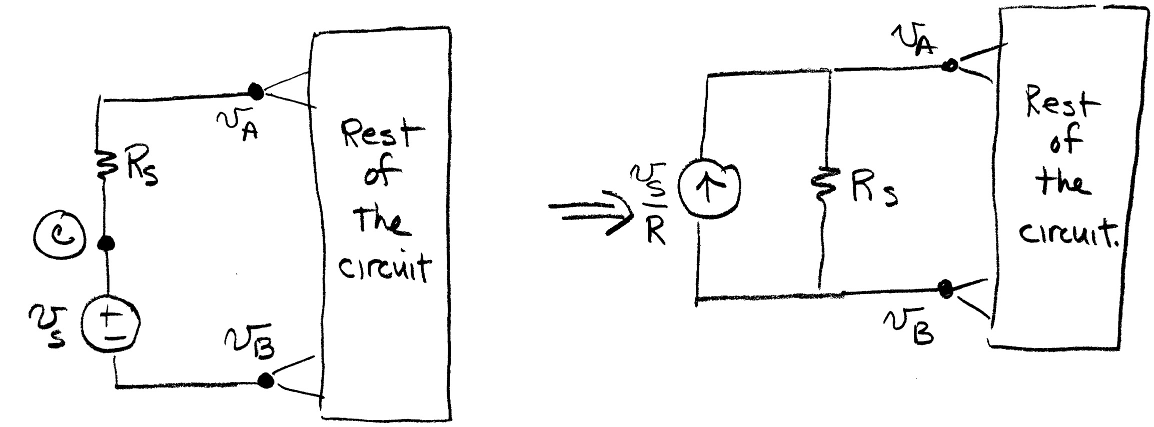
$$v_c = \frac{\Delta c}{\Delta} = \frac{\begin{bmatrix} 2.5 & -0.5 & i_s R \\ -0.5 & +3 & 0 \\ -2 & -0.5 & 0 \end{bmatrix}}{\begin{bmatrix} 2.5 & -0.5 & -2 \\ -0.5 & +3 & -0.5 \\ -2 & -0.5 & +3.5 \end{bmatrix}}$$

Expanding by co-factors:

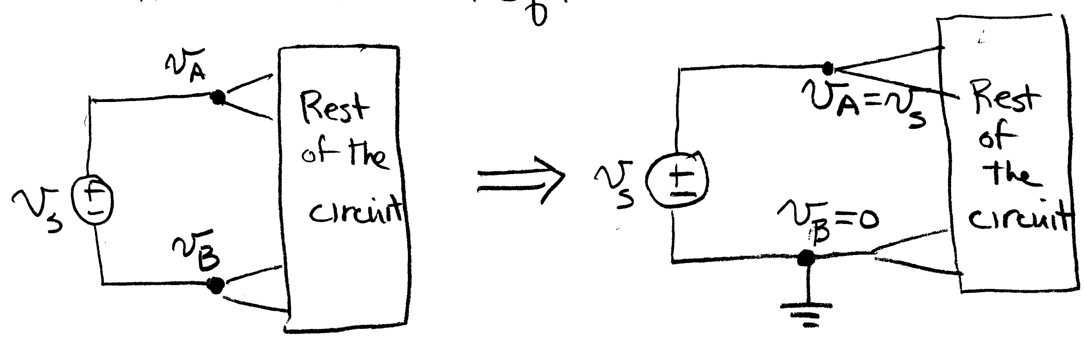
$$v_c = \frac{+i_s R \begin{bmatrix} -0.5 & +3 \\ -2 & -0.5 \end{bmatrix}}{\Delta \text{ (same as before)}}$$

$$v_c = \frac{i_s R \ (6.25)}{11.75} = 0.532 \ i_s R$$

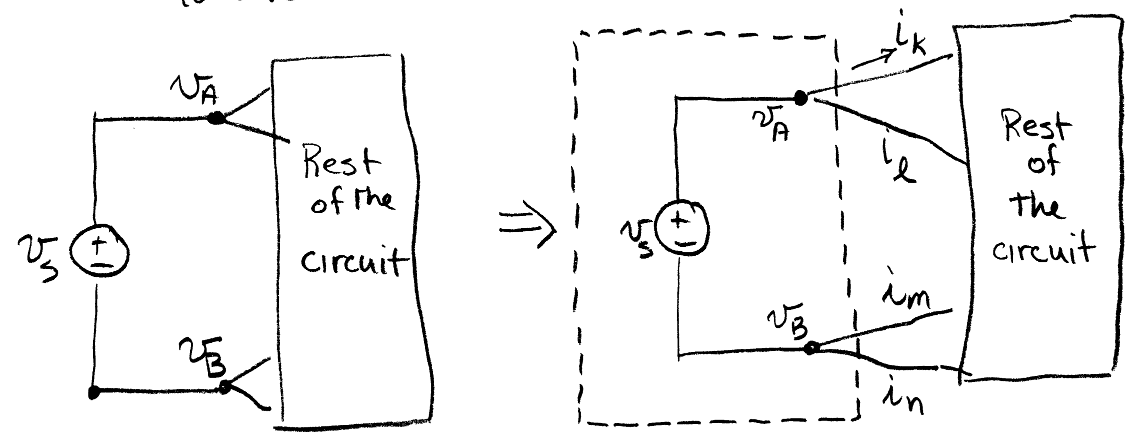
Three methods of dealing with voltage sources in node voltage technique. The problem is that we can't write KCL for a element such as a voltage source.



I. Whenever there is a series R with the voltage source transform to an equivalent current source.



II. If you can ground (i.e., select as the reference node) one of the battery terminals then you don't need KCL to write the node voltages since $v_B=0$ and $v_A=v_s$.



III. If you can't ground define a supernode which encloses the voltage source.

define a supernode

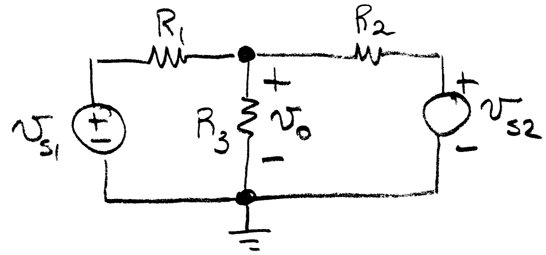
write $v_A - v_B = v_s$ for inside the supernode

$\sum_{tin} i \qquad i_k + i_l + i_m + i_n = 0$

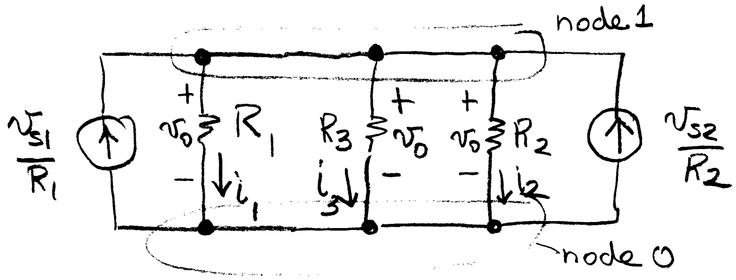
where these are all the currents entering and leaving the supernode

Example 3-4

Use node voltage analysis to find v_o in the circuit given below.



You could write node equations (4 nodes − 1 = 3).
However, node equations are hard to write for voltage sources.
We have resistors in series with voltage sources so we use source transformations to get the following circuit.



v_o is the voltage at node 1 and appears across R_1, R_2 and R_3

Since this circuit has only two nodes we have 2−1=1 equation.

KCL @ node 1 $\quad \sum_{\text{in}} i = 0 \qquad +\dfrac{v_{s1}}{R_1} - i_1 - i_3 - i_2 + \dfrac{v_{s2}}{R_2} = 0$

From the element relations $\quad i_1 = \dfrac{v_o}{R_1} \qquad i_2 = \dfrac{v_o}{R_2} \qquad i_3 = \dfrac{v_o}{R_3}$

Substituting

$$\frac{v_{s1}}{R_1} - \frac{v_o}{R_1} - \frac{v_o}{R_3} - \frac{v_o}{R_2} + \frac{v_{s2}}{R_2} = 0$$

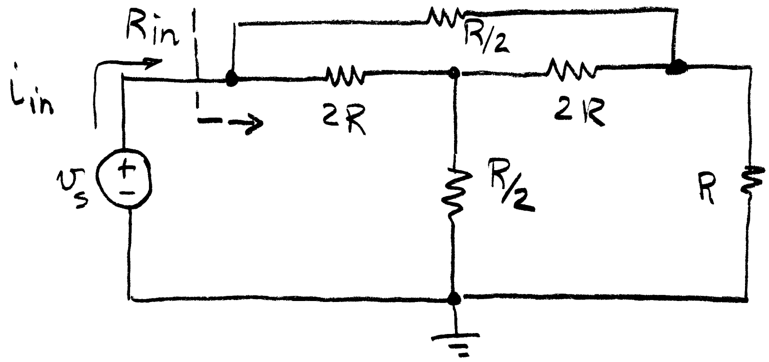
Solving for v_o

$$v_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = \frac{v_{s1}}{R_1} + \frac{v_{s2}}{R_2}$$

$$v_o = \frac{\dfrac{v_{s1}}{R_1} + \dfrac{v_{s2}}{R_2}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

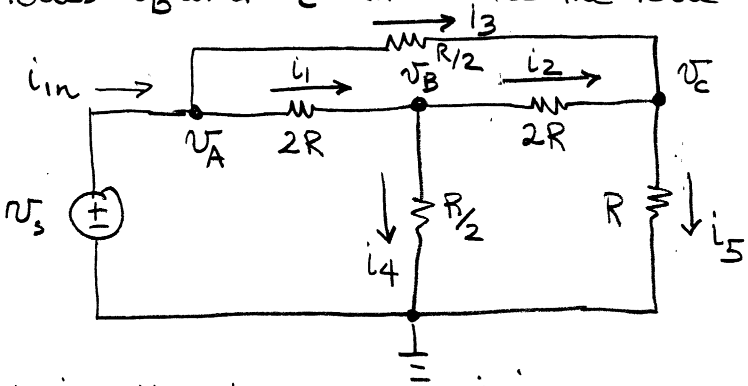
Example 3-5

Find the input resistance of the circuit given below.



$$R_{in} = \frac{v_s}{i_{in}}$$

We cannot write a node equation at the input because it is a voltage source. The source transformation method can't be used because there is no resistor in series with v_s. However, one terminal of v_s can be grounded making the node voltage at the input v_s. We can label the voltages at the other two nodes v_B and v_C and write the node equations.



Write the element equations

$$i_1 = \frac{v_A - v_B}{2R}$$

$$i_2 = \frac{v_B - v_C}{2R}$$

$$i_3 = \frac{v_A - v_C}{R/2}$$

$$i_4 = \frac{v_B - 0}{R/2}$$

$$i_5 = \frac{v_C - 0}{R}$$

and $v_A = v_s$

Then write the node equations at B and C.

$v_A = v_s$ so we only write two node equations.

$$@B \sum_{+in} i = 0 \quad +i_1 - i_2 - i_4 = 0$$

$$@C \sum_{+in} i = 0 \quad +i_3 + i_2 - i_5 = 0$$

We can substitute the element equations into the node equations and solve the problem.

@ B $\quad i_1 - i_2 - i_4 = 0$

$$\frac{v_A - v_B}{2R} - \frac{v_B - v_C}{2R} - \frac{v_B - 0}{R/2} = 0$$

$$\frac{1}{2R} v_A - \left(\frac{1}{2R} + \frac{1}{2R} + \frac{1}{R/2}\right) v_B + \left(\frac{1}{2R}\right) v_C = 0$$

@ C $\quad i_3 + i_2 - i_5 = 0$

$$\frac{v_A - v_C}{R/2} + \frac{v_B - v_C}{2R} - \left(\frac{v_C - 0}{R}\right) = 0$$

$$\left(\frac{1}{R/2}\right) v_A + \left(\frac{1}{2R}\right) v_B + \left(-\frac{1}{R/2} - \frac{1}{2R} - \frac{1}{R}\right) v_C = 0$$

Substituting $v_A = v_s$ and putting into standard form.

@ C $\quad \left(\frac{1}{2R}\right) v_B + \left(-\frac{1}{R/2} - \frac{1}{2R} - \frac{1}{R}\right) v_C = -\frac{1}{R/2} v_s$

@ B $\quad \left(-\frac{1}{2R} - \frac{1}{2R} - \frac{1}{R/2}\right) v_B + \left(\frac{1}{2R}\right) v_C = -\frac{1}{2R} v_s$

@ C $\quad \frac{1}{2R} v_B - \frac{7}{2R} v_C = -\frac{2}{R} v_s$

@ B $\quad -\frac{3}{R} v_B + \frac{1}{2R} v_C = -\frac{1}{2R} v_s$

$$\frac{1}{2} v_B - \frac{7}{2} v_C = -2 v_s$$

divided out all the R's

$$-3 v_B + \frac{1}{2} v_C = -\frac{1}{2} v_s$$

This can be solved by Cramer's Rule