

**CASE WESTERN RESERVE UNIVERSITY**  
Case School of Engineering  
Department of Electrical Engineering and Computer Science

**ENGR 210. Introduction to Circuits and Instruments (4)**

**Homework Set No. 13 SOLUTIONS**

Issued 4/20/04

References: [T&R4] sections 7-1 to 7-4.

**NOTE: This HW assignment was not to be turned in but was presented only as an example of what you should know about first-order circuits for the final exam.**

- 1)** (5 pts) Problem 7-2, p. 337. Basic first order linear differential circuit equation with initial condition.

Find the function that satisfies the following differential equation and initial condition.

$$10^{-4} \frac{di(t)}{dt} + 10^{-1} i(t) = 0, \quad i(0) = -20 \text{ mA}$$

**SOLUTION**

You know that this is a first order linear differential equation with an exponential solution  $i(t) = Ke^{st}$ . Substituting this solution into the given differential equation

gives  $10^{-4} \frac{d(Ke^{st})}{dt} + 10^{-1} Ke^{st} = 0$ . Taking the derivative gives

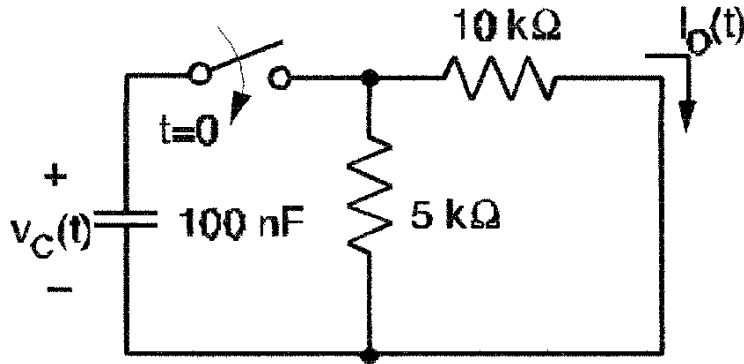
$10^{-4} sKe^{st} + 10^{-1} Ke^{st} = 0$ . Rearranging this gives the characteristic equation  $10^{-4} s + 10^{-1} = 0$  which can be solved to give  $s = -1000$ .

We now take the function  $i(t) = Ke^{-1000t}$  and use the initial condition to find K.

Evaluating for  $t=0$  gives  $i(0) = Ke^0 = -20 \text{ mA}$  so the function is  $i(t) = -20e^{-1000t} \text{ mA}$ .

2) (5 pts) Problem 7-5, p. 337. Basic switched circuit with initial condition.

The switch in the figure below is closed at  $t=0$ . The initial voltage on the capacitor is  $v_c(t)=15$  volts. Find  $v_c(t)$  and  $i_o(t)$  for  $t \geq 0$ .



SOLUTION.

As in all RC and RL problems we determine the equivalent resistance of the circuit as seen by the capacitor after the switch closes. After the switch closes the circuit simply consists of the  $5k\Omega$  and the  $10k\Omega$  resistor in parallel. The

equivalent resistance is then given by  $R_{EQ} = \frac{R_1 R_2}{R_1 + R_2} = \frac{(5k\Omega)(10k\Omega)}{5k\Omega + 10k\Omega} = 3.33k\Omega$

We can derive the differential equation for the circuit to determine  $v_c(t)$ . Using

KVL we can write the equation  $-v_c + i_R R_{EQ} = 0$ . We can use  $i_R = -i_C$ ,  $i_C = C \frac{dv_c}{dt}$

and  $R_{eq}=3.33k\Omega$  to rewrite this as  $-v_c - i_C R_{EQ} = -v_c - C \frac{dv_c}{dt} R_{EQ} = 0$ . Rearranging

this gives the differential equation for the state variable as  $R_{EQ} C \frac{dv_c}{dt} + v_c = 0$ .

We know the solution of this is  $v_c(t) = v_{C,forced} + v_{C,natural}$ . The forced response is the DC response. Since there is no battery or other voltage source in this circuit we know that the steady-state (or final) value is given by  $v_{C,forced} = v_c(\infty) = 0$ .

The transient (or natural) solution comes from  $R_{EQ} C \frac{dv_c}{dt} + v_c = 0$  and is given by

$v_{C,natural}(t) = K e^{st}$ . Substituting this into the differential equation gives

$R_{EQ} C \frac{d(K e^{st})}{dt} + K e^{st} = 0$ . Differentiating gives  $R_{EQ} C s K e^{st} + K e^{st} = 0$  or  $R_{EQ} C s + 1 = 0$ .

The solution is then  $v_c(t) = K e^{-\frac{t}{R_{EQ}C}}$  and we can identify the time constant

$T_C = R_{EQ}C$ . For the given values this is

$$T_C = R_{EQ}C = (3333)(100 \times 10^{-9}) = 0.000333 \text{ seconds.}$$

The solution is then given by  $v_C(t) = Ke^{-\frac{t}{0.000333}}$ . We can solve for K by using the initial condition  $v_C(0) = 15$  volts to give  $v_C(0) = 15 = Ke^0$ . The final solution is then  $v_C(t) = 15e^{-\frac{t}{0.000333}}$  for  $t \geq 0$ .

Let us compare this to finding the solution using the initial/final value theorem.

The initial value is given in the problem statement as  $v_C(0) = 15$ . The final value was determined to be  $v_C(\infty) = 0$ . The only thing to be determined is the time constant which we know is given by  $T_C = R_T C$  where  $R_T$  is the Thevenin resistance as seen by the capacitor. We computed this resistance above as  $R_T = 3333 \Omega$  so the time constant comes out as

$$T_C = R_{EQ}C = (3333)(100 \times 10^{-9}) = 0.000333$$

Using the initial/final value theorem we have

$$[state](t) = [initialvalue - finalvalue]e^{-\frac{t}{T_C}} + finalvalue$$

Substituting values gives

$$v_C(t) = [v_C(0) - v_C(\infty)]e^{-\frac{t}{0.000333}} + v_C(\infty) = [15 - 0]e^{-\frac{t}{0.000333}} + 0 = 15e^{-\frac{t}{0.000333}} \text{ for } t \geq 0.$$

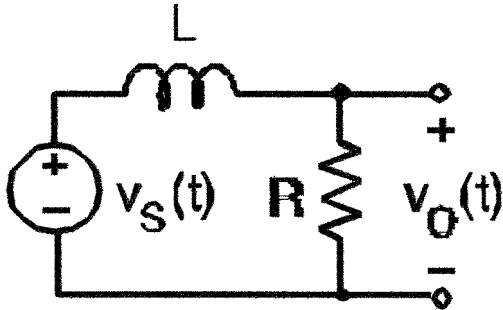
This certainly appears to be a simpler method for this problem.

The final part of the problem is to find  $i_o(t)$ . We always find the state variable response first and then determine other circuit variables such as  $i_o(t)$  in terms

of the state variable. In this case,  $i_o(t) = \frac{v_C(t)}{10k\Omega} = \frac{15e^{-\frac{t}{0.000333}}}{10 \times 10^3} = 1.5e^{-\frac{t}{0.000333}} \text{ mA.}$

3) (5 pts) Problem 7-10, p. 338. Forced and natural response.

The circuit shown below is in the zero state when the input  $v_s(t) = V_A u(t)$  is applied. Find  $v_o(t)$  for  $t \geq 0$ . Identify the forced and natural components.



### SOLUTION

As with the previous problem you can either do the differential equation which explicitly has forced and natural components or we can use the initial/final value theorem.

Let us use the initial/final value theorem to get practice with DC analysis of these circuits. Furthermore, let us remember that the theorem only applies to the state variable, i.e., the current through the inductor.

Since  $v_s(t) = 0$  for  $t < 0$  and we are not given any initial conditions for the inductor we may conclude that  $i_L(0) = 0$  for the initial value. The initial value is the value of the current through the inductor JUST BEFORE the input changes. The final value is the current through the inductor after all transients have gone to zero and the circuit is AGAIN in a steady-state condition. At  $t = \infty$  the terminal relation for the inductor voltage becomes  $v_L = L \frac{di_L}{dt} \rightarrow 0$  since the current is no longer changing, i.e., it is in steady state. If  $v_L = 0$  the entire source voltage appears across the resistor and  $v_o = v_s = A = i(\infty)R$ . Solving for R gives  $i_L(\infty) = i_R(\infty) = \frac{A}{R}$ .

All that remains is to determine the time constant. We simply turn OFF the source and find the Thevenin resistance seen by the inductor L. The voltage source becomes a short when turned OFF and the Thevenin resistance seen by the inductor is simply R. The time constant is then given by  $T_C = \frac{L}{R} = R$ .

The complete solution as given by the initial/final value theorem is then

$$[state](t) = [initialvalue - finalvalue]e^{-\frac{t}{T_c}} + finalvalue \text{ or}$$

$i_L(t) = [i_L(0) - i_L(\infty)]e^{-\frac{t}{T_c}} + i_L(\infty)$ . Using the circuit values gives

$$i_L(t) = \left[0 - \frac{A}{R}\right]e^{-\frac{t}{T_c}} + \frac{A}{R} = \frac{A}{R} - \frac{A}{R}e^{-\frac{t}{T_c}}$$

We can now use this state variable response to determine  $v_o(t)$ . By Ohm's Law

$$v_o(t) = i_L(t)R = \left(\frac{A}{R} - \frac{A}{R}e^{-\frac{t}{T_c}}\right)R = A - Ae^{-\frac{t}{T_c}} \text{ for } t \geq 0.$$

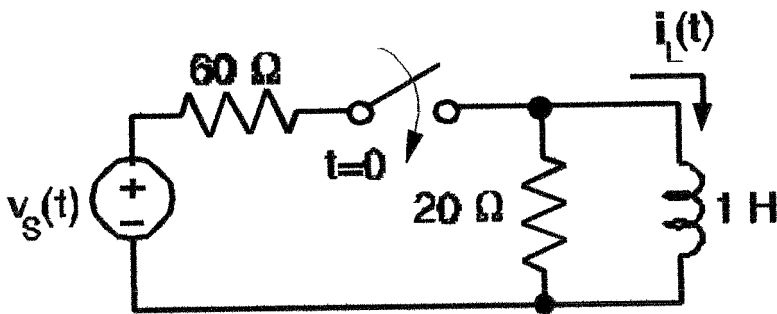
The forced response is the response AFTER all transients have gone to zero, i.e., at  $t = \infty$ . By simple evaluation  $v_{O,forced}(t) = A - Ae^{-\infty} = A$ .

The natural response is the transient part and is simply given as

$$v_{O,natural}(t) = v_o(t) - v_{O,forced}(t) = \left(A - Ae^{-\frac{t}{T_c}}\right) - A = -Ae^{-\frac{t}{T_c}}$$

4) (5 pts) Problem 7-17, p. 338. Switched inductive circuit.

The input in the figure below is  $v_s(t) = 20 \cos 5t$



#### SOLUTION

When you see a sinusoidal input you should think of solving the problem using phasors. When you see a switch you should probably think of using the initial/final value theorem. However, when you see a sinusoidal input AND a switch you almost always have to use the differential equation approach.

This problem illustrates one technique which is very important – that of Thevenizing the circuit seen by that inductor (or capacitor). In this case, we Thevenize the voltage source and the two resistors. Note that we do this for  $t \geq 0$  since  $i_L(t) = 0$  for  $t < 0$ . The Thevenin resistance is found by turning OFF the voltage source and is recognized to be two resistors in parallel. The Thevenin resistance is given by  $R_t = 60\Omega \parallel 20\Omega = \frac{(60)(20)}{60 + 20} = 15\Omega$ . The Thevenin voltage is the open circuit voltage seen by the inductor. This is a voltage divider with a Thevenin output voltage of  $V_t(t) = \frac{20}{20 + 60} V_s(t) = \frac{1}{4} V_s(t)$ .

We can then write a differential equation for the state variable  $i_L(t)$ . Using KVL we

can write  $-V_t + i_L R_t + v_L(t) = 0$ . We can use  $i_R = i_L$ ,  $v_L = L \frac{di_L}{dt}$  and the above values for  $V_t(t)$  and  $R_t$  to rewrite this as

$$-V_t(t) + i_L R_t + v_L(t) = -\frac{1}{4} V_s(t) + i_L R_t + L \frac{di_L}{dt} = 0. \text{ Rearranging this gives the}$$

$$\text{differential equation for the state variable } i_L(t) \text{ as } \frac{L}{R_t} \frac{di_L}{dt} + i_L = \frac{1}{4} V_s(t) = 5 \cos(5t).$$

We know the solution of this differential equation is  $i_L(t) = i_{L,forced} + i_{L,natural}$ .

The transient (or natural) solution comes from  $\frac{L}{R_t} \frac{di_L}{dt} + i_L = 0$  and is given by

$i_{L,natural}(t) = Ke^{st}$ . Substituting this into the differential equation gives

$$\frac{L}{R_t} \frac{d(Ke^{st})}{dt} + Ke^{st} = 0. \text{ Differentiating gives } \frac{L}{R_t} sKe^{st} + Ke^{st} = 0 \text{ or } \frac{L}{R_t} s + 1 = 0. \text{ The}$$

natural solution is then  $i_{L,natural}(t) = Ke^{-\frac{t}{L/R_t}}$

You probably do not recognize the forced solution since we did it several weeks ago. ☺ It is the solution of the differential equation for a sinusoidal input. We initially did it in Section 7.4 using algebra but the correct way to do it is using phasors as we did in Chapters 8 and 12. The differential equation for the

forced response is  $\frac{L}{R_t} \frac{di_L}{dt} + i_L = 5 \cos(5t)$ . We begin by converting all terms to

phasors, i.e.,  $5 \cos(5t) \rightarrow 5 \angle 0^\circ$ ,  $i_L(t) \rightarrow \underline{I_L}$  and  $\frac{L}{R_t} \frac{di_L(t)}{dt} \rightarrow \frac{L}{R_t} j\omega \underline{I_L}$  where the

underlining denotes a phasor. Using these quantities gives the algebraic phasor equation  $\frac{L}{R_t} j\omega \underline{I_L} + \underline{I_L} = 5$ . We use the numerical values for L and  $R_t$  to

get  $\frac{1}{15} j(5) \underline{I_L} + \underline{I_L} = 5$ . You might have missed the value of  $\omega$  which comes from  $20 \cos(5t)$ . Using these numbers we can re-arrange the equation to get

$$\underline{I_L} \left( j \frac{5}{15} + 1 \right) = 5 \text{ or } \underline{I_L} = \frac{5}{j \frac{5}{15} + 1} = \frac{5}{j \frac{1}{3} + 1} = \frac{15}{j + 3} = 4.5 - j1.5 = 4.743 \angle -18.4^\circ. \text{ The}$$

forced solution is then  $i_{L,forced}(t) = \text{Re} \left\{ (4.743 \angle -18.4^\circ) e^{j5t} \right\} = 4.743 \cos(5t - 18.4^\circ)$ .

The total solution is then  $i_L(t) = i_{L,forced} + i_{L,natural} = 4.743 \cos(5t - 18.4^\circ) + Ke^{-\frac{t}{15}}$  or

$$i_L(t) = 4.743 \cos(5t - 18.4^\circ) + Ke^{-\frac{t}{0.0667}}$$

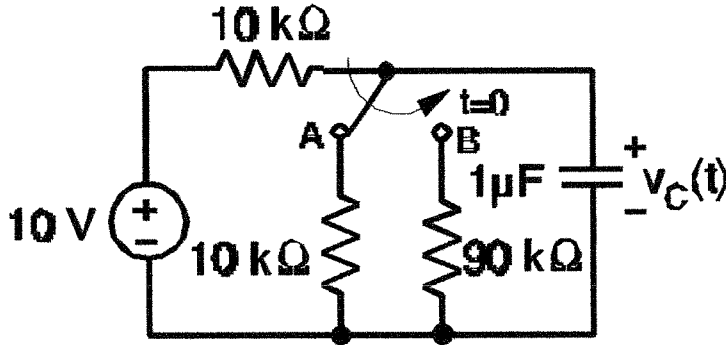
We can solve for K using the initial condition  $i_L(0) = 0$  (there is no current through the inductor before the switch closes) to give

$$i_L(0) = 4.743 \cos(0 - 18.4^\circ) + Ke^{-\frac{0}{0.0667}} = 4.500 + K = 0. \text{ We solve for K to get } K = -4.500.$$

The total solution is then  $i_L(t) = 4.743 \cos(5t - 18.4^\circ) - 4.500 e^{-\frac{t}{0.0667}}$  for  $t \geq 0$ .

5) (5 pts) Problem 7-19, p. 338. Switched capacitive circuit.

The switch in the figure below has been in position A for a long time and is moved to position B at  $t=0$ . Find  $v_C(t)$  for  $t \geq 0$  and sketch its waveform.



### SOLUTION

These switched circuits look more complicated than they really are. They are solved by using DC analysis to determine the initial and final values of the state variable and Thevenizing the circuit to determine the time constant. Once these three values are known the response is given by

$$[state](t) = [initialvalue - finalvalue] e^{-\frac{t}{T_c}} + finalvalue$$

In position A there is a voltage divider with the capacitor C across the lower  $10k\Omega$  resistor. The capacitor is in steady-state and fully charged so it looks like an open. This means that the initial voltage across the capacitor (the state variable) at  $t=0^-$  is given by  $v_C(0^-) = \frac{10k\Omega}{10k\Omega + 10k\Omega}(10) = 5$  volts.

When the switch changes to position B the voltage divider changes. In particular the lower resistor changes from  $10k\Omega$  to  $90k\Omega$ . We are interested in the steady state voltage (after all transients have gone to zero) so we now have the final voltage at  $t=\infty$  is given by  $v_C(\infty) = \frac{90k\Omega}{10k\Omega + 90k\Omega}(10) = 9$

You could get the same results by Thevenizing the source/resistors for each switch position.

We now know that the state variable ( $v_C$ ) changes from an initial value of 5 volts to a final value of 9 volts. What remains is to identify the time constant. You must be careful to recognize that the time constant is for the switch in position B. In position B we turn the voltage source OFF (replace it by a short) and then find the equivalent resistance of the two resistors. The resistors are in parallel



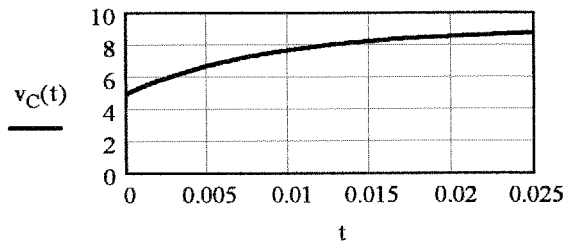
so their equivalent resistance is given by  $R_{EQ} = \frac{(10k\Omega)(90k\Omega)}{10k\Omega + 90k\Omega} = 9k\Omega$ . Since this is

a RC circuit we know the time constant is given by

$$T_C = R_{EQ}C = (9 \times 10^3)(1 \times 10^{-6}) = 0.009$$

The state-variable (the voltage across the capacitor) is then given as a function of

time by  $v_C(t) = [5 - 9]e^{-\frac{t}{0.009}} + 9 = 9 - 4e^{-\frac{t}{0.009}}$ . This function is plotted below.



6) (5 pts) Problem 7-25, p. 340. First order capacitive circuit response.

For  $t \geq 0$  the voltage across and the current through the capacitor in the RC circuit shown below are  $v_C(t) = 15 - 10e^{-2000t}$  volts and  $i_C(t) = 10e^{-2000t}$  mA.

- (a) Find the circuit time constant.
- (b) Find the initial value of the state variable.
- (c) Find  $v_s$ , R and C.
- (d) Find the energy stored in the capacitor at  $t=1\text{ms}$ .

SOLUTION

(a) The time constant is the denominator of the exponent, i.e.,  $e^{-\frac{t}{T_C}}$ . By inspection we see that  $\frac{1}{T_C} = 2000$  or  $T_C = \frac{1}{2000} = 0.5 \times 10^{-3}$  seconds.

(b) The state variable for a capacitor is  $v_C$  so this question is simply what is the initial  $v_C(0)$ . This can be obtained by evaluation as  $v_C(0) = 15 - 10e^0 = 15 - 10 = 5$  volts.

(c) We first find the inductance C from the terminal relationship for the capacitor, i.e.,  $i_C = C \frac{dv_C}{dt}$ . Solving for C gives

$$C = \frac{i_C}{\frac{dv_C}{dt}} = \frac{10e^{-2000t} \times 10^{-3}}{\frac{d(15 - 10e^{-2000t})}{dt}} = \frac{10e^{-2000t} \times 10^{-3}}{(-10)(-2000)(e^{-2000t})} = \frac{10^{-3}}{2000} = 0.5 \times 10^{-6}$$

Knowing C we can determine R from the time constant since  $T_C = RC$  for a capacitive circuit. Using the value from (a) and the just determined value of C gives  $R = \frac{T_C}{C} = \frac{0.5 \times 10^{-3}}{0.5 \times 10^{-6}} = 1000$  ohms. Note that you want to be very careful of keeping track of your powers of 10 for the time constant.

Once you know all the circuit elements determining the source voltage  $v_s(t)$  is pretty straightforward. Since these are exponential responses you know this is a response to a step input. The issue is that of determining the value of the step. The simplest way to determine the value of the step is to examine the response of the circuit at  $t=\infty$  after all the transients have gone to zero. At  $t=\infty$  the capacitor has zero current ( $i_C(\infty) = C \frac{dv_C}{dt} \rightarrow 0$ ) through it since the derivative

goes to zero (nothing is changing and the circuit is in steady state). We now do a DC analysis. Since there is no current through the capacitor there is also no current through R. That means that the voltage across R is zero and the voltage  $v_s(\infty)$  appears across C which looks like an open. Consequently,  $v_s = v_C(\infty) = 15$  volts. The input voltage is more formally  $v_L(t) = 15u(t)$  volts.

(d) Finding the energy stored in the capacitor is a simple matter of evaluating the energy formula  $W_C = \frac{1}{2}Cv^2$  using

$$v_C(t = 0.001) = 15 - 10e^{-2000(0.001)} = 15 - 10e^{-2} = 15 - 1.35 = 13.65 \text{ volts to get}$$

$$W_C(0.001) = \frac{1}{2}Cv^2(0.001) = \frac{1}{2}(0.5 \times 10^{-6})(13.65)^2 = 46.56 \times 10^{-6} \text{ joules.}$$

**7)** (5 pts) Problem 7-27, p. 340. First order inductive circuit response.

For  $t \geq 0$  the voltage across and the current through the inductor in the RL circuit shown below are  $i_L(t) = 5 - 10e^{-1000t}$  mA and  $v_L(t) = e^{-1000t}$  volts.

- (a) Find the circuit time constant.
- (b) Find the initial and final value of the state variable.
- (c) Find  $v_s$ , R and L.
- (d) Find the energy stored in the capacitor at  $t=0$  and  $t=\infty$ .

**SOLUTION**

(a) The time constant is the denominator of the exponent, i.e.,  $e^{-\frac{t}{T_c}}$ . By inspection we see that  $\frac{1}{T_c} = 1000$  or  $T_c = \frac{1}{1000} = 1\text{ms}$

(b) The state variable for an inductor is  $i_L$  so this question is simply what are the initial  $i_L(0)$  and final  $i_L(\infty)$  values. These can be obtained by evaluation as  $i_L(0) = 5 - 10e^0 = 5 - 10 = -5$  mA and  $i_L(\infty) = 5 - 10e^{-\infty} = 5$  mA.

(c) We first find the inductance L from the terminal relationship for the inductor, i.e.,  $v_L = L \frac{di_L}{dt}$ . Solving for L gives

$$L = \frac{v_L}{\frac{di_L}{dt}} = \frac{e^{-1000t}}{\frac{d(5 - 10e^{-1000t})}{dt} \times 10^{-3}} = \frac{e^{-1000t}}{(-10)(-1000)(e^{-1000t}) \times 10^{-3}} = \frac{1}{10000 \times 10^{-3}} = 0.1$$

Knowing L we can determine R from the time constant since  $T_c = \frac{L}{R}$  for an inductive circuit. Using the value from (a) and the just determined value of L gives  $R = \frac{L}{T_c} = \frac{0.1}{0.001} = 100$  ohms. Note that you want to be very careful of keeping track of your powers of 10 for the time constant.

Once you know all the circuit elements determining the source voltage  $v_s(t)$  is pretty straightforward. Since these are exponential responses you know this is a response to a step input. The issue is that of determining the value of the step. The simplest way to determine the value of the step is to examine the response of the circuit at  $t=\infty$  after all the transients have gone to zero. At  $t=\infty$

the inductor has zero voltage ( $v_L(\infty) = L \frac{di}{dt} \rightarrow 0$ ) across it since the derivative goes to zero (nothing is changing and the circuit is in steady state). We now do a DC analysis. Since there is no voltage across the inductor the only voltages in the circuit are  $v_s(t)$  and the voltage across R. Consequently,  
 $v_s = iR = i_L(\infty)R = (.005)(100) = 0.5$  volts. The input voltage is more formally  
 $v_L(t) = 0.5u(t)$  volts.

(d) Finding the energy stored in the inductor is a simple matter of evaluating the energy formula  $W_L = \frac{1}{2}Li^2$  using  $i_L(0) = -5$  mA and  $i_L(\infty) = 5$  mA to get

$$W_L(0) = \frac{1}{2}Li^2(0) = \frac{1}{2}(0.5)(-.005)^2 = 1.25 \times 10^{-6} \text{ joules and}$$

$$W_L(\infty) = \frac{1}{2}Li^2(\infty) = \frac{1}{2}(0.5)(+.005)^2 = 1.25 \times 10^{-6} \text{ joules.}$$