## Chapter 4 - Time and Frequency Response

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## Chapter 4

Warmups

1. For the circuit shown, determine the steady state inductance current $i_{L}$ for (a) the switch closed, and (b) the switch open.


By inspection,
(a) when the switch is closed the inductor acts like a short
$i_{L}=\frac{15}{4}=3.75$ Amperes
(b) when the switch is open $i_{L}=0$
2. For the circuit shown, determine the steady state capacitance voltage $V_{C}(t)$ for (a) the switch closed, and (b) the switch open.


The boundary condition for the capacitance voltage is $V_{C}\left(0^{-}\right)=V_{C}\left(0^{+}\right)$
(a) when the switch is closed $V_{C}=\frac{8}{8+4}(15 \mathrm{~V})=10 \mathrm{~V}$
(b) when the switch is open $V_{C}=0$ Volts since the capacitor discharges through the $8 \Omega$ resistor
3. For the circuit shown, determine the steady state inductance current $i_{L}$ for (a) the switch closed, and (b) the switch open.


Using a current divider $i_{L}=i_{T} \frac{8}{8+j}=\left(3.536 \angle 6.568^{\circ}\right)\left(0.992 \angle-7^{\circ}\right)=3.51 \angle 0.556^{\circ}$
(b) when the switch is open $i_{L}=0$
4. The circuit in problem 2 has had the switch closed for a long time. At $t=0$, the switch opens. Determine the capacitance voltage, $V_{C}(t)$, for $\mathrm{t}>0$.


When the switch opens we have only the RC circuit.


By Ohm's Law $V_{C}(t)=8 i_{C}(t)$
but $-i_{C}=C \frac{d V_{C}}{d t}$ for the capacitor. The - sign comes in because of the direction chosen for the capacitive current.
$V_{C}=8\left(-C \frac{d V_{C}}{d t}\right)=-8\left(1 \times 10^{-6}\right) \frac{d V_{C}}{d t}$
$\left(8 \times 10^{-6}\right) \frac{d V_{C}}{d t}+V_{C}=0$
$\frac{d V_{C}}{d t}+1.25 \times 10^{5} V_{C}=0$
This kind of equation always has a solution of the form $V_{C}=A e^{\alpha t}$
$\frac{d V_{C}}{d t}=A \alpha e^{\alpha t}$
Substituting this solution into (*)
$A \alpha e^{\alpha t}+1.25 \times 10^{5} A e^{\alpha t}=0$
$\alpha=-1.25 \times 10^{5}$
$V_{C}(t)=A e^{-1.25 \times 10^{5} t}$
at $\mathrm{t}=0+, \mathrm{VC}=10$ volts (due to the voltage divider)
$V_{C}(t)=10 e^{-1.25 \times 10^{5} t}$
5. The circuit in problem 2 has had the switch open for a long time. At $\mathrm{t}=0$, the switch closes. Determine the capacitance voltage $V_{C}(t)$ for $\mathrm{t}>0$.


Do KCL at node B with $\sum i=0(+$ into the node)
$\frac{15-V_{C}}{4}-\frac{V_{C}}{8}-i_{C}=0$
$\frac{15-V_{C}}{4}-\frac{V_{C}}{8}-\left(1 \times 10^{-6}\right) \frac{d V_{C}}{d t}=0$
$\frac{15}{4}-\frac{V_{C}}{4}-\frac{V_{C}}{8}-10^{-6} \frac{d V_{C}}{d t}=0$
$\frac{d V_{C}}{d t}+\frac{3}{8} \times 10^{6} V_{C}=\frac{15}{4} \times 10^{6}$
For the steady state $\frac{d V_{C}}{d t} \rightarrow 0$
$V_{C, s s}=\frac{\frac{15}{4} \times 10^{6}}{\frac{3}{8} \times 10^{6}}=10 \mathrm{Volts}$
The transient solution is of the form $V_{C}=A e^{a t}$
$\alpha A e^{\alpha t}+\frac{3}{8} \times 10^{6} A e^{\alpha t}=0$
$\alpha=-\frac{3}{8} \times 10^{6}$
$V_{C}(t)=V_{C, \text { transient }}+V_{C, s s}=A e^{-\frac{3}{8} \times 10^{6} t}+10$
$@ \mathrm{t}=0 V_{C}\left(0^{-}\right)=V_{C}\left(0^{+}\right)=0$
$0=A+10$ or $A=-10$
The final solution is then
$V_{C}(t)=10\left(1-e^{-\frac{3}{8} \times 10^{6} t}\right)$
6. A system is described by the equation
$12 \sin 2 t=\frac{d^{2} V}{d t^{2}}+5 \frac{d V}{d t}+4 V$
The initial conditions on the voltage are $V(0)=-5$ and $\frac{d V(0)}{d t}=2$.
Determine $V(t)$ for $\mathrm{t}>0$.
The transient solution comes from $\frac{d^{2} V}{d t^{2}}+5 \frac{d V}{d t}+4 V=0$
Assume a solution of the form $e^{s t}$
$s^{2}+5 s+4=0$
$s=\frac{-5 \pm \sqrt{25-4(4)}}{2}=\frac{-5 \pm \sqrt{9}}{2}=\frac{-5 \pm 3}{2}=-4,-1$
$V_{\text {transient }}=A e^{-4 t}+B e^{-t}$
To determine the steady state solution convert to phasors
$12 \sin 2 t=12 \cos \left(2 t-90^{\circ}\right) \rightarrow 12 e^{j \omega t-j \frac{\pi}{2}}=-j 12 e^{j \omega t}$
Assume a soluton of the form $C e^{j \omega t}$
$(j \omega)^{2} C e^{j \omega t}+5 j \omega C e^{j \omega t}+4 C e^{j \omega t}=-j 12 e^{j \omega t}$
$C\left(4-\omega^{2}+5 j \omega\right)=-j 12$
$C=-\frac{j 12}{4-\omega^{2}+5 j \omega}$
For $\omega=2($ since $\sin 2 \mathrm{t}) C=\frac{-j 12}{4-4+j 10}=-1.2$
$V_{\text {steady }- \text { state }}=\operatorname{Re}\left\{-1.2 e^{j 2 t}\right\}=-1.2 \cos 2 t$
$V(t)=-1.2 \cos 2 t+A e^{-4 t}+B e^{-t}$
$\frac{d V}{d t}=-1.2(2) \sin 2 t-4 A e^{-4 t}-B e^{-t}$
$V(0)=-5 @ \mathrm{t}=0 \quad-5=-1.2(1)+A+B$
$\frac{d V(0)}{d t}=2 @ \mathrm{t}=02=-1.2(2)(0)-4 A-B$
Solving simultaneously
$A+B=-5+1.2=-3.8$
$4 A+B=-2$
Subtracting gives $-3 A=-1.8$, or $A=0.6$
Using this result $0.6+B=-3.8$ or $B=-4.4$
The final solution is then
$V(t)=-1.2 \cos 2 t+0.6 e^{-4 t}-4.4 e^{-t}$
7. For the circuit shown, the current source is $i_{S}(t)=0.01 \cos 500 t$


Find the transfer function $G(s)=\frac{V_{C}(s)}{I_{S}(s)}$
Using KCL at node $\mathrm{A}\left(+\right.$ out) and that the voltage at A is $V_{C}$ :


$$
\begin{aligned}
& -I_{S}+\frac{V_{C}}{30}+0.001 s V_{C}+\frac{V_{C}}{10+0.25 s}=0 \\
& I_{S}=V_{C}\left(\frac{1}{30}+0.001 s+\frac{1}{10+0.25 s}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
& G(s)=\frac{V_{C}(s)}{I_{S}(s)}=\frac{V_{C}}{V_{C}\left(\frac{1}{30}+0.001 s+\frac{1}{10+0.25 s}\right)}=\frac{1}{\frac{10+0.25 s+0.03 s(10+0.25 s)+30}{30(10+0.25 s)}} \\
& G(s)=\frac{30(10+0.25 s)}{40+0.25 s+0.03 s(10+0.25 s)}=\frac{300+7.5 s}{40+0.55 s+0.0075 s^{2}} \\
& G(s)=\frac{7.5(s+40)}{0.0075\left(s^{2}+\frac{0.55}{0.0075} s+\frac{40}{0.0075}\right)} \\
& G(s)=1000 \frac{s+40}{s^{2}+73.33 s+5333}
\end{aligned}
$$

8. For a series RLC circuit of 10 ohms, 0.5 henrys, and $100 \mu \mathrm{f}$, determine the resonant frequency, the quality factor, and the bandwidth.


Z(s)

0
$Z(s)=s L+R+\frac{1}{s C}$
$Z(s)=\frac{1}{2} s+10+\frac{1}{s\left(10^{2} \times 10^{-6}\right)}$
$Z(s)=\frac{1}{2}\left[s+20+\frac{2 \times 10^{4}}{s}\right]=\frac{1}{2}\left[\frac{s^{2}+20 s+2 \times 10^{4}}{s}\right]$
$Z(s)=\frac{1}{2 s}\left[s^{2}+20 s+2 \times 10^{4}\right]$
The standard form is $s^{2}+\frac{\omega_{0}}{Q} s+\omega_{0}^{2}$
$\omega_{0}^{2}=2 \times 10^{4}$
$\omega_{0}=100 \sqrt{2}=141.2 \mathrm{rad} / \mathrm{sec}$
$\frac{\omega_{0}}{Q}=\frac{100 \sqrt{2}}{Q}=20$
$Q=\frac{100 \sqrt{2}}{20}=5 \sqrt{2} \cong 7.07$
$Q=\frac{\omega_{0}}{\Delta \omega}$
$\Delta \omega=\frac{\omega_{0}}{Q}=\frac{100 \sqrt{2}}{5 \sqrt{2}}=20$
9. For a parallel RLC circuit of 10 ohms, 0.5 henrys, and $100 \mu \mathrm{f}$, determine the resonant frequency, the quality factor, and the bandwidth.


Use the same approach as for a series RLC circuit but use $\mathrm{Y}(\mathrm{s})$ instead.
$Y(s)=\frac{1}{Z(s)}=\frac{1}{s L}+\frac{1}{R}+s C=\frac{1}{s\left(\frac{1}{2}\right)}+\frac{1}{10}+s\left(10^{2} \times 10^{-6}\right)$
$Y(s)=\frac{2}{s}+\frac{1}{10}+10^{-4} s=\frac{20+s+10-10^{-3} s^{2}}{10 s}$
$Y(s)=\frac{s^{2}+1000 s+20,000}{10^{4} s}$
As before the model is $s^{2}+\frac{\omega_{0}}{Q} s+\omega_{0}^{2}$
$\omega_{0}^{2}=20,000$
$\omega_{0}=141.42 \mathrm{rad} / \mathrm{sec}$
$\frac{\omega_{0}}{Q}=1000$
$Q=\frac{\omega_{0}}{1000}=0.141$
$Q=\frac{\omega_{0}}{\Delta \omega}$
$\Delta \omega=\frac{\omega_{0}}{Q}=\frac{141.42}{0.141}=1000$
10. The switch of the circuit shown has been open for a long time and is closed at $t=0$.


Determine the currents flowing in the two capacitances at the instant the switch closes. Give the magnitude of the capacitances and directions.

The initial circuit when the switch is open for a long time is very simple.


The initial conditions are $V_{C 1}\left(0^{-}\right)=0$ and $V_{C 2}\left(0^{-}\right)=+10 \mathrm{~V}$
When switching capacitors look like ideal voltages sources which retain their initial voltages, i.e., the circuit at $\mathrm{t}=0+$ looks like

which can be reduced to


By inspection (and being careful to follow the passive sign convention)
$i_{1}=\frac{10 \mathrm{~V}}{50 \Omega}=\frac{1}{5} \mathrm{Amp}$
$i_{2}=\frac{10 \mathrm{~V}}{100 \Omega}=\frac{1}{10} \mathrm{Amp}$
Then,
$i_{C 1}=\frac{1}{5}+\frac{1}{10}=0.3 \mathrm{Amp}$
$i_{C 2}=-i_{2}=-\frac{1}{10} A m p$

## CONCENTRATES

1. A system is described by the differential equation
$20 \sin 4 t=\frac{d^{2} i}{d t^{2}}+4 \frac{d i}{d t}+4 i$
The initial conditions are $i(0)=0$ and $\frac{d i(0)}{d t}=4$. Determine $i(t)$ for $\mathrm{t}>0$.
To find the steady state solution convert to phasors
$20 \sin 4 t=20 \cos \left(4 t-90^{\circ}\right) \rightarrow-j 20 e^{j \omega t}$
Then (1) becomes
$-j 20=-\omega^{2} i+4 j \omega+4 i$
Since $\omega=4$
$i=\frac{-j 20}{-\omega^{2}+4 j \omega+4}=\frac{-j 20}{-16+j 16+4}=\frac{-j 20}{-12+j 16}=1 \angle 141.3^{\circ}$
$i_{\text {steady-state }}=\cos \left(4 t+141.3^{\circ}\right)$
To find the transient solution set the forcing function to zero.
$\frac{d^{2} i}{d t^{2}}+4 \frac{d i}{d t}+4 i=0$
$s^{2}+4 s+4=0$
$s=\frac{-4 \pm \sqrt{16-4(4)}}{2}=-2$
Since this is a second order equation we MUST have two linearly independent solutions. we get the second solution by multiplying by t
$A e^{-2 t}+B t e^{-2 t}$
$i(t)=\cos \left(4 t+143.1^{\circ}\right)+A e^{-2 t}+B t e^{-2 t}$
$i(0)=0=\cos \left(143.1^{\circ}\right)+A$
$A=-\cos \left(143.1^{\circ}\right)=0.8$
$\frac{d i}{d t}=-2 A e^{-2 t}+B e^{-2 t}-2 B t e^{-2 t}-4 \sin \left(4 t+143.1^{\circ}\right)$
$\left.\frac{d i}{d t}\right|_{t=0}=4=-2(0.8)+B-4 \sin \left(143.1^{\circ}\right)$
$B=4 \sin \left(143.1^{\circ}\right)+1.6+4=8$
The final solution is then
$i(t)=\cos \left(4 t+143.1^{\circ}\right)+0.8 e^{-2 t}+8 t e^{-2 t}$
2. For the circuit shown, find (a) the system transfer function, (b) the response to a unit step at the input, and (c) the response to a unit pulse with a duration of millisecond.
$1000 \Omega$


Compute the parallel impedance of the $10 \mu \mathrm{f}$ capacitor and the $100 \Omega$ resistance and then use a voltage divider relationship.

Compute the parallel impedance
$Z_{\text {parallel }}=\frac{\frac{1}{s\left(10 \times 10^{-6}\right)} 100}{\frac{1}{s\left(10 \times 10^{-6}\right)}+100}=\frac{\frac{100}{10^{-5} s}}{\frac{1+10^{-3} s}{10^{-5} s}}=\frac{100}{1+10^{-3} s}$
(a) system transfer function
$\frac{V_{\text {out }}}{V_{\text {in }}}=\frac{\frac{100}{1+10^{-3} s}}{1000+\frac{100}{1+10^{-3} s}}=\frac{\frac{100}{1+10^{-3} s}}{\frac{1000+s+100}{1+10^{-3} s}}=\frac{100}{s+1100}$
(b) response to a unit step $\frac{1}{S}$
$V_{\text {out }}=\frac{100}{s+1100} \frac{1}{s}=\frac{A}{s}+\frac{B}{s+1100}$
$A s+A 1100+B s=100$
Equating the real parts $A 1100=100$ or $A=\frac{1}{11}$
Equating the imaginary parts $A s+B s=0$ or $B=-A=-\frac{1}{11}$
$V_{\text {out }}=\frac{\frac{1}{11}}{s}-\left(\frac{\frac{1}{11}}{s+1100}\right)=\frac{1}{11}\left(\frac{1}{s}\right)-\frac{1}{11}\left(\frac{1}{s+1100}\right)$
Inverse Laplace transforming
$V_{\text {out }}(t)=\frac{1}{11} u(t)-\frac{1}{11} e^{-1100 t} u(t)$
(c) response to a unit pulse with duration of 1 msec .
$V_{i n}(t)=u(t)-u\left(t-10^{-3}\right)$
Laplace transforming this gives $V_{\text {in }}(s)=\frac{1}{s}-\frac{e^{-0.001 s}}{s}$
The output is then given by

$$
\begin{aligned}
& V_{\text {out }}(s)=\frac{1}{11}\left(\frac{1}{s}-\frac{1}{s+1100}\right)-\frac{1}{11} e^{-0.001 s}\left(\frac{1}{s}-\frac{1}{s+1100}\right) \\
& \left.V_{\text {out }}(t)=\frac{1}{11}\left(1-e^{-1100 t}\right) u(t)-\frac{1}{11} e^{-1100\left(t-10^{-3}\right.}\right) u\left(t-10^{-3}\right)
\end{aligned}
$$

3. The circuit shown has had the switch closed for a long time. It is opened at $\mathrm{t}=0$. For the instant just after the switch opens $(\mathrm{t}=0+)$, determine the current $i_{1}$, the current $i_{2}$, the voltage $v$, and the time rate of change of the current $i_{3}$.


The values at $\mathrm{t}=0$ - are the steady-state "closed" values.


Find the current through the resistors, and the voltage across the capacitors.
$\frac{(30)(60)}{30+60}=\frac{1800}{90}=20 \Omega$
$i_{1}\left(0^{-}\right)=\frac{100 \mathrm{~V}}{30+20}=2 \mathrm{Amps}$
$i_{3}\left(0^{-}\right)=\frac{100 \mathrm{~V}}{5 \Omega}=20 \mathrm{Amps}$
$i_{2}\left(0^{-}\right)=0$ since there is no current through a charged capacitor
$V\left(0^{-}\right)=0$ since there is no voltage across a closed switch
$V_{5 u f}\left(0^{-}\right)=100 \frac{20}{20+30}=40$ Volts which is the same as the voltage across the parallel combination of the $60 \Omega$ and $30 \Omega$ resistors.
$i_{4}\left(0^{-}\right)=\frac{40 \mathrm{~V}}{30 \Omega}=\frac{4}{3} \mathrm{Amps}$

Now redraw the circuit as it is at $\mathrm{t}=0+$


Do KCL at A
$-i_{1}-20+i=0$
$i=i_{1}+20$
Do KVL through both capacitors going clockwise as shown above.
Use $i$ to be the current through the $20 \Omega$ resistor.
$+40-100+20\left(i_{1}+20\right)+i_{1}(30)=0$
$-60+20 i_{1}+400+30 i_{1}=0$
$50 i_{1}=-340$
$i_{1}=-6.8 \mathrm{Amps}$
To find $i_{2}$ use KCL at node B
$i_{2}=-6.8-\frac{4}{3}-\frac{2}{3}=-8.8 \mathrm{Amps}$
The voltage V is given by
$V=\left(40+i_{1} 30\right)-100=40+(-6.8) 30-100=40-204+100=-264$ Volts
To find the rate of change of $i_{3}$ use the inductance terminal relationship
$V_{L}=L \frac{d i_{3}}{d t}$
The voltage across the inductor is the same as the voltage across the $20 \Omega$ resistor
$V_{20 \Omega}=20(-6.8+20)=+264$ Volts
(Note that $i_{3}$ and $i$ are in opposite directions.
$V_{L}=-264=2 \frac{d i_{3}}{d t}$
Then,
$\frac{d i_{3}}{d t}=-132 \mathrm{Amps} / \mathrm{sec}$

1. The circuit shown is a phase-lead compensator. Together with the amplifier, it is to previde a voltage ratio (gain) of 1 and a phase lead of $\phi_{d}$ at a frequency of $\omega_{d}$. The amplifier has a voltage gain of $K$. Determine the values of $R_{1}, R_{2}$ and $C$ in terms of $K$, $\phi_{d}$ and $\omega_{d}$. Discuss the limitations on the angle and the desirable ranges of values.


This is a REALLY poorly worded problem.
The frequency response is determined entirely by the R-C network.

$$
\frac{V_{2}}{V_{\text {in }}}=\frac{R_{2}}{R_{2}+R_{1} \| \frac{1}{s C}}=\frac{R_{2}}{R_{2}+\frac{\frac{R_{1}}{s C}}{R_{1}+\frac{1}{s C}}}=\frac{R_{2}\left(R_{1}+\frac{1}{s C}\right)}{R_{1} R_{2}+\frac{R_{2}}{s C}+\frac{R_{1}}{s C}}=\frac{R_{1} R_{2} s C+R_{2}}{R_{1} R_{2} s C+R_{1}+R_{2}}
$$

By inspection there is a zero at $-\frac{1}{R_{1} C}$ and a pole at $-\frac{R_{1}+R_{2}}{R_{1} R_{2} C}$
Assuming the pole and zero are separated we have a quick sketch similar to


The transfer function is then given by

$$
\frac{V_{\text {out }}}{V_{\text {in }}}=K \frac{V_{2}}{V_{\text {in }}}=K \frac{R_{2}\left(1+s R_{1} C\right)}{\left(R_{1}+R_{2}\right)\left(1+s \frac{R_{1} R_{2} C}{R_{1}+R_{2}}\right)}
$$

Define $Z=\frac{1}{R_{1} C}$ and $P=\frac{R_{1}+R_{2}}{R_{1} R_{2} C}$. Then the transfer function can be rewritten as $\frac{V_{\text {out }}}{V_{\text {in }}}=K \frac{R_{2}}{R_{1}+R_{2}} \frac{1+\frac{s}{Z}}{1+\frac{s}{P}}=K \frac{R_{2}}{R_{1}+R_{2}} \frac{P}{Z} \frac{s+Z}{s+P}$
The term $\frac{s+Z}{s+P}$ determines the phase which is given as $\phi_{d}=\tan ^{-1}\left(\frac{\omega}{Z}\right)-\tan ^{-1}\left(\frac{\omega}{P}\right)$
4. Specify R2 and capacitances C1 and C2 to cause the following circuit to be a Butterworth low-pass filter with a -3 dB (corner) frequency of 150 Hz . The amplifier is to be considered an ideal unity gain amplifier (infinite input resistance and zero output resistance).

$\frac{V_{1}}{V_{\text {in }}}=\frac{\frac{1}{s C_{1}}}{\frac{1}{s C_{1}}+10^{4}}=\frac{1}{1+s 10^{4} C_{1}}$
$\frac{V_{o}}{V_{1}}=\frac{\frac{1}{s C_{2}}}{\frac{1}{s C_{2}}+\left(R_{2}+5\right)+s 10^{-2}}=\frac{1}{s^{2} 10^{-2} C_{2}+\left(R_{2}+5\right) s C_{2}+1}$
$\frac{V_{o}}{V_{\text {in }}}=\frac{V_{1}}{V_{\text {in }}} \frac{V_{o}}{V_{1}}=\frac{1}{1+s 10^{4} C_{1}} \times \frac{1}{s^{2} 10^{-2} C_{2}+\left(R_{2}+5\right) s C_{2}+1}$
The standard form for a Butterworth filter from Table 4-2 (3rd order) is
$\frac{V_{o}}{V_{i n}}=\frac{1}{1+\frac{s}{\omega_{o}}} \times \frac{1}{\left(\frac{s}{\omega_{o}}\right)^{2}+\left(\frac{s}{\omega_{o}}\right)+1}$
We want $\omega_{o}=2 \pi f_{o}=2 \pi(150)=300 \pi$
Identifying terms
$10^{4} C_{1}=300 \pi$, or $C_{1}=\frac{10^{-4}}{300 \pi}=106 n f$
$10^{-2} C_{2}=\frac{1}{(300 \pi)^{2}}$, or $C_{2}=\frac{10^{2}}{(300 \pi)^{2}}=113 \mu f$
$\left(R_{2}+5\right) C_{2}=\frac{1}{300 \pi}$, or $R_{2}=\frac{1}{300 \pi} \frac{(300 \pi)^{2}}{10^{2}}-5=4.4 \Omega$
5. Given the circuit shown where the amplifiers are ideal unity-gain voltage amplifiers (infinite input resistance and zero output resistance). Amplifier A3 is a summing amplifier, so its output is the sum of its two inputs. Determine the transfer function of the resulting filter, Identify it and its principal parameters.

$\frac{V_{1}}{V_{\text {in }}}=\frac{\frac{1}{s 10^{-6}}}{\frac{1}{s 10^{-6}}+50+s 3 \times 10^{-3}}=\frac{1}{1+50 \times 10^{-6} s+3 \times 10^{-3} \times 10^{-6} s^{2}}$
$\frac{V_{1}}{V_{\text {in }}}=\frac{1}{1+5 \times 10^{-5} s+3 \times 10^{-9} s^{2}}$
$\frac{V_{2}}{V_{i n}}=\frac{s 3 \times 10^{-3}}{\frac{1}{s 10^{-6}}+50+s 3 \times 10^{-3}}=\frac{3 \times 10^{-9} s^{2}}{1+5 \times 10^{-5} s+3 \times 10^{-9} s^{2}}$
$\frac{V_{o}}{V_{\text {in }}}=\frac{V_{1}}{V_{\text {in }}}+\frac{V_{2}}{V_{\text {in }}}=\frac{1+3 \times 10^{-9} s^{2}}{1+5 \times 10^{-5} s+3 \times 10^{-9} s^{2}}$
Next, we need to put it into a standard form
It is that of a band-reject filter $\frac{1+\frac{s^{2}}{\omega_{o}^{2}}}{1+\frac{s}{Q \omega_{o}}+\frac{s^{2}}{\omega_{o}^{2}}}$
Identify the equivalences,
$\frac{1}{\omega_{o}^{2}}=3 \times 10^{-9}$ gives $\omega_{o}^{2}=3.33 \times 10^{8}$, or $\omega_{o}=18257 \mathrm{rad} / \mathrm{sec} \quad($ about 2905 Hz$)$
$\frac{1}{Q \omega_{o}}=5 \times 10^{-5}$ gives $Q=\frac{1}{5 \times 10^{-5} \omega_{o}}=\frac{1}{5 \times 10^{-5}(18257)}=1.09$
6. A Butterworth filter has its transfer function given below. Determine its corner frequency, and its response to a 1 volt step input.

$$
G(s)=\frac{100,000}{s^{4}+26.13 s^{3}+341.4 s^{2}+2613 s+10,000}
$$

From Table 4-2, for $\mathrm{n}=4$
$\theta_{1}=22.5^{\circ},\left(\frac{s}{\omega_{o}}\right)^{2}+0.765 \frac{s}{\omega_{o}}+1$ and $\theta_{2}=67.5^{\circ},\left(\frac{s}{\omega_{o}}\right)^{2}+1.848 \frac{s}{\omega_{o}}+1$
We need to identify $\omega_{o}$ so divide by 10,000 to get 1 in denominator.

$$
G(s)=\frac{100,000 / 10,000}{\left(\frac{s}{10}\right)^{4}+\frac{26.13}{10}\left(\frac{s}{10}\right)^{3}+\frac{341.4}{100}\left(\frac{s}{10}\right)^{2}+\frac{2613}{1000}\left(\frac{s}{10}\right)+1}
$$

By inspection $\omega_{o}=10$
From Table 4-2, the poles are at $\pm 22.5^{\circ}$ and $\pm 67.5^{\circ}$
Putting this into standard form for a transfer function we have

$$
G(s)=\frac{10}{\left[\left(\frac{s}{10}\right)^{2}+0.765\left(\frac{s}{10}\right)+1\right]\left[\left(\frac{s}{10}\right)^{2}+1.848\left(\frac{s}{10}\right)+1\right]}
$$

The step response for this function is truely NASTY to compute. Do by partial fraction expansion.

$$
R(s)=G(s) \frac{1}{s}=\frac{A^{\prime}}{s}+\frac{B^{\prime} s+C}{\left(\frac{s}{10}\right)^{2}+0.765\left(\frac{s}{10}\right)+1}+\frac{D^{\prime} s+E}{\left(\frac{s}{10}\right)^{2}+1.848\left(\frac{s}{10}\right)+1}
$$

Put into more standard form for Laplace analysis.

$$
R(s)=\frac{A}{s}+\frac{B s+C}{s^{2}+7.65 s+100}+\frac{D s+E}{s^{2}+18.48 s+100}
$$

This transforms to an expression of the form
$r(t)=A u(t)+B^{\prime} e^{-a t} \cos b t+C^{\prime} e^{-a t} \sin b t+D^{\prime} e^{-c t} \cos d t+E^{\prime} e^{-c t} \sin d t$ where I did not bother to compute the coefficents.

1. Consider the OP-AMP circuit shown below.
(a) Determine the system transfer function.
(b) Sketch the gain of the circuit as a function of $\omega$. Assume that $\mathrm{R}_{1}=1000 \Omega$, $\mathrm{R}_{2}=200 \Omega, \mathrm{R}_{2}=1000 \Omega$ and $\mathrm{C}=1 \mu \mathrm{~F}$.
(c) What type of filter is this: low-pass, high-pass, etc.?


ANSWER:
(a) Applying KCl at the inverting input of the op-amp. Assuming that current into the node is positive we can write
$\frac{V_{s}-V_{o}}{R_{1}}+\frac{V_{s}-V_{o}}{\frac{1}{s C}}-\frac{V_{o}}{R_{2}}=0$
Solving for the transfer function gives
$\left(\frac{1}{R_{1}}+s C\right) V_{s}=\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}+s C\right) V_{o}$
$\frac{V_{o}}{V_{s}}=\frac{\frac{1}{R_{1}}+s C}{\frac{1}{R_{1}}+\frac{1}{R_{2}}+s C}=\frac{s+\frac{1}{R_{1} C}}{s+\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right) \frac{1}{C}}=\frac{s+\frac{1}{R_{1} C}}{s+\frac{1}{\left(R_{1} \| R_{2}\right) C}}$

This transfer function has a single pole and a single zero.
(b) For the given values, the transfer function evaluates to

$$
\frac{V_{0}}{V_{s}}=\frac{s+\frac{1}{(1000)\left(1 \times 10^{-6}\right)}}{s+\frac{1}{\left(\frac{(1000)(200)}{1000+200}\right)\left(1 \times 10^{-6}\right)}}=\frac{s+1000}{s+1250}=0.8 \frac{1+\frac{s}{1000}}{1+\frac{s}{1250}}
$$

$$
20 \log |H(j \omega)|=20 \log (0.8)+20 \log \left|1+\frac{s}{1000}\right|-20 \log \left|1+\frac{s}{1250}\right|
$$

The first term is approximately -1.9 dB .
(c) This is a high-pass filter.

2. Plot the gain and phase response of the system given by the transfer function:
$H(s)=\frac{(s+6500)(s+8500)}{(s+350)(s+100,000)}$

## SOLUTIONS:

Putting the expression into standard form
$H(s)=\frac{(6500)(8500)}{(350)(100,000)} \frac{\left(1+\frac{s}{6500}\right)\left(1+\frac{s}{8500}\right)}{\left(1+\frac{s}{350}\right)\left(1+\frac{s}{100,000}\right)}$
The gain is given by

$$
\begin{aligned}
20 \log |H(j \omega)|= & 20 \log |1.57|+20 \log \left|1+j \frac{\omega}{6500}\right|+20 \log \left|1+j \frac{\omega}{8500}\right| \\
& -20 \log \left|1+j \frac{\omega}{350}\right|-20 \log \left|1+j \frac{\omega}{100,000}\right|
\end{aligned}
$$

The first term is +4 db . The other terms are evaluated graphically..
The phase angle is given by
$\angle H(j \omega)=\angle 1.57+\operatorname{Tan}^{-1}\left(\frac{\omega}{6500}\right)+\operatorname{Tan}^{-1}\left(\frac{\omega}{8500}\right)-\operatorname{Tan}^{-1}\left(\frac{\omega}{350}\right)-\operatorname{Tan}^{-1}\left(\frac{\omega}{100,000}\right)$
The first term evaluates to zero degrees. The other terms are evaluated graphically.

GAIN:


PHASE:

$\omega$

## 1. Plot $|\mathrm{H}(\mathrm{j} \omega)|$ and $\angle \mathrm{H}(\mathrm{j} \omega)$, given that

$$
H(s)=\frac{10 s}{(1+s)\left(1+\frac{s}{10}\right)}
$$

## SOLUTIONS:




-     - s
$-\quad-\frac{1}{1+s}$
$-\quad-\frac{1}{1+\frac{8}{10}}$



$$
\begin{array}{lcl}
\text { — - } & 10 & \angle 10=0^{\circ} \\
\text { — - } & \mathrm{s} & \angle(\mathrm{j} \omega)=90^{\circ} \\
- & \angle\left(\frac{1}{1+\mathrm{j} \omega}\right) \rightarrow 45^{\circ} \text { breakpoint at } \omega=1 \\
- & \frac{1}{1+\mathrm{s}} & \angle\left(\frac{1}{1+\frac{\mathrm{j} \omega}{10}}\right) \rightarrow 45^{\circ} \text { breakpoint at } \omega=10
\end{array}
$$

## An Abbreviated List of Laplace Transform Pairs

| $\mathbf{f}(\mathbf{t})(\mathbf{t}>\mathbf{0}-)$ | TYPE | F(s) |
| :--- | :--- | :--- |
| $\mathrm{d}(\mathrm{t})$ | (impulse) | 1 |
| $\mathrm{u}(\mathrm{t})$ | (step) | $\frac{1}{\mathrm{~s}}$ |
| t | (ramp) | $\frac{1}{\mathrm{~s}^{2}}$ |
| $\mathrm{e}^{-\mathrm{at}}$ | (exponential) | $\frac{1}{\mathrm{~s}+\mathrm{a}}$ |
| $\sin (\omega \mathrm{t})$ | (sine) | $\frac{\omega}{\mathrm{s}^{2}+\omega^{2}}$ |
| $\cos (\omega \mathrm{t})$ | (cosine) | $\frac{\mathrm{s}}{\mathrm{s}^{2}+\omega^{2}}$ |
| te |  |  |
| $\mathrm{e}^{-a t}$ | (damped ramp) | $\frac{1}{(\mathrm{~s}+\mathrm{a})^{2}}$ |
| $\mathrm{e}^{-a \mathrm{t}} \cos (\omega \mathrm{t}(\omega \mathrm{t})$ | (damped sine) | $\frac{\omega}{(\mathrm{s}+\mathrm{a})^{2}+\omega^{2}}$ |

## An Abbreviated List of Operational Transforms

| f(t) | F (s) |
| :---: | :---: |
| Kf(t) | KF(s) |
| $\mathrm{f}_{1}(\mathrm{t})+\mathrm{f}_{2}(\mathrm{t})-\mathrm{f}_{3}(\mathrm{t})+\ldots$ | $\mathrm{F}_{1}(\mathrm{~s})+\mathrm{F}_{2}(\mathrm{~s})-\mathrm{F}_{3}(\mathrm{~s})+\ldots$ |
| $\frac{\mathrm{df}}{}(\mathrm{t})$ | $\mathrm{sF}(\mathrm{s})-\mathrm{f}\left(0^{-}\right)$ |
| dt |  |
| $\underline{\mathrm{d}^{2} \mathrm{f}(\mathrm{t})}$ | $s^{2} \mathrm{~F}(\mathrm{~s})-\mathrm{sf}\left(0^{-}\right)-\frac{\mathrm{df}\left(0^{-}\right)}{\mathrm{dt}}$ |
| $\overline{\mathrm{dt}^{2}}$ | $\mathrm{s}^{2} \mathrm{~F}(\mathrm{~s})-\mathrm{sf}(0)-\frac{\mathrm{dt}}{}$ |
| $\underline{d^{n} f(t)}$ | $s^{n} F(s)-s^{n-1} f\left(0^{-}\right)-s^{n-2}{ }^{\text {df }}\left(0^{-}\right) \quad s^{n-3} d^{2} f\left(0^{-}\right) \quad d^{n-1} f\left(0^{-}\right)$ |
| $\mathrm{dt}^{\mathrm{n}}$ | $\mathrm{s}^{\mathrm{n}} \mathrm{F}(\mathrm{s})-\mathrm{s}^{\mathrm{n}-1} \mathrm{f}\left(0^{-}\right)-\mathrm{S}^{\mathrm{n}-2} \frac{\mathrm{dt}^{\text {d }}}{\mathrm{dt}}-\mathrm{S}^{\mathrm{n}-3} \frac{\mathrm{dt}^{2}}{}-\ldots-\frac{\mathrm{dt}^{\mathrm{n}-1}}{}$ |
| $\int_{0}^{t} f(x) d x$ | $\underline{F}(\mathrm{~s})$ |
|  |  |
| $\mathrm{f}(\mathrm{t}-\mathrm{a}) \mathrm{u}(\mathrm{t}-\mathrm{a}), \mathrm{a}>0$ | $\mathrm{e}^{-\mathrm{as}} \mathrm{F}(\mathrm{s})$ |
| $\left.\mathrm{e}^{-\operatorname{at}(\mathrm{f}} \mathrm{t}\right)$ | $F(s+a)$ |
| $\mathrm{f}(\mathrm{at}), \mathrm{a}>0$ | $\frac{1}{\mathrm{a}} \mathrm{f}\left(\frac{\mathrm{S}}{\mathrm{a}}\right)$ |
| tf( t ) | dF(s) |
|  | $-\frac{\mathrm{ds}}{}$ |
| $\mathrm{t}^{\mathrm{n}} \mathrm{f}(\mathrm{t})$ | $(-1)^{\mathrm{n}} \frac{\mathrm{~d}^{\mathrm{n}} \mathrm{~F}(\mathrm{~s})}{\mathrm{dan}}$ |
|  |  |
| $\frac{\mathrm{f}(\mathrm{t})}{\mathrm{t}}$ | $\int F(u) d u$ |

