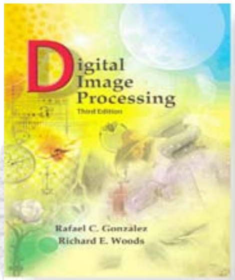
Lecture #8

- Fourier Transform
  - Fourier series
  - 2-D basis functions
  - 2-D Fourier Transform
- Sampling and band-limited functions
  - Aliasing



The Fourier Series

is the decomposition of a λ-periodic signal into a sum of sinusoids.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{\lambda} t\right) + B_n \sin\left(\frac{2\pi n}{\lambda} t\right)$$

periodic: $\exists \lambda \in \Re$ such that $f(t \pm n\lambda) = f(t)$.

$$A_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t)\left[\cos\left(\frac{2\pi n}{\lambda} t - \phi_n\right)\right] dt \text{ for } n \geq 0$$

$$B_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t)\left[\sin\left(\frac{2\pi n}{\lambda} t - \phi_n\right)\right] dt \text{ for } n \geq 0$$

The representation of a function by its Fourier Series is the sum of sinusoidal "basis functions" multiplied by coefficients.

Fourier coefficients are generated by taking the inner product of the function with the basis.

The Fourier Series

can also be written in terms of complex exponentials

$$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{+j\frac{2\pi n}{\lambda}t} = \sum_{n=-\infty}^{\infty} |C_n| e^{+j\left(\frac{2\pi n}{\lambda}t+\phi_n\right)}$$

$$=\sum_{n=-\infty}^{\infty} |C_n| \cos\left(\frac{2\pi n}{\lambda}t+\phi_n\right) + j\cdot|C_n|\sin\left(\frac{2\pi n}{\lambda}t+\phi_n\right)$$

$$j=\sqrt{-1}$$

$$C_n = |C_n| e^{+j\phi_n}$$

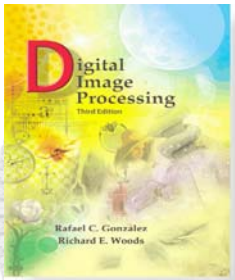
$$e^{\pm jx} = \cos x \pm j \sin x$$

$$Cn = |Cn| e^{+j\phi_n} = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(t)\, e^{-j\frac{2\pi n}{\lambda}t}\, dt$$

$$=\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(t)\left[\cos\left(\frac{2\pi n}{\lambda}t-\phi_n\right) - j\cdot\sin\left(\frac{2\pi n}{\lambda}t-\phi_n\right)\right]dt$$

$$f(t+n\lambda)=f(t)$$

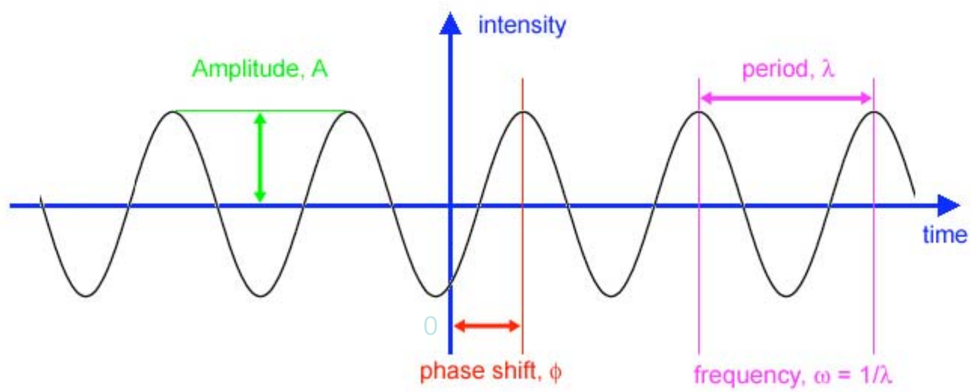
for all intergers n

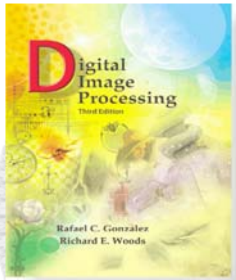


Why are Fourier Coefficients Complex Numbers?

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{+j\frac{2\pi n}{\lambda}t} \quad \text{where} \quad C_n = |C_n| \, e^{+j\phi_n}.$$

C_n represents the amplitude, $A=|C_n|$, and relative phase, ϕ, of that part of the original signal, $f(t)$, that is a sinusoid of frequency $\omega_n = n / \lambda$.



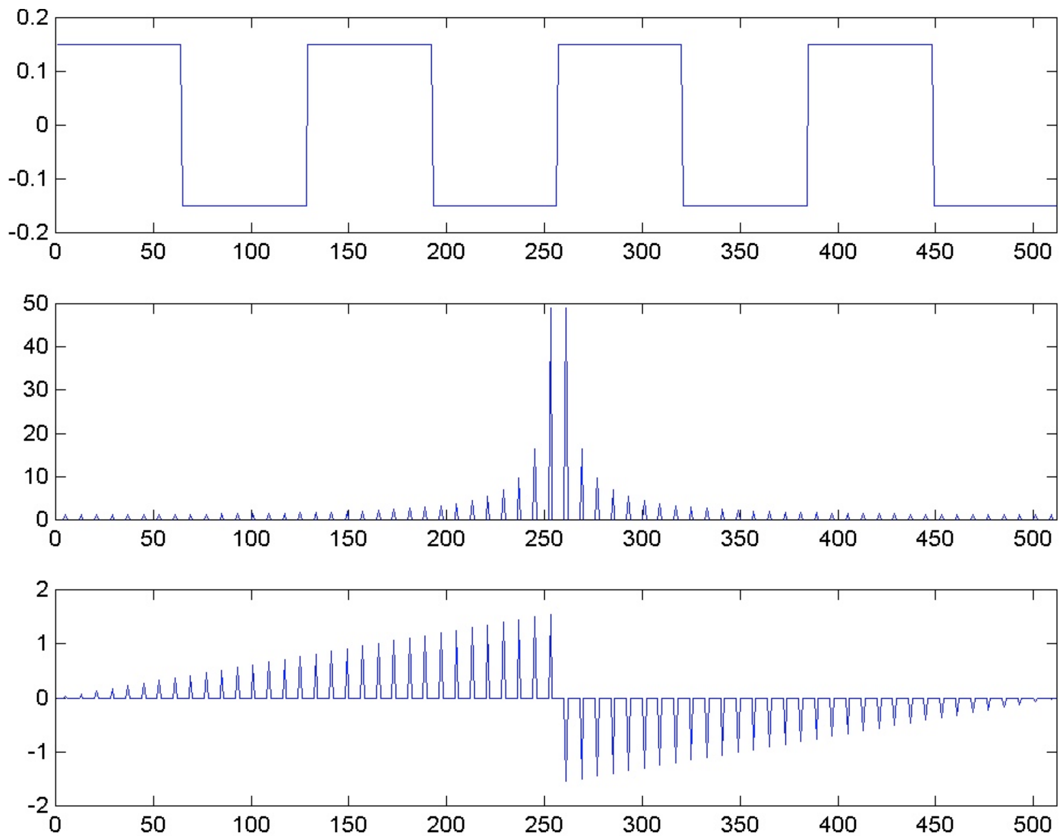


Fourier Series of a Square Wave

Periodic Time-domain signal

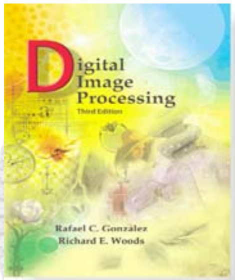
Fourier magnitude

Fourier phase



$$f(x) = \sum_{k=-\infty}^{+\infty} H_k e^{j\frac{2\pi}{T_o}kx}$$

$$H_k = \begin{cases} 1/2 & k = 0 \\ 0 & k \ \text{even} \\ 1/j\pi k & k \ \text{odd} \end{cases}$$



The Fourier Transform

is the decomposition of a *nonperiodic* signal into a continuous sum* of sinusoids.

$$F(\omega) = |F(\omega)|\, e^{j\Phi(\omega)} = \int_{-\infty}^{\infty} f(t)\, e^{j2\pi\omega t}\, dt$$

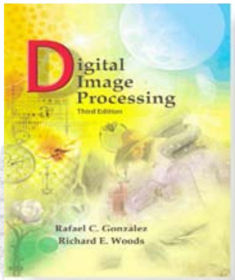
$$= \int_{-\infty}^{\infty} f(t)\left[\cos(2\pi\omega t) + j\sin(2\pi\omega t)\right] dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j2\pi\omega t}\, d\omega = \int_{-\infty}^{\infty} |F(\omega)|\, e^{-j(2\pi\omega t + \Phi(\omega))}\, d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega)\left[\cos(2\pi\omega t) - j\sin(2\pi\omega t)\right] d\omega$$

$$= \int_{-\infty}^{\infty} |F(\omega)|\left[\cos\left(2\pi\omega t + \Phi(\omega)\right) - j\sin\left(2\pi\omega t + \Phi(\omega)\right)\right] d\omega$$

* *i.e.*, an integral.



The Discrete Fourier Transform

A discrete signal, $\{ h_k \mid k = 0, 1, 2, \ldots, N-1 \}$, of finite length N can be represented as a weighted sum of N sinusoids, $\{ e^{-j2\pi kn/N} \mid n = 0, 1, 2, \ldots, N-1 \}$ through

$$h_k = \sum_{n=0}^{N-1} H_n e^{-j2\pi kn/N}$$

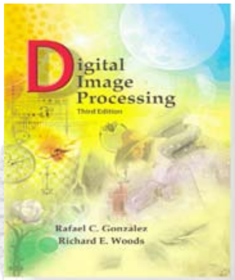
where the set, $\{ H_n \mid n = 0, 1, 2, \ldots, N-1 \}$ are the Fourier coefficients defined as the projection of the original signal onto sinusoid, n, given by:

$$H_n = \frac{1}{N} \sum_{k=0}^{N-1} h_k e^{+j2\pi kn/N}$$

The Two-Dimensional Fourier Transform

Primary Uses of the FT in Image Processing:

- For feature detection and enhancement, especially edge detection.
- Useful for certain types of noise reduction, deblurring, and other types of image restoration.
- Explains why sampling can add distortion to an image and shows how to avoid it.



The Fourier Transform: Discussion

The expressions

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\omega t}\, dt = \left\langle f(t),\ e^{+j2\pi\omega t} \right\rangle$$

continuous signals defined over all real numbers

and

$$H_n = \frac{1}{N} \sum_{n=0}^{N-1} h_k e^{-j2\pi k n/N} = \left\langle h_k,\ e^{+j2\pi k n/N} \right\rangle$$

discrete signals with N terms or samples.

for the Fourier coefficients are “inner products” which can be thought of as measures of the similarity between the functions $f(t)$ and $e^{+j2\pi\omega t}$ for $t \in (-\infty, \infty)$ or between the sequences $\{h_k\}_{k=0}^{N-1}$ and $\{e^{+j2\pi k n/N}\}_{k=0}^{N-1}$.

The Fourier Transform: Discussion (cont'd.)

In the context of inner products, the complex exponentials

$$\left\{ e^{-j2\pi\omega t} \,\middle|\, \omega \in \Re \text{ and } \omega \in (-\infty,\infty) \right\} \text{ and } \left\{ e^{-j2\pi kn/N} \,\middle|\, \ldots,-2,-1,0,1,2,\ldots \right\}$$

are called "orthogonal sets" since they have the property:

$$\left\langle e^{-j2\pi\omega_1 t},\; e^{-j2\pi\omega_2 t} \right\rangle = \int_{-\infty}^{\infty} e^{-j2\pi\omega_1 t} \cdot e^{+j2\pi\omega_2 t}\, dt = \begin{cases} \infty, & \text{if } \omega_1 = \omega_2 \\ 0, & \text{if } \omega_1 \neq \omega_2 \end{cases}$$

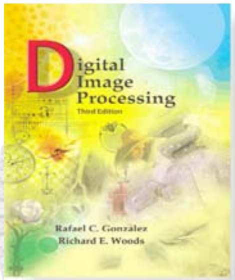
$$\left\langle e^{-j2\pi jn/N},\; e^{-j2\pi kn/N} \right\rangle = \sum_{n=0}^{N-1} e^{-j2\pi jn/N} \cdot e^{+j2\pi kn/N} = \begin{cases} c, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases},$$

The function sets are called "orthogonal basis sets"

They are called "basis sets" since for any function[1], $f(t)$, of a real variable there exists a complex-valued function F(w), and for any sequence[1], h_k, there exist complex numbers, H_n, such that

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j2\pi\omega t}\, d\omega \quad \text{and} \quad h_k = \sum_{n=0}^{N-1} H_n e^{-j2\pi kn/N}.$$

[1] with finite energy.



The 2D Fourier Transform of a Digital Image

Let $I(r,c)$ be a single-band (intensity) digital image with R rows and C columns. Then, $I(r,c)$ has Fourier representation

$$I(r,c) = \sum_{u=0}^{R-1} \sum_{v=0}^{C-1} \mathcal{I}(v,u)\, e^{+j2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)},$$

these complex exponentials are 2-D sinusoids.

where

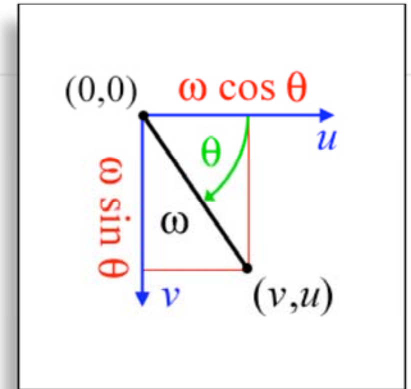
$$\mathcal{I}(v,u) = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} I(r,c)\, e^{-j2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

are the R x C Fourier coefficients.

Note that R and C take on the role of the period such as you would find in a 1-D time series

What are 2D sinusoids?

Since we are dealing with space (not time) ω is a spatial frequency



To simplify the situation assume $R = C = N$. Then

$$e^{\pm j2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)} = e^{\pm j\frac{2\pi}{N}(vr+uc)} = e^{\pm j\frac{2\pi\omega}{N}(r\sin\theta + c\sin\theta)},$$

where

$$v = \omega\sin\theta, \quad u = \omega\cos\theta, \quad \omega = \sqrt{v^2+u^2}, \quad \text{and} \quad \theta = \tan^{-1}\left(\tfrac{v}{u}\right).$$

Write

$$\lambda = \frac{N}{\omega},$$

Similarly, λ is a spatial frequency

Note: since images are indexed by row & col with r down and c to the right, θ, is positive in the counterclockwise direction.

Then by Euler's relation,

$$e^{\pm j2\pi\frac{1}{\lambda}(r\sin\theta\,+\,c\cos\theta)} = \cos\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right] \pm j\sin\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right].$$

Cont'd. on next page.

What are 2D sinusoids? (cont'd.)

Both the real part of this,

$$\mathrm{Re}\left\{ e^{\pm j2\pi\frac{1}{\lambda}(r\sin\theta + c\cos\theta)} \right\} = +\cos\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right]$$

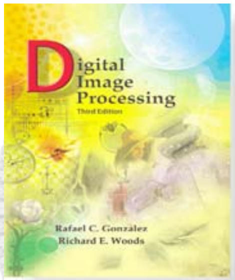
and the imaginary part,

$$\mathrm{Im}\left\{ e^{\pm j2\pi(r\sin\theta + c\cos\theta)} \right\} = \pm j\sin\left[\tfrac{2\pi}{\lambda}(r\sin\theta + c\cos\theta)\right]$$

are sinusoidal "gratings" of unit amplitude, period λ and direction θ.

Then $\dfrac{2\pi\omega}{N}$ is the radian frequency, and $\dfrac{\omega}{N}$ the frequency, of the wavefront

and $\lambda = \dfrac{N}{\omega}$ is the wavelength in pixels in the wavefront direction.

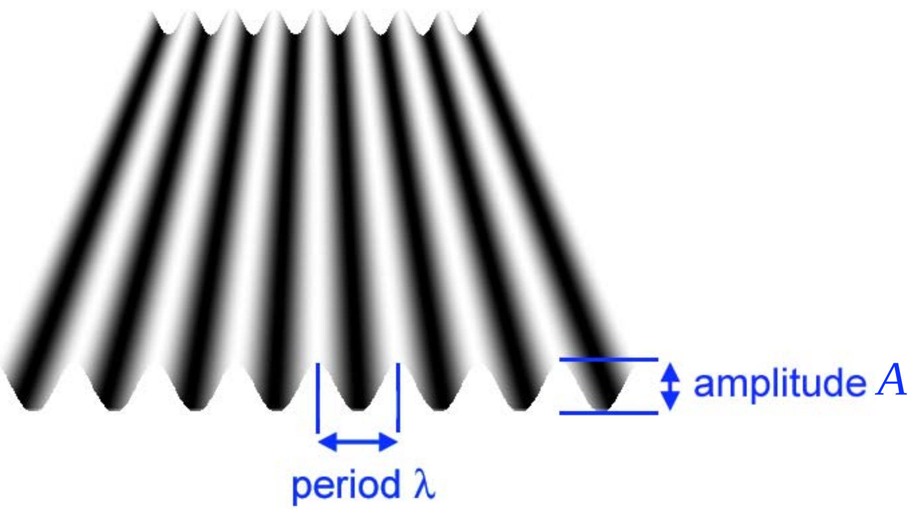


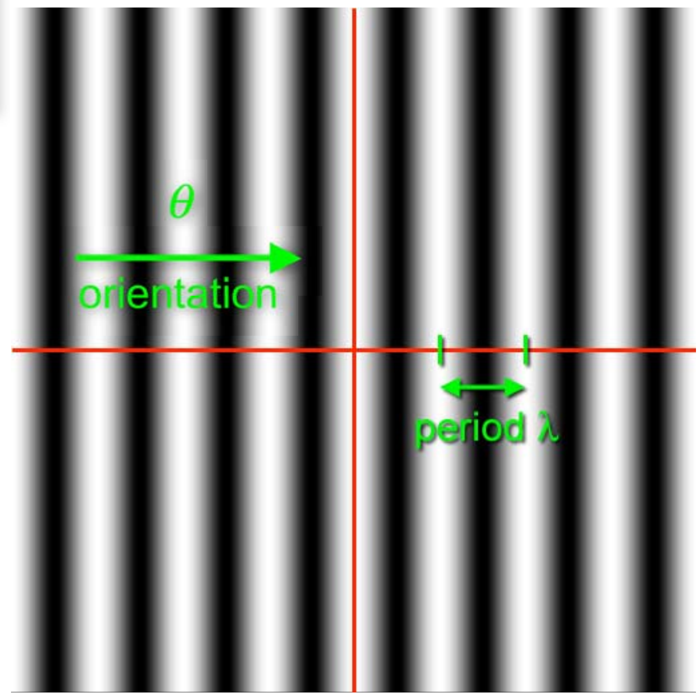
2D Sinusoids:

... are plane waves with grayscale amplitudes, periods in terms of lengths, ...

The 1 is added to make the range positive, [0,2] and dividing by 1/2 takes it back to [0,1]. The A is then the peak amplitude of the grating.

$$I(r,c)=\frac{A}{2}\left\{\cos\left[\frac{2\pi}{\lambda}\left(r\cdot\sin\theta+c\cdot\cos\theta\right)+\phi\right]+1\right\}$$

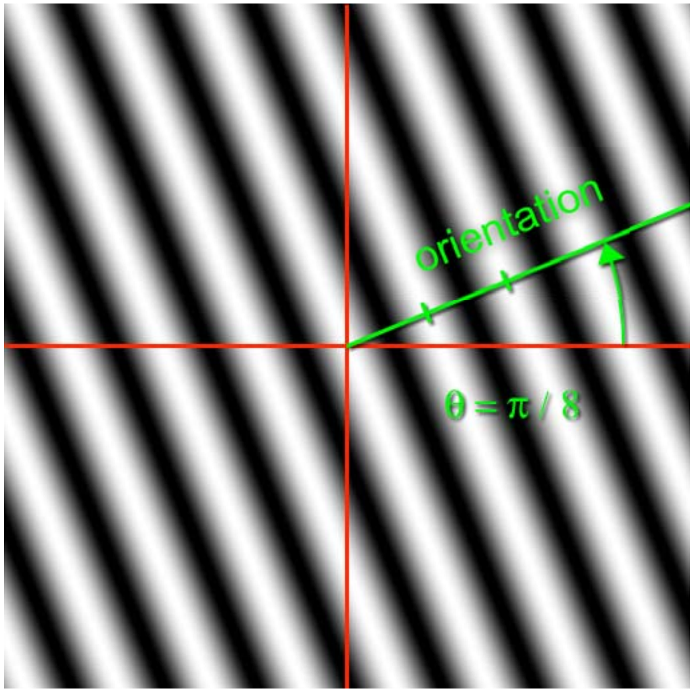


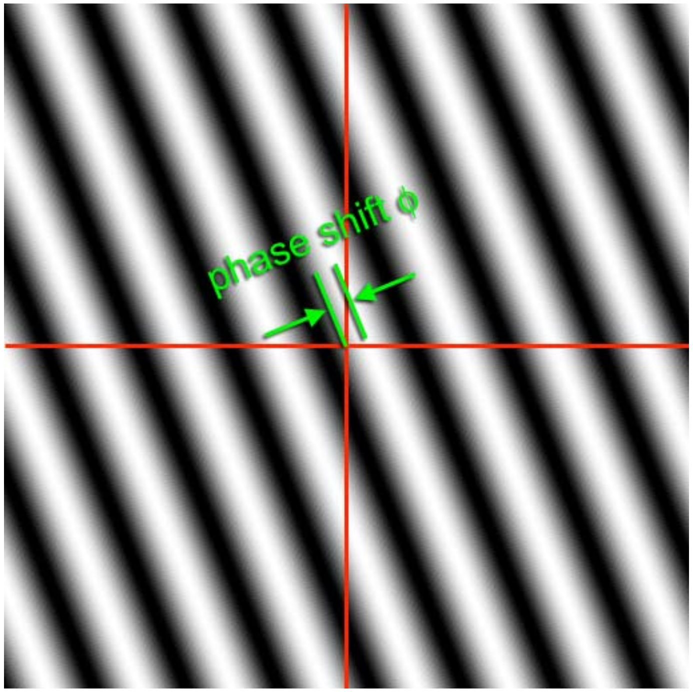


ϕ = phase shift

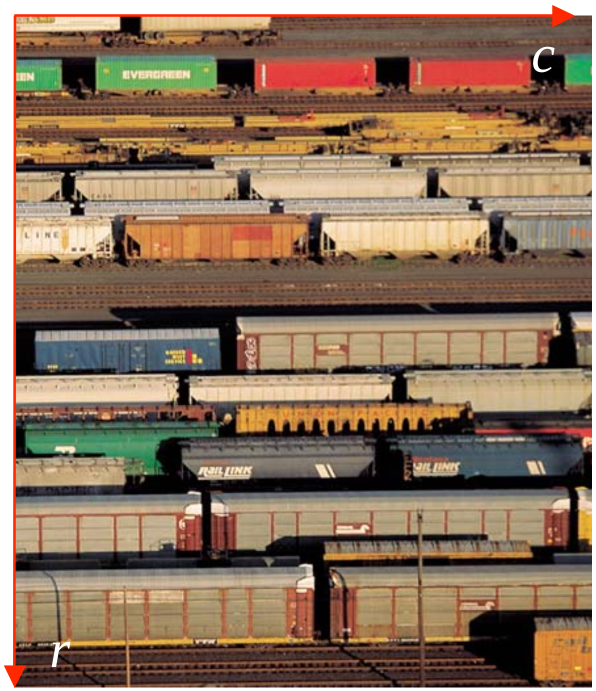
2D Sinusoids:

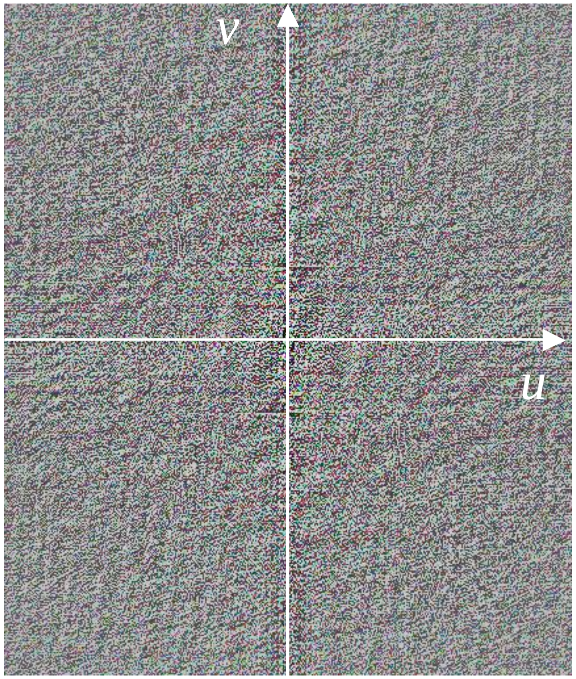
... have specific orientations, and phase shifts.

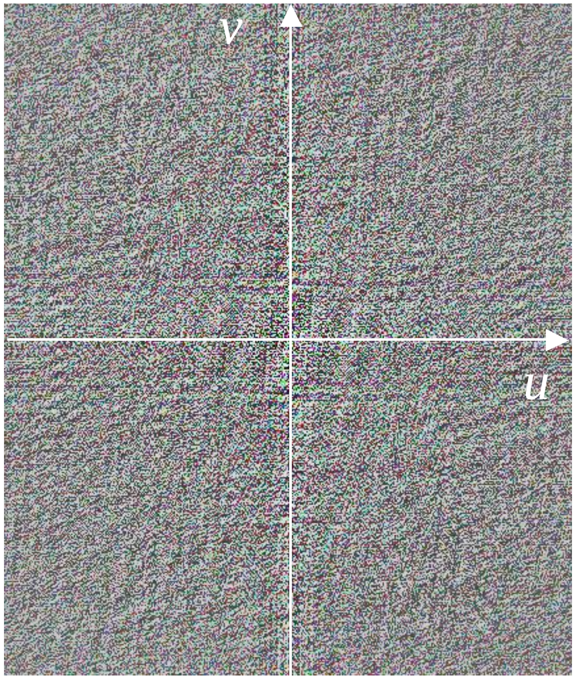




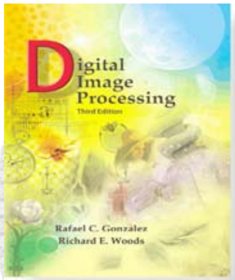
The Fourier Transform of an Image







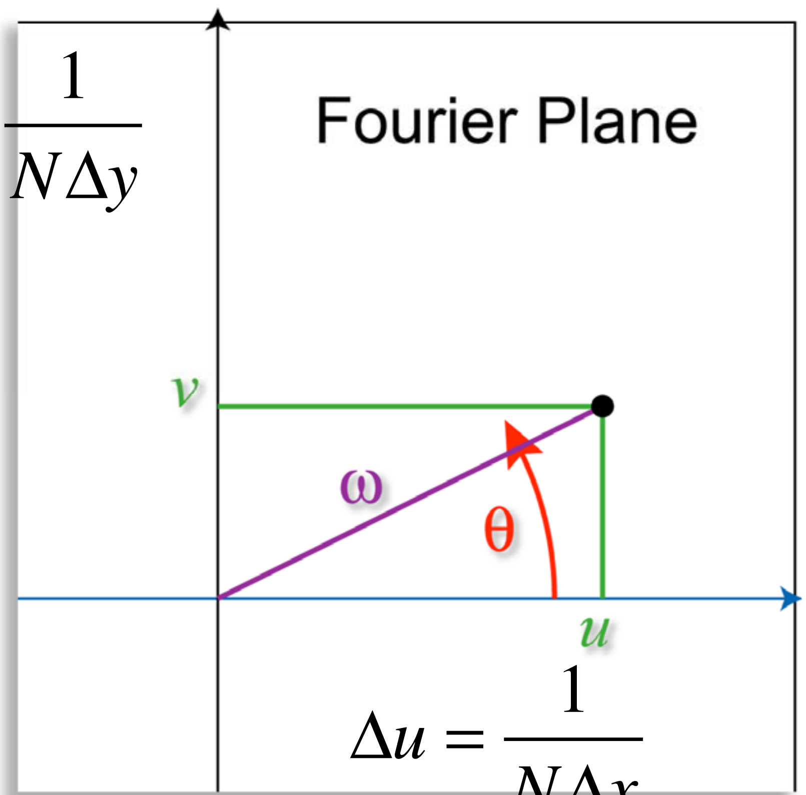
I $\qquad$ $\mathrm{Re}[\mathcal{F}\{I\}]$ $\qquad$ $\mathrm{Im}[\mathcal{F}\{I\}]$



Points on the Fourier Plane

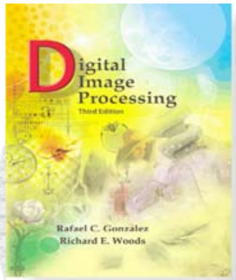
The point at column freq. u and row freq. v represents a sinusoid with wavelength ω and orientation θ (if R=C=N).

$$\Delta v = \frac{1}{N\Delta y}$$



$$\Delta u = \frac{1}{N\Delta x}$$

$\omega = N/\lambda$, where λ is the wavelength and R=C=N.



Points on the Fourier Plane (of a Digital Image)

In the Fourier transform of an $R \times C$ digital image the wavelengths, λ_u and λ_v represent a fraction of the R and C values. That is,

$$\lambda_u = \frac{C}{u} \quad \text{and} \quad \lambda_v = \frac{R}{v} \quad \text{pixels.}$$

The wavefront direction is given by

$$\theta_{\text{wf}} = \tan^{-1}\left(\frac{v\,\cancel{C}}{u\,\cancel{R}}\right),$$

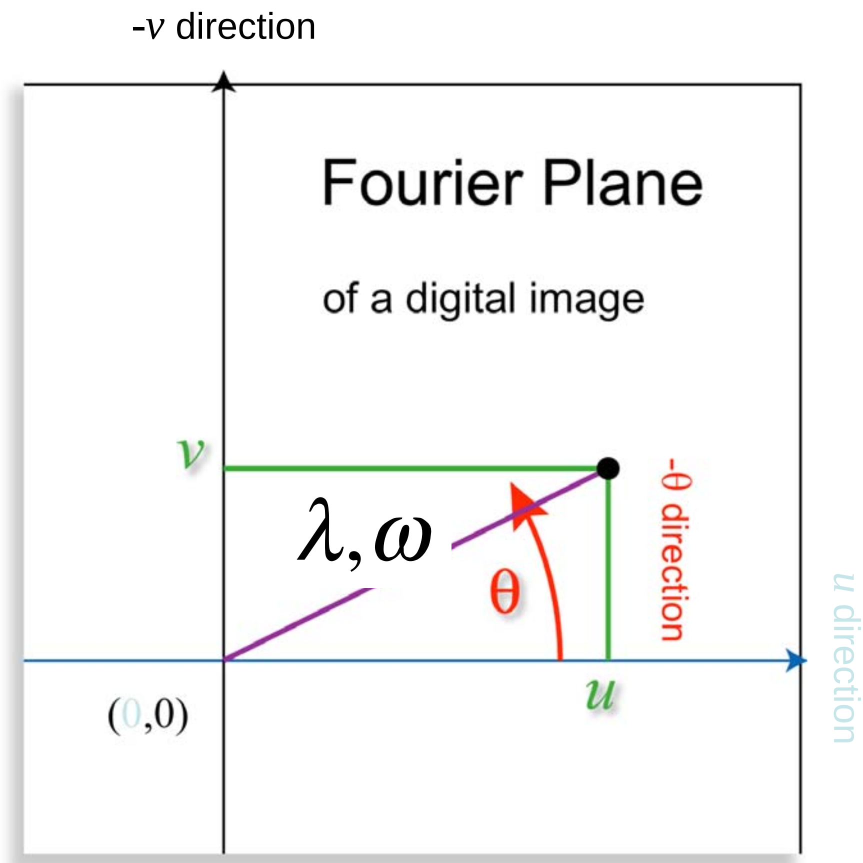
and the wavelength is

$$\lambda_{\text{wf}} = \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2}.$$

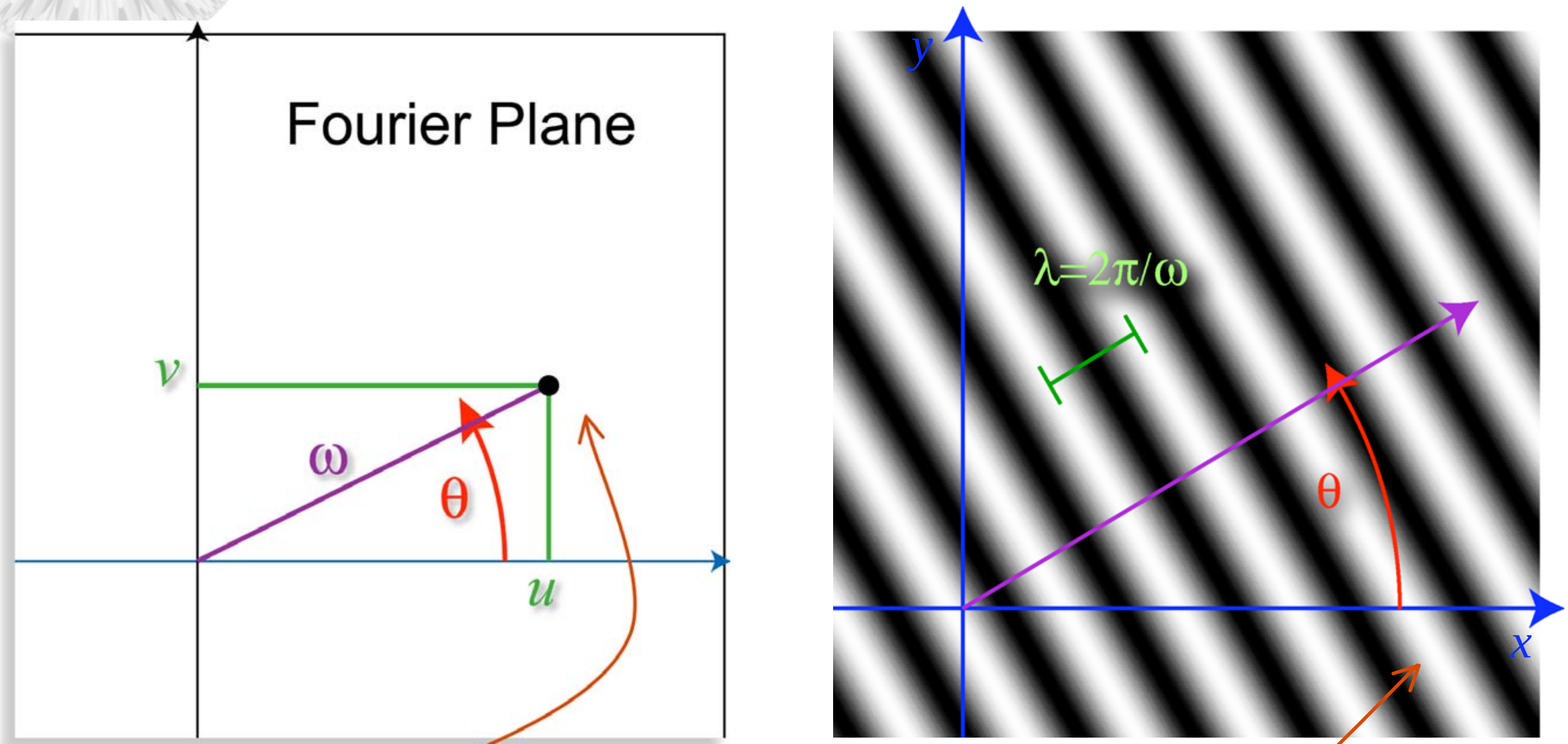
The frequencies represent fractions of R & C,

$$f_u = \frac{u}{C}\ ,\ f_v = \frac{v}{R}\ \text{, and}$$

$$f_{\text{wf}} = 1 \Big/ \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2} \quad \text{cycles.}$$



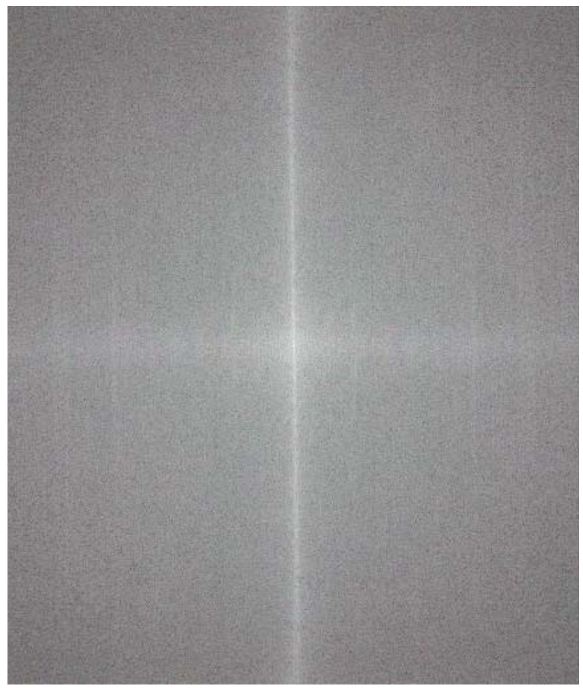
Points on the Fourier Plane



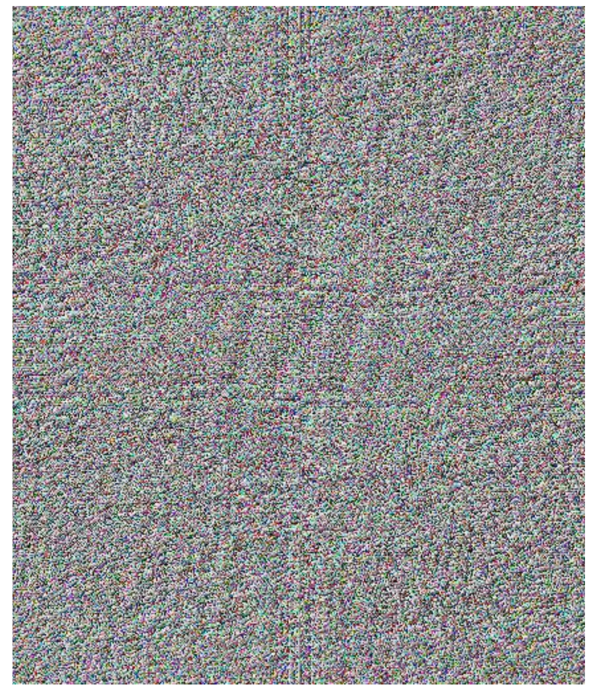
This point represents this particular sinusoidal grating

FT of an Image (Magnitude + Phase)

I

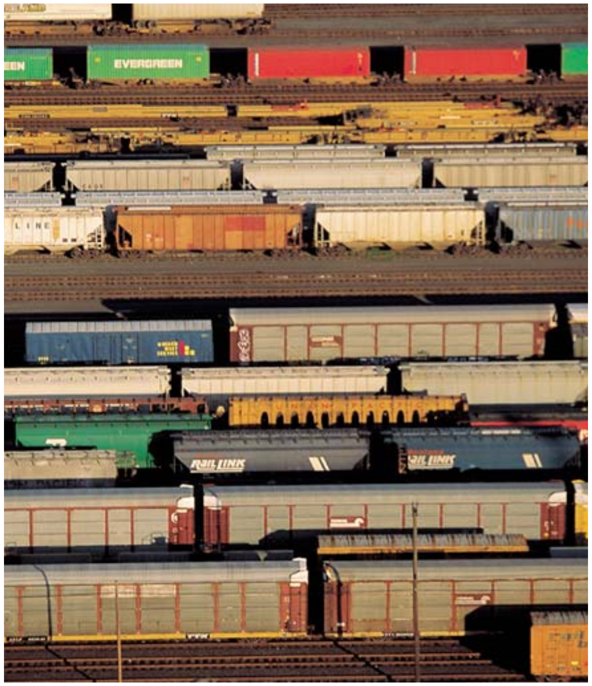


$\log\{|\mathcal{F}\{I\}|^2+1\}$

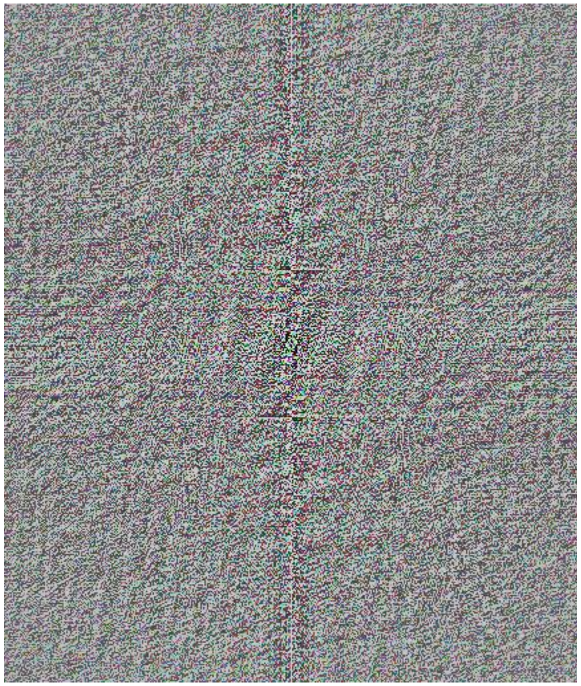


$\angle[\mathcal{F}\{I\}]$

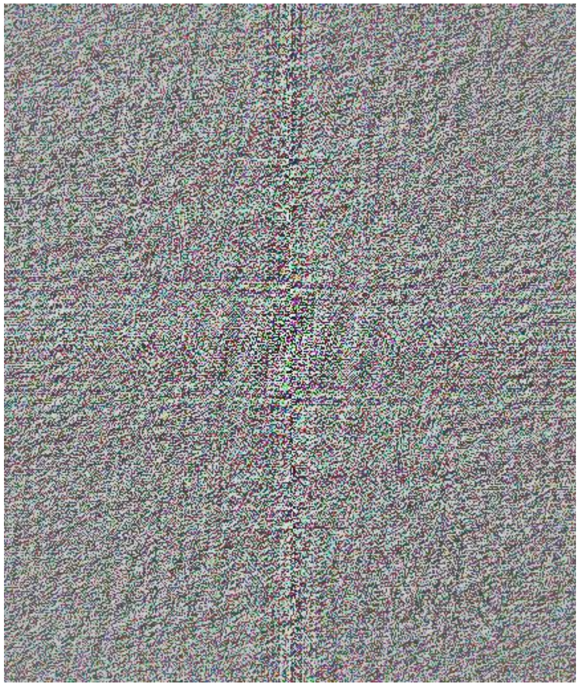
FT of an Image (Real + Imaginary)



I



$\mathrm{Re}[\mathcal{F}\{I\}]$



$\mathrm{Im}[\mathcal{F}\{I\}]$

FT of a Rect(t) Function

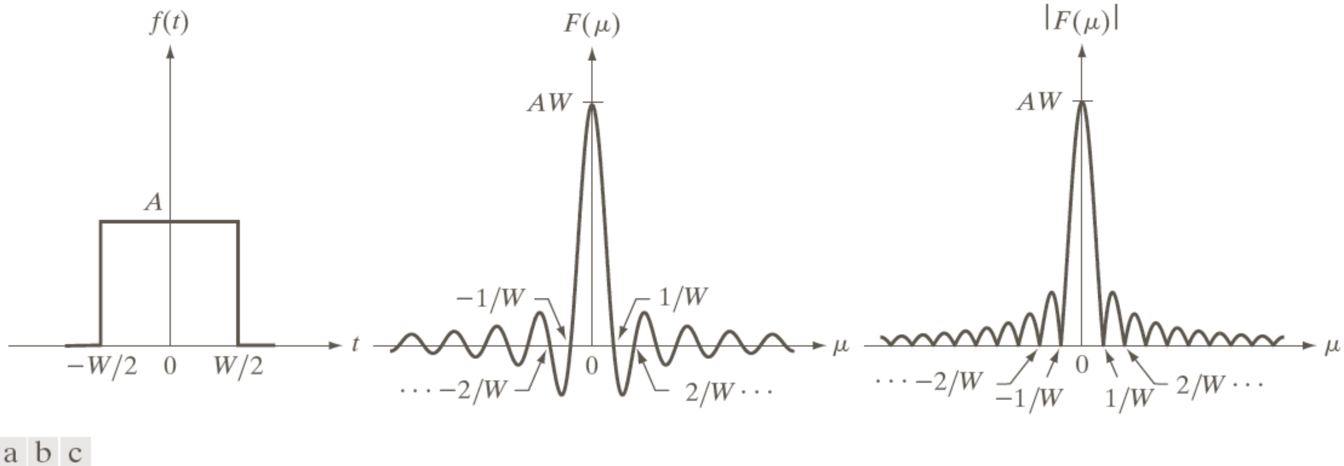
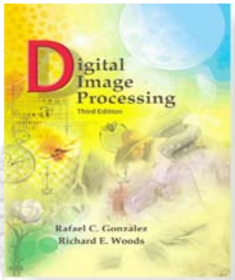


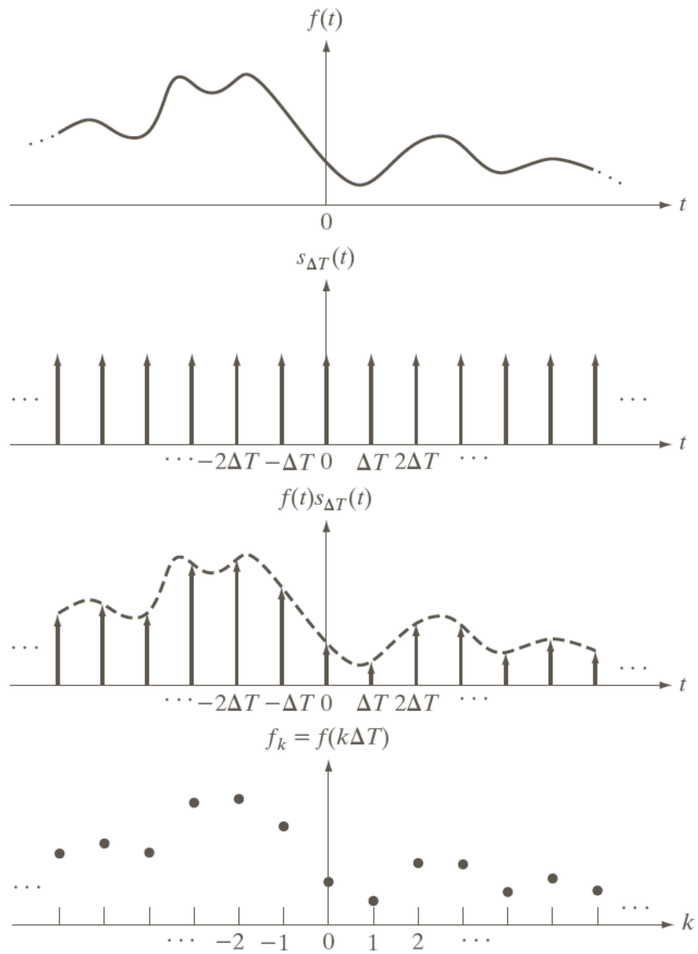
FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

A fundamental property of the Fourier transform relates to W. As the width W of the function increases in time t its corresponding Fourier transform becomes narrower indicating that the frequencies are becoming lower.



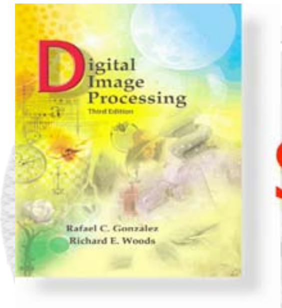
Sampling a Continuous Function

Real images are continuous but most modern sensors (and signal processing) are digital, i.e., the image is sampled.



a
b
c
d

FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

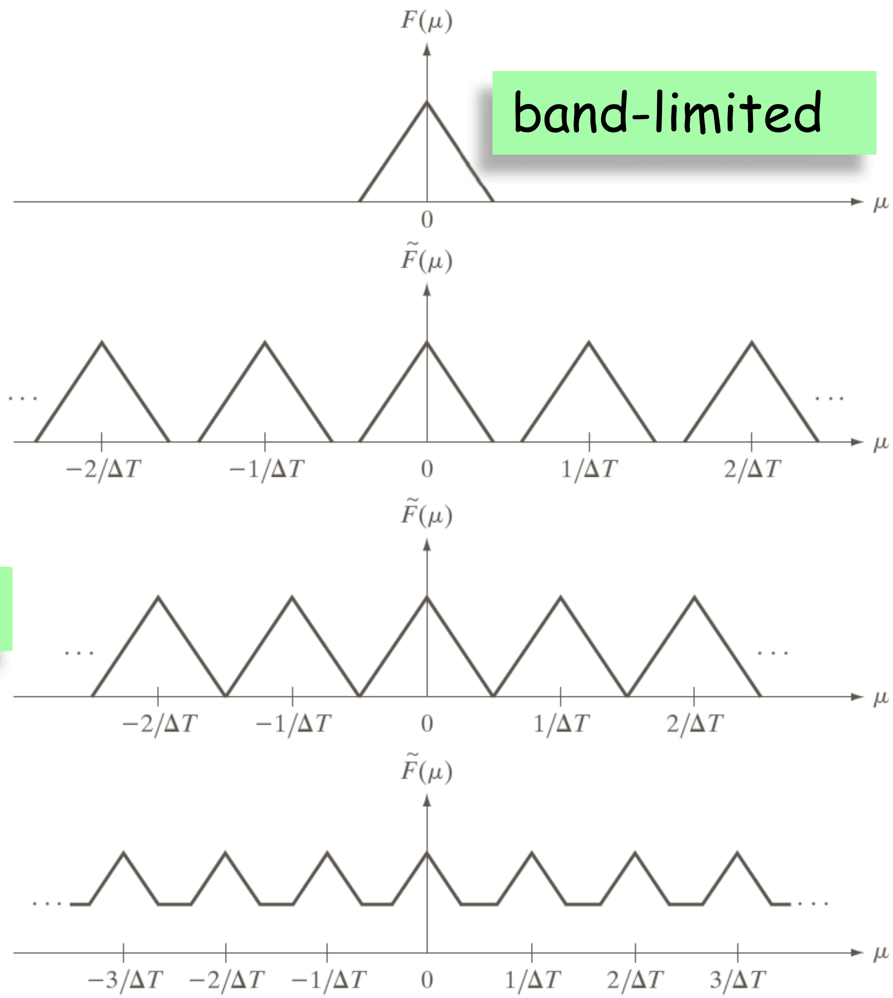


Sampling a Band-limited Function

The rate at which we sample the original function is very important ... sampling replicates the original spectrum — these can overlap!

band-limited

critically sampled



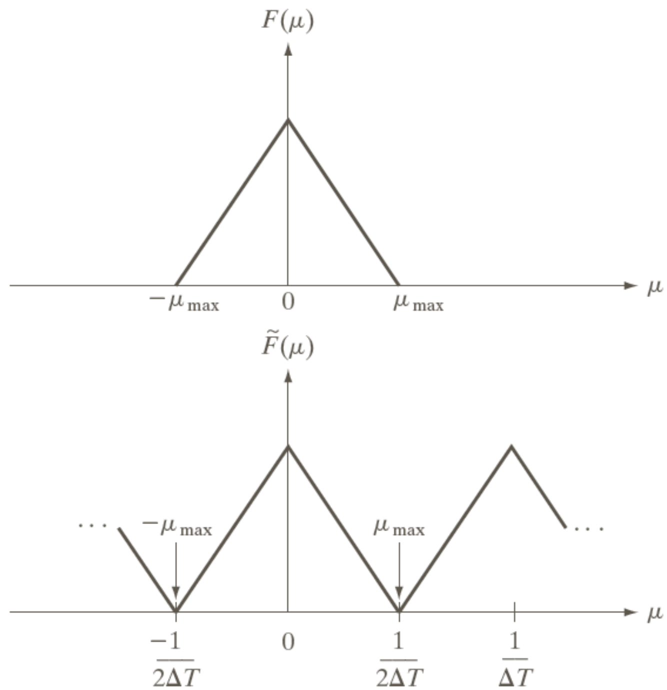
a
b
c
d

FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

Sampling Theorem

Sampling theorem requires that we sample at a rate ΔT given by:

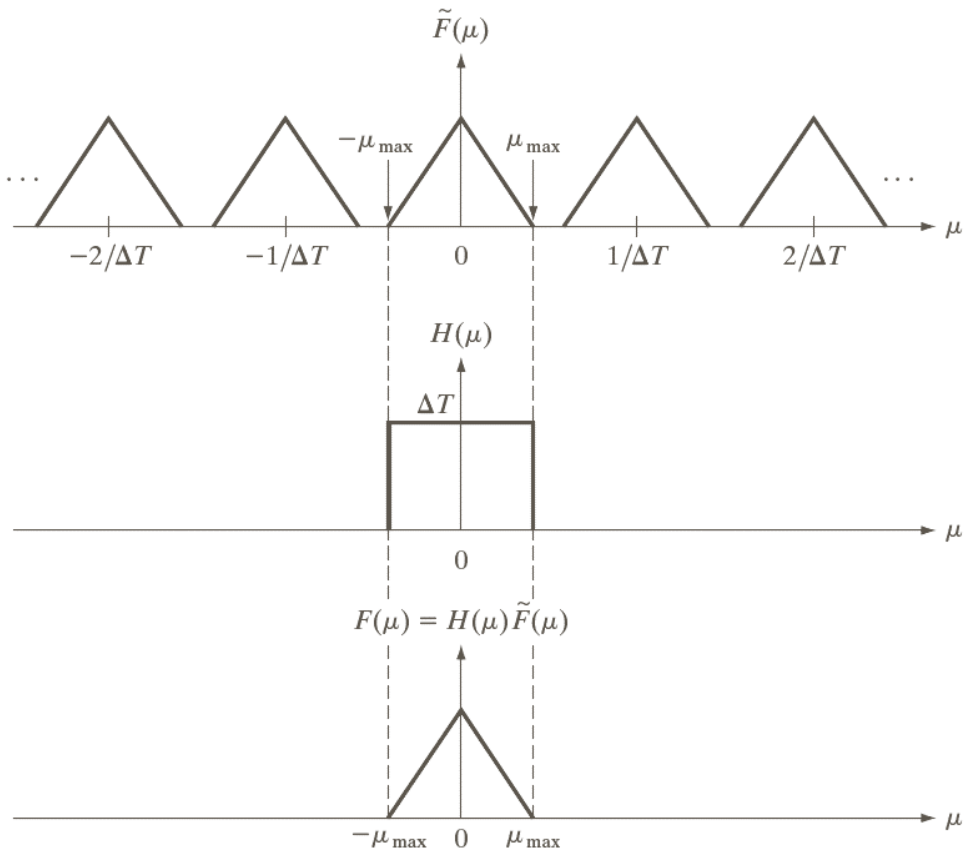
$$\frac{1}{\Delta T} > 2\mu_{\max}$$



a
b

FIGURE 4.7 (a) Transform of a band-limited function. (b) Transform resulting from critically sampling the same function.

Reconstruction Filtering

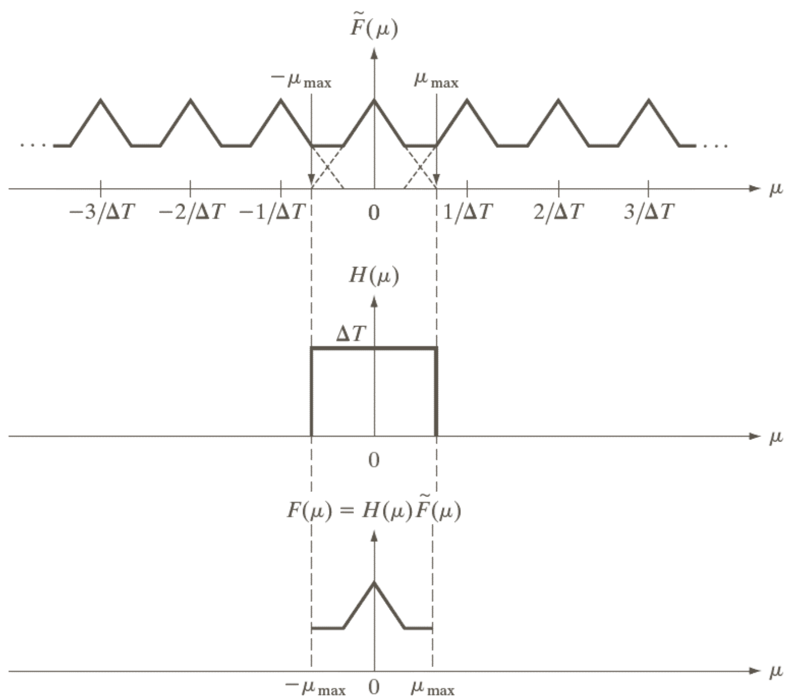


a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

Undersampling → Aliasing

Frequency domain view of undersampling.



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Undersampling → Aliasing

Time domain view of undersampling.

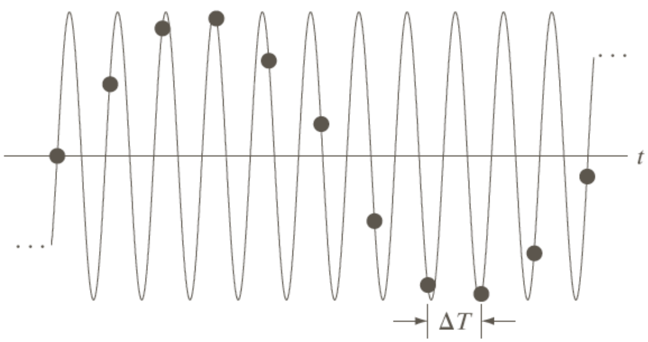


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

2-D Discrete Impulse

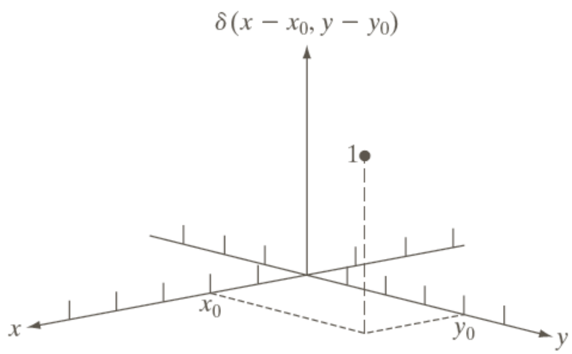


FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).

We sample pictures using a 2-D discrete impulse function.

2-D Sampling Grid

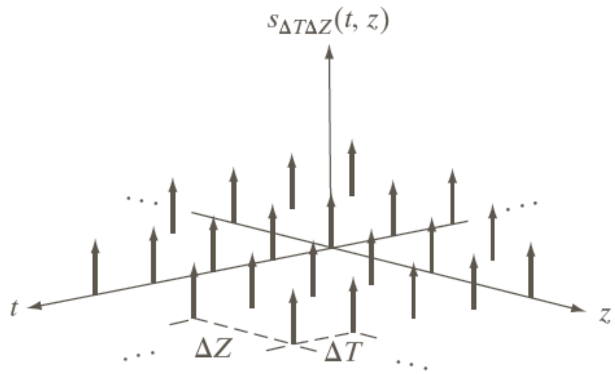
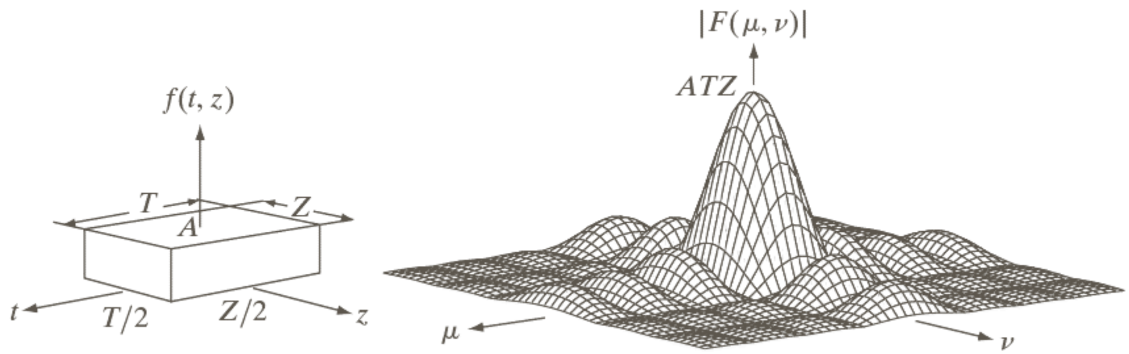


FIGURE 4.14 Two-dimensional impulse train.

We use 2-D arrays of 2-D impulse functions to sample a continuous image to give a discrete image.

2-D DFT of Rect(x,y)



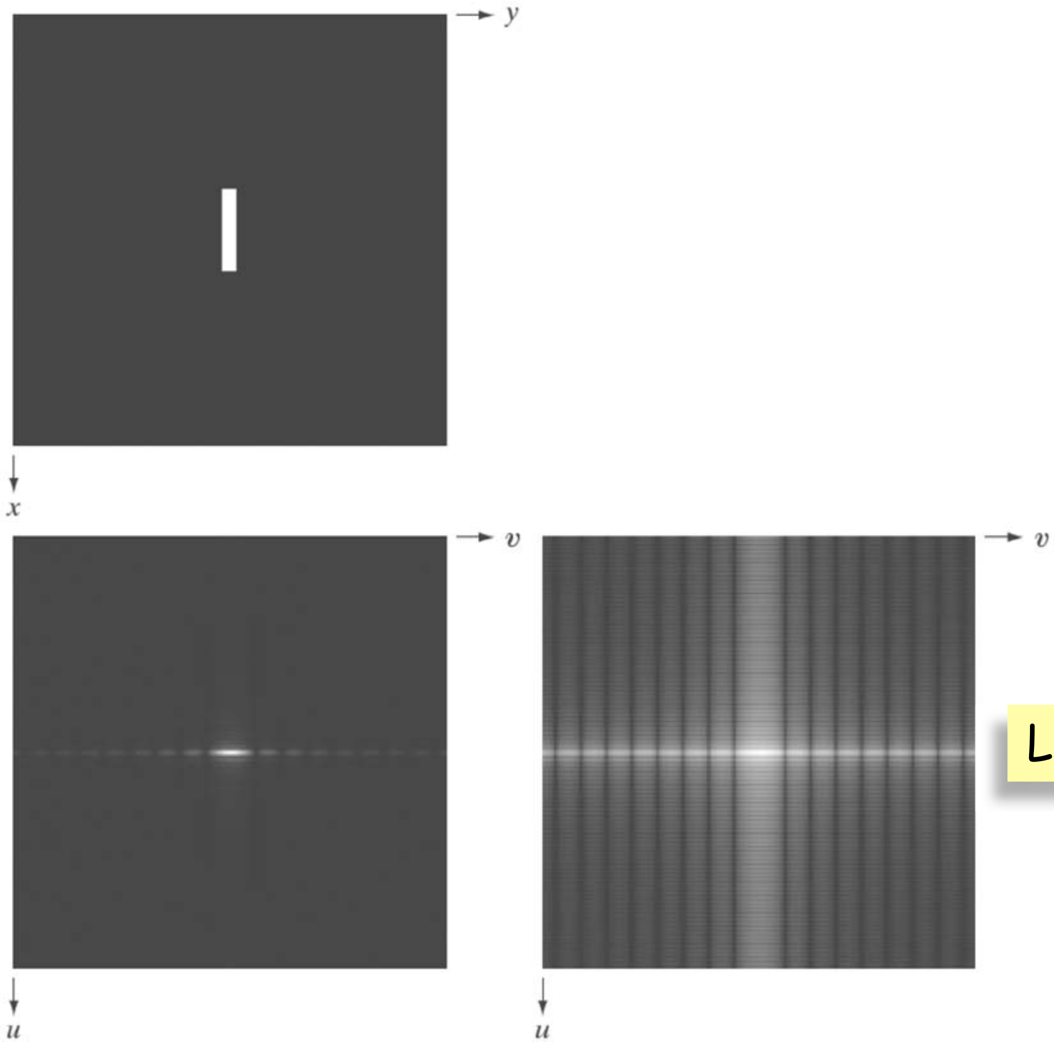
a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

2-D DFT of Rect(x,y)

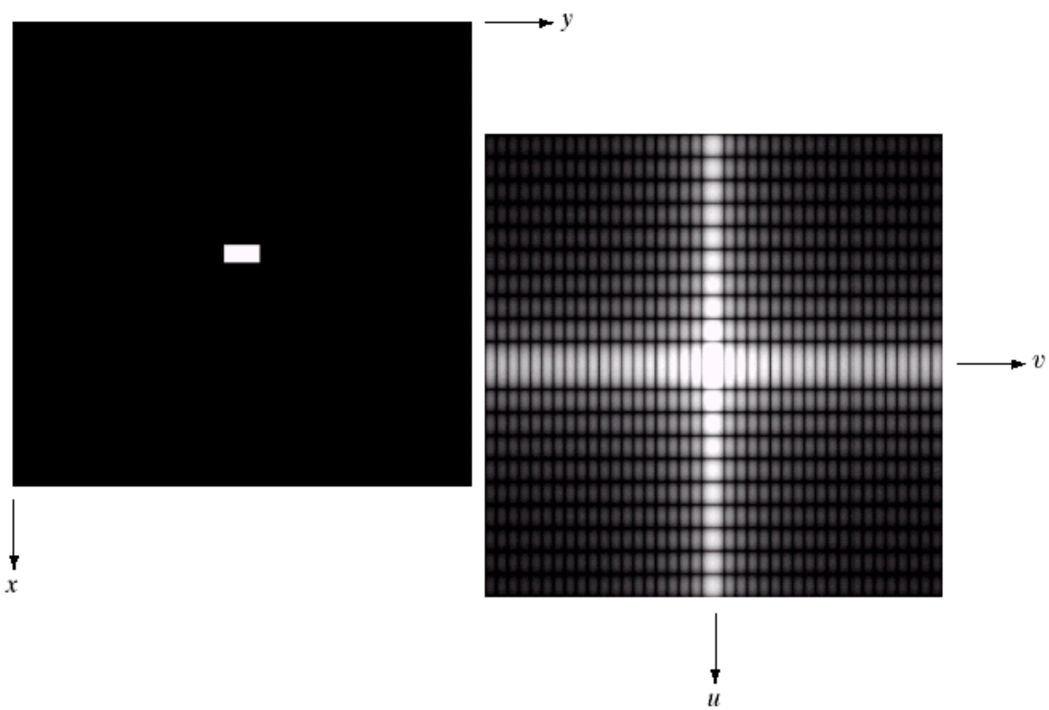
Rect(x,y)

$|\mathcal{F}[\text{Rect}(x,y)]|$



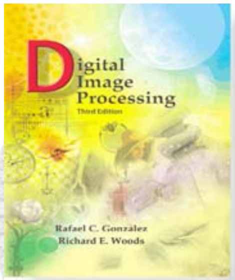
Log transformed

Chapter 4
Image Enhancement in the Frequency Domain

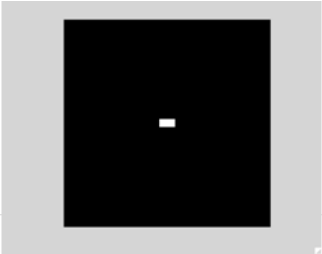


a b

FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

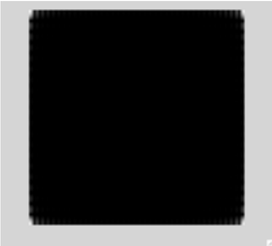


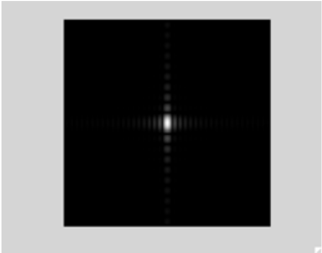
MATLAB/Image Processing Toolbox

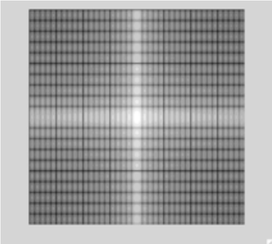


MATLAB Fourier transforms

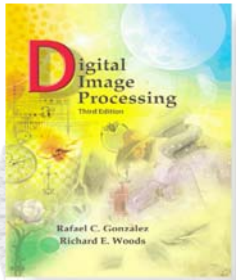
```
>> f=imread('Figure_Rectangle.jpg');  % load in spatial rectangle
>> F=fft2(double(f));                 % do 2D FFT
>> S=abs(F);                          % calculate magnitude for display
>> imshow(S, [ ])                     % shows in four corners of display
% [] indicates that MATLAB should scale the image's minimum and
% maximum values to 0 and 255 respectively
>> Fc=fftshift(F);                    % shift FFT to center
>> imshow(abs(Fc), [ ]);              % show magnitude of FFT in center

% much tougher to do display transform
>> g=uint8(log(1+double(abs(Fc))));
>> imshow(g, [ ])
% double converts the image to double precision floating point
% uint8 brings the values back to the range [0,255]
```







SEE GWE, Section 4.2 Computing and Visualizing the 2-D DFT in MATLAB
GWE, Section 3.2.2 Logarithmic and Contrast Stretching Transformations



Undersampling in 2-D

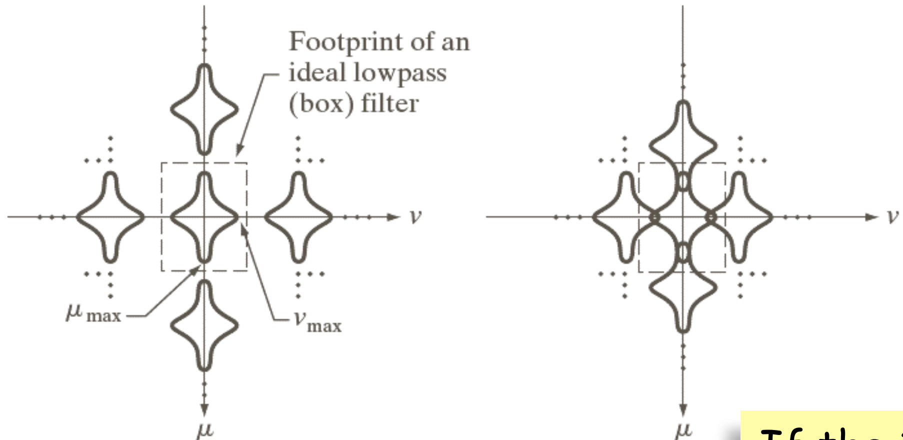


FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

If the 2-D samples impulses are too far apart (undersampling) their 2-D Fourier transforms will overlap.

Undersampling in 2-D

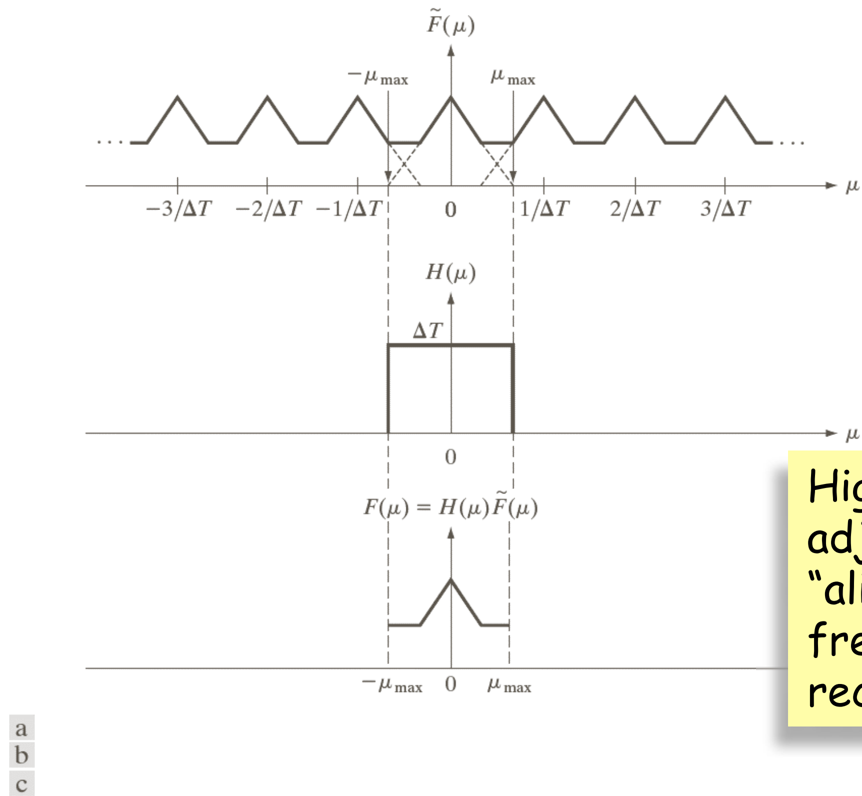
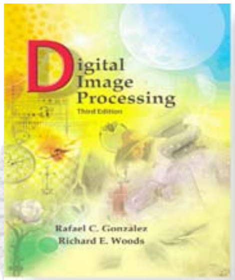


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

High frequencies from adjacent spectra will "alias" as lower frequencies in the reconstructed image



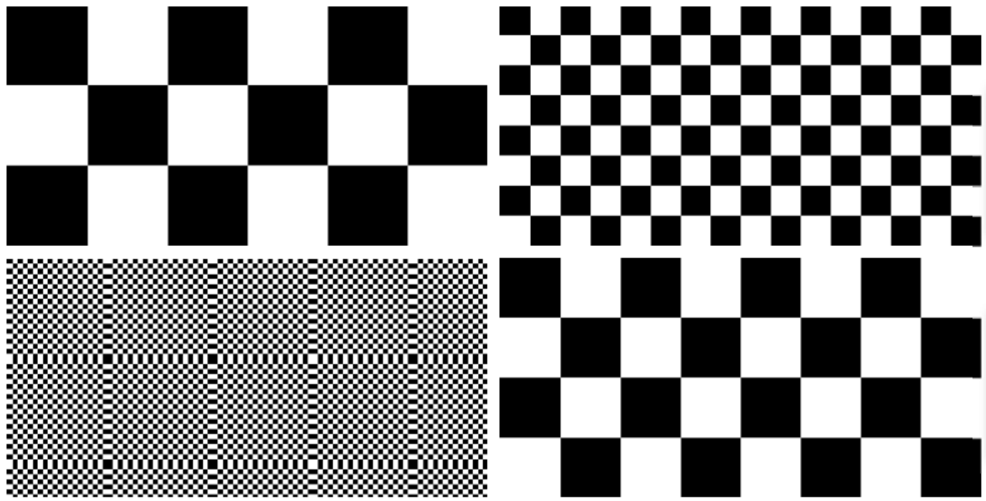
Undersampling in 2-D

16 pixel squares

6 pixel squares

0.9174 pixel squares

0.4798 pixel squares



a b
c d

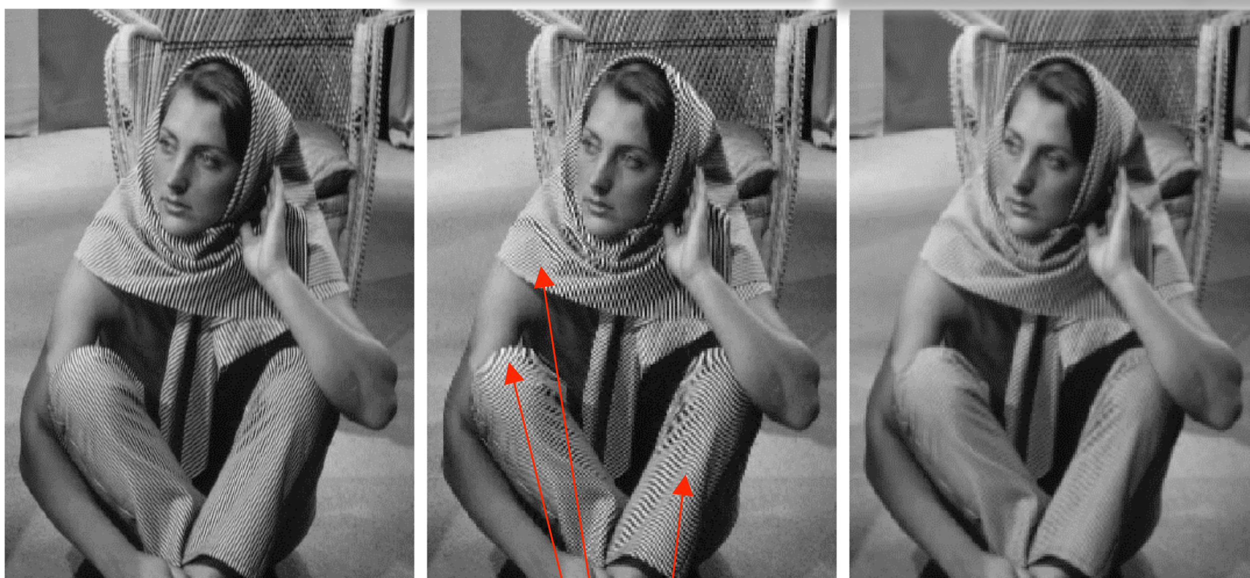
FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

Results of checkboard images sampled) with a 96x96 pixel resolution sensor. Sensor max resolution=1 pixel.

Aliasing

50% pixel deletion

3x3 averaging prior to sampling to band-limit image



a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3 × 3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Note the aliasing