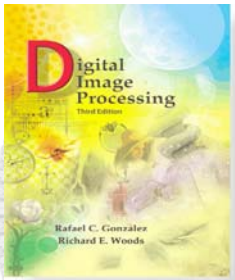
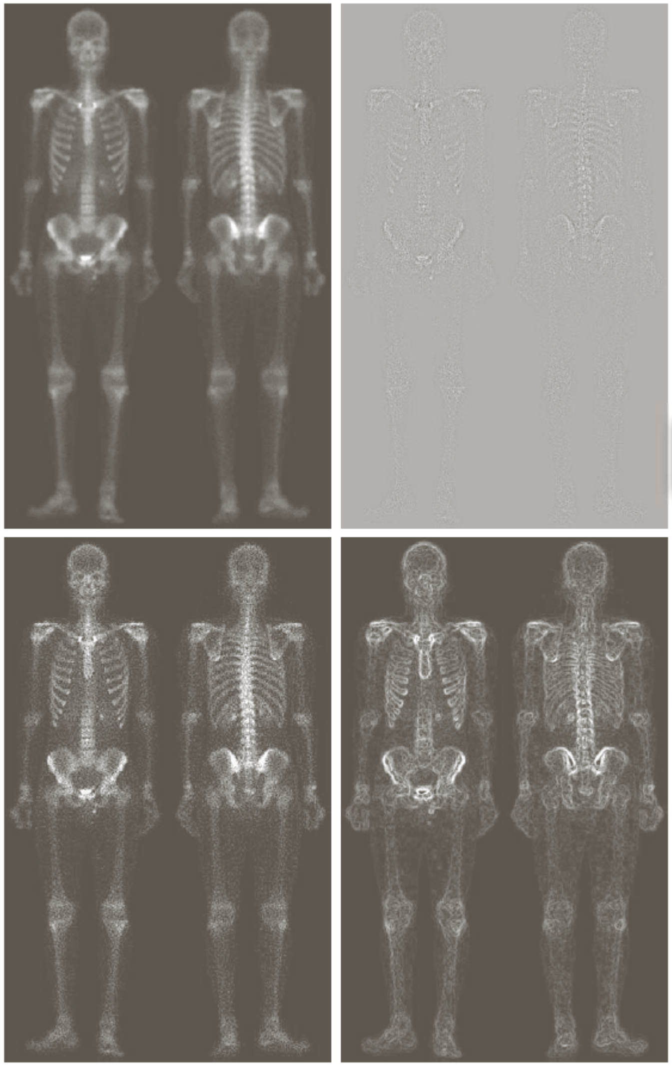


Lecture #7

- Image Processing Example
- Fuzzy logic
  - Basics
  - Image processing examples
- Fourier Transform
  - Inner product, basis functions
  - Fourier series

Image Processing Example



a b
c d

FIGURE 3.43 (a) Image of whole body bone scan. (b) Laplacian of (a). (c) Sharpened image obtained by adding (a) and (b). (d) Sobel gradient of (a).

original image

Laplacian of image

(c) Sharpened by adding Laplacian

Sobel gradient image

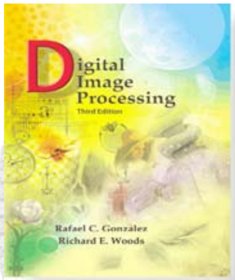
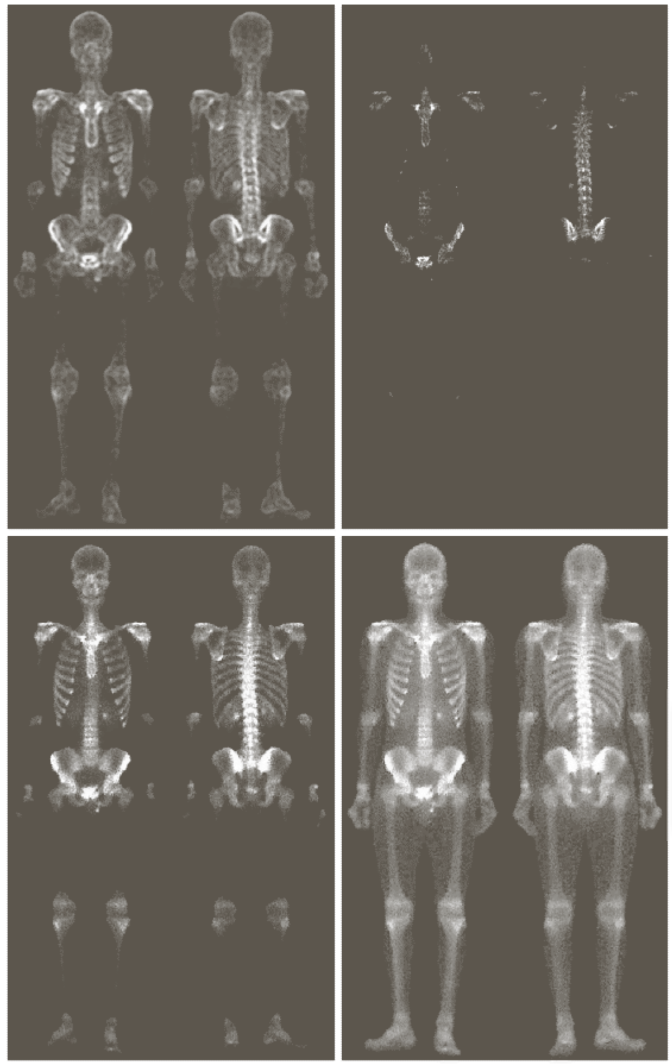


Image Processing Example

(e) Blurred Sobel gradient image

Add Mask to original



e f
g h

FIGURE 3.43
(Continued)
(e) Sobel image smoothed with a 5 … fi… image formed by the product of (c) and (e). (g) Sharpened image obtained by the sum of (a) and (f). (h) Final result obtained by applying a power-law transformation to (g). Compare (g) and (h) with (a). (Original image courtesy of G.E. Medical Systems.)

Mask = (c) x (e)

Power Law Intensity Transform

Basic Fuzzy Logic

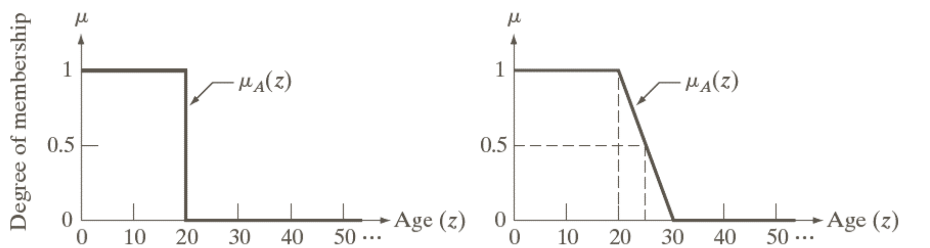


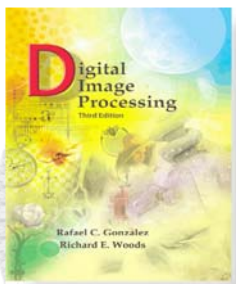
FIGURE 3.44 Membership functions used to generate (a) a crisp set, and (b) a fuzzy set.

"crisp" membership function

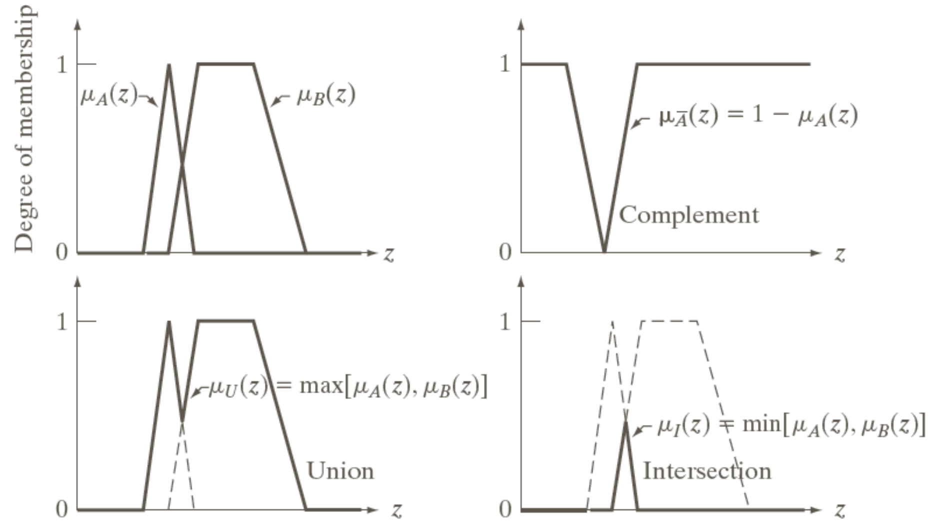
"fuzzy" membership function

Probability: there is a 50% chance that a particular person is young
Fuzzy logic: a person's membership with the set of young people is 0.5

Basically everyone is "young" to some degree. A membership function represents that degree.



Basic Fuzzy Logic



a b
c d

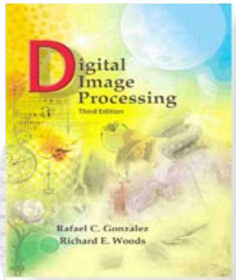
FIGURE 3.45 (a) Membership functions of two sets, *A* and *B*. (b) Membership function of the complement of *A*. (c) and (d) Membership functions of the union and intersection of the two sets.

$\mu_\cup = \max[\mu_A(z), \mu_B(z)]$

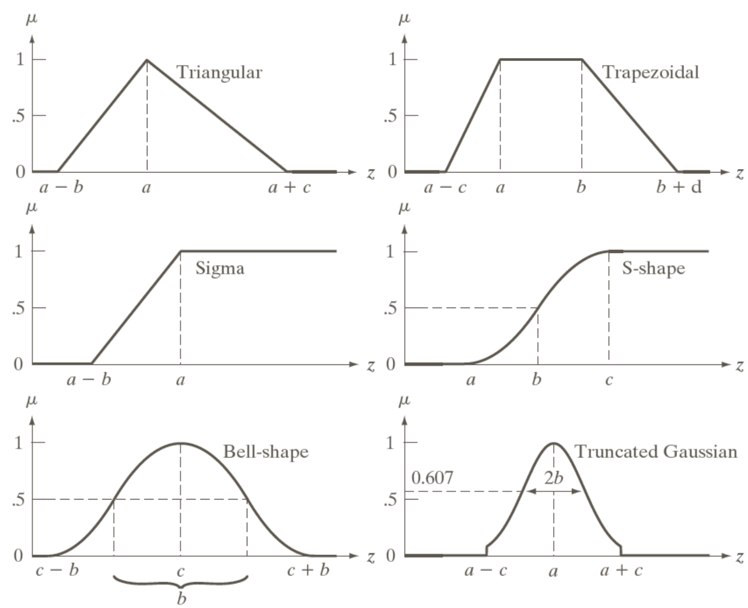
OR

$\mu_\cap = \min[\mu_A(z), \mu_B(z)]$

AND



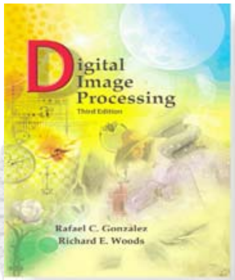
Basic Fuzzy Logic



a b
c d
e f

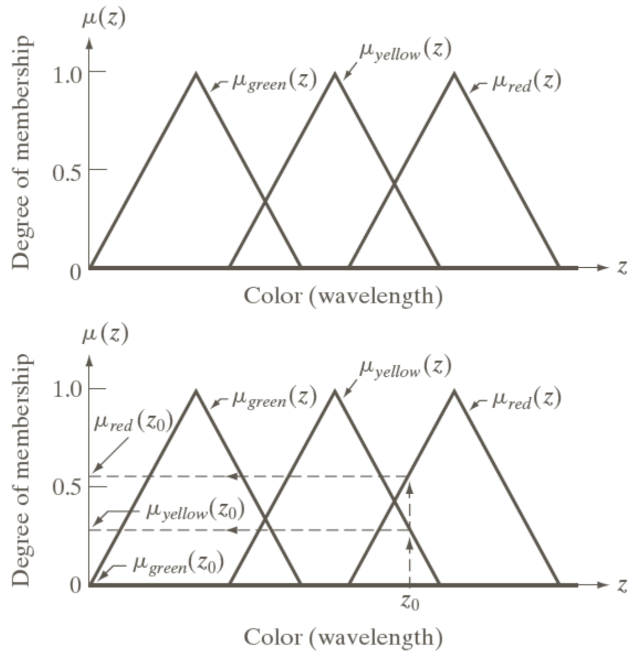
FIGURE 3.46 Membership functions corresponding to Eqs. (3.8-6)–(3.8-11).

Commonly used membership functions used to describe inputs and outputs.



Fuzzy Input Variables

We will use a single color to describe a fruit with a color that changes from green to yellow to red as it ripens.



a
b

FIGURE 3.47 (a) Membership functions used to fuzzify color. (b) Fuzzifying a specific color z_0. (Curves describing color sensation are bell shaped; see Section 6.1 for an example. However, using triangular shapes as an approximation is common practice when working with fuzzy sets.)

A particular color z_o has a membership value $\mu_{green}(z_o)$, $\mu_{yellow}(z_o)$, and $\mu_{red}(z_o)$ in <u>all three</u> input membership functions.

Fuzzy Output Variables

The fruit can be verdant(unfit to eat), half-mature (ripening), and mature (ripe)

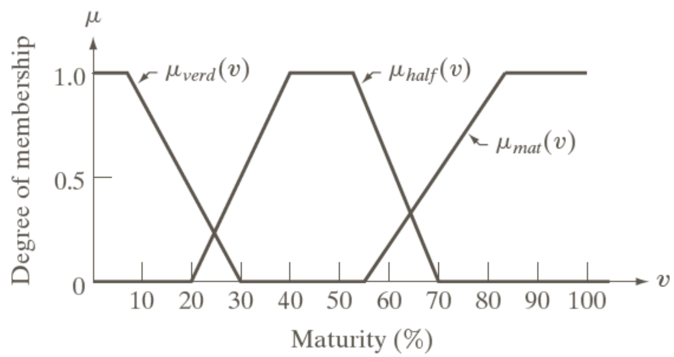
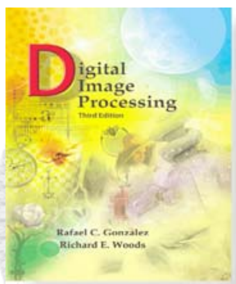


FIGURE 3.48 Membership functions characterizing the outputs *verdant*, *half-mature*, and *mature*.

The output variable is maturity which is hard to quantify



Fuzzy System

We now need to relate the input membership functions to the output membership functions — this is called implication

input membership

output membership

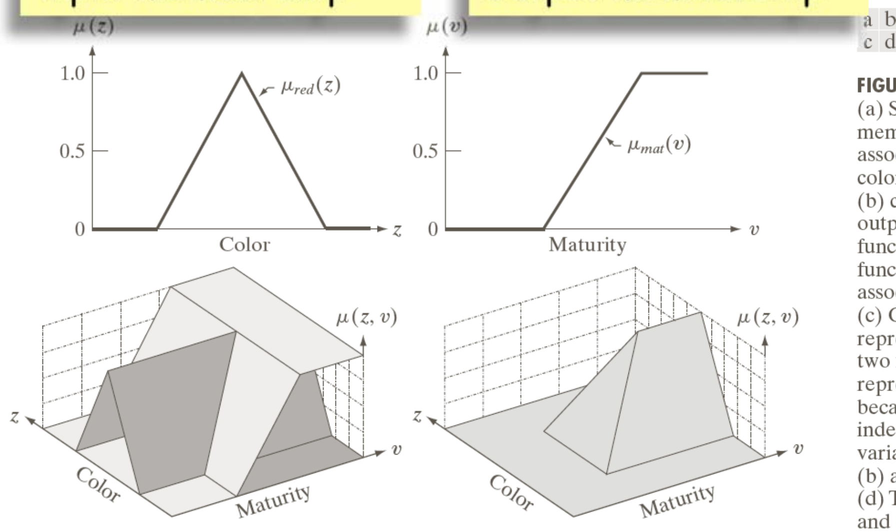
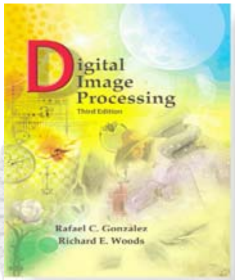


FIGURE 3.49 (a) Shape of the membership function associated with the color red, and (b) corresponding output membership function. These two functions are associated by rule R_3. (c) Combined representation of the two functions. The representation is 2-D because the independent variables in (a) and (b) are different. (d) The AND of (a) and (b), as defined in Eq. (3.8-5).

This is a simple plot of the relationship between the two membership functions

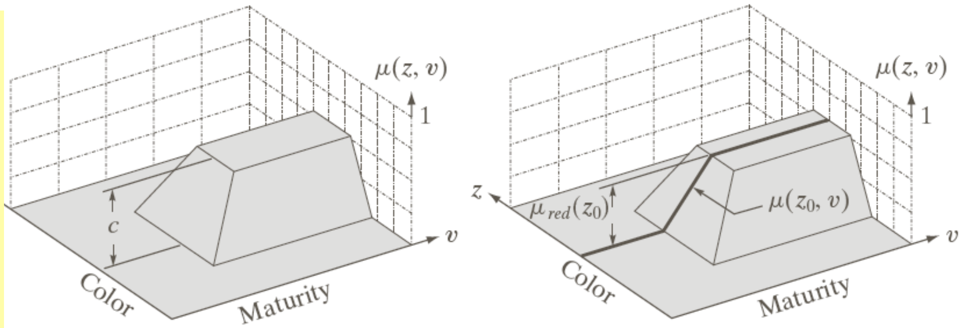
This is simply the membership function for red AND mature or red ∩ mature

$$\mu_{red \cap mature}(z,v)=\min[\mu_{red}(z),\mu_{mature}(v)]$$



A fuzzy output for a given input

We now need to evaluate each output membership function for the given input value



a b

FIGURE 3.50 (a) Result of computing the minimum of an arbitrary constant, c, and function $\mu_3(z, v)$ from Eq. (3.8-12). The minimum is equivalent to an AND operation. (b) Cross section (dark line) at a specific color, z_0.

$Q_3(v) = \mu_{red}(z_o)$ AND $\mu_{red \cap mature}(z_o, v)$
$= \min[[\mu_{red}(z_o), \mu_{red \cap mature}(z_o, v)]$

$\mu_{red}(z_o)$ is a constant c which clips the output membership function as shown above

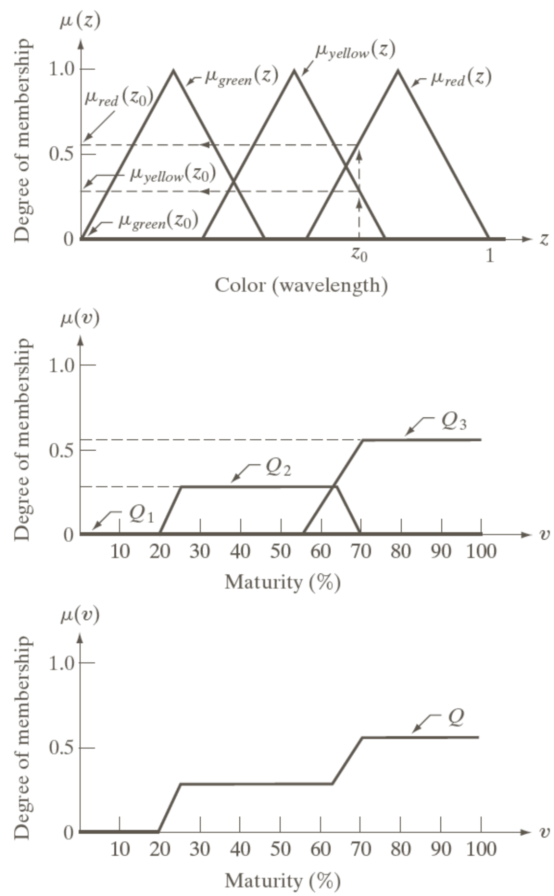
Q_3 is still a membership function!

All fuzzy outputs for a given input

There are 3 different input membership functions each of which can be ANDed with mature.

This gives three different output membership functions.

The system output is the maximum value at each point or $Q = Q_1$ OR Q_2 OR Q_3.

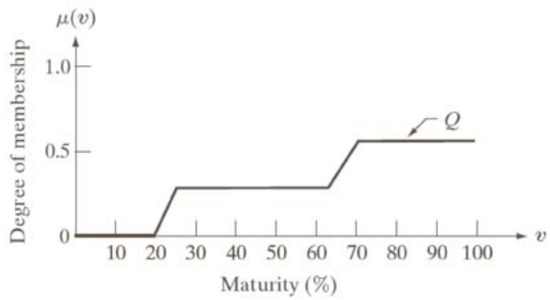


a
b
c

FIGURE 3.51 (a) Membership functions with a specific color, z_0, selected. (b) Individual fuzzy sets obtained from Eqs. (3.8-13)–(3.8-15). (c) Final fuzzy set obtained by using Eq. (3.8-16) or (3.8-17).

This is the mature membership function for a specific color z_0

Defuzzification



The output is still a set. The actual membership value is the center of gravity of the output set.

$$v_0 = \frac{\sum_{v=1}^{K} vQ(v)}{\sum_{v=1}^{K} Q(v)}$$

The Entire Process

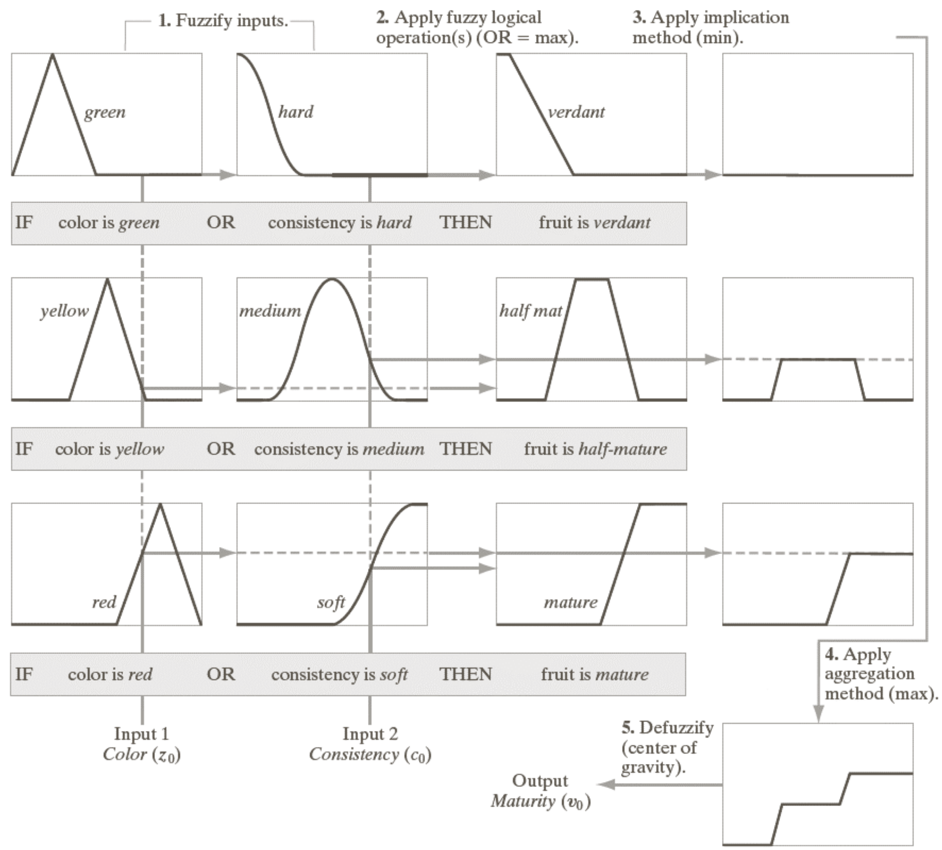
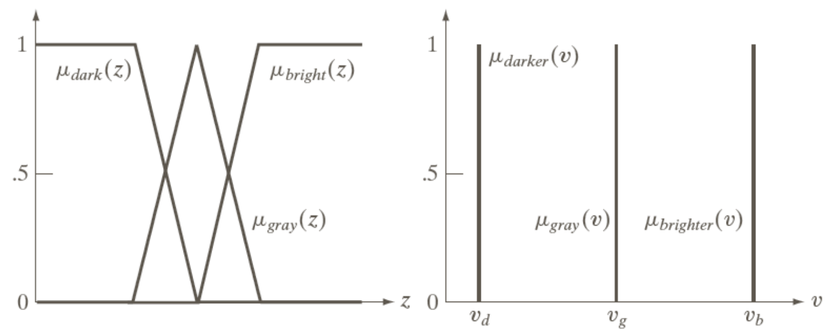


FIGURE 3.52 Example illustrating the five basic steps used typically to implement a fuzzy, rule-based system: (1) fuzzification, (2) logical operations (only OR was used in this example), (3) implication, (4) aggregation, and (5) defuzzification.

Fuzzy Contrast Enhancement



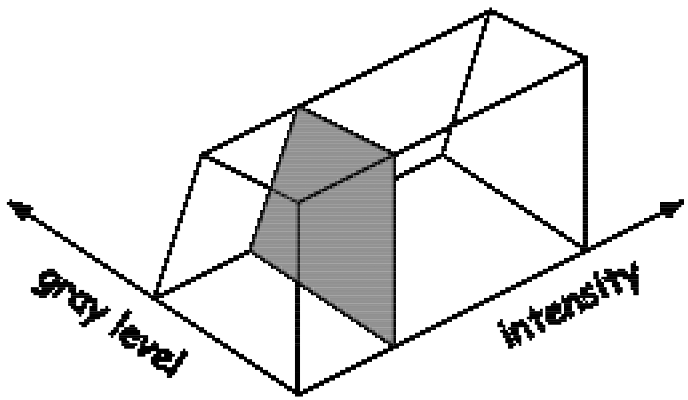
a b

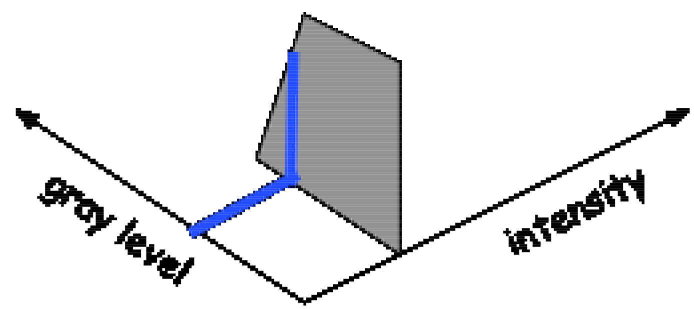
FIGURE 3.53 (a) Input and (b) output membership functions for fuzzy, rule-based contrast enhancement.

IF a pixel is dark THEN make it darker
IF a pixel is gray THEN make it gray
IF a pixel is bright THEN make it brighter

The output memberships are only three values.

Contrast Enhancement





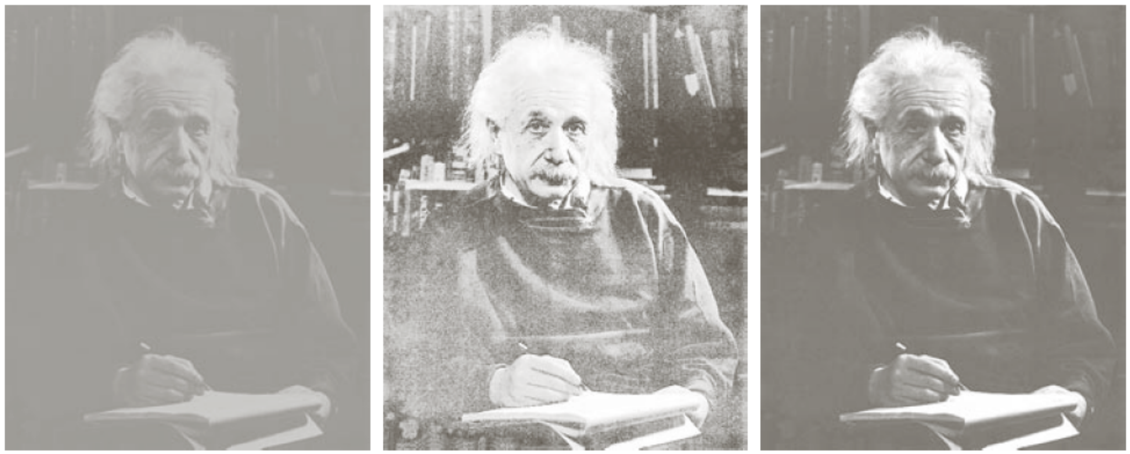
1. Compute the input membership function AND the output membership function

2. For a specific value of input gray level we map onto a single output plane. The membership is 1 for deep blacks and gradually decreases to zero. Do this for each output.

3. Determine the total membership function and compute the center of gravity of the output

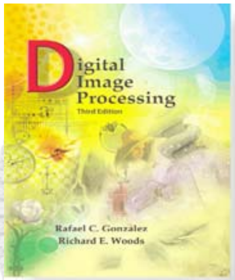
$$v_0 = \frac{\mu_{dark}(z_0)\times v_{dark} + \mu_{gray}(z_0)\times v_{gray} + \mu_{bright}(z_0)\times v_{bright}}{\mu_{dark}(z_0) + \mu_{gray}(z_0) + \mu_{bright}(z_0)}$$

Fuzzy Contrast Enhancement



a b c

FIGURE 3.54 (a) Low-contrast image. (b) Result of histogram equalization. (c) Result of using fuzzy, rule-based contrast enhancement.



Fuzzy Contrast Enhancement

Original histogram

Equalized histogram

Fuzzy membership functions

Fuzzy contrast enhanced histogram

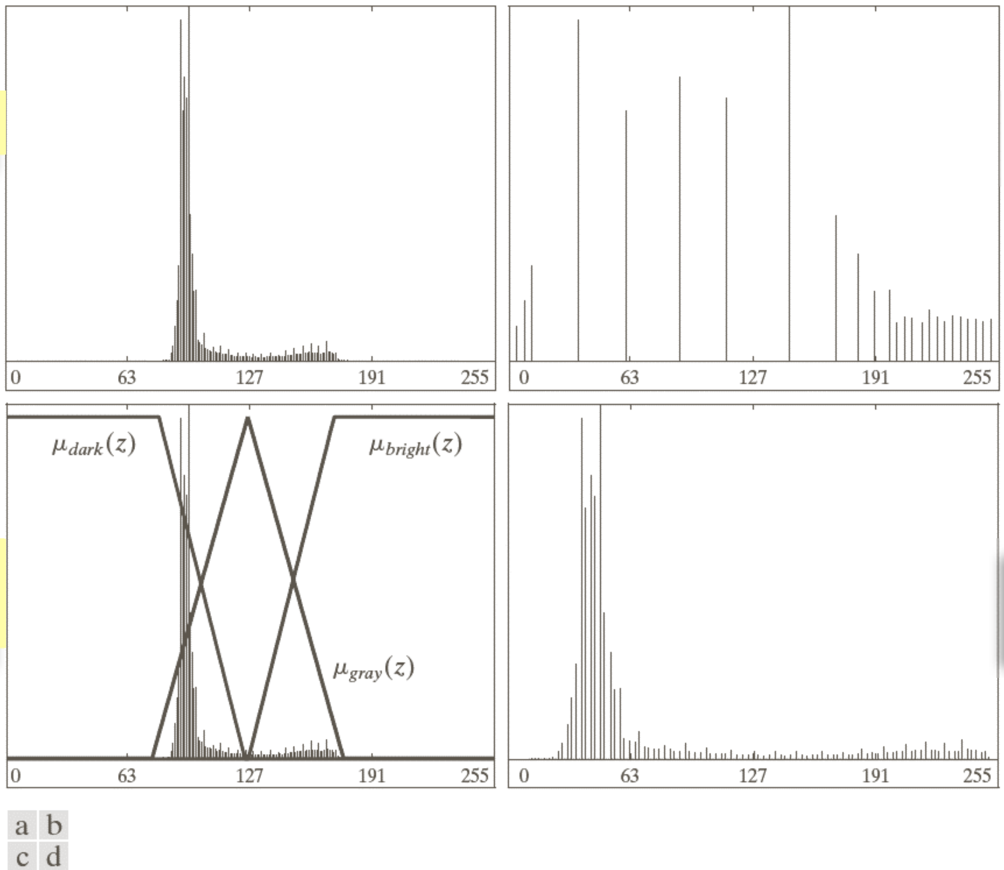
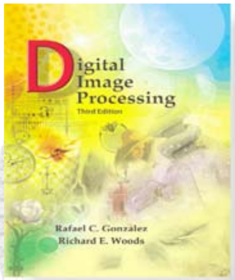


FIGURE 3.55 (a) and (b) Histograms of Figs. 3.54(a) and (b). (c) Input membership functions superimposed on (a). (d) Histogram of Fig. 3.54(c).



Fuzzy Boundary Extraction

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

Pixel neighborhood

d_1	d_2	d_3
d_4	0	d_6
d_7	d_8	d_9

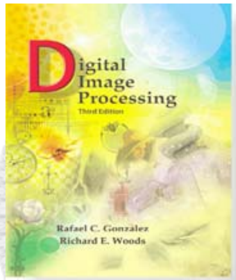
Intensity differences

a b

FIGURE 3.56 (a) A 3 × 3 pixel neighborhood, and (b) corresponding intensity differences between the center pixels and its neighbors. Only d_2, d_4, d_6, and d_8 were used in the present application to simplify the discussion.

IF a pixel belongs to a uniform region THEN make it white ELSE make it black

IF d_2 is zero AND d_6 is zero THEN z_5 is white
IF d_6 is zero AND d_8 is zero THEN z_5 is white
IF d_8 is zero AND d_4 is zero THEN z_5 is white
IF d_4 is zero AND d_2 is zero THEN z_5 is white
ELSE z_5 is black



Fuzzy Boundary Extraction Rules

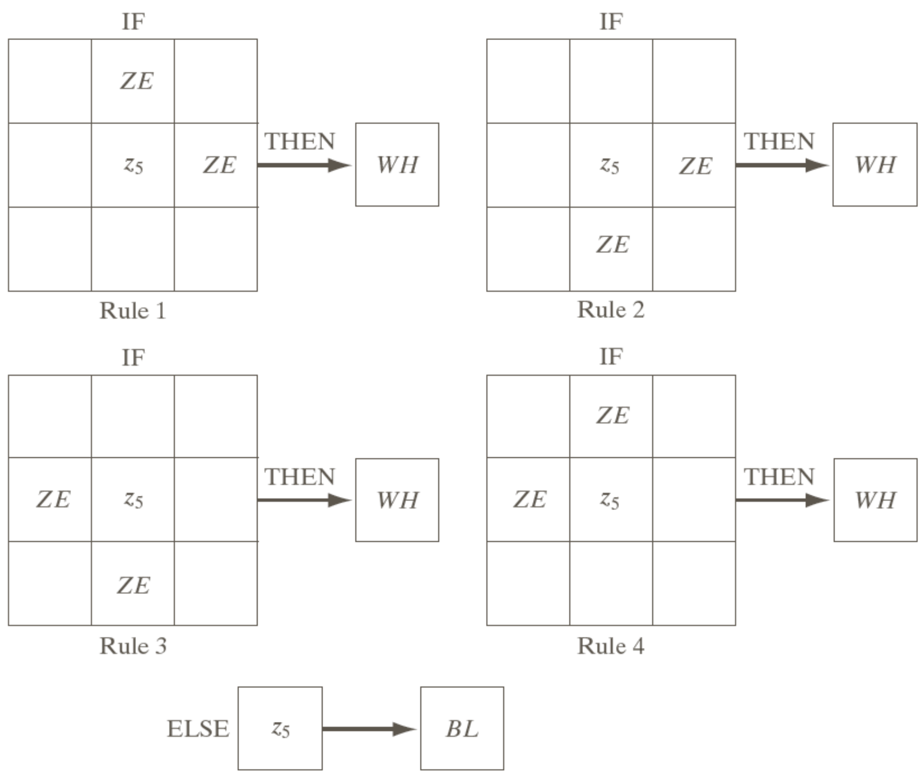
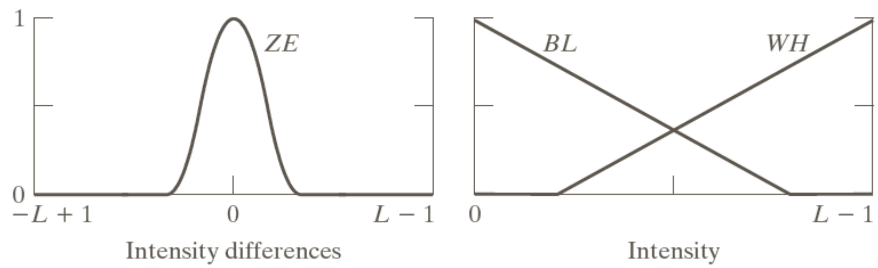


FIGURE 3.58 Fuzzy rules for boundary detection.

Fuzzy Boundary Extraction

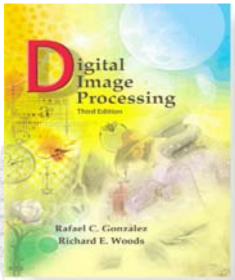


a b

FIGURE 3.57 (a) Membership function of the fuzzy set *zero*. (b) Membership functions of the fuzzy sets *black* and *white*.

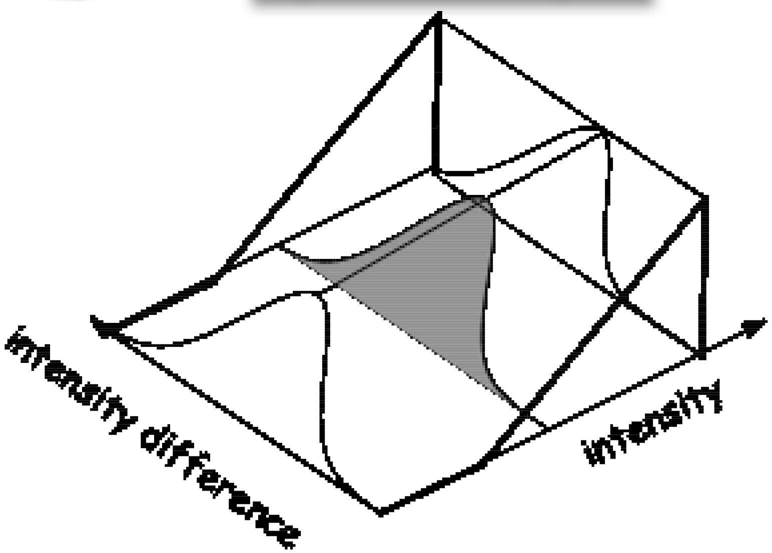
Input membership function for ZERO intensity differences

Output membership function for black and white

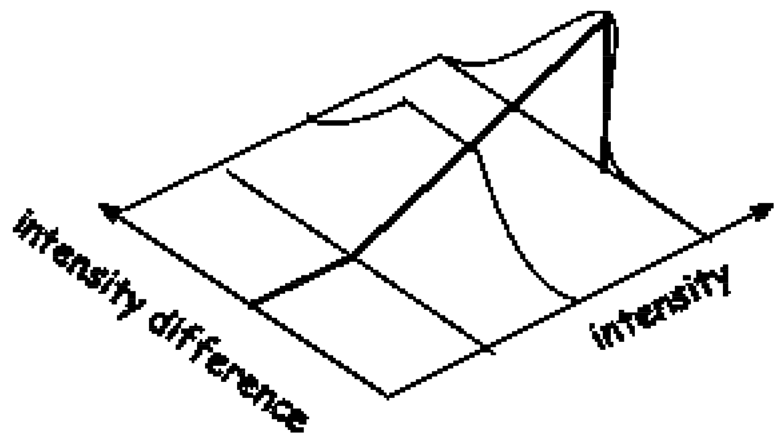


Fuzzy System

input membership



output membership

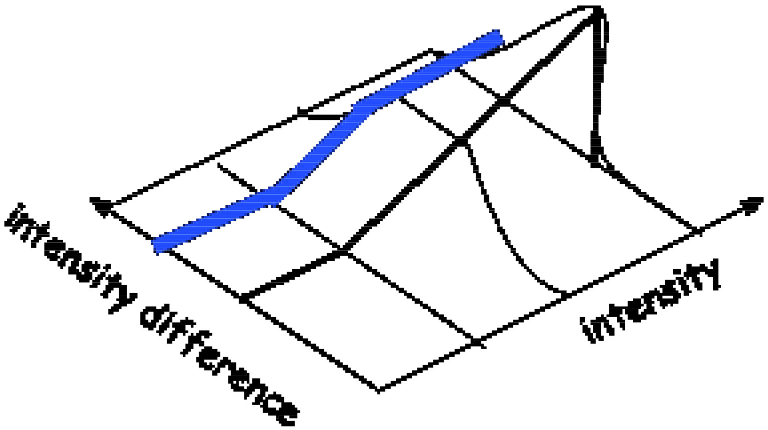


This is a plot of the relationship between the two membership functions

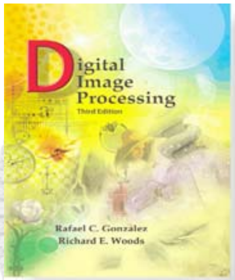
This is the membership function for difference AND white

A fuzzy output for a given input

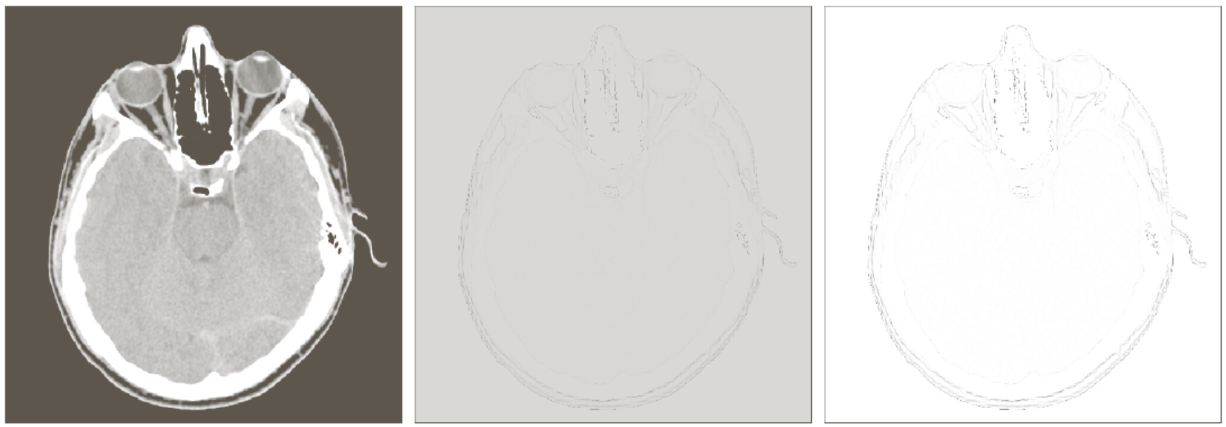
We now need to evaluate each output membership function for the given input value



This would be the output membership for a specific intensity difference input AND white



Fuzzy Boundary Extraction



a b c

FIGURE 3.59 (a) CT scan of a human head. (b) Result of fuzzy spatial filtering using the membership functions in Fig. 3.57 and the rules in Fig. 3.58. (c) Result after intensity scaling. The thin black picture borders in (b) and (c) were added for clarity; they are not part of the data. (Original image courtesy of Dr. David R. Pickens, Vanderbilt University.)

Sum of Functions

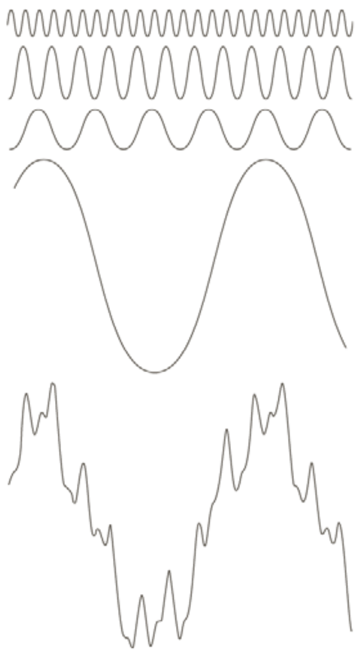
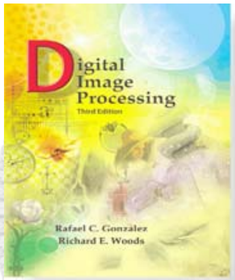


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.



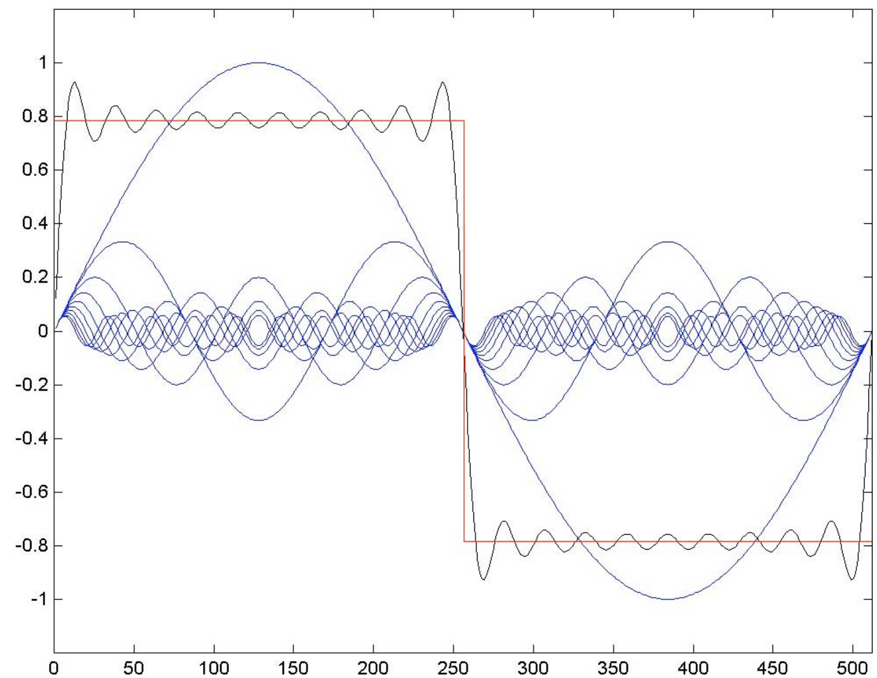
Fact: Any Real Signal has a Frequency-Domain Representation

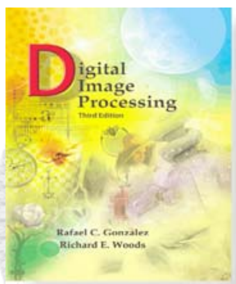
Odd-order harmonics

$$\mathrm{sq}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin\left[\frac{2\pi}{\lambda}(2n+1)t\right]$$

The terms shown (blue) sum to the rippling square wave (black).

As the number of terms in the sum becomes large, it approaches a square wave (red).





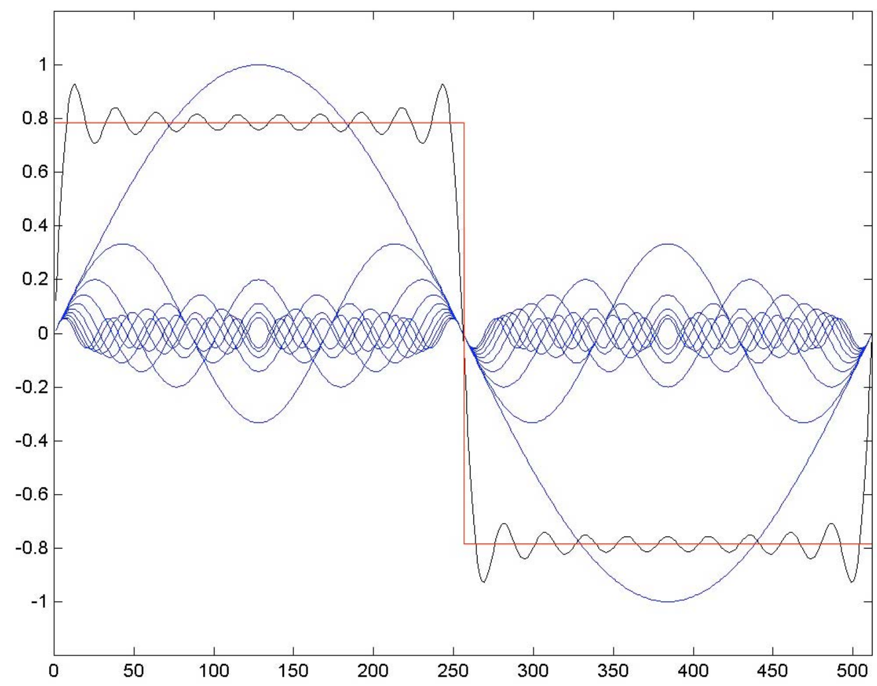
Frequency-Domain Representation

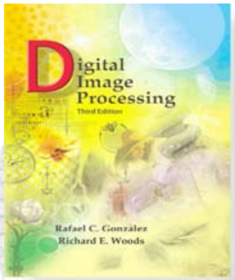
Any periodic signal can be described by a sum of sinusoids.

$$\mathrm{sq}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin\left[\frac{2\pi}{\lambda}(2n+1)t\right]$$

The sinusoids are called "basis functions".

The multipliers are called "Fourier coefficients".





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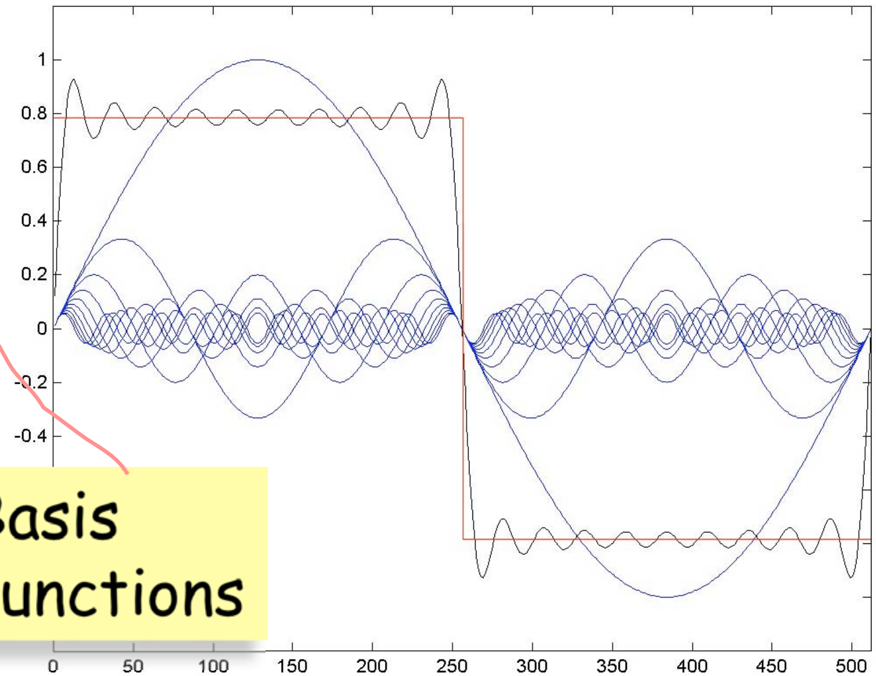
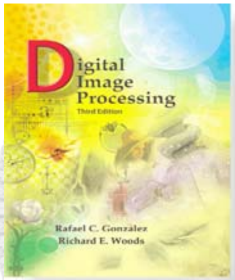
Basis functions





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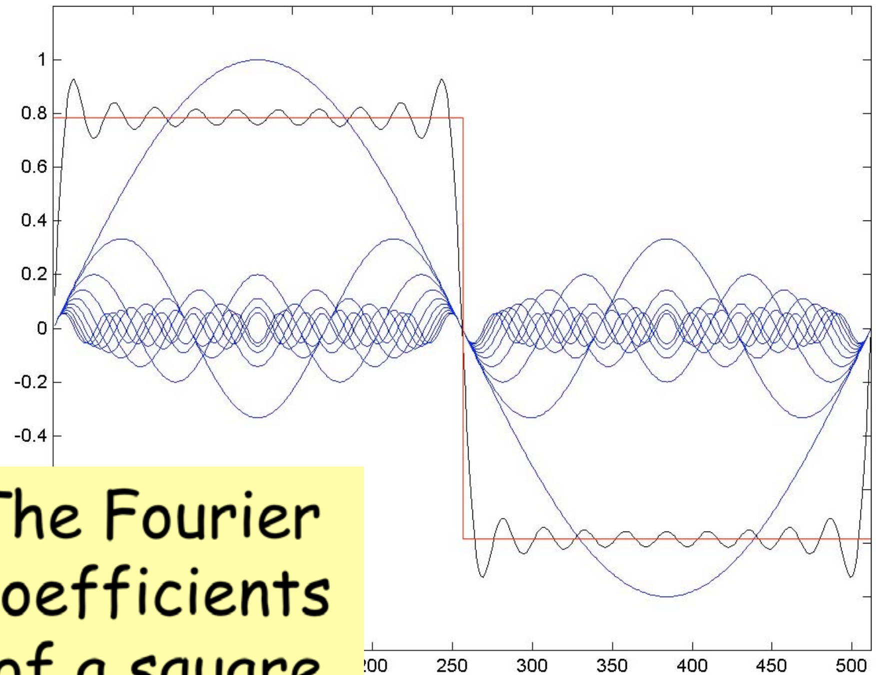
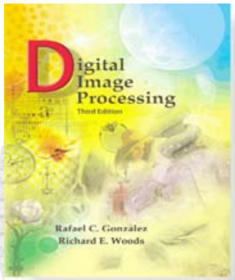


The Fourier coefficients (of a square wave).



The Inner Product: a Measure of Similarity

The similarity between functions f and g on the interval $\left(-\lambda/2, \lambda/2\right)$ can be defined by

$$\left\langle f, g \right\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\, g^{*}(t)\, dt$$

where $g^{*}(t)$ is the complex conjugate of $g(t)$.

This number, called the *inner product* of f and g, can also be thought of as the amount of g in f or as the projection of f onto g.

If f and g have the same energy, then their inner product is maximal if $f = g$. On the other hand if $\left\langle f, g \right\rangle = 0$, then f and g have nothing in common.

Inner Product of a Periodic Function and a Sinusoid

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \sin\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \cos\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\begin{aligned}\langle f, g \rangle &= \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\tfrac{2\pi}{\lambda} t\right) - j \sin\left(\tfrac{2\pi}{\lambda} t\right)\right] dt \\ &= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-j\frac{2\pi}{\lambda} t} \, dt \\ &= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-j\omega t} \, dt\end{aligned}$$

3 different representations

$$e^{-j\frac{2\pi}{\lambda} t} = \cos\left(\tfrac{2\pi}{\lambda} t\right) - j \sin\left(\tfrac{2\pi}{\lambda} t\right)$$

$$\omega = \frac{2\pi}{\lambda}$$

Inner Product of a Periodic Function and a Sinusoid

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\sin\left(\tfrac{2\pi}{\lambda}t\right)dt$$

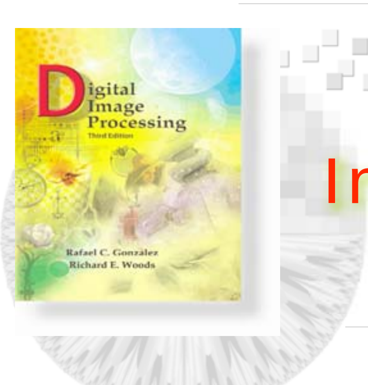
$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\cos\left(\tfrac{2\pi}{\lambda}t\right)dt$$

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t \ldots dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\frac{2\pi}{\lambda}t}dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\omega t}dt$$

real number results yield the amplitude of that sinusoid in the function.



Inner Product of a Periodic Function and a Sinusoid

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\sin\left(\tfrac{2\pi}{\lambda}t\right)dt$$

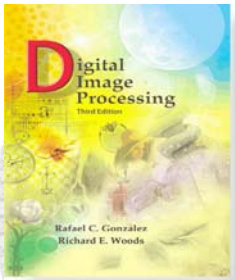
$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\cos\left(\tfrac{2\pi}{\lambda}t\right)dt$$

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\left[\cos\left(\tfrac{2\pi}{\lambda}t\right) - i\sin\left(\tfrac{2\pi}{\lambda}t\right)\right]dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\frac{2\pi}{\lambda}t}\,dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\omega t}\,dt$$

Complex number result yields the amplitude and phase of that sinusoid in the function.



The Fourier Series

is the decomposition of a λ-periodic signal into a sum of sinusoids.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{\lambda} t\right) + B_n \sin\left(\frac{2\pi n}{\lambda} t\right)$$

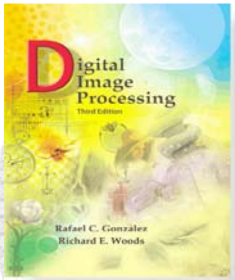
periodic: $\exists \lambda \in \Re$ such that $f(t \pm n\lambda) = f(t)$.

$$A_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t)\left[\cos\left(\frac{2\pi n}{\lambda} t - \phi_n\right)\right] dt \text{ for } n \geq 0$$

$$B_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t)\left[\sin\left(\frac{2\pi n}{\lambda} t - \phi_n\right)\right] dt \text{ for } n \geq 0$$

The representation of a function by its Fourier Series is the sum of sinusoidal “basis functions” multiplied by coefficients.

Fourier coefficients are generated by taking the inner product of the function with the basis.



The Fourier Series

can also be written in terms of complex exponentials

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{+j\frac{2\pi n}{\lambda}t} = \sum_{n=-\infty}^{\infty} |C_n| e^{+j\left(\frac{2\pi n}{\lambda}t + \phi_n\right)}$$

$$= \sum_{n=-\infty}^{\infty} |C_n| \cos\left(\frac{2\pi n}{\lambda}t + \phi_n\right) + j \cdot |C_n| \sin\left(\frac{2\pi n}{\lambda}t + \phi_n\right)$$

$$j = \sqrt{-1}$$

$$C_n = |C_n| e^{+j\phi_n}$$

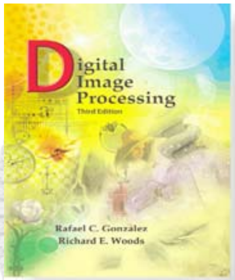
$$e^{\pm jx} = \cos x \pm j \sin x$$

$$Cn = |Cn| e^{+j\phi_n} = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) e^{-j\frac{2\pi n}{\lambda}t} dt$$

$$= \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\frac{2\pi n}{\lambda}t - \phi_n\right) - j \cdot \sin\left(\frac{2\pi n}{\lambda}t - \phi_n\right)\right] dt$$

$$f(t + n\lambda) = f(t)$$

for all intergers n



Why are Fourier Coefficients Complex Numbers?

$$f(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{+j\frac{2\pi n}{\lambda}t} \quad \text{where} \quad C_n = |C_n|\, e^{+j\phi_n}.$$

C_n represents the amplitude, $A=|C_n|$, and relative phase, ϕ, of that part of the original signal, $f(t)$, that is a sinusoid of frequency $\omega_n = n / \lambda$.