Random Signal Processing

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The sum we seek has a lower limit of 1 instead of 0, so we should calculate the sum using Eq. 5.1.6 and then subtract the term at $n = 0$. With $a = 0.81$ this gives

$$\sum_{n=1}^{\infty} x^2(n) = 4\sum_{n=1}^{\infty} (0.81)^n = \frac{4(0.81)}{1-0.81} = 17.0526$$

Therefore the energy is double this number plus $x^2(0)$.

$$E = 2(17.0526) + 4 = 38.1052$$

The power is 0, as we see from the following sequence of sums:

For $N = 5$, $$P = \frac{1}{11}\sum_{n=-5}^{5} x^2(n) = 2.383$$

For $N = 10$, $$P = \frac{1}{21}\sum_{n=-10}^{10} x^2(n) = 0.5471$$

For $N = 100$, $$P = \frac{1}{201}\sum_{n=-100}^{100} x^2(n) = 0.0552$$

In the limit, $P = 0$. Because the power is 0 and the energy is greater than 0 but finite, this is an energy signal. (The formula for calculating finite sums is derived in the next section; see Eq. 5.2.6.)

We will use only those signals that can be classified into one of the two categories of energy or power signals, according to the value of P or E from these definitions. Notice once again that a power signal must last forever, so these definitions apply only to their mathematical models.

Review

For continuous-time signals Eqs. 5.1.2 and 5.1.3 define power and energy, respectively. Equations 5.1.4 and 5.1.5 define power and energy for discrete-time signals. You should understand how to apply these formulas to find power and energy in a signal. You should also understand that these concepts apply equally to deterministic or random signals.

5.2 Time Correlation Functions

Preview

Correlation is a binary operation. This means that we operate on two waveforms and produce a third waveform. Convolution does the same thing, and convolution and correlation are almost identical, differing only in the details of calculation and in their use. Convolution is a mathematical description of the operation of linear systems on input signals, while correlation has many practical applications.

Initial applications were to radar and pulse communication, where a pulse of known shape is either present or absent during each specified period. The receiver uses a correlation receiver to determine (guess) whether or not the signal is present during that time interval. Almost all data and long-distance voice communication is handled in this way.

Pattern recognition illustrates another application. Despite all the theoretical and experimental work done to improve automatic pattern-recognition systems, most systems use correlation to decide (guess) which pattern is present. One familiar and very successful system is the bank check readers now used by every bank in the United States. The numbers and symbols printed across the bottom of your checks are specially designed for a correlation detector. A scanning device produces a signal that is sampled nine times during the span of each symbol. Therefore each symbol is represented by a digital signal of nine samples. There are 14 different symbols used in this system, 10 numbers and 4 special characters, so there are 14 different matched filters used to detect each symbol by correlation.

Automatic page scanners use correlation. These devices read and enter a page full of text into a computer automatically, a great labor-saving device in many offices. Correlation decides which letter or number is present at each possible symbol location. Page scanners are reliable for one particular font and type size, but as you can imagine, they are of little value for reading arbitrary type styles and sizes without adjustment. This is an example of two-dimensional correlation. We match a template for each class to the unknown symbol and make the decision on the basis of the highest correlation value. This is called template matching in pattern recognition.

These examples illustrate only some of the many applications of correlation to technical problems. For a better understanding of this general technique we need to describe the conditions under which we may apply the various forms of correlation. In this chapter we look at the four forms of time correlation. In later chapters we look at statistical correlation. In this section we define correlation for each of the four types of signals: continuous-time power and energy signals, and discrete-time power and energy signals. After completing this section you should be able to find and plot the correlation function for any two given signals when both are the same type.

Correlation is a binary operation. The black box with two input lines and one output line in Fig. 5.2.1 shows a binary operation. The chief requirement for this box is that the same type of things that go in come out. If the two input terms are numbers, then the output must be a number. If the two input terms are functions, then the output must be a function. (I suppose if the two inputs are elephants, then the output must be an elephant, but we'll stick to mathematics.) Ordinary addition of two numbers is a binary operation. We add two numbers x and y to produce the sum $z = x + y$. Convolution is a binary operation, where the two input terms are functions and the output

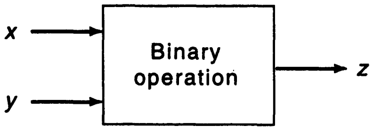

Fig. 5.2.1. A black box.

term is a function. Both statistical and time correlations are binary operations, and as we will see, time correlation is closely related to convolution.

The four forms of the correlation operation correspond to the four signal types we deal with. These four signal types are

Type 1. Continuous-time energy signals
Type 2. Continuous-time power signals
Type 3. Discrete-time energy signals
Type 4. Discrete-time power signals

See Section 5.1 for a discussion of these signal types. There is a different correlation formula for each type. We discuss each formula in turn below.

Type 1. Continuous-Time Energy Signals

The correlation $r_{xy}(t)$ between two continuous-time energy signals $x(t)$ and $y(t)$ is given by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t - \tau)\, dt \tag{5.2.1a}$$

$$= \int_{-\infty}^{\infty} x(t + \tau)\, y(t)\, dt \tag{5.2.1b}$$

If $x(t) = y(t)$, this is called the *autocorrelation* and is written $r_{xx}(\tau)$. If $x(t) \neq y(t)$, this is called the *cross correlation* between x and y. Notice that $r_{xy}(\tau) \neq r_{yx}(\tau)$. The two formulas in Eq. 5.2.1 give the same result because a change of variable relates the two integrals. Let $\lambda = t - \tau$ in the first form to obtain the second form. We will use lowercase r for time correlation, and uppercase R for statistical correlation.

Figure 5.2.2 shows correlation as a binary operation. The input terms are continuous-time signals x and y, so the output $r_{xy}(\tau)$ must also be a continuous-time signal. When the input terms are discrete-time signals, the correlation is a discrete-time signal. Here are some examples.

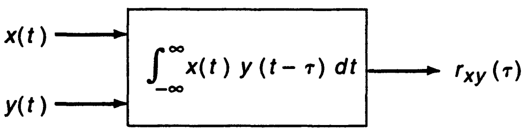

Fig. 5.2.2

EXAMPLE 5.2.1. Find the correlation function $r_{xy}(\tau)$ for the two signals in Figs. 5.2.3a and 5.2.3b.

SOLUTION: Figure 5.2.4 shows a plot of $x(t - \tau)$ and $y(t)$ for various values of shift τ. In Fig. 5.2.4a no overlap occurs between $x(t - \tau)$ and $y(t)$, so the correlation from Eq. 5.2.1a is 0. Notice that the abscissa is labeled t in the diagram, and the variable of integration in Eq. 5.2.1 is t. You should plot functions versus the variable of integration for both correlation and convolution so that they will provide a picture of the integration process. Figures 5.2.4b and 5.2.4c show the picture for $-1 < \tau < 0$, and for $0 < \tau < 1$. The picture for $1 < \tau$ is not shown, but there is no overlap so the correlation is 0 in that interval.

Figure 5.2.5 shows the correlation function $r_{xy}(\tau)$ plotted versus τ. Since $x(t) = y(t)$ this is also the autocorrelation function for signal $x(t)$. Notice that $r_{xx}(0)$ is the energy in the signal $x(t)$, which we calculated in Example 5.1.1.

EXAMPLE 5.2.2. Find the correlation function $r_{xy}(\tau)$ for the two signals in Fig. 5.2.6a and 5.2.6b.

SOLUTION: Figure 5.2.7 shows the answer. You should be able to arrive at the answer by using the same techniques for evaluating the correlation integral as in the previous example.

Type 2. Continuous-Time Power Signals

The correlation $r_{xy}(\tau)$ between two continuous-time power signals $x(t)$ and $y(t)$ is given by

$$r_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,y(t-\tau)\,dt \tag{5.2.2a}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\,y(t)\,dt \tag{5.2.2b}$$

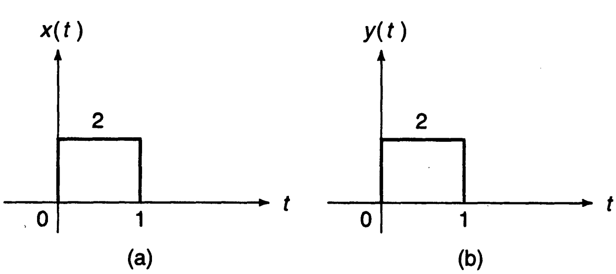

Fig. 5.2.3

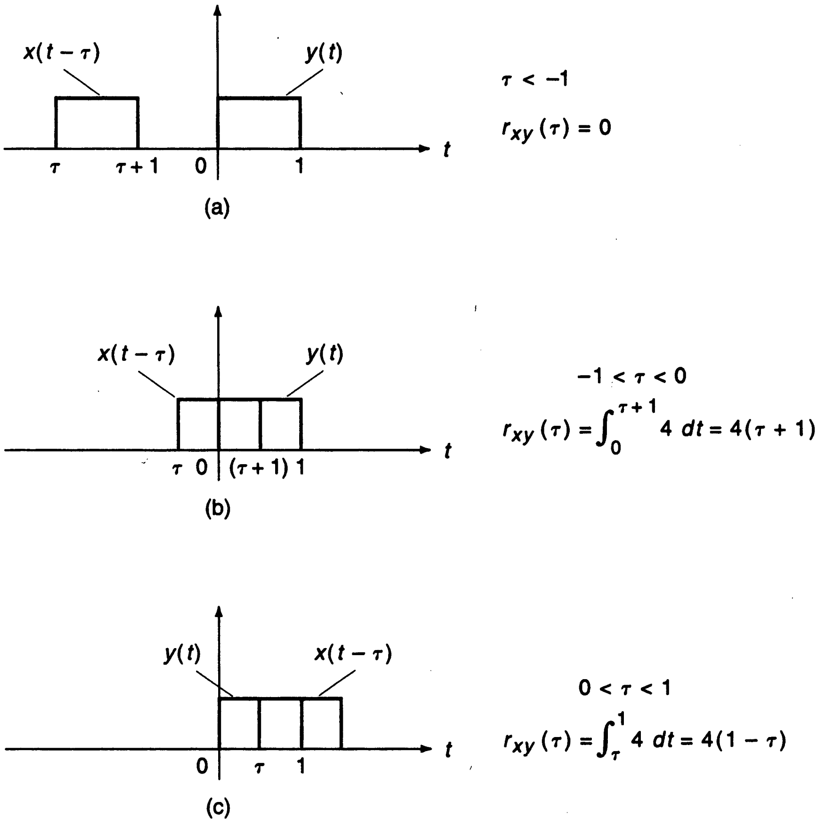

Fig. 5.2.4

This is similar to our definition for mean square value or power (see Eq. 5.1.2), because we have the limit as $T \to \infty$ in the formula.

If $x(t)$ and $y(t)$ are periodic with the same period T, we may replace Eq. 5.2.2 by the simpler form given by

$$r_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)\,y(t-\tau)\,dt \tag{5.2.3a}$$

$$= \frac{1}{T}\int_0^T x(t+\tau)\,y(t)\,dt \tag{5.2.3b}$$

EXAMPLE 5.2.3. Find the autocorrelation function $r_{xx}(\tau)$ for the signal in Fig. 5.2.8a.

SOLUTION: Figure 5.2.8b shows $x(t-\tau)$ for $0 < \tau < T/2$. Figure 5.2.8c shows the product $x(t) \times x(t-\tau)$. Integrate over one period and divide

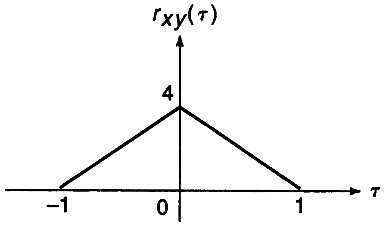

Fig. 5.2.5. The correlation of $x(t)$ with $y(t)$.

by T according to Eq. 5.2.3a to get

$$r_{xx}(\tau) = \frac{1}{T}\int_{\tau}^{T/2} E^2\,dt = \frac{E^2}{T}\left(\frac{T}{2} - \tau\right), \qquad 0 < \tau < \frac{T}{2}$$

Figure 5.2.9 is like Fig. 5.2.8, but for $T/2 < \tau < T$. Figure 5.2.9a shows the original function $x(t)$. Figure 5.2.9b shows $x(t - \tau)$ for τ approximately equal to $3T/2$, and Figure 5.2.9c shows the product. Now integrate over one period to get

$$r_{xx}(\tau) = \frac{1}{T}\int_{0}^{\tau-T/2} E^2\,dt = \frac{E^2}{T}\left(\tau - \frac{T}{2}\right), \qquad \frac{T}{2} < \tau < T$$

If this process is duplicated for all values of τ, the result is the periodic autocorrelation function shown in Fig. 5.2.10.

This example illustrates that the result of correlating two periodic signals with the same period is a periodic correlation function with the same period. Do not, however, draw the incorrect conclusion that the result of correlating two power signals is a power correlation function. In this example, the correlation $r_{xx}(\tau)$ is periodic, and is therefore a power function. But that is only in this example. The correlation function for the random power signal in Section 1.2 is an energy function.

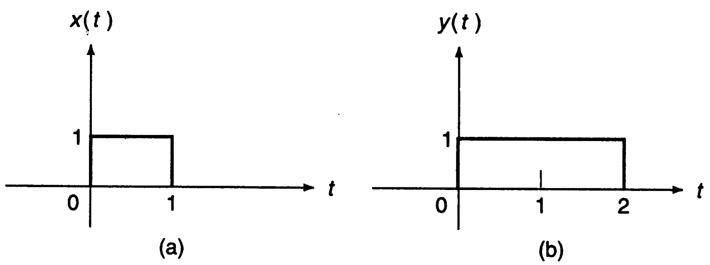

Fig. 5.2.6

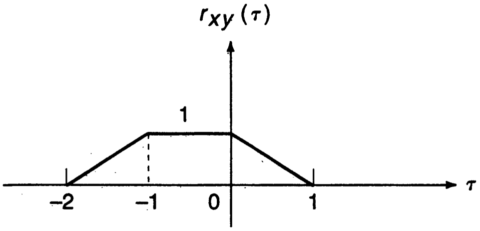

Fig. 5.2.7. The solution to Example 5.2.2.

Type 3. Discrete-Time Energy Signals

The correlation $r_{xy}(t)$ between two discrete-time energy signals $x(n)$ and $y(n)$ is given by

$$r_{xy}(\tau) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-\tau) \tag{5.2.4a}$$

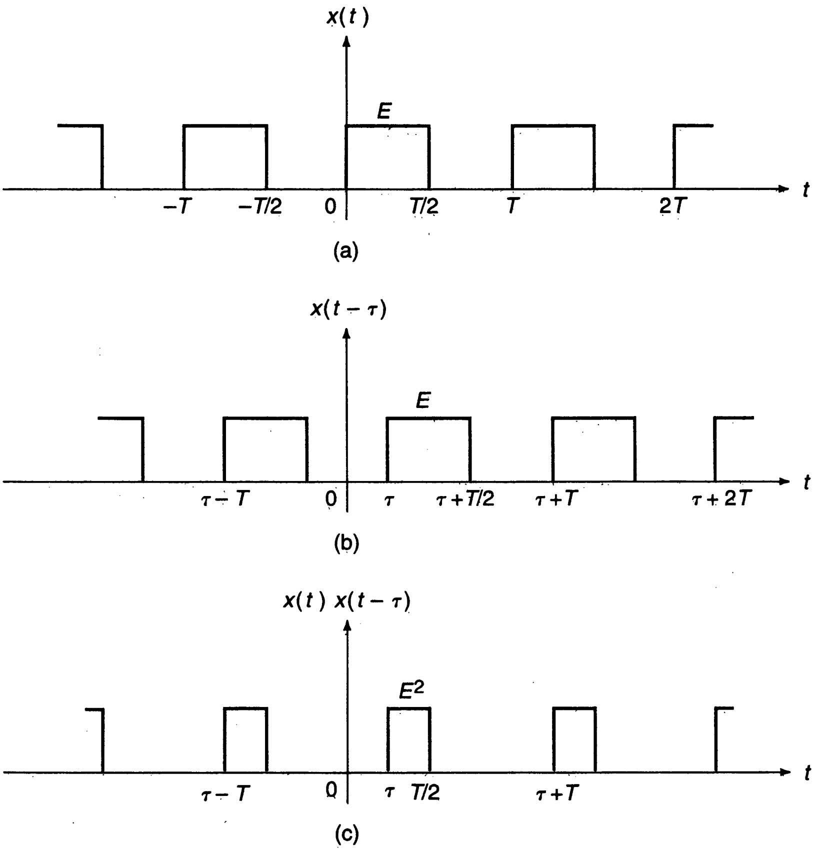

Fig. 5.2.8. Calculating correlation for $0 < \tau < T/2$.

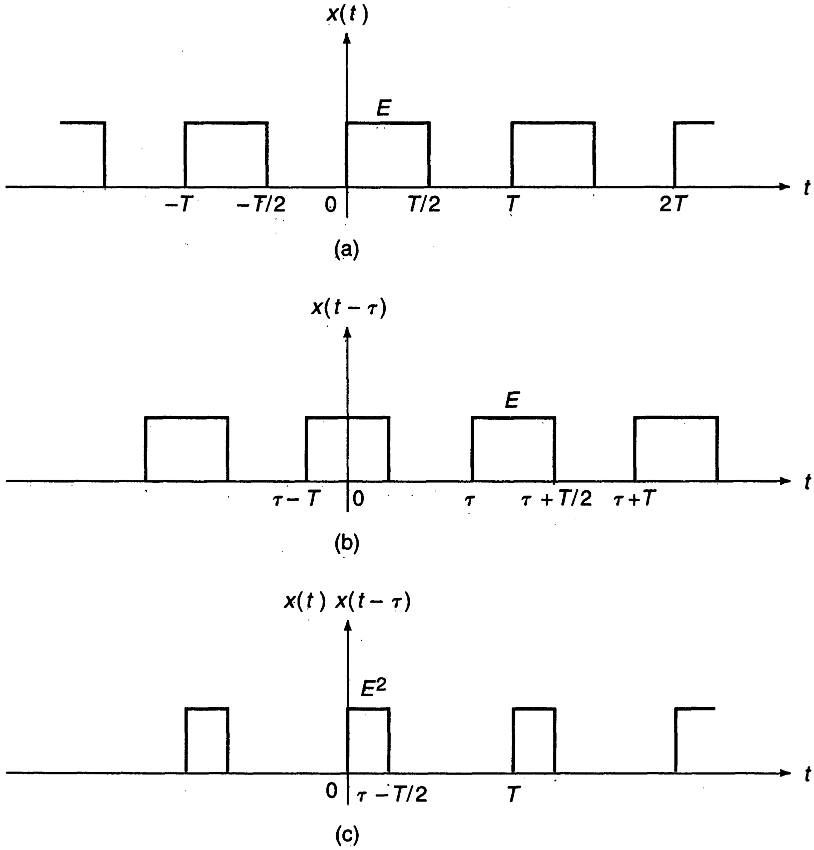

Fig. 5.2.9. Calculating correlation for $T/2 < \tau < T$.

$$= \sum_{n=-\infty}^{\infty} x(n + \tau)\, y(n) \tag{5.2.4b}$$

We will use τ for the shift parameter for both continuous- and discrete-time signals. You should be aware, however, that τ is a continuous variable in

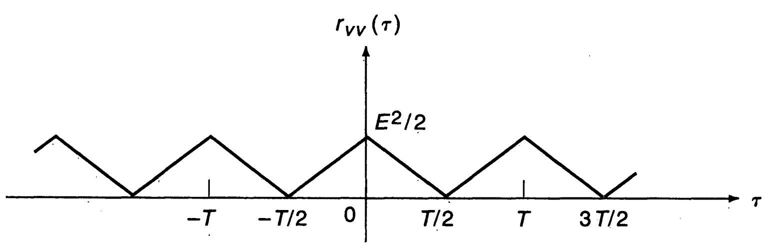

Fig. 5.2.10. The autocorrelation function for Example 5.2.3.

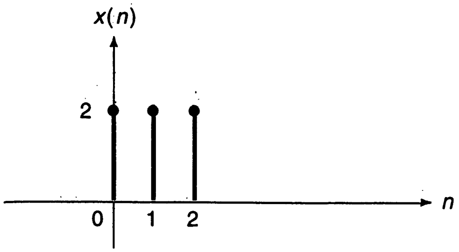

Fig. 5.2.11

Eqs. 5.2.1 through 5.2.3, and τ is a discrete variable here for digital signals.

EXAMPLE 5.2.4. Find the autocorrelation function $r_{xx}(\tau)$ for the signal in Fig. 5.2.11.

SOLUTION: This is the energy signal from Example 5.1.4. Figure 5.2.12a shows $x(n - \tau)$ for $\tau < 0$, along with $x(n)$. You can see that as τ increases, overlap occurs at $\tau = -2$. The sum in Eq. 5.2.4 therefore has the values shown in Fig. 5.2.12b.

EXAMPLE 5.2.5. Find the autocorrelation function $r_{xx}(\tau)$ for the exponential energy signal shown in Fig. 5.2.13.

SOLUTION: The problem here is to evaluate the sum

$$r_{xx}(\tau) = \sum_{n=-\infty}^{\infty} x(n - \tau)\, x(n)$$

for each value of τ. We begin with $\tau < 0$, as shown in Fig. 5.2.14a. After multiplying the functions $x(n - \tau)$ and $x(n)$, we have two regions with different terms in the product.

Region 1: $(n < 0)$, product $= 0$.
Region 2: $(n \geq 0)$, product $= (0.9)^{(n-\tau)}(0.9)^n = (0.9)^{-\tau}(0.9)^{2n}$.

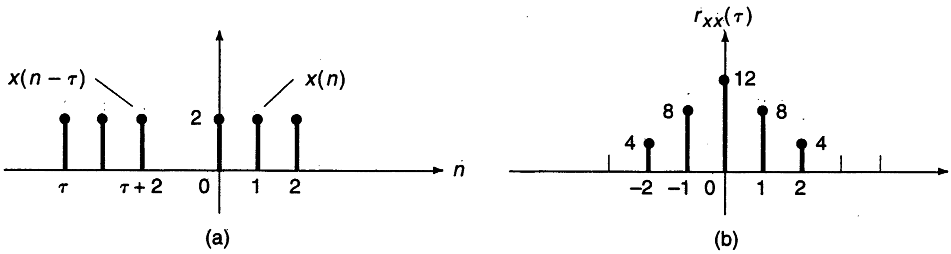

Fig. 5.2.12

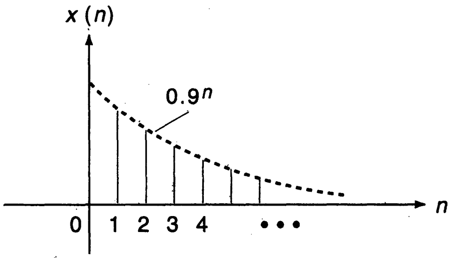

Fig. 5.2.13

The sum over these two regions gives the autocorrelation function for $\tau < 0$.

$$r_{xx}(\tau) = \sum_{n=-\infty}^{-1} 0 + \sum_{n=0}^{\infty} (0.9)^{-\tau}(0.9)^{2n} = (0.9)^{-\tau} \sum_{n=0}^{\infty} (0.81)^n$$

The sum $\Sigma_{n=0}^{\infty} (0.81)^n$ is a geometric series. This familiar series has the value given by

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \qquad |a| < 1 \tag{5.2.5}$$

To see this, divide $1 - a$ into 1 by long division. With $a = 0.81$ the autocorrelation function is

$$r_{xx}(\tau) = \frac{1}{0.19}(0.9)^{-\tau} \qquad \text{for } T < 0$$

Figure 5.2.14b shows the picture for $\tau > 0$. For this situation the autocorrelation sum is given by

$$r_{xx}(\tau) = \sum_{n=-\infty}^{\tau-1} 0 + \sum_{n=\tau}^{\infty} (0.9)^{(n-\tau)}(0.9)^n = (0.9)^{-\tau} \sum_{n=\tau}^{\infty} (0.81)^n$$

If the lower limit on the sum was 0, we could use Eq. 5.2.5 to find its value. Since the lower limit is τ, we can write

$$\sum_{n=\tau}^{\infty} a^n = \sum_{n=0}^{\infty} a^n - \sum_{n=0}^{\tau-1} a^n$$

Therefore we need a closed form expression for this last term. To find this, write

$$S = \sum_{n=0}^{\tau-1} a^n = 1 + a + a^2 + \cdots + a^{\tau-1}$$

$$aS = a \sum_{n=0}^{\tau-1} a^n = a + a^2 + a^3 + \cdots + a^{\tau}$$

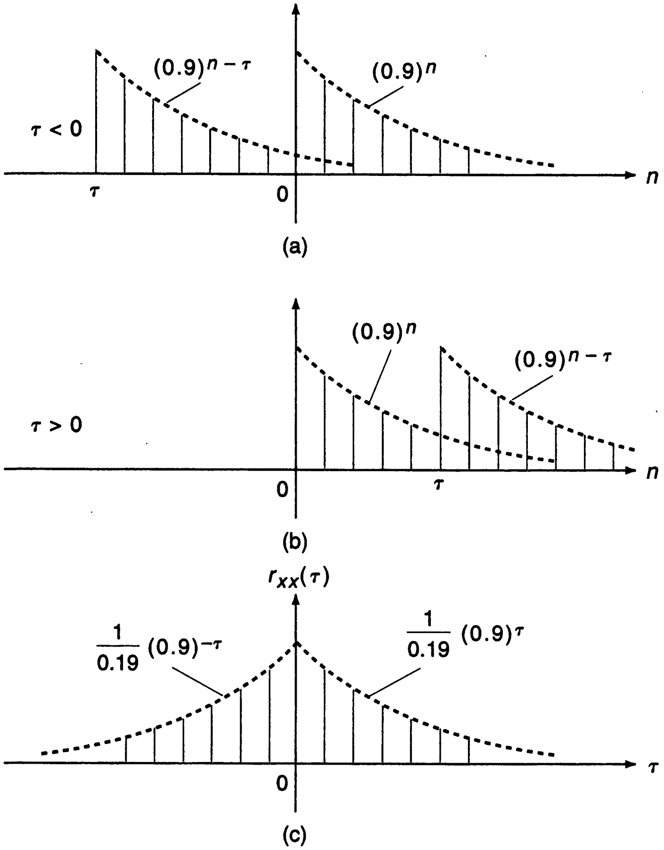

Fig. 5.2.14

Now subtract the bottom expression from the top one to get

$$S - aS = 1 - a^{\tau}$$

or

$$S = \sum_{n=0}^{\tau-1} a^n = \frac{1 - a^{\tau}}{1 - a}, \qquad |a| \neq 1 \tag{5.2.6}$$

For our sum, $a = 0.81$. Note that $(0.81)^{\tau} = (0.9)^{2\tau}$. From all this we can write

$$r_{xx}(\tau) = \frac{1}{0.19}(0.9)^{\tau} \qquad \text{for } \tau \geq 0$$

This result is plotted in Fig. 5.2.14c.

Type 4. Discrete-Time Power Signals

The correlation $r_{xy}(\tau)$ between two discrete-time power signals $x(n)$ and $y(n)$ is given by

$$r_{xy}(\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y(n-\tau) \tag{5.2.7a}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+\tau)\, y(n) \tag{5.2.7b}$$

If $x(n)$ and $y(n)$ are periodic with the same period N, we may replace Eq. 5.2.7 by the simpler form given by

$$r_{xy}(\tau) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, y(n-\tau) \tag{5.2.8a}$$

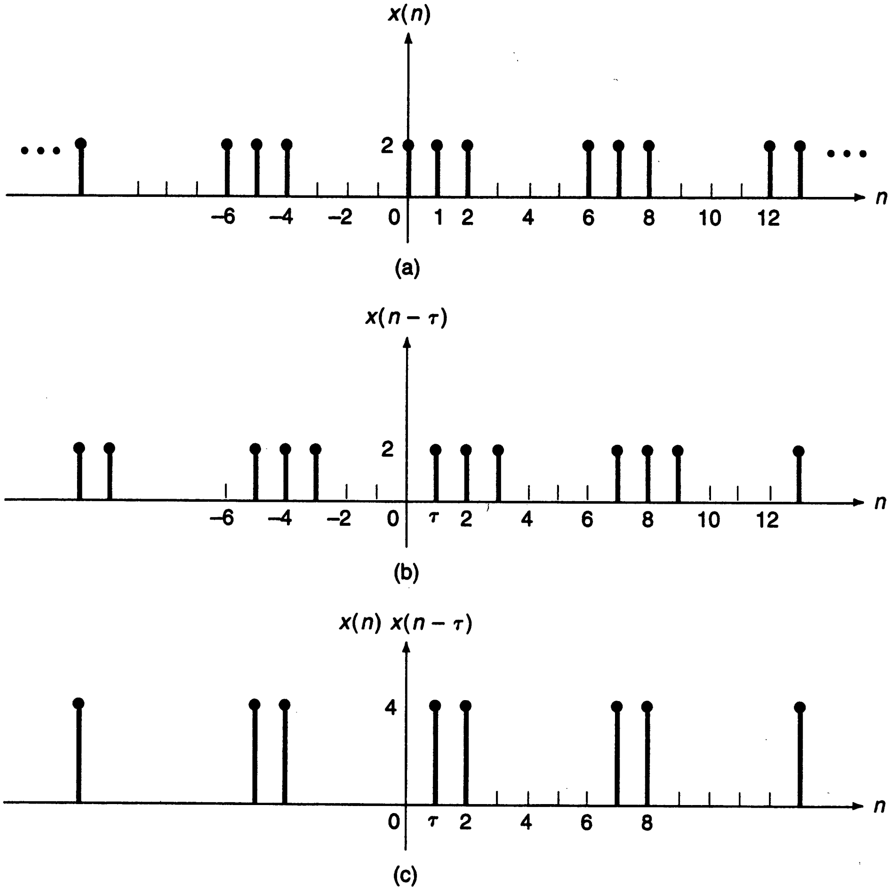

Fig. 5.2.15

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n + \tau)\, y(n) \qquad (5.2.8b)$$

EXAMPLE 5.2.6. Find the autocorrelation function for the periodic signal from Example 5.1.5, shown in Fig. 5.2.15a.

SOLUTION: We multiply the shifted function $x(n - \tau)$ in Fig. 5.2.15b by $x(n)$ to produce the product in Fig. 5.2.15c. In this picture, $\tau = 1$, but the general idea is to shift $x(n)$ by any amount τ and multiply to produce the product. For each value of τ the average in Eq. 5.2.8 gives the correlation function plotted in Fig. 5.2.16.

As with continuous-time signals, periodic signals produce periodic autocorrelation functions. Also notice that $r_{xx}(0)$ is the mean square value of the signal. This is a general property of autocorrelation functions. The value at $\tau = 0$ is either the power or the energy in the signal, depending on whether we have a power or an energy signal.

Random Signals

The time correlation function that we have defined for each of the four types of signals applies to both random and deterministic time functions. Therefore we could use these formulas for the coin-toss function in Section 1.2. The trouble is that these formulas are accurate only for $T \to \infty$. The correlation functions in Section 1.2 for the coin-tossing experiment were calculated using the statistical relation between samples, rather than the formulas for time correlation functions that we have presented in this section. Let us now calculate the time correlation functions for the signal $x_0(n)$ for comparison. See Section 1.2 for a description of this signal. Figure 1.2.2 shows a plot of the first 100 samples of $x_0(n)$.

The idea expressed in Eq. 5.2.7 for calculating the autocorrelation function is to multiply $x(n + \tau)$ by $x(n)$, sum over all terms, and divide by the number

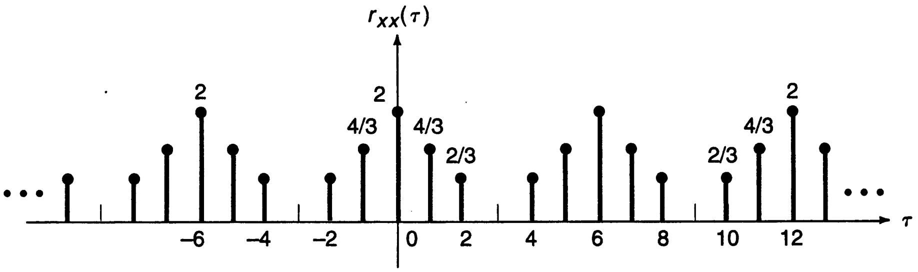

Fig. 5.2.16

Recursive filtering, which is patterned after Eq. 5.2.10, plays an important part in applications. We will discuss Kalman (recursive) filtering in Section 8.6.

Here is a listing of procedure corr, which implements Eq. 5.2.9. The input to this procedure is the signal x, and it returns the correlation $r(\tau)$ for $0 \leq \tau < M$.

```
procedure corr(x)   /* find autocorrelation of x */
define x(N), r(M)
       τ = 0
       while τ < M
       begin
              n = 1
              sum = 0
              while n ≤ (N−τ)
              begin
                     sum = sum + x(n)*x(n+τ)
                     n = n + 1
              end for n
              r(τ) = sum/(N−τ)
              τ = τ + 1
       end for τ
       return(r)
end corr
```

EXAMPLE 5.2.7. In Section 1.3 we promised to find the correlation between successive numbers from the random number generator there. The autocorrelation function for $\tau = 1$ gives this value. The plot in Fig. 5.2.20 shows the normalized time autocorrelation function averaged over 500 values of this number generator, which was found by the procedure above. At $\tau = 1$ the value of the function is 0.046, a relatively small number compared with 1. If we could trust the results from only one sample of length 500 from the random number generator, we could say that this is a reasonable generator.

Noisy Periodic Sequences

Suppose a signal is periodic with added noise,

$$x(n) = s(n) + w(n) \tag{5.2.11}$$

where $s(n)$ is the signal, $w(n)$ is the noise, and $x(n)$ is the received signal. Because the autocorrelation function of a periodic signal is periodic, the autocorrelation function of $x(n)$ reveals periodic tendencies in the signal. To show why, consider the autocorrelation function of $x(n)$.

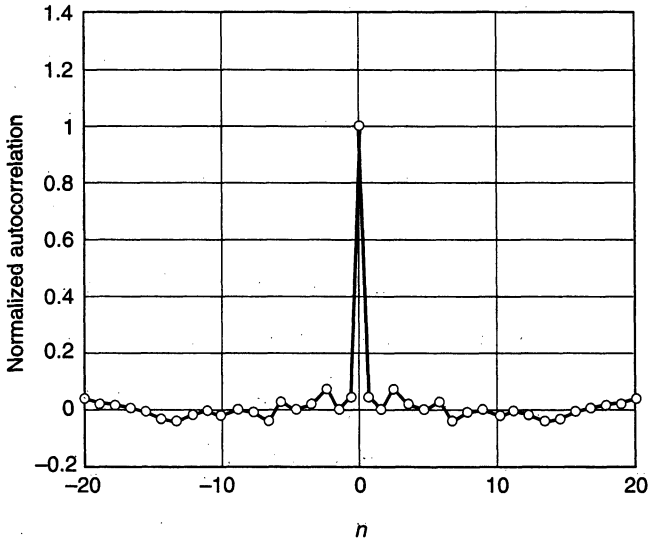

Fig. 5.2.20. Normalized autocorrelation function for the random number generator.

$$r_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x(n-\tau) \qquad (5.2.12)$$

Substitute Eq. 5.2.11 into this expression to get

$$\begin{aligned} r_{xx}(\tau) &= \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} [s(n) + w(n)][s(n-\tau) + w(n-\tau)] \\ &= r_{ss}(\tau) + r_{sw}(\tau) + r_{ws}(\tau) + r_{ww}(\tau) \end{aligned} \qquad (5.2.13)$$

The first factor is the autocorrelation function of $s(n)$. If $s(n)$ is periodic, this function is periodic. The second and third terms represent cross correlations between signal and noise. If there is no relation between signal and noise, these are 0 for all τ. The last term represents the noise correlation, and this is often 0 for all $\tau \neq 0$. Therefore, under these conditions we can detect periodic components in a received signal by finding its autocorrelation function.

Review

Basically the same operation calculates correlation functions for each class of signal. Integration applies to continuous-time signals, and summation applies to discrete-time signals. For energy signals we integrate or sum over all time, from $-\infty$ to $+\infty$. For power signals we integrate or sum over one period, and then divide by the period length. This works well for periodic signals, but has little meaning for aperiodic signals. This business of letting the period approach ∞

makes good sense until we try to put it into practice. That is when we begin to appreciate the statistical approach for power signals.

You should now be able to calculate the time correlation function for two signals from the same class.

5.3 Time–Frequency Relations

Preview

Some people become set in their ways as they grow older, and some become eccentric. Jean Baptiste Joseph Fourier (1768–1830) became eccentric. He pursued a military career in his youth, spending time in Egypt with Napoleon Bonaparte, and also pursued his mathematical interests. He was too good a mathematician to stay in military service, but his stay in Egypt exposed him to the benefits of desert heat. Believing heat essential to health, Fourier spent his later years with his residence overheated and himself swathed in layers of clothes. Perhaps he was right: Isaac Asimov said that Fourier died of a fall down the stairs. (The mathematical historian E.T. Bell said he died of heart disease.)

Fourier's theorem, announced in 1807, states that a periodic function can be expressed as the sum of sinusoidal components. This we now call the Fourier series for continuous-time periodic power signals, and is one of four formulas we call the Fourier transform. There is a Fourier transform for each of the four types of signals, but before defining these four transforms, let us discuss transforms in general.

A transform is a special type of function. In the beginning, the concept of a function applied to numbers. The domain and codomain consisted of numbers. Now the term function applies to the relationship between any two sets if the two properties for a function are satisfied. These are: (1) for each element in the first set (the domain), there corresponds *at least* one element in the second set (the codomain); and (2) for each element in the domain, there corresponds *at most* one element in the codomain. We combine these into one statement by saying that a function is a relationship between two sets such that for each element in the first set there corresponds *exactly* one element in the second set. Here is an example of a function where neither the domain nor the codomain have anything to do with numbers.

Suppose I have a basket full of cards, and on each card is written an instruction. You reach into the basket, select a card, and carry out the instruction written on it. This is a function. The domain is the basket of cards with their instructions. The codomain is the set of all possible actions you could perform. The set of actions that might actually be performed (those on the cards) is called the range, and this is a subset of the codomain.

This distinction between the codomain and range occurred recently in mathematics. We previously called both sets the range, which led to some confusion. In modern parlance the term *range* refers to those elements in the codomain that