2 Stationary Magnetic Fields

2.1 Introduction

Magnetic effects have many similarities to electric effects, but there are also important differences. Magnetic forces were first observed through the attraction of iron to naturally occurring magnetic materials such as lodestone. The compass, apparently developed in China, was introduced into Europe around A.D. 1190, and had a profound effect upon navigation thereafter. In 1600 William Gilbert, physician to Queen Elizabeth I, published an important book, *De Magnete*, presenting a rational and thorough summary of the magnetic effects known to that date, with discussions of some of the similarities to and differences from the electric effects then known. Had discoveries stopped at that point, we could immediately adapt the development of the preceding chapter to magnetic fields, the two kinds of magnetic "charges" being called north and south *poles*. The important difference is that magnetic charges have so far been found only in pairs, not isolated, so that we would be concerned with fields from dipoles, as in Ex. 1.8d.

Discoveries did not stop, however. In 1820, Hans Christian Oersted, during a class demonstration of an electric battery, observed that the electric current in a wire caused a nearby compass needle to be deflected, thus establishing clearly the first of several important relationships between electric and magnetic effects. André-Marie Ampère very quickly extended the experiments and developed a quantitative law for the phenomenon. Others who contributed both to the understanding and to the practical use of electromagnets within a very short period were Jean-Baptiste Biot, Felix Savart, Joseph Henry, and Michael Faraday. The force produced by magnetic fields (either from permanent magnets or from electromagnets) on electric currents was also clearly established through these many experiments. These relationships between electric currents and magnetic fields will constitute the starting point for our development of magnetic fields in this chapter. The relationships are somewhat more complicated than those of the preceding chapter, primarily because both the current that acts as the source of field and the current element acting as a probe to measure it are vectors whose directions must be introduced into the laws.

As with electric fields, the distributions studied in this chapter, although called "static," are applicable to many time-varying phenomena. These "quasistatic" problems are among the most important uses of the laws and, in some cases, are valid for extremely rapid rates of change. Still we must remember that other phenomena enter—and are likely to be important—when the fields change with time. These are studied in the following chapter.

Before beginning the detailed development, let us look briefly at a few examples of important static or quasistatic magnetic field problems. There was the prompt application of Oersted's observation to useful electromagnets. One of Henry's early magnets supported more than a ton of iron, with the current driven only by a small battery. Electromagnets are now routinely used in loading or unloading scrap iron and many other applications. The development of practical superconductors in the 1960s has made possible magnets with high fields in large volumes with additional advantages of stability and light weight. Large currents can be made to flow in the magnet winding since there is no voltage drop and no heating. The need to refrigerate is compensated sufficiently for a number of special applications. Superconductors are used extensively in high-energy physics, where the need is for large volumes of strong field. Fusion research depends on massive superconductive magnets for containment of the ionized gases of a plasma. Motors and generators for special applications such as ship propulsion are being made lighter and smaller by using superconductors.[1]

Moving charges constitute currents and magnetic fields produce forces on them as they travel through a vacuum or a semiconductor. Thus magnetic field coils are used for deflection and focusing of beams of electrons in television picture tubes and electron microscopes. The magnetic deflection of flowing charge carriers in a semiconductor is known as the *Hall effect*; it is used for measurement of the semiconductor properties or, with a known semiconductor, may be used as a probe for measurement of magnetic field.

Coils are used to provide the inductance needed for high-frequency circuits and the magnetic fields can be found from the currents as in static calculations when the sizes involved are small compared with wavelength. (However, current distributions are complicated at high frequencies by distributed capacitance in the windings.) Just as we noted in Sec. 1.1 for electric fields, the distribution of magnetic field in the cross section of a transmission line is essentially the same as calculated using static field concepts, even though the fields can actually be varying at billions of times per second.

[1] *More details on superconductors can be found in Sec. 13.4.*

Static Magnetic Field Laws and Concepts

2.2 Concept of a Magnetic Field

As with the electrostatic fields of the preceding chapter, we use the measurable quantity, force, to define a magnetic field. We noted in Sec. 2.1 that magnetic forces may arise either from permanent magnets or from current flow. Since the approach from currents is more general—and on the whole more important—we start by consideration of the force between current elements. Permanent magnets may then be included, at least conceptually, by considering the effects of these as arising from atomic currents of the magnetic materials.

The force arising from the interaction of two current elements depends on the magnitude of the currents, the medium, and the distance between currents analogously to the force between electric charges. However, current has direction so the force law between the two currents will be more complicated than that for charges. Consequently, it is convenient to proceed by first defining the quantity we will call the magnetic field and then, in another section, give the law (Ampère's) that describes how currents contribute to that magnetic field. A vector field quantity **B**, usually known as the *magnetic flux density*, is defined in terms of the force **df** produced on a small current element of length **dl** carrying current I, such that

$$df = I\, dl\, B \sin\theta \tag{1}$$

where θ is the angle between **dl** and **B**. The direction relations of the vectors are so defined that the vector force **df** is along a perpendicular to the plane containing **dl** and **B**, and has the sense determined by the advance of a right-hand screw if **dl** is rotated into **B** through the smaller angle (Fig. 2.2). It is convenient to express this information more compactly through the use of the *vector product*. The vector product (also called *cross product*) of two vectors (denoted by a cross) is defined as a vector having a magnitude equal to the product of the magnitudes of the two vectors and the sine of the angle between them, a direction perpendicular to the plane containing the two vectors, and a sense given by the advance of a right-hand screw if the first is rotated

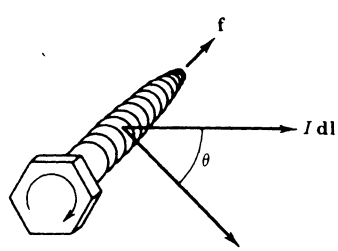

FIG. 2.2 Right-hand screw rule for force on a current element in a magnetic field.

into the second through the smaller angle. Relation (1) may then be written

$$\mathbf{df} = I\,\mathbf{dl} \times \mathbf{B} \tag{2}$$

The quantity known as the magnetic field vector or magnetic field intensity is denoted **H** and is related to the vector **B** defined by the force law (2) through a constant of the medium known as the *permeability*, μ:

$$\mathbf{B} = \mu \mathbf{H} \tag{3}$$

Many technologically important materials such as iron and ferrite are nonlinear and/or anisotropic, in which case μ is not a scalar constant, but to keep this introductory treatment simple, the medium will first be assumed to be homogeneous, isotropic, and linear. A somewhat more general form of (3) will be given in Sec. 2.3.

In SI units, force is in newtons (N). Current is in amperes (A), and magnetic flux density B is in tesla (T), which is a weber per square meter or volt second per square meter and is 10^4 times the common cgs unit, gauss. Magnetic field **H** is in amperes per meter and μ is in henrys (H) per meter. Conversion factors to other cgs units are in Appendix I. The value of μ for free space is

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

2.3 AMPÈRE'S LAW

Ampère's law, deduced experimentally from a series of ingenious experiments,[2] describes how the magnetic field vector defined in Sec. 2.2 is calculated from a system of direct currents. Consider an unbounded, homogeneous, isotropic medium with a small line element of length dl' carrying a current I' located at a point in space defined by a vector $\mathbf{r}'$ from an arbitrary origin as in Fig. 2.3*a*. The magnitude of the magnetic field at some other point P in space defined by the vector $\mathbf{r}$ from the origin is

$$dH(\mathbf{r}) = \frac{I'(\mathbf{r}')\, dl' \sin \phi}{4\pi R^2}$$

where $R = |\mathbf{r} - \mathbf{r}'|$, the distance from the current element to the point of observation. The angle ϕ is that between the direction of the current defined by $\mathbf{dl}'$ and the vector

[2] *For a description, see J. C. Maxwell,* A Treatise on Electricity and Magnetism, *3rd ed., Part IV, Chap. 2, Oxford Univ. Press, Oxford, 1892. The law is now more frequently named after Biot and Savart, but the assignment remains somewhat arbitrary. Following Oersted's announcement of the effect of currents on permanent magnets in 1820, Ampère immediately announced similar forces of currents on each other. Biot and Savart presented the first quantitative statement for the special case of a straight wire; Ampère later followed with his formulation for more general current paths. The form given here is a derived form borrowing from all that work. For more of the history see E. T. Whittaker,* A History of the Theories of the Aether and Electricity, Am. Inst. Physics, New York, 1987, or P. F. Mottelay, Bibliographical History of Electricity and Magnetism, *Ayer Co. Publishers, Salem, NH, 1975.*

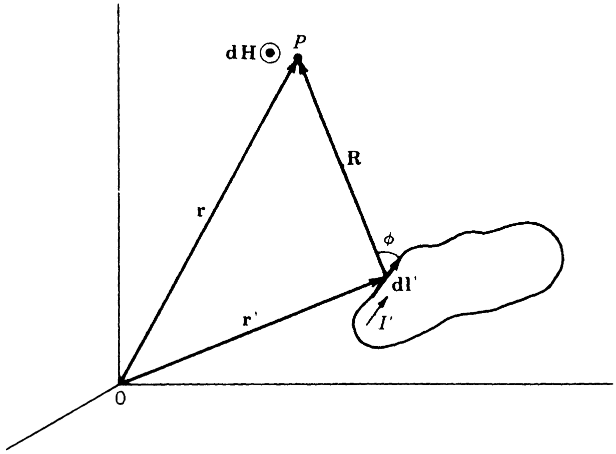

FIG. 2.3*a* Coordinates for calculation of magnetic field from current element.

$\mathbf{R} = \mathbf{r} - \mathbf{r}'$ from the current element to the point of observation. The direction of $\mathbf{dH}(\mathbf{r})$ is perpendicular to the plane containing $\mathbf{dl}$ and $\mathbf{R}$, and the sense is determined by the advance of a right-hand screw if $\mathbf{dl}$ is rotated through the smaller angle into the vector $\mathbf{R}$. Thus, with the current direction shown in Fig. 2.3*a*, $\mathbf{dH}$ at P is outward from the page. We see then that the cross product can be used to write the vector form of Ampère's law:

$$\mathbf{dH}(\mathbf{r}) = \frac{I'(\mathbf{r}')\,\mathbf{dl}' \times \mathbf{R}}{4\pi R^3} \tag{1}$$

To obtain the total magnetic field of the current elements along a current path, (1) is integrated over the path

$$\mathbf{H}(\mathbf{r}) = \int \frac{I'(\mathbf{r}')\,\mathbf{dl}' \times \mathbf{R}}{4\pi R^3} \tag{2}$$

It is of interest to examine further the relation between $\mathbf{B}$ and $\mathbf{H}$. We see that the field $\mathbf{H}$ is directly related to the currents, without regard for the nature of the medium as long as it fills all space homogeneously. The force on a current element was seen in Sec. 2.2 to depend upon magnetic flux density. The influence of the medium in relating $\mathbf{B}$ and $\mathbf{H}$ comes about in the following way. The electronic orbital and spin motions in the atoms can be thought of as circulating currents on which a force is exerted by $\mathbf{B}$ and which produce a field $\mathbf{M}$ (called *magnetization*) that adds to $\mathbf{H}$. This is analogous to the response of a dielectric medium shown in Fig. 1.3*c*. Then $\mathbf{B}$ is related to $\mathbf{H}$ as though there were only free space but with the added field of the atomic currents

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{3}$$

Magnetization $\mathbf{M}$ may have a permanent contribution (to be considered in Sec. 2.15), but here we neglect this and assume the material isotropic so that $\mathbf{M}$ is parallel to $\mathbf{H}$.

We can then write

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} = \mu_r\mu_0\mathbf{H}$$

where χ_m is called the *magnetic susceptibility*, μ is the permeability introduced in Sec. 2.2, and μ_r is the relative permeability. Many materials have nonlinear behavior so χ_m and μ are, in general, functions of the field strength. For diamagnetic materials $\chi_m < 0$, and for paramagnetic, ferromagnetic, and ferrimagnetic materials $\chi_m > 0$. Most materials commonly considered to be dielectrics or metals have either diamagnetic or paramagnetic behavior and typically $|\chi_m| < 10^{-5}$ so we treat them as free space, taking $\mu = \mu_0$. Ferromagnetic and ferrimagnetic materials usually have χ_m and μ/μ_0 much greater than unity and in some cases are anisotropic, that is, dependent upon direction of the field. All of these aspects are considered in more detail in Chapter 13.

Example 2.3a
FIELD ON AXIS OF CIRCULAR LOOP

As an example of the application of the law, the magnetic field is computed for a point on the axis of a circular loop of wire carrying dc current I (Fig. 2.3*b*). The element $\mathbf{dl}'$ has magnitude $a\,d\phi'$ and is always perpendicular to $\mathbf{R}$. Hence the contribution dH from an element is

$$dH = \frac{Ia\,d\phi'}{4\pi(a^2 + z^2)} \tag{4}$$

As one integrates about the loop, the direction of $\mathbf{R}$ changes, and so the direction of

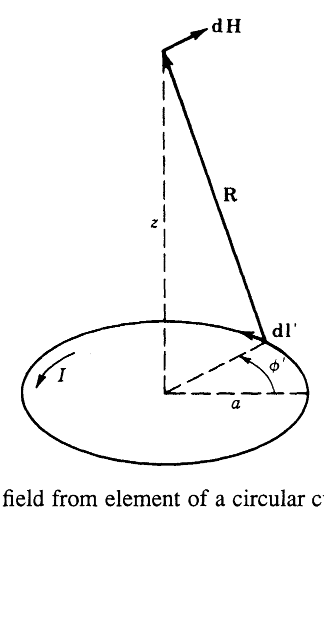

FIG. 2.3*b* Magnetic field from element of a circular current loop (Ex. 2.3a).

dH changes, generating a conical surface as ϕ goes through 2π radians (rad). The radial components of the various contributions cancel, and the axial components add. Using (4)

$$dH_z = dH \sin\theta = \frac{a\, dH}{(a^2 + z^2)^{1/2}}$$

Integrating in ϕ amounts to multiplying by 2π; thus

$$H_z = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} \tag{5}$$

Note that for a point at the center of the loop, $z = 0$,

$$H_z\Big|_{z=0} = \frac{I}{2a} \tag{6}$$

Example 2.3b
Field of a Finite Straight Line of Current

Let us find the magnetic field **H** at a point P a perpendicular distance r from the center of a finite length of current I, as shown in Fig. 2.3c. It is easy to see from the right-hand rule that there is only an H_ϕ component. Its magnitude is given by the integral of (1) over the length $2a$

$$H_\phi = \int_{-a}^{a} \frac{I \sin\phi\, dz}{4\pi R^2}$$

We can see from Fig. 2.3c that $\sin\phi = r/R$ and $R = (r^2 + z^2)^{1/2}$. Thus,

$$H_\phi = \frac{Ir}{4\pi}\int_{-a}^{a} \frac{dz}{(r^2 + z^2)^{3/2}} = \frac{I}{2\pi r}\,\frac{1}{[(r/a)^2 + 1]^{1/2}} \tag{7}$$

which becomes $I/2\pi r$ if $|a| \to \infty$. This same result is found in Ex. 2.4a by a different method.

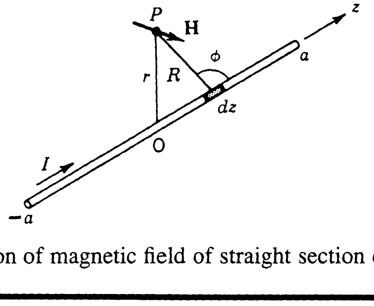

Fig. 2.3c Calculation of magnetic field of straight section of current (Ex. 2.3b).

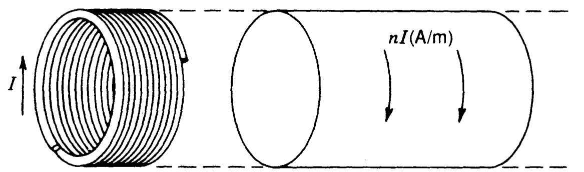

FIG. 2.3d Tightly wound solenoid of n turns per meter and its representation by a current sheet of nI A/m (Ex. 2.3c).

Example 2.3c
FIELD IN AN INFINITE SOLENOID

Let us here model the long, tightly wound solenoid shown in Fig. 2.3*d* by an equivalent current sheet to facilitate calculation of the magnetic field inside. We assume that though the wire makes a small helical angle with a cross-sectional plane, we can adequately model it with a circumferential current. The current flowing around the solenoid per meter is nI, where n is the number of turns per meter and I is the current in each turn. Then, in a differential length of the sheet model, there is a current $nIdz$. We will calculate, for simplicity, the field on the axis. But one can show, by means that will come later (see Ex. 2.4d) that the field for an infinitely long solenoid is uniform throughout the inside of the solenoid. We can adapt (4) for the present calculation by taking I in (4) to be $nIdz$. Then the total field on the axis for the infinitely long solenoid is given by

$$H_z = \int_{-\infty}^{\infty} \frac{nIa^2\,dz}{2(a^2 + z^2)^{3/2}} \tag{8}$$

In evaluating the integral in (8), one first takes symmetrical finite limits as in (7) and then lets the limits go to infinity with the result

$$H_z = nI \tag{9}$$

For a solenoid of finite length, it is easy to modify (8) to obtain on-axis fields (Prob. 2.3c) but difficult to perform the integrals for fields not on the axis.

2.4 THE LINE INTEGRAL OF MAGNETIC FIELD

Although Ampère's law describes how magnetic field may be computed from a given system of currents, other derived forms of the law may be more easily applied to certain types of problems. In this and the following sections, some of these forms are presented, with examples of their application. The sketch of the derivations of these forms, because they are more complex than for the corresponding electrostatic forms, will be left to Appendix 3.

One of the most useful forms of the magnetic field laws derived from Ampère's law is that which states that a line integral of static magnetic field taken about any given closed path must equal the current enclosed by that path. In the vector notation,

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} = I \tag{1}$$

Equation (1) is often referred to as *Ampère's circuital law.* The sign convention for current on the right side of (1) is taken so that it is positive if it has the sense of advance of a right-hand screw rotated in the direction of circulation chosen for the line integration. This is simply a statement of the well-known right-hand rule relating directions of current and magnetic field.

Equation (1) is rather analogous to Gauss's law in electrostatics in the sense that it is an important general relation and is also useful for problem solving if there is sufficient symmetry in the problem. If the product $\mathbf{H} \cdot \mathbf{dl}$ is constant along some path, $\mathbf{H}$ can be found simply by dividing I by the path length.

Example 2.4a
MAGNETIC FIELD ABOUT A LINE CURRENT

An important example is that of a long, straight, round conductor carrying current I. If an integration is made about a circular path of radius r centered on the axis of the wire, the symmetry reveals that magnetic field is circumferential and does not vary with angle as one moves about the path. Hence the line integral is just the product of circumference and the value of H_ϕ. This must equal the current enclosed

$$\oint \mathbf{H} \cdot \mathbf{dl} = 2\pi r H_\phi = I$$

or

$$H_\phi = \frac{I}{2\pi r} \quad \text{A/m} \tag{2}$$

as was found by a different method in Ex. 2.3b. The sense relations are given in Fig. 2.4*a*.

Example 2.4b
MAGNETIC FIELD BETWEEN COAXIAL CYLINDERS

A coaxial line (Fig. 2.4*b*) carrying current I on the inner conductor and $-I$ on the outer (the return current) has the same type of symmetry as the isolated wire, and a circular path between the two conductors encloses current I, so that the result (1) applies directly for the region between conductors:

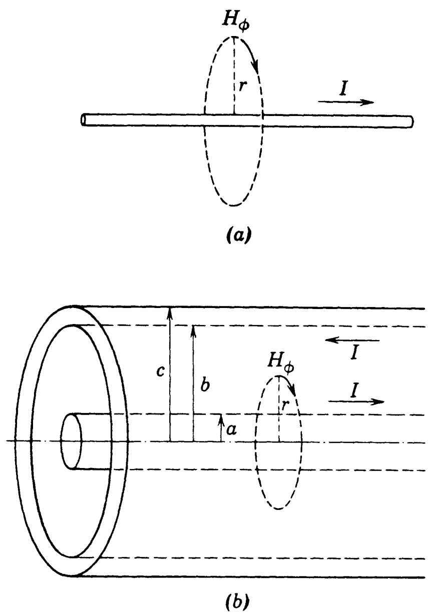

FIG. 2.4 (*a*) and (*b*) Magnetic field about line current and between coaxial cylinders (Exs. 2.4a and b).

$$H_\phi = \frac{I}{2\pi r} \qquad a < r < b \tag{3}$$

Outside the outer conductor, a circular path encloses both the going and return currents, or a net current of zero. Hence the magnetic field outside is zero.

Example 2.4c
MAGNETIC FIELD INSIDE A UNIFORM CURRENT

Let us find the magnetic field inside the round inner conductor in Fig. 2.4*b* assuming a uniform distribution of current. We will apply (2) but with I replaced by $I(r)$, the current enclosed by a circle at radius r. The total current in the wire is $I(a) = I$ and the current density is $I/\pi a^2$. The current $I(r)$ is

$$I(r) = \left(\frac{r}{a}\right)^2 I \tag{4}$$

and using (2),

$$H_\phi(r) = \frac{I(r)}{2\pi r} = \frac{Ir}{2\pi a^2} \tag{5}$$

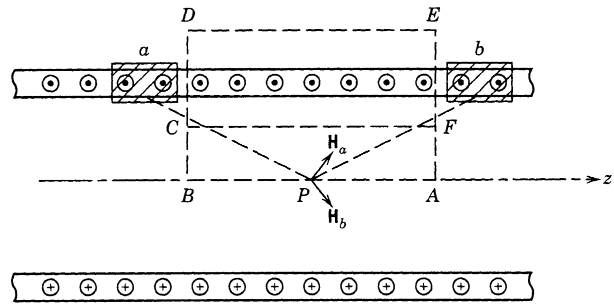

FIG. 2.4c Section through axis of infinite solenoid for Ex. 2.4d showing contributions to **H** on axis from two symmetrically spaced elements.

Example 2.4d
MAGNETIC FIELD OF A SOLENOID

In Ex. 2.3c we showed that the magnetic field H_z on the axis of an infinitely long solenoid of n turns per meter carrying a current I A is nI. Now let us use the integral relation (1) to show that the field outside is zero and that inside is uniformly nI. Figure 2.4c shows the section through the solenoid in a plane containing the axis. Let us consider the integration paths shown by broken lines to be 1 m long in the z direction for simplicity of notation. Any radial component of **H** produced by a current element is canceled by that of a symmetrically located element. This is illustrated in Fig. 2.4c for the fields $\mathbf{H}_a$ and $\mathbf{H}_b$ from elements a and b located equal distances from the point P. Thus, $\mathbf{H} \cdot \mathbf{dl}$ is zero along the sides BD and AE.

Taking the line integral around path $ABDEA$ and setting it equal to the enclosed current gives

$$\oint \mathbf{H} \cdot \mathbf{dl} = nI + \int_D^E \mathbf{H} \cdot \mathbf{dl} = nI \tag{6}$$

since H on the axis is nI. From (6) the integral from D to E is zero. Since the placement of the outside path DE is arbitrary, external H must be zero.

The line integral around path $ABCFA$ encloses no current so the integral along the arbitrarily positioned path CF must be equal in magnitude to, and of opposite sign from, that along AB. Thus, the internal field is everywhere z-directed and has the value

$$H_z = nI \tag{7}$$

Note that these symmetry arguments cannot be made for a solenoid of finite length, but the results given here are reasonably accurate for a solenoid having a length much greater than its diameter, except near the ends.

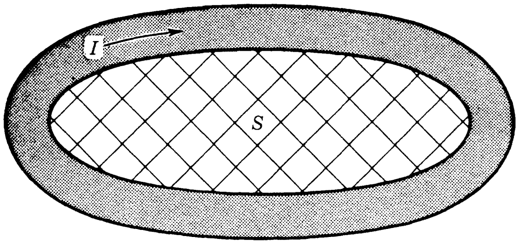

FIG. 2.5a Loop of wire. Cross-hatching shows surface used for calculation of external inductance.

2.5 INDUCTANCE FROM FLUX LINKAGES: EXTERNAL INDUCTANCE

The important circuit element which describes the effect of magnetic energy storage for an electric circuit is the inductor. It is of primary concern for dynamic, that is, time-varying, problems, but the inductance calculated from static concepts is often useful up to very high frequencies. This is the quasistatic use discussed in the introduction to this chapter. In a manner similar to the capacitance definition of Sec. 1.9, inductance can be defined in terms of flux linkage by

$$L = \frac{1}{I}\int_S \mathbf{B} \cdot \mathbf{dS} \tag{1}$$

where the surface S must be specified. Consider, for example, the loop of wire shown in Fig. 2.5*a*. The current I produces magnetic flux in the cross-hatched area S bounded by the loop. Also, some of the flux produced by the current is inside the wire itself. It is convenient to separate the inductances related to these two components of flux and call them, respectively, *external inductance* and *internal inductance*. Examples of calculations of external inductance for simple structures are given below and an example of an internal inductance calculation is presented in Sec. 2.17.

Example 2.5a
EXTERNAL INDUCTANCE OF A PARALLEL-PLANE TRANSMISSION LINE

Here we find the external inductance for a unit length of a parallel-plane structure (Fig. 2.5*b*) which is wide enough compared with the conductor spacing that the fields between the conductors are, to a reasonable degree of accuracy, those of infinite parallel planes, as suggested in Fig. 2.5*c*. Note that the flux tubes (bounded by the field lines) spread out greatly outside the edges of the conductors. Thus, there is a strong reduction of flux density **B** and, therefore, also **H**. The line integral of **H** around one of the conductors

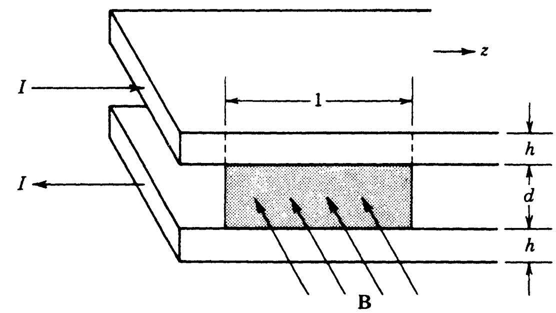

FIG. 2.5b Surface for calculation of external inductance of a parallel-plane transmission line.

has its predominant contribution from the field H_0 between the conductors,

$$I = \oint \mathbf{H} \cdot \mathbf{dl} \cong H_0 w \tag{2}$$

where I is the total current in one conductor and w is the conductor width. This result applies to any path in the cross-sectional plane (Fig. 2.5c) between and parallel to the conductors, so H_0 can be considered approximately uniform.

The external inductance for a unit length is found by applying (1) to the surface between the conductors which is shown shaded in Fig. 2.5b. Since I is independent of z and H_0 is nearly constant through the space between the conductors and is perpendicular to the shaded surface, (1) becomes

$$L = \frac{1}{I}\mu_0\left(\frac{I}{w}\right)d = \mu_0 \frac{d}{w} \quad \text{H/m} \tag{3}$$

This relation is based on the neglect of fringing fields and is most accurate for small d/w.

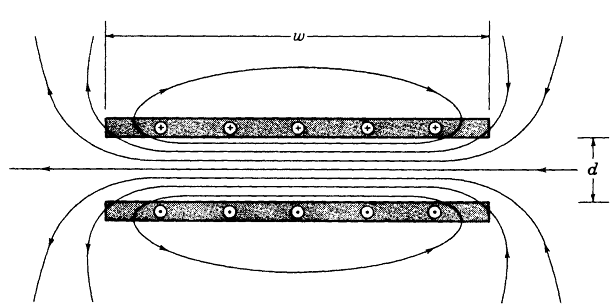

FIG. 2.5c Cross-section of parallel-plane transmission line of finite width showing general character of magnetic field lines.

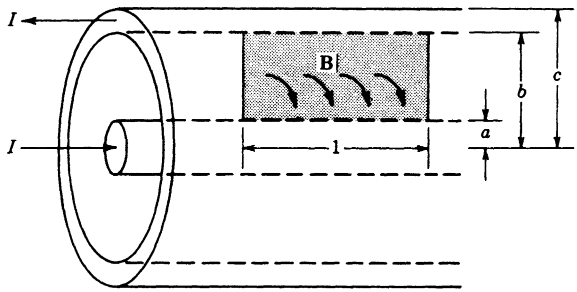

FIG. 2.5*d* Surface for calculation of external inductance of a coaxial transmission line.

Example 2.5b
EXTERNAL INDUCTANCE OF A COAXIAL TRANSMISSION LINE

For a coaxial line as pictured in Fig. 2.5*d* with axial current I flowing in the inner conductor and returning in the outer, the magnetic field is circumferential and, for $a < r < b$, is (Ex. 2.4b)

$$H_\phi = \frac{I}{2\pi r} \tag{4}$$

For a unit length the magnetic flux between radii a and b is, by integration over the shaded area in Fig. 2.5*d*,

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \int_a^b \mu\left(\frac{I}{2\pi r}\right) dr = \frac{\mu I}{2\pi} \ln \frac{b}{a} \tag{5}$$

So, from (1), the inductance per unit length is

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} \quad \text{H/m} \tag{6}$$

For high frequencies, there is not much penetration of fields into conductors as will be seen in Chapter 3, so this is then the main contribution to inductance. The internal inductance for low frequencies will be considered in Sec. 2.17.

Differential Forms for Magnetostatistics and the Use of Potential

2.6 The Curl of a Vector Field

To write differential equation forms for laws having to do with line integrals, it will be necessary to make use of the vector operation called *curl*. This is defined in terms of a line integral taken around an infinitesimal path, divided by the area enclosed by that path. It is seen to have some similarities to the operation of divergence of Sec. 1.11, which was defined as the surface integral taken about an infinitesimal surface divided by the volume enclosed by that surface. Unlike the divergence, however, the curl operation results in a vector because the orientation of the surface element about which the integral is taken must be defined. This additional complication seems to be enough to make curl a more difficult concept for a beginning student. The student should attempt to obtain as much physical significance as possible from the definitions to be given, but at the same time should recognize that full appreciation of the operation will come only with practice in its use.

The curl of a vector field is defined as a vector function whose component at a point in a particular direction is found by orienting a small area normal to the desired direction at that point, and finding the limit of the line integral divided by the area:

$$[\text{curl } \mathbf{F}]_i \triangleq \lim_{\Delta S_i \to 0} \frac{\oint \mathbf{F} \cdot d\mathbf{l}}{\Delta S_i} \tag{1}$$

where i denotes a particular direction, ΔS_i is normal to that direction, and the line integral is taken in the right-hand sense with respect to the positive i direction. In rectangular coordinates, for example, to compute the z component of the curl, the small area $\Delta S = \Delta x \, \Delta y$ is selected in the x–y plane to be normal to the z direction (Fig. 2.6a). The right-hand sense of integration about the path with respect to the positive z direction is as shown by the arrows of the figure. The line integral is then

$$\oint \mathbf{F} \cdot d\mathbf{l} = \Delta y F_y \bigg|_{x+\Delta x} - \Delta x F_x \bigg|_{y+\Delta y} - \Delta y F_y \bigg|_{x} + \Delta x F_x \bigg|_{y}$$

We find F_y at $x + \Delta x$ and F_x at $y + \Delta y$ by truncated Taylor series expansions

$$F_x \bigg|_{y+\Delta y} \cong F_x \bigg|_{y} + \Delta y \frac{\partial F_x}{\partial y}\bigg|_{y}; \qquad F_y \bigg|_{x+\Delta x} \cong F_y \bigg|_{x} + \Delta x \frac{\partial F_y}{\partial x}\bigg|_{x} \tag{2}$$

So

$$\oint \mathbf{F} \cdot d\mathbf{l} \cong \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \Delta x \, \Delta y$$

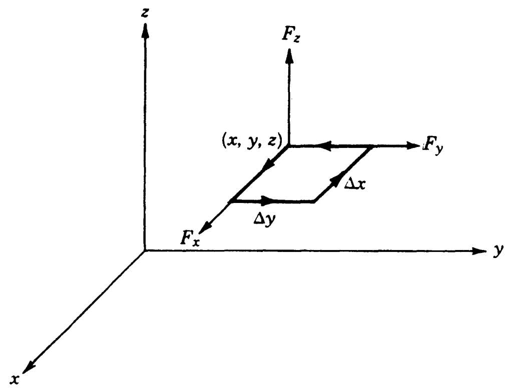

FIG. 2.6a Path for line integral in definition of curl.

Then using the definition (1), we get

$$[\text{curl }\mathbf{F}]_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{3}$$

because the expansions (2) become exact in the limit. Similarly, by taking the elements of area in the y–z plane and x–z plane, respectively, we find

$$[\text{curl }\mathbf{F}]_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \tag{4}$$

$$[\text{curl }\mathbf{F}]_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \tag{5}$$

These components may be multiplied by the corresponding unit vectors and added to form the vector representing the curl:

$$\text{curl }\mathbf{F} = \hat{\mathbf{x}}\left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right] + \hat{\mathbf{y}}\left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right] + \hat{\mathbf{z}}\left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right] \tag{6}$$

If this form is compared with the form of the cross product and the definition of the vector operator ∇, Eq. 1.10(7), the above can logically be written as

$$\text{curl }\mathbf{F} \equiv \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{7}$$

In deriving curl for other coordinate systems, the variation of line elements with coordinates must be considered, just as the variation of surface elements with coordinates in spherical coordinates was considered in Sec. 1.11. (See Appendix 2.) Results for circular cylindrical and spherical coordinates are given on the inside front cover.[3]

The name *curl* (or *rotation* as it is sometimes called) has some physical significance in the sense that a finite value for the line integral taken in the vicinity of a point is obtained if the curl is finite. The name should not be associated with the curvature of the field lines, however, for a field consisting of closed circles may have zero curl nearly everywhere, and a straight-line field varying in certain ways may have a finite curl. The following examples illustrate these points.

Example 2.6a
CURL-FREE FIELD WITH CIRCULAR FIELD LINES

The magnetic field in the region surrounding a current in a long straight round wire was seen in Eq. 2.4(2) to be $H_\phi = I/2\pi r$. If we write this in rectangular coordinates using $\sin\phi = y/r$, $\cos\phi = x/r$, and $r^2 = x^2 + y^2$, we get

$$H_x = -H_\phi \sin\phi = -\frac{I}{2\pi}\frac{y}{x^2 + y^2} \tag{8}$$

$$H_y = H_\phi \cos\phi = \frac{I}{2\pi}\frac{x}{x^2 + y^2} \tag{9}$$

$$H_z = 0 \tag{10}$$

as can be seen from Fig. 2.6*b*. Since there is no z component and no dependence on z, (6) shows immediately that the x and y components of the curl are zero. Substituting (8) and (9) into (6) with $\mathbf{F} = \mathbf{H}$ we obtain

$$\text{curl } \mathbf{H} = \hat{\mathbf{z}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = 0 \tag{11}$$

This result is found more naturally and directly for this problem using the expression for the curl in cylindrical coordinates found inside the front cover:

$$\nabla \times \mathbf{H} = \hat{\mathbf{r}}\left[\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right] + \hat{\mathbf{z}}\left[\frac{1}{r}\frac{\partial (rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial \phi}\right] \tag{12}$$

Since there is only an H_ϕ and no z dependence, the first two components vanish. The r and ϕ components are the transverse ones corresponding to x and y components. Since there is no H_r and rH_ϕ does not depend upon r, we see that $\nabla \times \mathbf{H} = 0$ as shown above.

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[3] *As with the divergence (footnote 4 of Chapter 1), one cannot take the cross product of* ∇ *and the vector to obtain the curl in a curvilinear coordinate system, but must use the basic definition (1).*

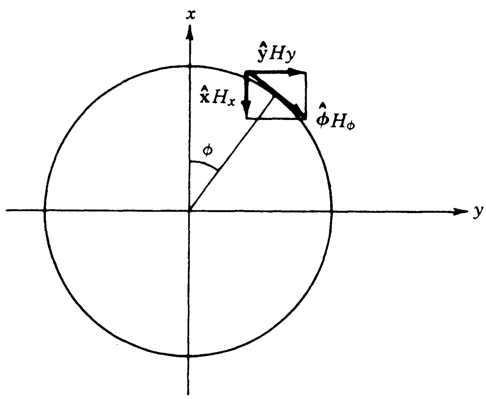

FIG. 2.6b Resolution of H_ϕ of a line current into rectangular components (Ex. 2.6a).

Example 2.6b
FIELD WITH NONVANISHING CURL

The magnetic field inside a uniform current with circular symmetry was seen in Ex. 2.4c to be $H_\phi(r) = Ir/2\pi a^2$. As in the preceding example, we see that the symmetries indicate the presence of only the z component of the curl in (12). Also, the second term in the z component is zero. Thus

$$\nabla \times \mathbf{H} = \hat{\mathbf{z}}\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} = \hat{\mathbf{z}}\frac{I}{\pi a^2} \tag{13}$$

Example 2.6c
NONVANISHING CURL IN FIELD OF STRAIGHT PARALLEL VECTORS

A theoretically stable electron flow in a type of microwave electron tube called a planar magnetron has an electron velocity distribution described by $\mathbf{v} = \hat{\mathbf{z}}y$ and is shown in Fig. 2.6c. We see there a vector function with all vectors straight and parallel. It is immediately evident by substitution of $\mathbf{v}$ in (6) that

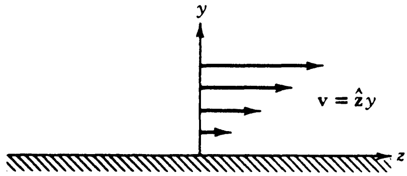

FIG. 2.6c Electron flow in planar magnetron (Ex. 2.6c).

$$\text{curl } \mathbf{v} = \hat{\mathbf{x}}[\text{curl } \mathbf{v}]_x = \hat{\mathbf{x}} \neq 0 \tag{14}$$

It is instructive to see, by using the line-integral definition of the curl (1) why this result obtains. All vectors and their spatial variations are in the y–z plane, and (6) shows there can be only an x component of the curl. Then we can write for a small area $\Delta S = \Delta y\, \Delta z$

$$[\text{curl } \mathbf{v}]_x = \lim_{\Delta S \to 0} \frac{[v_z(y + \Delta y) - v_z(y)]\Delta z}{\Delta y\, \Delta z} \tag{15}$$

We see that the curl is nonzero because the velocity is larger on one side of the loop than on the other.

Example 2.6d
CURL OF THE GRADIENT OF A SCALAR

Here we show the useful fact that the curl of the gradient of a scalar is zero. If we write

$$\mathbf{F} = \nabla \xi = \hat{\mathbf{x}} \frac{\partial \xi}{\partial x} + \hat{\mathbf{y}} \frac{\partial \xi}{\partial y} + \hat{\mathbf{z}} \frac{\partial \xi}{\partial z}$$

and substitute it in (6), we get

$$\nabla \times \mathbf{F} = \hat{\mathbf{x}}\left(\frac{\partial^2 \xi}{\partial y\, \partial z} - \frac{\partial^2 \xi}{\partial z\, \partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 \xi}{\partial z\, \partial x} - \frac{\partial^2 \xi}{\partial x\, \partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 \xi}{\partial x\, \partial y} - \frac{\partial^2 \xi}{\partial y\, \partial x}\right) \tag{16}$$

Since the order of the partial derivative operations is arbitrary $\nabla \times \mathbf{F} = 0$. A particularly important example is the electrostatic field. The fact that $\nabla \times \mathbf{E} = 0$ follows immediately from either $\mathbf{E} = -\nabla\Phi$ or $\oint \mathbf{E} \cdot \mathbf{dl} = 0$. We shall see in Chapter 3 that these properties of $\mathbf{E}$ do not apply for time-varying fields.

2.7 CURL OF MAGNETIC FIELD

Now let us use the formulations of the last two sections to derive a new relation for magnetic field. The line integral of $\mathbf{H}$ around an area ΔS_i is substituted in the definition of the curl, Eq. 2.6(1), to get

$$[\text{curl } \mathbf{H}]_i = \lim_{\Delta S_i \to 0} \frac{\oint \mathbf{H} \cdot \mathbf{dl}}{\Delta S_i}$$

But $\oint \mathbf{H} \cdot \mathbf{dl}$ is the current through the area ΔS_i by Eq. 2.4(1) so

$$[\text{curl } \mathbf{H}]_i = \lim_{\Delta S_i \to 0} \frac{\int_{\Delta S_i} \mathbf{J} \cdot \mathbf{dS}}{\Delta S_i} = J_i \tag{1}$$

This relation holds for all three orthogonal components. If these are multiplied by the corresponding unit vectors and added, we get the vector relation

$$\text{curl } \mathbf{H} \triangleq \nabla \times \mathbf{H} = \mathbf{J} \tag{2}$$

This can be thought of as the equivalent of Eq. 2.4(1) for a differential path taken around a point. Note that the curl **H** found in Eq. 2.6(13) is the current density, as required by (2).

Example 2.7
Current Density at Superconductor Surface

If a sheet of superconductive material[4] is in a magnetic field $\mathbf{H} = \hat{\mathbf{z}}H_z$ parallel to its surface, there is a penetration of **H** only a very short distance into the superconductor as shown in Fig. 2.7. The decay of H_z with distance is given by

$$H_z = H_0 e^{-x/\lambda_s} \tag{3}$$

where H_0 is the value at the surface and λ_s, called the *penetration depth*, is a property of the material. We can find the corresponding current density using (2) and the

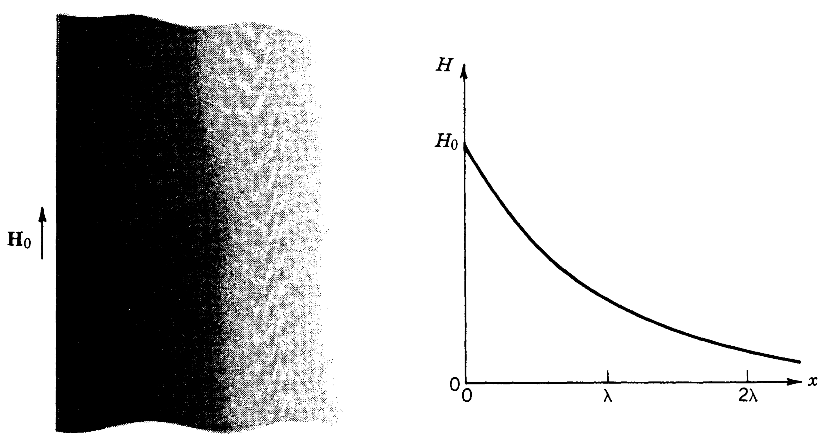

Fig. 2.7 Penetration of magnetic field into a thick sheet of superconducting material.

[4] *Superconductors include lead, tin, niobium, and numerous other elements, alloys, and compounds. They have zero dc resistance and other special properties below their critical temperatures. See, for example, V. Z. Kresin and S. A. Wolf,* Fundamentals of Superconductivity, *Plenum Press, New York, 1990.*

expansion in Eq. 2.6(6):

$$J_y = [\text{curl } \mathbf{H}]_y = -\frac{\partial H_z}{\partial x} = \frac{H_0}{\lambda_s} e^{-x/\lambda_s}$$

Thus, the current also is found only near the surface.

2.8 Relation Between Differential and Integral Forms of the Field Equations

The differential form relating magnetic field to current density was derived from the integral form through the definition of curl. One can proceed in reverse by using Stokes's theorem, which states that for a vector function **F**,

$$\oint \mathbf{F} \cdot \mathbf{dl} = \int_S (\text{curl } \mathbf{F}) \cdot \mathbf{dS} \equiv \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} \tag{1}$$

This theorem is made plausible by looking at a general surface as in Fig. 2.8*a*, breaking it into elemental areas. For each differential area, the contribution $(\nabla \times \mathbf{F}) \cdot \mathbf{dS}$ gives the line integral about that area by the definition of curl. If contributions from infinitesimal areas are summed over the surface, the line integral must disappear for all internal areas, since a boundary is first traversed in one direction and then later in the opposite direction in determining the contribution from an adjacent area. The only places where these contributions do not disappear are along the outer boundary, so that

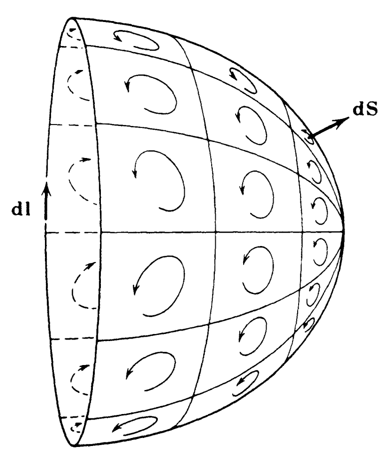

Fig. 2.8*a* Subdivision of arbitrary surface for proof of Stokes's theorem.

the result of the summation is then the line integral of the vector around the boundary as stated in (1). It is recognized that the process is similar to the transformation from the differential to the integral form of Gauss's law through the divergence theorem in Sec. 1.11. Then writing Stokes's theorem for magnetic field, we have

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{H}) \cdot \mathbf{dS} \tag{2}$$

But, by Eq. 2.7(2), the curl may be replaced by the current density:

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{3}$$

The right side represents the current flow through the surface of which the path for the line integration on the left is a boundary. Hence (3) is exactly equivalent to Eq. 2.4(1).

Example 2.8a
DEMONSTRATION OF STOKES'S THEOREM

Let us demonstrate Stokes's theorem for a magnetic field that is part of an electromagnetic wave in a certain kind of transmission structure. The field at a particular instant of time is described by

$$\mathbf{H} = \hat{\mathbf{y}} A \cos \frac{\pi x}{a} \tag{4}$$

We will apply (2) to the area shown in Fig. 2.8*b* where the field distribution (4) is illustrated. The line integral of (4) along the broken path is

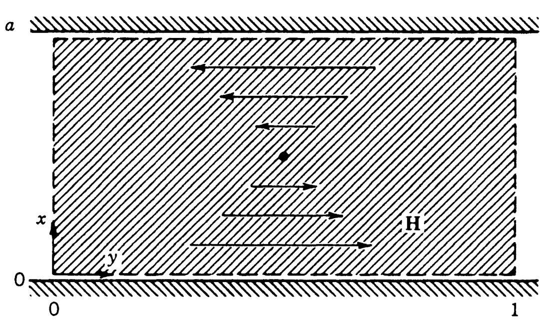

FIG. 2.8*b* Area for integration of field **H** to demonstrate the validity of Stokes's theorem (Ex. 2.8a).

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_0^a H_x\, dx + \int_0^1 H_y\, dy + \int_a^0 H_x\, dx + \int_1^0 H_y\, dy$$
$$= 0 + A \cos \pi + 0 - A \cos 0 = -2A \tag{5}$$

where the facts that $H_x = 0$ and $H_y \neq f(y)$ are used.

The curl of **H** in rectangular coordinates is

$$\nabla \times \mathbf{H} = \hat{\mathbf{z}} \frac{\partial H_y}{\partial x} = -\hat{\mathbf{z}} A \frac{\pi}{a} \sin \frac{\pi x}{a} \tag{6}$$

The integral of (6) over the surface bounded by the broken line in Fig. 2.8*b* is

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^a -A \frac{\pi}{a} \sin \frac{\pi x}{a}\, dx = A \cos \frac{\pi x}{a} \bigg|_0^a$$
$$= -2A \tag{7}$$

Since (5) and (7) give the same results, Stokes's theorem (1) is illustrated.

Example 2.8b
PROOF THAT $\nabla \cdot \nabla \times \mathbf{F} = 0$

That $\nabla \cdot \nabla \times \mathbf{F} = 0$ can be proved by using the expressions in rectangular coordinates as was done for $\nabla \times \nabla\psi$ in Ex. 2.6d. Here we take a different approach that uses Stokes's theorem. Since Stokes's theorem applies to any surface, we may treat the surface shown in Fig. 2.8*c* and let the bounding line shrink to zero so the surface becomes a closed one. Then the line integral on the left side of (2) vanishes and we have

$$\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0 \tag{8}$$

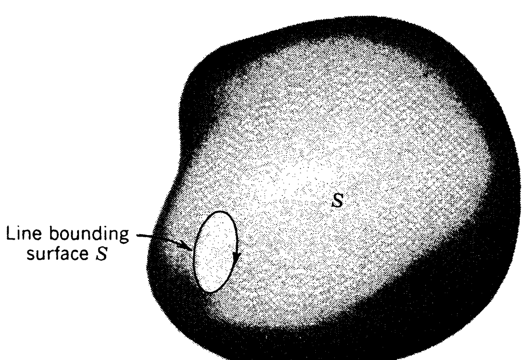

FIG. 2.8c Surface used in Ex. 2.8b.

We may then apply the divergence theorem (Sec. 1.11) to the vector $\nabla \times \mathbf{F}$:

$$\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_V \nabla \cdot \nabla \times \mathbf{F}\, dV \tag{9}$$

Since we saw in (8) that the left side is zero with an arbitrary choice of surface, the integrand on the right side must vanish,

$$\nabla \cdot \nabla \times \mathbf{F} = 0 \tag{10}$$

which was to be shown. This is a useful relation in the study of electromagnetic fields.

2.9 Vector Magnetic Potential

We introduce here another potential, which is often used as a conveniently calculated quantity from which the magnetic field can be found. An integral expression for the flux density can be obtained from Eq. 2.3(2) by multiplying by μ for homogeneous media:

$$\mathbf{B}(\mathbf{r}) = \int \frac{\mu I'(\mathbf{r}')\, d\mathbf{l}' \times \mathbf{R}}{4\pi R^3} \tag{1}$$

It is shown in Appendix 3 that this can be broken into two steps by making use of certain vector equivalences. The result gives

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \tag{2}$$

where

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu I'(\mathbf{r}')\, d\mathbf{l}'}{4\pi R} \tag{3}$$

The current may be given as a vector density $\mathbf{J}$ in current per unit area spread over a volume V'. Then, since $I = J\, dS$, where dS is the differential area element perpendicular to $\mathbf{J}$, and $\mathbf{dl}$ is in the direction of $\mathbf{J}$, $dS\, dl$ forms a volume element dV and the equivalent to (3) is

$$\mathbf{A}(\mathbf{r}) = \int_V \frac{\mu \mathbf{J}(\mathbf{r}')\, dV'}{4\pi R} \tag{4}$$

In both (3) and (4), R is the distance from a current element of the integration to the point at which $\mathbf{A}$ is to be computed. The function $\mathbf{A}$, introduced as an intermediate step, is computed as an integral over the given currents from (3) or (4) and then differentiated in the manner defined by (2) to yield the magnetic field. Function $\mathbf{A}$ is called the *magnetic vector potential.* Note that each element of $\mathbf{A}$ has the direction of the current element producing it. It is analogous to the potential function of electrostatics, which is found in terms of an integral over charges and then differentiated in a certain way to yield the electric field. The magnetic potential $\mathbf{A}$ is different, however, because it is a vector, and does not have the simple physical significance of work done in moving through the field that electrostatic potential has. Some physical pictures can be formed but the student should not worry about these until more familiarity with the function has been developed through certain examples.

Example 2.9a
VECTOR POTENTIAL AND MAGNETIC FIELD OF A CURRENT ELEMENT

Here we show that the magnetic flux density of a current element found using (3) and (2), in that order, is the same as the integrand of (1), which expresses Ampère's law. The magnetic vector potential **A** exists throughout the region surrounding the given current element, as shown in Fig. 2.9*a*. From (3) we find

$$\mathbf{A} = \hat{\mathbf{z}}A_z = \hat{\mathbf{z}}\frac{\mu I\,dz}{4\pi r} \tag{5}$$

since the origin of coordinates is positioned at the current element. As noted earlier **dA** is parallel to the current element producing it. It is most convenient to use spherical coordinates in this example. From the figure we see that $A_r = A_z \cos\theta$ and $A_\theta = -A_z \sin\theta$. The curl in spherical coordinates (from inside the front cover) reduces to

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] \tag{6}$$

since $A_\phi = 0$ and $\partial/\partial\phi = 0$, by symmetry. Substituting A_r and A_ϕ using (5), we find

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\boldsymbol{\phi}}\left(\frac{\mu I\,dz}{4\pi}\right)\frac{\sin\theta}{r^2} \tag{7}$$

Note that $\mathbf{dl}' \times \mathbf{R}$ is $\hat{\boldsymbol{\phi}}\,dz\,r\sin\theta$, so (7) is equivalent to the integrand in (1).

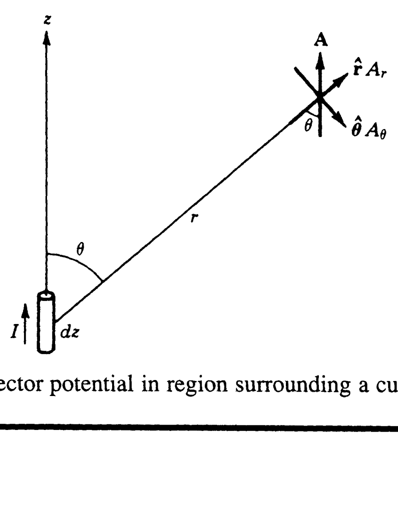

FIG. 2.9*a* Vector potential in region surrounding a current element.

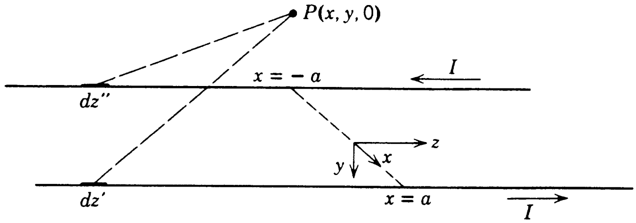

FIG. 2.9b Parallel-wire transmission line.

Example 2.9b
VECTOR POTENTIAL AND FIELD OF A PARALLEL-WIRE TRANSMISSION LINE

Let us consider a parallel-wire transmission line of infinite length carrying current I in one conductor and its return in the other distance $2a$ away. The coordinate system is set up as in Fig. 2.9*b*. Since the field quantities do not vary with z, it is convenient to calculate them in the plane $z = 0$. The conductors will first be taken as extending from $z = -L$ to $z = L$ to avoid indeterminacies in the integrals. Since current is only in the z direction, **A** by (3) will be in the z direction also. The contribution to A_z from both wires is

$$A_z = \int_{-L}^{L} \frac{\mu I\, dz'}{4\pi\sqrt{(x-a)^2 + y^2 + z'^2}} - \int_{-L}^{L} \frac{\mu I\, dz''}{4\pi\sqrt{(x+a)^2 + y^2 + z''^2}}$$
$$= \frac{2\mu}{4\pi}\left[\int_0^L \frac{I\, dz'}{\sqrt{(x-a)^2 + y^2 + z'^2}} - \int_0^L \frac{I\, dz''}{\sqrt{(x+a)^2 + y^2 + z''^2}}\right]$$

The integrals may be evaluated[5]:

$$A_z = \frac{I\mu}{2\pi}\{\ln[z' + \sqrt{(x-a)^2 + y^2 + z'^2}]$$
$$- \ln[z'' + \sqrt{(x+a)^2 + y^2 + z''^2}]\}_0^L$$

Now, as L is allowed to approach infinity, the upper limits of the two terms cancel. Hence

$$A_z = \frac{I\mu}{4\pi}\ln\left[\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}\right] \tag{8}$$

[5] *Most integrals of this text can be found in standard handbooks such as the* CRC Handbook of Chemistry and Physics *(any recent edition); M. R. Spiegel,* Mathematical Handbook, *Schaum's Outline Series, McGraw-Hill, 1968; or M. Abramowitz and I. A. Stegun (Eds.),* Handbook of Mathematical Functions, *National Bureau of Standards Applied Mathematics, Dover, New York, 1964. One of the most complete listings is I. S. Gradshteyn and I. M. Ryzhik,* Table of Integrals, Series, and Products *(A. Jeffrey, Trans.), Academic Press, San Diego, CA, 1980.*

If (2) is then applied, using the expression for curl in rectangular coordinates, we find

$$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} = \frac{I}{2\pi}\left[\frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2+y^2}\right] \tag{9}$$

$$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} = \frac{I}{2\pi}\left[\frac{(x-a)}{(x-a)^2+y^2} - \frac{(x+a)}{(x+a)^2+y^2}\right] \tag{10}$$

2.10 Distant Field of Current Loop: Magnetic Dipole

The magnetic field on the axis of a loop of current was derived in Ex. 2.3a. Here we will find the magnetic vector potential and field at locations not restricted to the axis but distant from the loop. The arrangement to be analyzed is shown in Fig. 2.10. For any point (r, θ, ϕ) at which **A** is to be found, some current elements $I\,\mathbf{dl}'$ are oriented such that they produce components of **A** in directions other than the ϕ direction. However, by the symmetry of the loop, equal and opposite amounts of such components exist. As a result **A** is ϕ directed and is independent of the value of ϕ at which it is to be found. For convenience, we choose to calculate **A** at the point $(r, \theta, 0)$. The ϕ-directed contribution of a differential element of current is

$$dA_\phi = \frac{\mu I\, dl' \cos\phi'}{4\pi R} \tag{1}$$

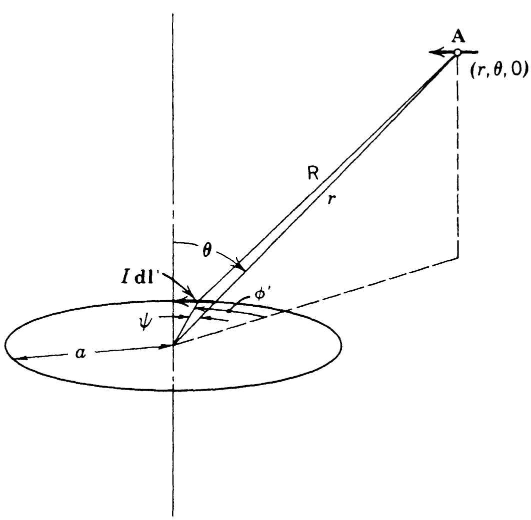

Fig. 2.10 Coordinates for calculation of magnetic-dipole fields.

where R is the distance from the element dl' to $(r, \theta, 0)$. The total is found as the integral around the loop

$$A_\phi = \frac{\mu I}{4\pi} \oint \frac{dl' \cos \phi'}{R} = \frac{\mu I a}{4\pi} \int_0^{2\pi} \frac{\cos \phi' \, d\phi'}{R} \tag{2}$$

where a is the radius of the loop. The distance R can be expressed in terms of the radius from the origin to $(r, \theta, 0)$ as

$$R^2 = r^2 + a^2 - 2ra \cos \psi \tag{3}$$

To get $ra \cos \psi$ we note that $r \cos \psi$ is the projection of r onto the extension of the radius line to dl'. Therefore

$$ra \cos \psi = ra \sin \theta \cos \phi' \tag{4}$$

Substituting (4) into (3) and assuming $r >> a$, we find

$$R \approx r\left(1 - 2\frac{a}{r} \sin \theta \cos \phi'\right)^{1/2}$$

or

$$R^{-1} \approx r^{-1}\left(1 + \frac{a}{r} \sin \theta \cos \phi'\right) \tag{5}$$

Utilizing this expression in (2), we find

$$\begin{aligned} A_\phi &= \frac{\mu I a}{4\pi r} \int_0^{2\pi} \left(\cos \phi' + \frac{a}{r} \sin \theta \cos^2 \phi'\right) d\phi' \\ &= \frac{\mu I a}{4\pi r} \frac{a\pi \sin \theta}{r} = \frac{\mu(I\pi a^2) \sin \theta}{4\pi r^2} \end{aligned} \tag{6}$$

As was noted at the outset the result applies to any value of ϕ. The components of $\mathbf{B}$, found by substituting (6) in Eq. 2.9(2), are

$$B_r = \frac{\mu I \pi a^2}{2\pi r^3} \cos \theta \tag{7}$$

$$B_\theta = \frac{\mu I \pi a^2}{4\pi r^3} \sin \theta \tag{8}$$

$$B_\phi = 0 \tag{9}$$

The group of terms $I\pi a^2$ can be given a special significance by comparison of (7)–(9) with the fields of an electric dipole, Eq. 1.10(10). The identity of the functional form of the fields has led to defining the magnitude of the *magnetic dipole moment* as

$$m = I\pi a^2 \tag{10}$$

The dipole direction is along the $\theta = 0$ axis in Fig. 2.10 for the direction of I shown.

The vector potential can be written in terms of the magnetic dipole moment **m** as

$$\mathbf{A} = \frac{-\mu_0}{4\pi} \mathbf{m} \times \nabla\left(\frac{1}{r}\right) \tag{11}$$

where the partial derivatives in the gradient operation are with respect to the point of observation of **A**.

2.11 Divergence of Magnetic Flux Density

As given by Eq. 2.9(2) (derived in Appendix 3), the magnetic flux density **B** can be expressed as the curl of another vector **A** when the sources of **B** are currents. We have shown in Ex. 2.8b that the divergence of the curl of any vector is zero. Thus,

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

A major difference between electric and magnetic fields is now apparent. The magnetic field must have zero divergence everywhere. That is, when the magnetic field is due to currents, there are no sources of magnetic flux which correspond to the electric charges as sources of electric flux. Fields with zero divergence such as these are consequently often called *source-free fields.*

Magnetic field concepts are often developed from an exact parallel with electric fields by considering the concept of isolated magnetic poles as sources of magnetic flux, corresponding to the charges of electrostatics. The result of zero divergence then follows because such poles have so far been found in nature only as equal and opposite pairs. Physicists continue to search for isolated magnetic poles; if they are found, a magnetic charge density ρ_m will simply be added to the equations giving a finite $\nabla \cdot \mathbf{B}$.

2.12 Differential Equation for Vector Magnetic Potential

The differential equation for magnetic field in terms of current density was developed in Sec. 2.7:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

If the relation for **B** as the curl of vector potential **A** is substituted,

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} \tag{1}$$

This may be considered a differential equation relating **A** to current density. It is more common to write it in a different form utilizing the Laplacian of a vector function defined in rectangular coordinates as the vector sum of the Laplacians of the three scalar components:

$$\nabla^2 \mathbf{A} = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z \tag{2}$$

It may then be verified that, for rectangular coordinates

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) \tag{3}$$

For other than rectangular coordinate systems, separation in the form (2) cannot be done so simply and (3) may be taken as the definition of ∇^2 of a vector.

With $\nabla \cdot \mathbf{A} = 0$, as shown in Appendix 3 for statics, (3) and (1) give

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{4}$$

This is a vector equivalent of the Poisson equation first met in Sec. 1.12. It includes three component scalar equations which are exactly of the Poisson form.

Example 2.12
VECTOR POTENTIAL AND FIELD OF UNIFORM CURRENT DENSITY FLOWING AXIALLY

Let us show that the appropriate form for the vector potential in a uniform flow of z-directed current in a circularly cylindrical system is

$$A_z = -\frac{\mu J_0}{4}(x^2 + y^2) \tag{5}$$

From (4) and (2),

$$J_z = -\frac{1}{\mu}\nabla^2 A_z = -\frac{1}{\mu}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right) = J_0 \tag{6}$$

From this we see that (5) is the appropriate form for vector potential in a cylindrical conductor carrying a current of constant density J_0. The magnetic field found from (5) is

$$\mathbf{H} = \frac{1}{\mu}(\nabla \times \mathbf{A}) = \frac{-J_0}{2}(\hat{\mathbf{x}}y - \hat{\mathbf{y}}x) \tag{7}$$

In cylindrical coordinates, this is

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\frac{J_0 r}{2} \tag{8}$$

which is the value of Eq. 2.4(5).

2.13 SCALAR MAGNETIC POTENTIAL FOR CURRENT-FREE REGIONS

In many problems concerned with the finding of magnetic fields, at least a part of the region is current-free. The curl of the magnetic field vector $\mathbf{H}$ is then zero for such current-free regions [Eq. 2.7(2)]. Any vector with zero curl may be represented as the gradient of a scalar (see Ex. 2.6d). Thus the magnetic field can be expressed for such points as

$$\mathbf{H} = -\nabla\Phi_m \tag{1}$$

where the minus sign is conventionally taken only to complete the analogy with electrostatic fields. The vector potential applies to both current-carrying and current-free regions, but it is usually more convenient for the latter to use this scalar potential.

Since the divergence of the magnetic flux density **B** is everywhere zero,

$$\nabla \cdot \mu\nabla\Phi_m = 0 \tag{2}$$

Thus, for a homogeneous medium, Φ_m satisfies Laplace's equation

$$\nabla^2\Phi_m = 0 \tag{3}$$

It will be observed from (1) that

$$\Phi_{m2} - \Phi_{m1} = -\int_1^2 \mathbf{H} \cdot \mathbf{dl} \tag{4}$$

Thus, if the path of integration encircles a current, Φ_m does not have a unique value. For if 1 and 2 are the same point in space and the path of integration encloses a current I, two values of Φ_m, differing by I, will be assigned to the point. To make the scalar magnetic potential unique, we must restrict attention to regions which do not entirely encircle currents. Suitable regions are called "simply connected" because any two paths connecting a pair of points in the region form a loop which does not enclose any exterior points. An example of a simply connected region between coaxial conductors is shown in Fig. 2.13. The restriction to a simply connected region is not a serious limitation once it is understood.

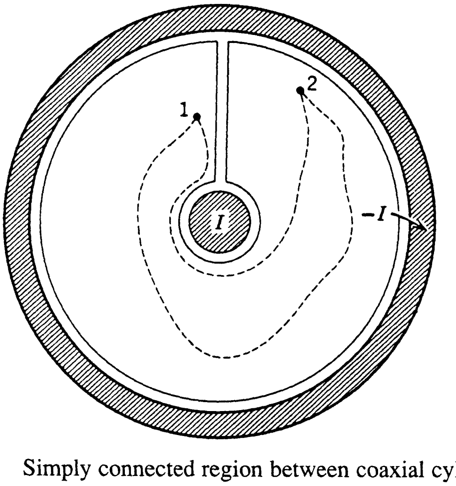

FIG. 2.13 Simply connected region between coaxial cylinders.

The importance of the scalar potential for current-free regions is that it satisfies Laplace's equation, for which exist numerous methods of solution. The graphical and numerical methods given in Chapter 1 for electrostatic fields are directly applicable, as are the more powerful numerical methods, conformal transformations, and method of separation of variables to be studied in Chapter 7.

2.14 Boundary Conditions for Static Magnetic Fields

The boundary conditions at an interface between two regions with different permeabilities can be found in the same way as was done for static electric fields in Sec. 1.14. Consider a volume in the shape of a pillbox enclosing the boundary between the two media as shown in Fig. 2.14. The surfaces ΔS of the volume are considered to be arbitrarily small so that the normal flux density B_n does not vary across the surface. Also, the thickness of the pillbox is vanishingly small so that there is negligible flux flowing through the side wall. The net outward flux from the box is

$$B_{n1}\,\Delta S = B_{n2}\,\Delta S \qquad \text{or} \qquad B_{n1} = B_{nz} \tag{1}$$

where the sense of B_n is as shown in the figure.

The relation between transverse magnetic fields may be found by integrating the magnetic field $\mathbf{H}$ along a line enclosing the interface plane as shown in Fig. 2.14,

$$\oint \mathbf{H} \cdot \mathbf{dl} = H_{t1}\,\Delta l - H_{t2}\,\Delta l = J_s\,\Delta l \tag{2}$$

where J_s is a surface current in amperes per meter width flowing in the direction shown.

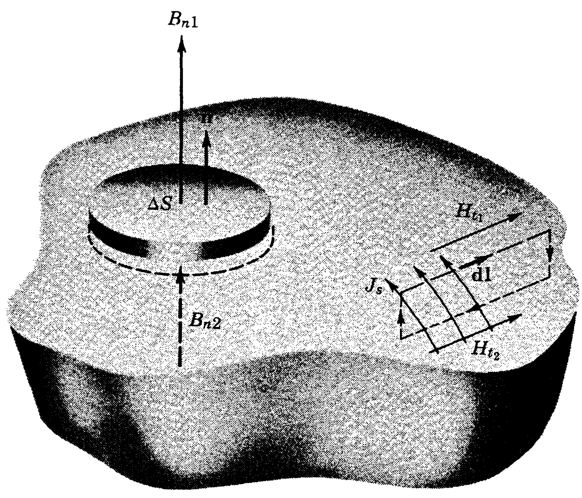

Fig. 2.14 Magnetic fields at boundary between two different media.

The lengths Δl of the sides are arbitrarily small so H_t may be considered uniform. The other legs of the integration path are effectively reduced to zero length. From (2)

$$H_{t1} - H_{t2} = J_s \tag{3}$$

There is a discontinuity of the tangential field at the boundary between two regions equal to any surface current which may exist on the boundary. With direction information included, where $\hat{\mathbf{n}}$ is the unit vector normal to the surface,

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \tag{4}$$

Although the concept of a surface current is an idealization, it is useful when the depth of current penetration into a conductor is small, as in the skin effect to be studied later. In problems involving the scalar magnetic potential, continuity of H_t where $J_s = 0$ is ensured by taking Φ_m to be continuous across the boundary. Where surface currents exist, (4) leads to

$$\hat{\mathbf{n}} \times (\nabla\Phi_{m2} - \nabla\Phi_{m1}) = \mathbf{J}_s \tag{5}$$

as may be seen by combining (3) with the definition of Φ_m, Eq. 2.13(1).

2.15 Materials with Permanent Magnetization

Permanent magnets have a remnant value of magnetization [defined in Eq. 2.3(3)] when all applied fields are removed. Magnetic materials are discussed in more detail in Chapter 13, but here we consider some examples with permanent magnetization M_0 and no true current flow. There are two ways of analyzing such problems: through the scalar magnetic potential and through the vector potential.

Use of Scalar Magnetic Potential Since current density $\mathbf{J}$ is zero, we may derive $\mathbf{H}$ from a scalar potential as in Sec. 2.13:

$$\mathbf{H} = -\nabla\Phi_m \tag{1}$$

Now using the definition of magnetization from Eq. 2.3(3),

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{2}$$

If the divergence of (2) is taken, with $\nabla \cdot \mathbf{B} = 0$ utilized, we can write

$$\nabla^2\Phi_m = -\frac{\rho_m}{\mu_0} \tag{3}$$

where

$$\rho_m = -\mu_0\nabla \cdot \mathbf{M} \tag{4}$$

In this formulation we see that we have a Poisson equation for potential Φ_m, with an equivalent magnetic charge density in the region proportional to the divergence of magnetization. For a uniform magnetization, the divergence is zero and Φ_m satisfies Laplace's equation. At the boundaries of the magnet, however, integration of (3) would show that there is an equivalent magnetic surface charge density ρ_{sm} given by

$$\rho_{sm} = \mu_0 \hat{\mathbf{n}} \cdot \mathbf{M} \tag{5}$$

The arguments for this are similar to those for surface charge density ρ_s when there is a discontinuity in **D**, as explained in Sec. 1.14. We will illustrate this through an example after giving a formulation using the vector potential.

Use of Vector Magnetic Potential If we write **B** as curl of vector potential **A** as in Eq. 2.9(2) and use the definition of magnetization,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \nabla \times \mathbf{A} \tag{6}$$

we can take the curl of this equation, using $\nabla \times \mathbf{H} = 0$ since $\mathbf{J} = 0$, to write

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}_{\text{eq}} \tag{7}$$

where

$$\mathbf{J}_{\text{eq}} = \nabla \times \mathbf{M} \tag{8}$$

So by comparison with Eq. 2.12(1), the problem is equivalent to one with internal currents in free space proportional to the curl of magnetization. Inside a region of uniform magnetization, the curl is zero and there are no internal currents. At the boundary of the magnetic material, a surface integral of (8) over the area enclosed by the path used to get Eq. 2.14(2) and application of Stokes's theorem give

$$\oint \mathbf{M} \cdot d\mathbf{l} = \int_S \mathbf{J}_{\text{eq}} \cdot d\mathbf{S}$$

In the same way as for Eq. 2.14(4), this gives an equivalent surface current

$$(\mathbf{J}_{\text{eq}})_S = \mathbf{M} \times \hat{\mathbf{n}} \tag{9}$$

since $\mathbf{M} = 0$ outside the magnetic material. So in this formulation the magnet is replaced by a system of volume and surface currents from which magnetic field may be found through use of the vector potential, or directly by using Ampère's law. Example 2.15b illustrates this procedure.

Example 2.15a
UNIFORMLY MAGNETIZED SPHERE

Consider first a sphere of magnetic material with uniform magnetization M_0 in the z direction as in Fig. 2.15a using the method with scalar magnetic potential. Since M is uniform, there is no volume charge by (4), but if space surrounds the sphere, there is

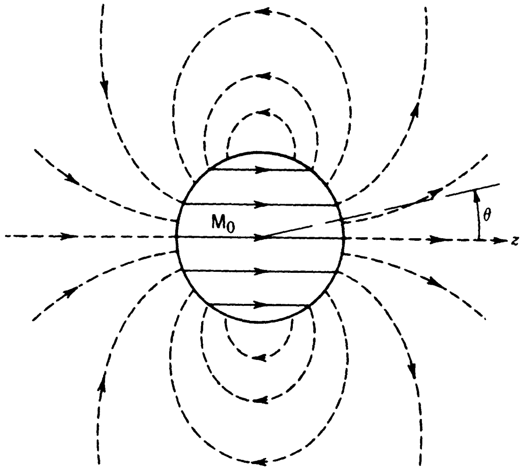

FIG. 2.15a Sphere of radius a with uniform magnetization $\hat{\mathbf{z}}M_0$. Field lines (**H** or **B**) outside the sphere shown dashed.

a surface magnetic charge density at $r = a$ given by

$$\rho_{sm} = \mu_0 M_0 \cos\theta \tag{10}$$

Solutions of (3), in spherical coordinates with a variation corresponding to (10) and $\rho_m = 0$, are

$$\begin{aligned} \Phi_{m1} &= \frac{Cr}{a}\cos\theta \qquad r < a \\ \Phi_{m2} &= \frac{Ca^2}{r^2}\cos\theta \qquad r > a \end{aligned} \tag{11}$$

as can be verified by substitution in the expression for $\nabla^2\Phi = 0$ in spherical coordinates on the inside front cover. The surface magnetic charge given by (10) gives a discontinuity in derivative,

$$\mu_0\left[\frac{\partial\Phi_{m2}}{\partial r} - \frac{\partial\Phi_{m1}}{\partial r}\right]_{r=a} = -\mu_0 M_0 \cos\theta \tag{12}$$

from which

$$C = \frac{M_0 a}{3} \tag{13}$$

Thus for $r < a$, using (1) in spherical coordinates

$$\mathbf{H} = -\frac{M_0}{3}[\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta] = -\hat{\mathbf{z}}\frac{M_0}{3} \tag{14}$$

which is a uniform field within the sphere. For $r > a$,

$$\mathbf{H} = \frac{M_0 a^3}{3r^3}[2\hat{\mathbf{r}} \cos \theta + \hat{\boldsymbol{\theta}} \sin \theta] \tag{15}$$

which are curves (shown dashed in Fig. 2.15*a*) similar to those outside a magnetic dipole (Sec. 2.10).

Example 2.15b
Round Rod with Uniform Magnetization

A circular cylindrical bar magnet of length l having uniform magnetization in the axial direction is shown in Fig. 2.15*b*. Using the second formulation given above, we see from (8) that there are no equivalent volume currents since $\nabla \times \mathbf{M} = 0$, but there is a surface current at the discontinuous boundary $r = a$:

$$\mathbf{J}_s = \hat{\boldsymbol{\phi}} M_0 \tag{16}$$

We see that this problem is then identical to that of the solenoid of length l with current per unit length given by (16) insofar as the calculation of **A** (and hence **B**) is concerned. As noted in Ex. 2.3c, it is difficult to calculate field lines for an off-axis point, but **B** lines will appear somewhat as shown dashed in Fig. 2.15*b*. Lines of magnetic field **H** will be of the same form outside the magnet, but will be of different form inside through the vector addition $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$.

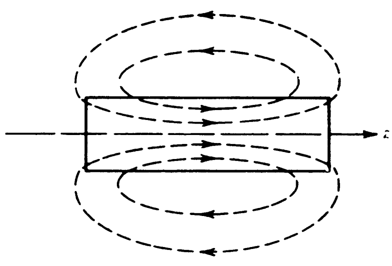

Fig. 2.15*b* Cylinder of radius a and length l with magnetization $\hat{\mathbf{z}}M_0$. Flux density lines **B** shown dashed. (**H** lines are of the same form outside the magnet.)

Magnetic Field Energy

2.16 Energy of a Static Magnetic Field

In considering the energy of a magnetic field, it would appear by analogy with Sec. 1.22 that we should consider the work done in bringing a group of current elements together from infinity. This point of view is correct in principle, but not only is it more difficult to carry out than for charges because of the vector nature of currents, but it also requires consideration of time-varying effects as shown in references deriving the relation from this point of view.[6] We will consequently set down the result at this point, waiting for further discussion until we derive a most important general energy relationship in Chapter 3. The general relation for nonlinear materials, corresponding to 1.22(9) for electric fields, is

$$dU_H = \int_V \mathbf{H} \cdot \mathbf{dB}\, dV \tag{1}$$

where dU_H is the energy added to the system when **B** is changed by a differential amount (possibly different amounts for different positions within the volume). For linear materials, **H** is proportional to **B** so (1) may be integrated over **B** to give

$$U_H = \frac{1}{2}\int_V \mathbf{B} \cdot \mathbf{H}\, dV = \int_V \frac{\mu}{2} H^2\, dV \tag{2}$$

The analogy to Eq. 1.22(6) is apparent, and here also we interpret the energy of a system of sources as actually stored in the fields produced by those sources. The result is consistent with the inductive circuit energy term, $\frac{1}{2}LI^2$, when circuit concepts hold and will be utilized in the following section.

Example 2.16a
Energy Storage in Superconducting Solenoid

It has been proposed that energy stored in large superconducting coils be used to meet peaks in electric power demand. Superconducting coils are chosen because their zero dc resistance allows very large currents to be carried with zero power loss (though, of course, refrigeration power must be supplied). To be useful such a storage system must be capable of providing about 50 MW for 6 hours, that is, storing an energy of about 10^6 MJ. Let us assume the coil is a solenoid and that the field is uniform, and we wish to find the required coil properties and current. The field from Eq. 2.4(7) is $H_z = nI$ so

[6] *J. A. Stratton,* Electromagnetic Theory, *pp. 118–124, McGraw-Hill, New York, 1941.*

$B_z = \mu nI$. The energy from (2), for volume V, is

$$U_H = \tfrac{1}{2}\mu(nI)^2V$$

For a realistic current of 1000 A and flux density of 15 T, a coil of 27-m diameter and 20-m length with 1.2×10^4 turns/m would give the required energy. The most promising coil shape is actually a toroid but it would have dimensions and currents of the same magnitude as calculated in this example.

Example 2.16b
ENERGY DISSIPATION IN HYSTERETIC MATERIALS

We will see here how energy loss in hysteretic materials can be explained in terms of their nonlinear B–H relations. A typical hysteretic relation is shown in Fig. 2.16. We will assume an isotropic material so that $\mathbf{B} \cdot \mathbf{H} = BH$. The energy required for one traversal of the loop by varying H from a large negative value to a large positive value and back again can be found from (1). The differential energy is shown as a shaded bar on the hysteresis loop in Fig. 2.16. When the field is decreased, a portion of the energy indicated by the part of the bar outside the loop is returned to the field. The result of integrating around the loop is that total expended energy per unit volume is equal to the area of the loop.

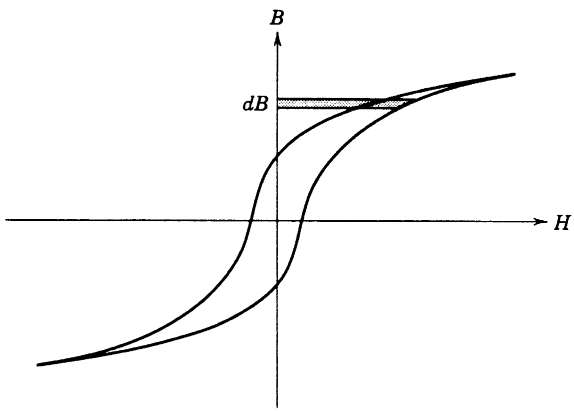

FIG. 2.16 Hysteretic, nonlinear B–H relation.

2.17 INDUCTANCE FROM ENERGY STORAGE; INTERNAL INDUCTANCE

It was shown in Sec. 2.16 that the magnetic energy may be found by integrating an energy density of $\frac{1}{2}\mu H^2$ throughout the volume of significant fields. From a circuit point of view, this is known to be $\frac{1}{2}LI^2$, where I is the instantaneous current flow through the inductance. Equating these two forms gives

$$\tfrac{1}{2}LI^2 = \int_V \frac{\mu}{2} H^2 \, dV \tag{1}$$

The form of (1) is useful as an alternate to the flux linkage method of calculating inductance given in Sec. 2.5. It is especially convenient for problems that would require consideration of partial linkages if done by the method of flux linkages. Problems of calculating internal inductance, defined in Sec. 2.5, are of this nature.

Example 2.17
INTERNAL INDUCTANCE OF CONDUCTORS WITH UNIFORM CURRENT DISTRIBUTION IN A COAXIAL TRANSMISSION LINE

As an example of the use of the energy method of inductance calculation, we will find the internal inductances for the two conductors of a coaxial transmission line under the assumption that the current is distributed uniformly in the conductors. The result for the inner conductor applies more generally to any straight, round wire with a uniform current distribution. The magnetic field in the inner conductor of Fig. 2.4*b* (Ex. 2.4c) is

$$H_\phi = \frac{Ir}{2\pi a^2} \qquad r < a \tag{2}$$

For a unit length, utilizing (1),

$$\tfrac{1}{2}LI^2 = \int_0^a \frac{\mu}{2}\left(\frac{Ir}{2\pi a^2}\right)^2 2\pi r \, dr = \frac{\mu I^2}{4\pi a^4}\cdot\frac{a^4}{4} \tag{3}$$

or

$$L = \frac{\mu_0}{8\pi} \quad \text{H/m} \tag{4}$$

The magnetic field in the outer conductor (Prob. 2.4a) is

$$H(r) = \frac{I}{2\pi(c^2 - b^2)}\left(\frac{c^2}{r} - r\right) \tag{5}$$

Substituting (5) in (1) we find

$$L = \frac{\mu}{2\pi}\left[\frac{c^4 \ln c/b}{(c^2 - b^2)^2} + \frac{b^2 - 3c^2}{4(c^2 - b^2)}\right] \quad \text{H/m} \tag{6}$$

For frequencies low enough to assume uniform current distribution in the conductors, the total inductance per unit length for the coaxial line is the sum of (4), (6), and Eq. 2.5(6).

PROBLEMS

2.2a Assuming that each electron constituting the current in a differential length of conductor is acted on by a force $-e\mathbf{v} \times \mathbf{B}$, show that the total force is equal to that given by Eq. 2.2(1). How is the force on the electrons transferred to the structure of the wire?

2.2b The Hall effect uses motion of charges in crossed fields within a semiconductor as shown in Fig. P2.2*b* to measure important properties of a semiconductor. Consider a p-type material so that the charge carriers are holes of charge $+e$. Electric field E_0 applied in the x direction causes a current $I_x = wd\sigma E_0$ to flow. The magnetic field causes a buildup of positive charge on the plate at $y = 0$ and an equal negative charge on the top plate because of the velocity $\mu_H E_0$ of the holes. The field produced by these charges on the bottom and top plates E_H is exactly of the magnitude to counteract the $e\mathbf{v} \times \mathbf{B}_0$ force on the holes so that, in steady state, the flow is only in the x direction. Show how the Hall mobility μ_H can be determined from measurement of I_x and V_H.

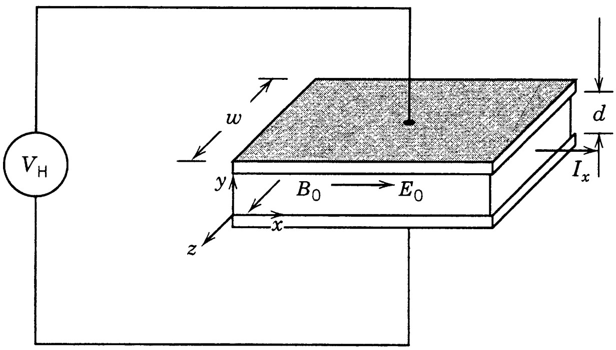

FIG. P2.2b

2.2c Show the following:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

2.2d* Cycloidal motion can occur when a particle of charge q and mass m is placed in crossed electric and magnetic fields. To demonstrate this, take a uniform electric field E_0 in the y direction and uniform magnetic flux density B_0 in the x direction. The charge starts at the coordinate origin at time $t = 0$ with zero velocity. Show that the trajectory can be written in the form $(z - R\omega_0 t)^2 + (y - R)^2 = R^2$, where $R = E_0/\omega_0 B_0$ and $\omega_0 = qB_0/m$. Explain the motion.

2.3a A loop of wire is formed by two semicircles, the inner of radius a and the outer of radius b, joined by radial line segments at $\phi = 0$ and $\phi = \pi$ (Fig. P2.3a). Find the magnetic field at the origin.

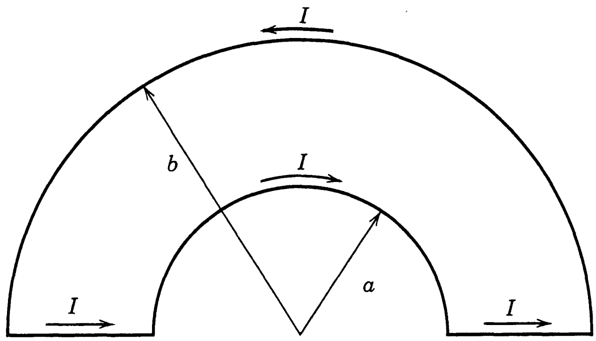

Fig. P2.3a

2.3b Direct current I_0 flows in a square loop of wire having sides of length $2a$. Find the magnetic field on the axis at a point z from the plane of the loop.

2.3c Represent a solenoid of finite length L and radius a having n turns per meter by a continuous sheet of circumferential current. Find the axial magnetic field at the center of the solenoid and determine the length for which the field is one-half that of the infinite solenoid.

2.3d Show that the magnetic field on axis of a long solenoid at the ends is half the value for an infinite solenoid.

2.3e An arrangement that can provide a region of relatively uniform fields consists of a pair of parallel, coaxial loops; the uniform-field region is on the axis midway between the loops. Show that the axial magnetic field, expressed as a Taylor series expansion along the axis about the point midway between the coils, will have zero first, second, and third derivatives if the loop radii a are equal to the spacing d of the loops. This is the so-called *Helmholtz* configuration.

2.4a For the coaxial line of Fig. 2.4b, find the magnetic field for $b < r < c$, assuming that current is distributed uniformly over the conductor cross section.

2.4b A certain kind of electron beam of circular cross section contains a current density $J_z = J_0[1 - (r/a)^4]$. Find $H_\phi(r)$ inside the beam.

2.4c Express the magnetic field about a long line current in rectangular coordinate components, taking the wire axis as the z axis, and evaluate $\oint \mathbf{H} \cdot \mathbf{dl}$ about a square path in the x–y plane from $(-1, -1)$ to $(1, -1)$ to $(1, 1)$ to $(-1, 1)$ back to $(-1, -1)$. Also evaluate the integral about the path from $(-1, 1)$ to $(1, 1)$ to $(1, 2)$ to $(-1, 2)$ back to $(-1, 1)$. Comment on the two results.

2.4d A long thin wire carries a current I_1 in the positive z direction along the axis of a cylindrical coordinate system as shown in Fig. P2.4d. A thin rectangular loop of wire lies in a plane containing the axis. The loop contains the region $0 \le z \le b$, $R - a/2 \le r \le R + a/2$ and carries a current I_2 which has the direction of I_1 on the side nearest the

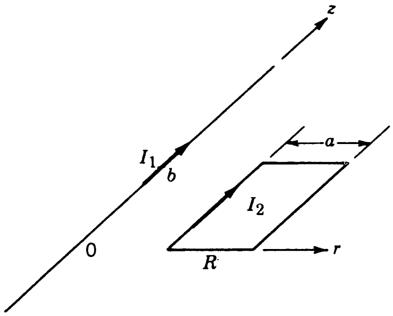

Fig. P2.4d

axis. Find the vector force on each side of the loop and the resulting force on the entire loop.

2.4e* Consider a round straight wire carrying a uniform current density J throughout, except for a round cylindrical void parallel with the wire axis so that the cross section is constant. Call the radius of the wire c, the radius of the hole b, and the distance of the center of the hole from the center of the wire a. Take $b < a < c$ and $b < c - a$. Use superposition to find the field H as a function of position along a radial line through the center of the hole for all values of radius from the center of the wire.

2.4f A demonstration can be given that a thin metal tube can be crushed by magnetic forces by passing current through it. Take the radius of the tube to be 2 cm and the magnetic field at which failure of the metal occurs as 9 Wb/m^2. (i) What is the maximum current that could flow axially along the tube before it would be crushed by the magnetic forces arising from this current? (ii) What is the force per unit area on the surface of the tubing under this condition?

2.4g For an infinitely long cylindrical hollow pipe of any cross section carrying current along the pipe, magnetic field within the hollow portion is zero. Show why.

2.5 A coaxial transmission line with inner conductor of radius a and outer conductor of radius b has a coaxial cylindrical ferrite of permeability μ_1 extending from $r = a$ to $r = d$ (with $d < b$), and air from radius d to b. Find the external inductance per unit length.

2.6a Find the curl of a vector field $\mathbf{F} = \hat{\mathbf{x}}x^2z^2 + \hat{\mathbf{y}}y^2z^2 + \hat{\mathbf{z}}x^2y^2$.

2.6b By using the rectangular coordinate forms show that

$$\nabla \times (\psi\mathbf{F}) = \psi\nabla \times \mathbf{F} - \mathbf{F} \times \nabla\psi$$

where $\mathbf{F}$ is any vector function and ψ any scalar function.

2.6c Derive the expression for curl in the spherical coordinate system.

2.7 For the coaxial line of Fig. 2.4*b*, express the magnetic field found in Ex. 2.4b and Prob. 2.4a in rectangular coordinates and find the curl in the four regions, $r < a$, $a < r < b$, $b < r < c$, $r > c$. Comment on the results.

2.8 Show that $\nabla \times \nabla\psi \equiv 0$ by integrating over an arbitrary surface and applying Stokes's theorem.

2.9a Check the results Eqs. 2.9(9) and (10) by adding vectorially the magnetic field from the individual wires, using the result of Ex. 2.4a.

2.9b* A square loop of thin wire lies in the x–y plane extending from $(-a, -a)$ to $(a, -a)$ to (a, a) to $(-a, a)$ back to $(-a, -a)$ and carries current I in that sense of circulation. Find $\mathbf{A}$ and H_x for any point (x, y, z).

2.9c* A circular loop of thin wire carries current I. Find $\mathbf{A}$ for a point distance z from the plane of the loop, and radius r from the axis, for $r/z << 1$. Use this to find the expression for magnetic field on the axis.

2.9d Show that the line integral of vector potential $\mathbf{A}$ about a closed path is equal to the magnetic flux enclosed,

$$\oint \mathbf{A} \cdot \mathbf{dl} = \int_S \mathbf{B} \cdot \mathbf{dS}$$

Apply this to find the form of $\mathbf{A}$ inside the long solenoid of Ex. 2.4d.

2.9e For an infinite single-wire line of current, show that A_z as calculated in Ex. 2.9b is infinite. Then show that if vector potential is calculated for a finite length $-L < z < L$ and $\mathbf{B}$ calculated from this before letting L approach infinity, the correct value of $\mathbf{B}$ is obtained.

2.9f As an exercise in using the vector potential, consider a very long thin conducting sheet having a width b carrying a uniformly distributed direct current I in the direction of its length. Show that if the sheet is assumed to lie in the x–z plane with the z axis along its centerline, the magnetic field about the strip will be given by

$$H_x = -\frac{I}{2\pi b}\left(\tan^{-1}\frac{b/2 + x}{y} + \tan^{-1}\frac{b/2 - x}{y}\right)$$

$$H_y = \frac{I}{4\pi b}\ln\left[\frac{(b/2 + x)^2 + y^2}{(b/2 - x)^2 + y^2}\right]$$

2.10 Show that the torque on a small loop of current can be expressed as $\tau = \mathbf{m} \times \mathbf{B}$.

2.12a Show that $\nabla^2\mathbf{A} = 0$ for the vector potential around a pair of currents, Eq. 2.9(8).

2.12b Use the rectangular coordinate forms to prove Eq. 2.12(3).

2.12c A certain current density is said to produce within itself a vector potential having the form $\mathbf{A} = \hat{\mathbf{z}}Cr^{-2}$ in circular cylindrical coordinates where C is a constant. Find the divergence of $\mathbf{A}$, current density, and magnetic field, assuming the medium to be free space.

2.12d We saw in Ex. 2.7 that magnetic field in a superconductor decays from the surface as

$$H_z = H_0 e^{-x/\lambda}$$

where z is parallel to the plane of the surface and x is perpendicular to the surface. Find the corresponding vector potential $\mathbf{A}$, and from it the current density comparing with the result of Ex. 2.7.

2.13a Show whether either of the following vector fields can be obtained from a scalar potential, and give the potential function if applicable:

$$\mathbf{F} = \hat{\mathbf{x}}3y^2z + \hat{\mathbf{y}}6xyz + \hat{\mathbf{z}}3y^2x$$

$$\mathbf{F} = \hat{\mathbf{x}}3y + \hat{\mathbf{y}}2x + \hat{\mathbf{z}}4$$

2.13b Find the form of scalar magnetic potential for the region between conductors as shown in Fig. 2.13, defined for the region $0 \leq \phi < 2\pi$; similarly for the region outside the outer conductor. Current I flows in the inner conductor and the return current in the outer one.

2.14* Consider the boundary between free space and a plane superconductor with nearby parallel line current I at $x = d$. It is the nature of a superconductor that when placed in a weak magnetic field, currents flow in such a way as to eliminate flux inside the superconductor so that B_n at the surface is zero, as is the tangential H inside the superconductor. Show that fields in the free-space region $x > 0$ can be found by replacing the superconductor with an image current at $x = -d$ carrying current $-I$. Find the magnetic field at $x = 0+$ and from this the surface current density $\mathbf{J}_s$.

2.15 For the problem of Fig. 2.15*b*, what magnetic charge distribution would be obtained for the formulation in terms of equivalent magnetic charges? How would this be modified if magnetization is inhomogeneous as defined below?

$$\mathbf{M} = \hat{\mathbf{z}} M_0(1 + kz)$$

2.16a Show that Eq. 2.16(1) leads to (2) for linear, isotropic materials.

2.16b Assume that the material having the B–H relation shown in Fig. 2.16 saturates at $B = 1000$ G and estimate graphically the energy per unit volume for one complete traversal of the hysteresis loop.

2.17a Find the external inductance per unit length for the arrangement of Prob. 2.5 from energy considerations.

2.17b Find the internal inductance per unit length for the parallel-plane transmission line of Fig. 2.5*c* if current is assumed of uniform density in each of the conductors.