are small compared with wavelength. For any such region small compared with wavelength, the wave equation will reduce to Laplace's equation so that low-frequency analyses neglecting any tendency toward wave propagation are applicable.

The presence of losses in the guide below cutoff causes the phase constant to change from the zero value for an ideal guide to a small but finite value, and modifies slightly the formula for attenuation. These modifications are most important in the immediate vicinity of cutoff, for with losses there is no longer a sharp transition but a more gradual change from one region to another. It should be emphasized again that the approximate formulas developed in previous sections may become extremely inaccurate in this region. For example, the approximate formulas for attenuation caused by conductor or dielectric losses would yield an infinite value at $f = f_c$. The actual value is large compared with the minimum attenuation in the pass range since it is approaching the relatively larger magnitude of attenuation in the cutoff regime, but it is nevertheless finite. Previous formulas have also shown an infinite value of phase velocity at cutoff, and with losses it too will be finite.

8.16 Dispersion of Signals Along Transmission Lines and Waveguides

We have in several instances noted the dispersive properties of transmission systems when phase velocity, group velocity, or both vary with frequency. In Chapter 5 we considered a simple two-frequency group in a dispersive system, but we now wish to be more general, using the Fourier integral of Sec. 7.11. There are two classes of problems of concern. One is that of a *base-band* signal, in which the detailed signal is of concern. Examples are audio or video signals, or electrical pulses from a computer, before being placed on other *carrier* frequencies. The other is that of modulated signals in which the base-band signal is placed on a high-frequency carrier. For the latter case we shall consider amplitude modulation and examine the distortion of the envelope.

Base-Band Signals Given an audio signal, series of pulses, or similar electrical waveform, we can express it as a Fourier integral as in Eq. 7.11(15). For a time function $f(t)$, the transform pair may be written

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j\omega t}\,d\omega \tag{1}$$

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{2}$$

If each frequency component is delayed in phase by βz in propagating distance z along the transmission system, (1) gives the delayed function at z as

$$f(t, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j(\omega t - \beta z)}\,d\omega \tag{3}$$

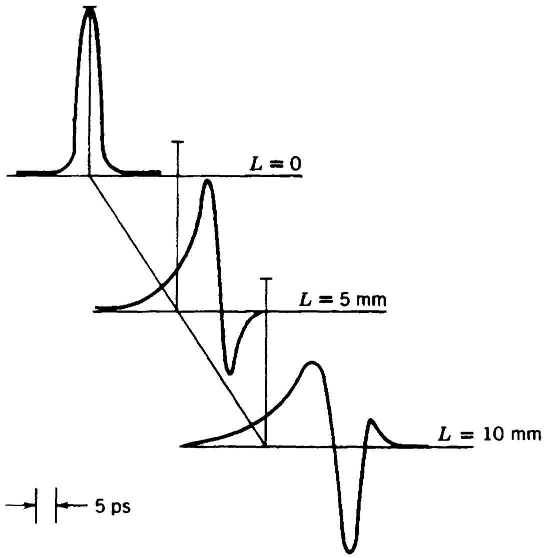

FIG. 8.16a Propagation of a 5-ps gaussian pulse along a microstrip line. Strip width = 0.32 mm, dielectric thickness = 0.4 mm, and $\varepsilon_r = 6.9$. Reproduced by permission from K. K. Li, G. Arjavalingam, A. Dienes, and J. R. Whinnery, *IEEE Trans.* **MTT-30**, 1270 (1982). © 1982 IEEE.

Now if

$$\beta = \frac{\omega}{v_p} \tag{4}$$

with v_p independent of ω, (3) is

$$f(t, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{j\omega(t - z/v_p)}\,d\omega = f\left(t - \frac{z}{v_p}\right) \tag{5}$$

Thus the original function maintains its shape and propagates at the phase velocity, as we have assumed in many wave problems. But any dispersion in v_p modifies the function, at least to some degree.

Transmission lines are often used for base-band signals and have some dispersion through loss terms and internal inductance as affected by skin effect. Some lines, as the microstrip line of Sec. 8.6, have additional dispersion from the presence of multiple dielectrics. Figure 8.16*a* shows the result of a numerical calculation from (3), using the dispersion relation of Eq. 8.6(18), for the change in shape of a 5-ps gaussian pulse in propagating along a typical microstrip used with short electrical pulses.[11]

[11] *K. K. Li, G. Arjavalingam, A. Dienes, and J. R. Whinnery,* IEEE Trans. ***MTT-30****, 1270 (1982).*

Modulated Signals If the signal (1) is used to amplitude modulate a carrier of amplitude V_c and angular frequency ω_c, the resulting modulated wave may be written

$$v_m(t) = \text{Re}\{V_c e^{j\omega_c t}[1 + mf(t)]\} \tag{6}$$

where m is a modulation coefficient. In substituting (1) in (6), we use ω_m for the frequency of the modulating (base-band) signal, and assume that its significant frequency components extend only over a band $-\omega_B \le \omega \le \omega_B$:

$$v_m(t, 0) = \text{Re}\left\{V_c e^{j\omega_c t}\left[1 + \frac{m}{2\pi}\int_{-\omega_B}^{\omega_B} g(\omega_m)e^{j\omega_m t}\, d\omega_m\right]\right\} \tag{7}$$

Or letting $\omega = \omega_c + \omega_m$,

$$v_m(t, 0) = \text{Re}\left\{V_c e^{j\omega_c t} + \frac{mV_c}{2\pi}\int_{\omega_c - \omega_B}^{\omega_c + \omega_B} g(\omega - \omega_c)e^{j\omega t}\, d\omega\right\} \tag{8}$$

Frequencies above ω_c in the integral in (8) correspond to upper sideband terms and those below ω_c to lower sideband terms. Each frequency component propagates according to its appropriate phase constant β. Let us expand β as a Taylor series about ω_c:

$$\beta(\omega) = \beta(\omega_c) + (\omega - \omega_c)\left.\frac{d\beta}{d\omega}\right|_{\omega_c} + \frac{(\omega - \omega_c)^2}{2}\left.\frac{d^2\beta}{d\omega^2}\right|_{\omega_c} + \cdots \tag{9}$$

So the modulated signal, after propagating a distance z, is

$$\begin{aligned} v_m(t, z) = \text{Re}\Bigg\{ & V_c e^{j[\omega_c t - \beta(\omega_c)z]} \\ & \times \left[1 + \frac{m}{2\pi}\int_{-\omega_B}^{\omega_B} g(\omega_m)e^{j[\omega_m(t - z/v_g) - (\omega_m^2/2)z(d^2\beta/d\omega^2) + \cdots]}\, d\omega_m\right]\Bigg\} \end{aligned} \tag{10}$$

where

$$\frac{1}{v_g} = \left.\frac{d\beta}{d\omega}\right|_{\omega_c} \tag{11}$$

Now if $d^2\beta/d\omega^2$ and higher terms are negligible, (10) is interpreted as

$$v_m(t, z) = \text{Re}\left\{V_c e^{j[\omega_c t - \beta(\omega_c)z]}\left[1 + mf\left(t - \frac{z}{v_g}\right)\right]\right\} \tag{12}$$

so the envelope propagates without distortion at group velocity v_g (though the carrier inside moves at a generally different phase velocity). But if the higher-order terms are not negligible, the envelope is distorted and there is said to be *group dispersion.* For a gaussian envelope,

$$f(t) = Ce^{-(2t/\tau)^2} \tag{13}$$

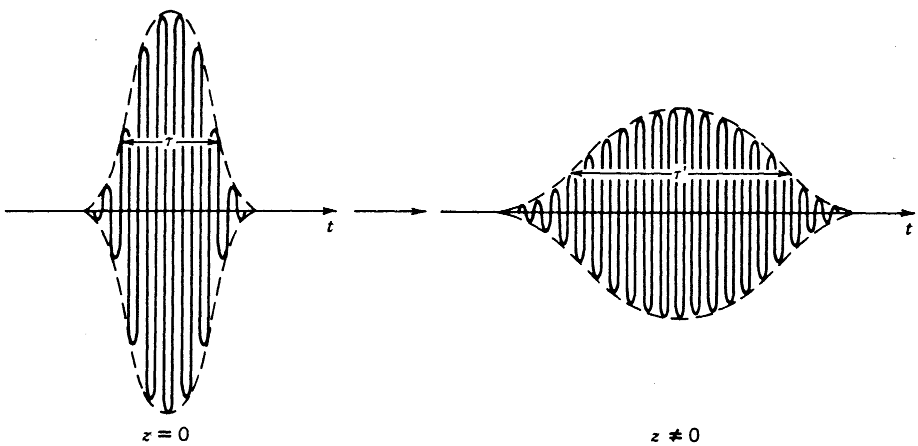

FIG. 8.16b Illustration of the spread of the modulated envelope of a pulse as it travels down a system with group dispersion.

It can be shown (Prob. 8.16c) that the term $d^2\beta/d\omega^2$ causes the envelope to spread to a width τ' after propagating distance z, with τ' given by

$$\tau' = \tau\left[1 + \left(\frac{8z}{\tau^2}\frac{d^2\beta}{d\omega^2}\right)^2\right]^{1/2} \tag{14}$$

The spread of a gaussian envelope, illustrated in Fig. 8.16*b*, clearly limits data rates as pulses begin to overlap their neighbors. Although a factor in some waveguide problems (Prob. 8.16a) the limitation is most important for optical fibers and will be met again in Chapter 14.

PROBLEMS

8.2a As we will see later, one mode of a rectangular waveguide is a TM wave with $H_z = 0$ and $E_z = A\sin(\pi x/a)\sin(\pi y/b)$ with z and t dependence assumed to be $e^{j(\omega t - \beta z)}$. Find expressions for the transverse field components. At a given plane what are the phase relations among the transverse components and between them and E_z.

8.2b The division into TM and TE classes is not the only way of classifying guided waves, as noted in Sec. 8.2. Another frequently useful division employs longitudinal-section electric (LSE) with $E_x = 0$ but all other components present and longitudinal-section magnetic (LSM) with $H_x = 0$ but all other components present. Find the relations between E_z and H_z for each of these classes.

8.3a Add induced charges and current flows, with attention to sign, to the pictures of Figs. 8.3*b* and *c* for the positively traveling TM_1 and TE_1 waves. Repeat for negatively traveling waves.

8.3b Calculate cutoff frequency for TE_1, TE_2, TE_3, TM_1, TM_2, TM_3 waves between planes 1.5 cm apart with air dielectric. Repeat for a glass dielectric with $\varepsilon'/\varepsilon_0 = 4$. Suppose excitation at 8 GHz is provided at a cross section of the air-filled line and all waves are excited. Which wave(s) will propagate without attenuation? At what distance from the excitation plane will each of the nonpropagating waves be attenuated to $1/e$ of its value at the excitation plane?

8.3c The slope of an electric field line in the x–z plane is $dx/dz = E_x/E_z$. Show that the curve for an electric field line of a TM_1 wave, obtained from the expressions for E_x and E_z of the wave, is defined by

$$\cos \beta z = [\cos \pi x_0/a][\cos(\pi x/a)]^{-1}$$

where x_0 is the value of x for a given curve at $z = 0$. Plot one or two lines to verify the form shown in Fig. 8.3*b*. [*Hint:* First express fields as real functions of z.)

8.3d Similarly to Prob. 8.3c, derive the expression defining magnetic field lines for a TE_1 wave and plot one or two lines to verify the form shown in Fig. 8.3*c*.

8.3e* Find the expression for electric field lines for a TM_2 wave, plot one or two lines, and sketch the remainder to give a plot similar to Fig. 8.3*b*. Similarly, plot and sketch magnetic field lines for a TE_2 wave.

8.3f Show that the expression for energy velocity as derived for TM_m waves [Eq. 8.3(37)] also applies to TE_m waves.

8.4a Calculate the angle θ as defined in Fig. 8.4*a* for ray directions of a TM_1 mode between planes 1.5 cm apart with glass dielectric, $\varepsilon'/\varepsilon_0 = 4$, for frequencies of 5, 6, 10, and 30 GHz.

8.4b Obtain the expressions for wave impedance of TM and TE waves, using the picture of uniform plane waves reflecting at an angle.

8.4c* By suitably changing coordinates as in Ex. 8.4, show that the expressions 6.09(18)–(20) for a wave polarized with electric field normal to the plane of incidence striking a conductor at an angle correspond exactly to the field expressions for a TE_m wave.

8.5a Find average power transfer and conductor loss for a TE mode between parallel planes to verify the expression for attenuation, Eq. 8.5(12).

8.5b Calculate attenuation in decibels per meter for a TM_1 wave between copper planes 1.5 cm apart with air dielectric. Frequency is 12 GHz. For the same frequency and spacing, a glass dielectric with $\varepsilon'/\varepsilon_0 = 4$, $\varepsilon''/\varepsilon' = 2 \times 10^{-3}$ is introduced. Calculate attenuation from both dielectric and conductor losses.

8.5c Prove that the frequency of minimum attenuation for a TM_m mode, from conductor losses, is $\sqrt{3} f_c$, where f_c is cutoff frequency. Give the expression for the minimum attenuation and calculate for silver conductors 2 cm apart and air dielectric for the $m = 1, 2,$ and 3 modes.

8.5d Show that the transmission-line formula for attenuation constant, Eq. 5.9(7), gives precisely the same result as the approximate wave analysis of Sec. 8.5 for the TEM wave.

8.5e Derive the approximate formula for attenuation constant due to dielectric losses by using $\alpha = w_L/2W_T$.

8.5f* Since E_z is equal and opposite at top and bottom conductors for TEM wave in the parallel-plane line, it is reasonable to assume a linear variation between the two values:

$$E_z = (1 + j)\frac{R_s E_0}{\eta}\left(1 - \frac{2x}{a}\right)$$

Find the modification in the distribution for E_x to satisfy the divergence equation for **E**. Find the corresponding modification in H_y from Maxwell's equations. Describe qualitatively the average Poynting vector as a function of position in the guide.

8.6a For a symmetric stripline as in Fig. 8.6*a* with $w = 1$ mm, $d = 2$ mm, $\varepsilon_r = 2.7$, and thickness t negligible (but larger than several penetration depths), calculate Z_0 and phase velocity of the TEM mode and the cutoff frequency of the next higher mode. [Note that tables of elliptic integrals are required.]

8.6b For $\varepsilon_r = 1$, the lossless microstrip of Fig. 8.6*b* can propagate a true TEM wave at the velocity of light. Find inductance and capacitance per unit length for a 50-Ω line with such a dielectric and the required w/d for this from Fig. 8.6*c*. Calculate the difference due to fringing fields between the capacitance per unit length found above and that given by the parallel-plane approximation, and express this as an equivalent extra width, $\Delta w/d$. Now maintaining w/d constant, assuming inductance is independent of ε_r, and transmission-line equations applicable, repeat for other values of ε_r and plot the extra equivalent $\Delta w/d$ due to fringing as a function of ε_r.

8.6c Calculate the characteristic impedance for a copper microstrip line with an alumina (Al_2O_3 ceramic) dielectric and air above the line. The dimensions should be $w/d = 8$ and $d = 0.2$ mm. Compare the results obtained using the formulas with the graphical data in Sec. 8.6. Find the fractional change of Z_0 between $f = 0$ and $f = 3$ GHz. Calculate the maximum frequency at which the static approximation should be used.

8.6d Design a stripline with the same materials and substrate thickness d and having the characteristic impedance found in Prob. 8.6c for the microstrip line. Calculate and compare the attenuations in the microstrip and stripline at 3 GHz assuming conductor thicknesses of 0.01 mm. Neglect dielectric losses.

8.6e It is desired to make a 15-Ω stripline with the maximum possible delay achievable with no more than 3 dB attenuation at 10 GHz. Consider two possible lines. One is to be made with copper conductors with $w = 100$ μm and alumina (Al_2O_3 ceramic) dielectric and is to be used at room temperature. The other is made with superconducting niobium conductors with $w = 100$ μm and undoped silicon dielectric, having $\varepsilon_r = 11.7$ and loss tangent $\tan\delta_\varepsilon = 10^{-5}$ at 4.2 K, at which temperature the line is to be used. Take $R_s = 10^{-5}$ Ω for niobium at 4.2 K and the strip thickness to be 5 μm for copper and 1 μm for niobium. Find the maximum delay achievable with each of the lines.

8.6f Consider the coplanar waveguide strip transmission line shown in Fig. 8.6*f*. Assuming the line is on an infinitely thick dielectric substrate, the electric fields are distributed symmetrically above and below the line.

(i) Argue that this leads to an effective dielectric constant $\varepsilon_{\text{eff}} = (\varepsilon_r + 1)/2$.
(ii) Find the dimensions to give a line with $Z_0 = 50$ Ω using $\varepsilon_r = 3.78$ and the following design formula[4]

$$\frac{w}{a} = \tanh^2\left(\frac{\pi\eta_0}{8Z_{00}} - \frac{\ln 2}{2}\right)$$

where Z_{00} is the characteristic impedance when the dielectric constant is $\varepsilon_r = 1$

everywhere, and w is the width of the strip located in the center of a gap of width a.

8.6g The various frequency components in a signal (e.g., a pulse) propagate at phase velocities determined by the effective dielectric constants at those frequencies. As will be discussed in Sec. 8.16, this variation of velocity leads to dispersion of signals. The fractional variation of phase velocity with frequency in a coplanar waveguide is lower at low frequencies than it is in microstrip. Consider the following 50-Ω lines with copper conductors and 0.635-mm-thick alumina (Al_2O_3 ceramic) substrates. The coplanar line has a strip width w of 0.266 mm and gaps s of 0.117 mm each. The strip width in the microstrip line is 0.598 mm.

(i) Plot the fractional change of phase velocity of the quasi-TEM mode as a function of frequency in the range $0 < f < 50$ GHz for the coplanar guide and $0 < f < 35$ GHz for the microstrip. For the microstrip, mark f_{max}, the limit of applicability of the static formulation, and also the cutoff frequency of the next higher mode, $(f_c)_{HE1} = cZ_0/2\eta_0 d$. Also mark the cutoff frequency $f_{TE} = c/4d\sqrt{\varepsilon_r - 1}$ of the next higher mode for the coplanar waveguide.

(ii) Find the fractional change of the phase velocity at the cutoff frequency of the next higher mode for the coplanar waveguide.

8.6h Compare the total attenuation at 3 GHz in nepers/meter for the two lines described in Prob. 8.6g and explain the physical reason why the higher one is higher.

8.7a For a rectangular waveguide with inner dimensions 3 × 1.5 cm and air dielectric, calculate the cutoff frequencies of the TE_{10}, TE_{20}, TE_{11}, TE_{12}, TE_{21}, TE_{22}, TM_{11}, TM_{22} modes. Repeat for a glass dielectric with $\varepsilon'/\varepsilon_0 = 4$. Find lengths to the $1/e$ distances for the nonpropagating modes excited at 10 GHz.

8.7b Derive the expression for magnetic lines in the transverse plane of a TM_{11} wave and plot one or two such lines, comparing with Table 8.7. (See approach in Prob. 8.3c.)

8.7c Derive the expression for electric field lines in the transverse plane of a TE_{11} wave and plot one or two such lines, comparing with Table 8.7. (See approach in Prob. 8.3c.)

8.7d Show that the expression for attenuation because of conductor loss for a TM_{mn} mode in the rectangular guide is as given by Eq. 8.7(14).

8.7e Show that the expression for attenuation because of conductor loss for a TE_{mn} mode (neither m nor n zero) in the rectangular guide is as given by Eq. 8.7(26). Explain why this does not apply to $m = 0$ or $n = 0$ case.

8.7f Recalling that surface resistivity R_s is a function of frequency, find the frequency of minimum attenuation for a TM_{mn} mode. Show that the expression for attenuation of a TE_{mn} mode must also have a minimum.

8.7g* Of the wave types studied so far, those transverse magnetic to the axial direction were obtained by setting $H_z = 0$; those transverse electric to the axial direction were obtained by setting $E_z = 0$. For the rectangular waveguide, obtain the lowest-order mode with $H_x = 0$ but all other components present. This may be called a wave transverse magnetic to the x direction. Show that it may also be obtained by superposing the TM and TE waves given previously of just sufficient amounts so that H_x from the two waves exactly cancel. This is a longitudinal-section wave as discussed in Prob. 8.2.

8.7h* Repeat Prob. 8.7g for a wave transverse electric to the x direction.

8.7i From the form of Eqs. 8.2(9)–(12), show that for a TM wave, imposition of the condition $E_z = 0$ on a perfectly conducting boundary of a cylindrical guide causes the other tangential component of **E** also to be zero along that boundary.

8.8a For $f = 3$ GHz, design a rectangular waveguide with copper conductor and air dielectric so that the TE_{10} wave will propagate with a 30% safety factor ($f = 1.30f_c$) but also so that the wave type with next higher cutoff will be 20% below its cutoff frequency. Calculate the attenuation due to copper losses in decibels per meter.

8.8b For Prob. 8.8a, calculate the attenuation in decibels per meter of the three modes with cutoff frequencies closest to that of the TE_{10} mode, neglecting losses.

8.8c Design a guide for use at 3 GHz with the same requirements as in Prob. 8.8a except that the guide is to be filled with a dielectric having a permittivity four times that of air. Calculate the increase in attenuation due to copper losses alone, assuming that the dielectric is perfect. Calculate the additional attenuation due to the dielectric, if $\varepsilon''/\varepsilon' = 0.01$.

8.8d Find the maximum power that can be carried by a 6-GHz TE_{10} wave in an air-filled guide 4 cm wide and 2 cm high, taking the breakdown field in air at that frequency as 2×10^6 V/m.

8.8e The transmission-line analogy can be applied to the transverse field components, the ratios of which are constants over guide cross sections and are given by wave impedances, just as in the case of plane waves in Chapter 6. A rectangular waveguide of inside dimensions 4 × 2 cm is to propagate a TE_{10} mode of frequency 5 GHz. A dielectric of constant $\varepsilon_r = 3$ fills the guide for $z > 0$, with an air dielectric for $z < 0$. Assuming the dielectric-filled part to be matched, find the reflection coefficient at $z = 0$ and the standing wave ratio in the air-filled part.

8.8f Find the length and dielectric constant of a quarter-wave matching section to be placed between the air and given dielectric of Prob. 8.8e.

8.9a Derive the set of Eqs. 8.9(1)–(4) by utilizing Maxwell's equations in circular cylindrical coordinates and assuming propagation as $e^{-j\beta z}$.

8.9b What inner radius do you need for an air-filled round pipe to propagate the TE_{11} wave at 6 GHz with operating frequency 20% above the cutoff frequency? What is the guide wavelength for this mode? Find the attenuation in decibels per meter of the TM_{01} mode at this frequency, neglecting losses for that calculation.

8.9c Show that the expression for attenuation from conductor losses of a TM_{nl} mode is

$$\alpha_c = \frac{R_s}{a\eta\sqrt{1 - (\omega_c/\omega)^2}}$$

At what value of ω/ω_c is this a minimum?

8.9d* Show that the expression for attenuation from conductor losses of a TE_{nl} mode is

$$\alpha_c = \frac{R_s}{a\eta\sqrt{1 - (\omega_c/\omega)^2}}\left[\left(\frac{\omega_c}{\omega}\right)^2 + \frac{n^2}{p'^2_{nl} - n^2}\right]$$

8.9e For a circular air-filled guide with copper conductor, select a radius so that the TE_{01} mode has attenuation of 0.3 dB/km for a frequency of 4 GHz. Estimate the number of modes (counting only the symmetric ones with $n = 0$) that have cutoff frequencies below the operating frequency.

8.10 Use the asymptotic forms of Bessel functions in Eqs. 8.10(1) and (2) for TM and TE waves, respectively, to show that for large $k_c r_i$ and r_0/r_i near unity, the cutoff wavelength of the $n = 0, p = 1$ modes is approximately twice the spacing between conductors.

8.11a Sketch examples of mode couplings by each of the six methods described in Sec. 8.11 using for each a system different from the one utilized in Fig. 8.11 to illustrate it.

8.11b Plot fraction of power coupled from a coaxial line into a waveguide (Fig. 8.11*g*) as a function of frequency from 10 to 11 GHz if probe radius is 1.5 mm and other dimensions are as stated in the figure caption.

8.12a Demonstrate that, although in a TEM wave **E** does satisfy Laplace's equation in the transverse plane and so may be considered a gradient of a scalar insofar as variations in the transverse plane are concerned, **E** is not the gradient of a scalar when variations in all directions (x, y, and z) are included.

8.12b Two perfectly conducting cylinders of arbitrary cross-sectional shapes are parallel and separated by a dielectric of conductivity σ and permittivity ε. Show that the ratio of electrostatic capacitance per unit length to dc conductance per unit length is ε/σ.

8.12c If the conductors are perfect but the dielectric has conductivity σ as well as permittivity ε, show that γ must have the following value for a TEM wave to exist ($E_z = 0$, $H_z = 0$):

$$\gamma = \pm[j\omega\mu(\sigma + j\omega\varepsilon)]^{1/2}$$

Explain why the distribution of fields may be a static distribution as in the loss-free line, unlike the case for a lossy conducting boundary.

8.12d How many linearly independent TEM waves may exist on a three-conductor transmission line? Describe current relations for a basic set. Complete the proof that there can be no static field, and hence no TEM wave, inside a single infinite cylindrical conductor.

8.13a Show that the circuit of Fig. P8.13*a* may be used to represent the propagation characteristics of the transverse magnetic wave, if the characteristic wave impedance and propagation constant are written by analogy with transmission-line results in terms of an impedance Z_1 and an admittance Y_1 per unit length, and the medium is μ_1, ε_1.

$$Z_{\text{TM}} = \sqrt{\frac{Z_1}{Y_1}}, \qquad \gamma = \sqrt{Z_1 Y_1}$$

Note the similarity between this and the circuits of conventional filter sections, remembering of course that all constants in this circuit are in reality distributed constants.

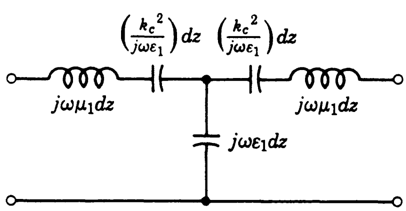

Fig. P8.13*a*

8.13b Show that all field components for a TM wave may be derived from the axial component of the vector potential **A**. Obtain the expressions relating E_x, H_x, and so on to A_z, the differential equation for A_z, and the boundary conditions to be applied at a perfect conductor. Repeat using the axial component of the Hertz potential defined in Prob. 3.19b.

8.13c Show for a TM wave that the magnetic field distribution in the transverse plane can be derived from a scalar flux function, and relate this to E_z. With transverse electric field derivable from a scalar potential function and transverse magnetic field derivable from a scalar flux function, does it follow that both are static-type distributions as in the TEM wave? Explain.

8.13d Show that energy velocity equals group velocity for the TM modes in a lossless waveguide of general cross section.

8.13e* Show that $E_z(x, y)$ for a general TM wave in a perfectly conducting guide satisfies the equation

$$k_c^2 = \left[\int_S (\nabla_t E_z)^2 \, dS\right]\left[\int_S E_z^2 \, dS\right]^{-1}$$

where ∇_t represents the transverse gradient and the integral is over the cross section of the guide. From this argue that k_c^2 is real and positive for waves in which phase is constant over the transverse plane.

8.13f* Numerical methods can be used to find the propagation constants for waveguides of arbitrary cross section. Following the procedures used in solving the Laplace or Poisson equations in Sec. 1.21 to get a difference equation solution for the scalar Helmholtz equation $\nabla^2\psi + k_c^2\psi = 0$, one finds the residual at the kth step to be $R^{(k)}(x, y) = \psi^{(i)}(x, y + h) + \psi^{(i)}(x, y - h) + \psi^{(i)}(x + h, y) + \psi^{(i)}(x - h, y) - (4 - k_c^2h^2)/\psi^{(k-1)}(x, y)$. The change of variable from one iteration step to the next in the successive overrelaxation method is governed by $\psi^{(k)} = \psi^{(k-1)} + \Omega R^{(k)}/(4 - k_c^2h^2)$. Apply the equations with Ω set to 1.0 for convenience to make anumerical evaluation of k_c^2 for a TM_{11} mode in a rectangular waveguide. Assume a rectangular guide with side ratio 1:2. The Helmholtz equation to be solved is Eq. 8.13(1). Divide the waveguide into a grid of 18 squares and number the interior points 1–10 left to right, top to bottom. A reasonable initial guess for the product $k_c^2h^2 = u^2h^2$ can be formed assuming a one-dimensional variation in the smallest dimension; here take $k_c^2h^2 = 1.1$. Start with E_z having the following values at the grid points as a first guess: for points 1, 5, 6, and 10, $E_z = 30$; for points 2, 4, 7, and 9, $E_z = 50$; for points 3 and 8, $E_z = 70$. Use simple relaxation twice to improve the values of E_z for the given $k_c^2h^2$. Then calculate an improved value of $k_c^2h^2$ using the relation

$$k_c^2h^2 = \frac{\sum E_z(x, y)[E_{zN} + E_{zE} + E_{zS} + E_{zW} - 4E_z(x, y)]}{\sum E_z^2(x, y)}$$

where N, E, S, W indicate the points surrounding the grid point at (x, y) and the summations are over all grid points. Next make two more steps of relaxation to adjust the fields to the new $k_c^2h^2$. Then use the above formula to get a second correction to $k_c^2h^2$. Compare the result with the value of $k_c^2h^2$ found using differential equations in Sec. 8.7.

8.14a Derive the equivalent circuit for a TE wave analogous to that of a TM wave given in Prob. 8.13a.

8.14b Show that fields satisfying Maxwell's equations in a homogeneous charge-free, current-free dielectric may be derived from a vector potential **F**:

$$\mathbf{E} = -\frac{1}{\varepsilon}\nabla \times \mathbf{F}$$

$$\mathbf{H} = \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F}) - j\omega\mathbf{F}$$

$$(\nabla^2 + k^2)\mathbf{F} = 0$$

Obtain expressions for all field components of a TE wave from the axial component F_z of the above potential function, and give the differential equation and boundary conditions for F.

8.14c Show that if one utilizes the potential function **A** instead of the **F** of Prob. 8.14b for derivation of a TE wave, more than one component is required.

8.14d Show for a TE mode that transverse distribution of electric field can be derived from a scalar flux function. How is this related to H_z?

8.14e Show that the energy velocity equals the group velocity for the TE modes in a lossless waveguide of general cross section.

8.14f Show for a TM wave in any shape of guide passing from one dielectric material to another, that at one frequency the change in cutoff factor may cancel the change in η, and the wave may pass between the two media without reflection. Identify this condition with the case of incidence at polarizing angle in Sec. 6.13. Determine the requirement for a similar situation with TE waves, and show why it is not practical to obtain this.

8.15 A particular waveguide attenuator is circular in cross section with radius 1 cm. Plot attenuation in decibels per meter for the TE_{11} mode over the frequency range 1–4 GHz. Also plot attenuation of the mode with next nearest cutoff frequency.

8.16a For a hollow-pipe waveguide, with β given by Eq. 8.13(9), find the group dispersion term $d^2\beta/d\omega^2$. Find the length of waveguide for which the width of a gaussian pulse with $\tau = 1$ ns is doubled if frequency is 10 GHz and $\omega_c/\omega = 0.85$.

8.16b Find $d^2\beta/d\omega^2$ for a transmission line with series resistance R and shunt conductance G independent of frequency, where $R/\omega L$ and $G/\omega C$ are small compared with unity. Repeat for a coaxial line with $G = 0$ and R governed by skin effect. Is the resulting group dispersion likely to be significant in usual applications?

8.16c* Start with a gaussian function $f(t)$ given by Eq. 8.16(13) and find its $g(\omega)$. Using this in Eq. 8.16(10), show that the envelope broadens with z as given by Eq. 8.16(14).

8.16d* From the solution of Prob. 8.16c find phase ϕ at z for the high-frequency pulse with gaussian envelope and find the frequency "chirp," defined as $d\phi/dt$.