in problems which include the axis $r = 0$ in the range over which the solution is to apply. The solution to the z equation (7) when $T^2 = -\tau^2$ is

$$Z = C_3' \sin \tau z + C_4' \cos \tau z \tag{18}$$

Summarizing, either of the following forms satisfies Laplace's equation in the two cylindrical coordinates r and z:

$$\Phi(r, z) = [C_1 J_0(Tr) + C_2 N_0(Tr)][C_3 \sinh Tz + C_4 \cosh Tz] \tag{19}$$

$$\Phi(r, z) = [C_1' I_0(\tau r) + C_2' K_0(\tau r)][C_3' \sin \tau z + C_4' \cos \tau z] \tag{20}$$

As was the case with the rectangular harmonics, the two forms are not really different since (19) includes (20) if T is allowed to become imaginary, but the two separate ways of writing the solution are useful, as will be demonstrated in later examples. The case with no assumed symmetries is discussed in the following section.

7.14 Bessel Functions

In Sec. 7.13 an example of a Bessel function was shown as a solution of the differential equation 7.13(8) which describes the radial variations in Laplace's equation for axially symmetric fields where a product solution is assumed. This is just one of a whole family of functions which are solutions of the general Bessel differential equation.

Bessel Functions with Real Arguments For certain problems, as, for example, the solution for field between the two halves of a longitudinally split cylinder, it may be necessary to retain the ϕ variations in the equation. The solution may be assumed in product form again, $RF_\phi Z$, where R is a function of r alone, F_ϕ of ϕ alone, and Z of z alone, Z has solutions in hyperbolic functions as before, and F_ϕ may also be satisfied by sinusoids:

$$Z = C \cosh Tz + D \sinh Tz \tag{1}$$

$$F_\phi = E \cos \nu\phi + F \sin \nu\phi \tag{2}$$

The differential equation for R is then slightly different from the zero-order Bessel equation obtained previously:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(T^2 - \frac{\nu^2}{r^2}\right)R = 0 \tag{3}$$

It is apparent at once that Eq. 7.13(8) is a special case of this more general equation, that is, $\nu = 0$. A series solution to the general equation carried through as in Sec. 7.13 shows that the function defined by the series

$$J_\nu(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{\nu+2m}}{m!\Gamma(\nu + m + 1)} \tag{4}$$

is a solution to the equation.

$\Gamma(\nu + m + 1)$ is the gamma function of $(\nu + m + 1)$ and, for ν integral, is equivalent to the factorial of $(\nu + m)$. Also for ν nonintegral, values of this gamma function are

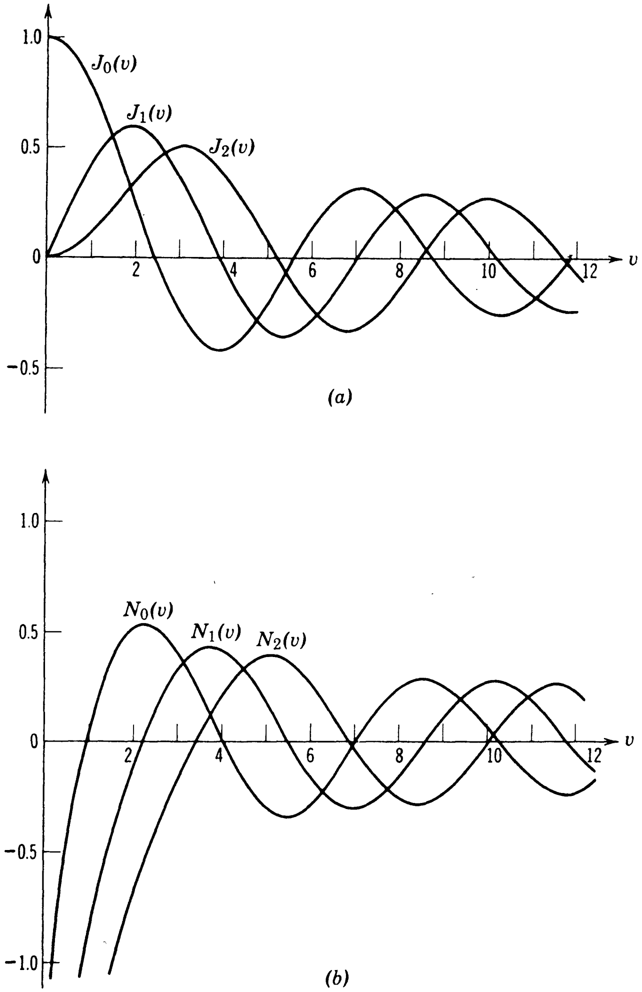

FIG. 7.14 (*a*) Bessel functions of the first kind. (*b*) Bessel functions of the second kind.

tabulated. If ν is an integer n,

$$J_n(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{n+2m}}{m!(n+m)!} \tag{5}$$

It can be shown that $J_{-n} = (-1)^n J_n$. A few of these functions are plotted in Fig. 7.14*a*. Similarly, a second independent solution[13] to the equation is

$$N_\nu(Tr) = \frac{\cos \nu\pi J_\nu(Tr) - J_{-\nu}(Tr)}{\sin \nu\pi} \tag{6}$$

[13] *If ν is nonintegral, $J_{-\nu}$ is not linearly related to J_ν, and it is then proper to use either $J_{-\nu}$ or N_ν as the second solution; for ν integral, N_ν must be used. Equation (6) is indeterminate for ν integral but is subject to evaluation by usual methods.*

and $N_{-n} = (-1)^n N_n$. As may be noted in Fig. 7.14*b* these are infinite at the origin. A complete solution to (3) may be written

$$R = AJ_\nu(Tr) + BN_\nu(Tr) \tag{7}$$

The constant ν is known as the order of the equation. J_ν is then called a Bessel function of first kind, order ν; N_ν is a Bessel function of second kind, order ν. Of most interest for this chapter are cases in which $\nu = n$, an integer.

It is useful to keep in mind that, in the physical problem considered here, ν is the number of radians of the sinusoidal variation of the potential per radian of angle about the axis.

The functions $J_\nu(v)$ and $N_\nu(v)$ are tabulated in the references.[14,15] Some care should be observed in using these references, for there is a wide variation in notation for the second solution, and not all the functions used are equivalent, since they differ in the values of arbitrary constants selected for the series. The $N_\nu(v)$ is chosen here because it is the form most common in current mathematical physics and also the form most commonly tabulated. Of course, it is quite proper to use any one of the second solutions throughout a given problem, since all the differences will be absorbed in the arbitrary constants of the problem, and the same final numerical result will be obtained; but it is necessary to be consistent in the use of only one of these throughout any given analysis.

It is of interest to observe the similarity between (3) and the simple harmonic equation, the solutions of which are sinusoids. The difference between these two differential equations lies in the term $(1/r)(dR/dr)$ which produces its major effect as $r \to 0$. Note that for regions far removed from the axis as, for example, near the outer edge of Fig. 1.19*a*, the region bounded by surfaces of a cylindrical coordinate system approximates a cube. For these reasons, it may be expected that, away from the origin, the Bessel functions are similar to sinusoids. That this is true may be seen in Figs. 7.14*a* and *b*. For large values of the arguments, the Bessel functions approach sinusoids with magnitude decreasing as the square root of radius, as will be seen in the asymptotic forms, Eqs. 7.15(1) and 7.15(2).

Hankel Functions It is sometimes convenient to take solutions to the simple harmonic equation in the form of complex exponentials rather than sinusoids. That is, the solution of

$$\frac{d^2Z}{dz^2} + K^2Z = 0 \tag{8}$$

can be written as

$$Z = Ae^{+jKz} + Be^{-jKz} \tag{9}$$

[14] *E. Jahnke, F. Emde, and F. Lösch,* Tables of Higher Functions, *6th ed. revised by F. Lösch, McGraw-Hill, New York, 1960.*

[15] *M. Abramowitz and I. A. Stegun (Eds.),* Handbook of Mathematical Functions, *Dover, New York, 1964.*

where

$$e^{\pm jKz} = \cos Kz \pm j \sin Kz \tag{10}$$

Since the complex exponentials are linear combinations of cosine and sine functions, we may also write the general solution of (8) as

$$Z = A'e^{jKz} + B' \sin Kz$$

or other combinations.

Similarly, it is convenient to define new Bessel functions which are linear combinations of the $J_\nu(Tr)$ and $N_\nu(Tr)$ functions. By direct analogy with the definition (10) of the complex exponential, we write

$$H_\nu^{(1)}(Tr) = J_\nu(Tr) + jN_\nu(Tr) \tag{11}$$

$$H_\nu^{(2)}(Tr) = J_\nu(Tr) - jN_\nu(Tr) \tag{12}$$

These are called Hankel functions of the first and second kinds, respectively. Since they both contain the function $N_\nu(Tr)$, they are both singular at $r = 0$. Negative and positive orders are related by

$$H_{-\nu}^{(1)}(Tr) = e^{j\pi\nu}H_\nu^{(1)}(Tr)$$

$$H_{-\nu}^{(2)}(Tr) = e^{-j\pi\nu}H_\nu^{(2)}(Tr)$$

For large values of the argument, these can be approximated by complex exponentials, with magnitude decreasing as square root of radius. For example,

$$\underset{Tr\to\infty}{H_\nu^{(1)}(Tr)} = \sqrt{\frac{2}{\pi Tr}}\, e^{j(Tr - \pi/4 - \nu\pi/2)}$$

This asymptotic form suggests that Hankel functions may be useful in wave propagation problems as the complex exponential is in plane-wave propagation. It is also sometimes convenient to use Hankel functions as alternate independent solutions in static problems. Complete solutions of (3) may be written in a variety of ways using combinations of Bessel and Hankel functions.

Bessel and Hankel Functions of Imaginary Arguments If T is imaginary, $T = j\tau$, and (3) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\tau^2 + \frac{\nu^2}{r^2}\right)R = 0 \tag{13}$$

The solution to (3) is valid here if T is replaced by $j\tau$ in the definitions of $J_\nu(Tr)$ and $N_\nu(Tr)$. In this case $N_\nu(j\tau r)$ is complex and so requires two numbers for each value of the argument, whereas $j^{-\nu}J_\nu(j\tau r)$ is always a purely real number. It is convenient to replace $N_\nu(j\tau r)$ by a Hankel function. The quantity $j^{\nu-1}H_\nu^{(1)}(j\tau r)$ is also purely real and so requires tabulation of only one value for each value of the argument. If ν is not an integer, $j^\nu J_{-\nu}(j\tau r)$ is independent of $j^{-\nu}J_\nu(j\tau r)$ and may be used as a second solution.

Thus, for nonintegral ν two possible complete solutions are

$$R = A_2 J_\nu(jTr) + B_2 J_{-\nu}(jTr) \tag{14}$$

and

$$R = A_3 J_\nu(jTr) + B_3 H_\nu^{(1)}(jTr) \tag{15}$$

where powers of j are included in the constants. For $\nu = n$, an integer, the two solutions in (14) are not independent but (15) is still a valid solution.

It is common practice to denote these solutions as

$$I_{\pm\nu}(v) = j^{\mp\nu} J_{\pm\nu}(jv) \tag{16}$$

$$K_\nu(v) = \frac{\pi}{2} j^{\nu+1} H_\nu^{(1)}(jv) \tag{17}$$

where $v = Tr$.

As is noted in Sec. 7.15 some of the formulas relating Bessel functions and Hankel functions must be changed for these modified Bessel functions. Special cases of these functions were seen as $I_0(Tr)$ and $K_0(Tr)$ in Sec. 7.13 for the axially symmetric field. The forms of $I_\nu(Tr)$ and $K_\nu(Tr)$ for $\nu = 0, 1$ are shown in Fig. 7.14*c*. As is suggested by these curves, the asymptotic forms of the modified Bessel functions are related to growing and decaying real exponentials, as will be seen in Eqs. 7.15(5) and 7.15(6). It is also clear from the figure that $K_\nu(Tr)$ is singular at the origin.

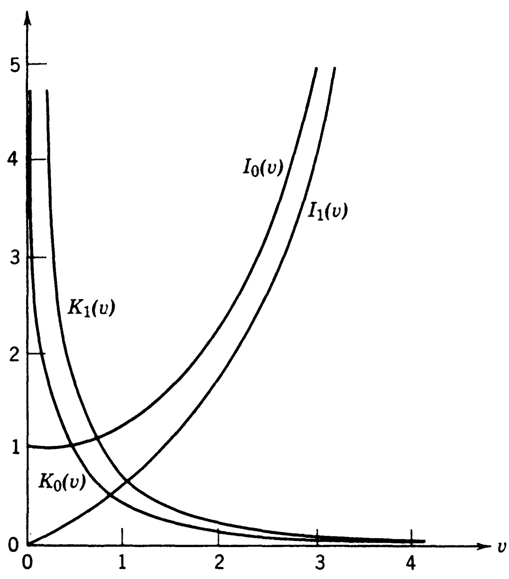

FIG. 7.14c Modified Bessel functions.

7.15 Bessel Function Zeros and Formulas[16]

The first several zeros of the low-order Bessel functions and of the derivatives of Bessel functions are given in Tables 7.15a and 7.15b, respectively.

Table 7.15a
Zeros of Bessel Functions

J_0	J_1	J_2	N_0	N_1	N_2
2.405	3.832	5.136	0.894	2.197	3.384
5.520	7.016	8.417	3.958	5.430	6.794
8.654	10.173	11.620	7.086	8.596	10.023

Table 7.15b
Zeros of Derivatives of Bessel Functions

J_0'	J_1'	J_2'	N_0'	N_1'	N_2'
0.000	1.841	3.054	2.197	3.683	5.003
3.832	5.331	6.706	5.430	6.942	8.351
10.173	8.536	9.969	8.596	10.123	11.574

Asymptotic Forms

$$\underset{v\to\infty}{J_\nu(v)} \to \sqrt{\frac{2}{\pi v}}\cos\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{1}$$

$$\underset{v\to\infty}{N_\nu(v)} \to \sqrt{\frac{2}{\pi v}}\sin\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{2}$$

$$\underset{v\to\infty}{H_\nu^{(1)}(v)} \to \sqrt{\frac{2}{\pi v}}\, e^{j[v-(\pi/4)-(\nu\pi/2)]} \tag{3}$$

$$\underset{v\to\infty}{H_\nu^{(2)}(v)} \to \sqrt{\frac{2}{\pi v}}\, e^{-j[v-(\pi/4)-(\nu\pi/2)]} \tag{4}$$

$$\underset{v\to\infty}{j^{-\nu}J_\nu(jv)} = \underset{v\to\infty}{I_\nu(v)} \to \sqrt{\frac{1}{2\pi v}}\, e^{v} \tag{5}$$

$$\underset{v\to\infty}{j^{\nu+1}H_\nu^{(1)}(jv)} = \frac{2}{\pi}\underset{v\to\infty}{K_\nu(v)} \to \sqrt{\frac{2}{\pi v}}\, e^{-v} \tag{6}$$

[16] *More extensive tabulations are found in the sources given in footnotes 14 and 15.*

Derivatives The following formulas which may be found by differentiating the appropriate series, term by term, are valid for any of the functions $J_\nu(v)$, $N_\nu(v)$, $H_\nu^{(1)}(v)$, and $H_\nu^{(2)}(v)$. Let $R_\nu(v)$ denote any one of these, and R_ν' denote $(d/dv)[R_\nu(v)]$.

$$R_0' = -R_1(v) \tag{7}$$

$$R_1'(v) = R_0(v) - \frac{1}{v} R_1(v) \tag{8}$$

$$vR_\nu'(v) = \nu R_\nu(v) - vR_{\nu+1}(v) \tag{9}$$

$$vR_\nu'(v) = -\nu R_\nu(v) + vR_{\nu-1}(v) \tag{10}$$

$$\frac{d}{dv}[v^{-\nu}R_\nu(v)] = -v^{-\nu}R_{\nu+1}(v) \tag{11}$$

$$\frac{d}{dv}[v^{\nu}R_\nu(v)] = v^{\nu}R_{\nu-1}(v) \tag{12}$$

Note that

$$R_\nu'(Tr) = \frac{d}{d(Tr)}[R_\nu(Tr)] = \frac{1}{T}\frac{d}{dr}[R_\nu(Tr)] \tag{13}$$

For the I and K functions, different forms for the foregoing derivatives must be used. They may be obtained from these formulas by substituting Eqs. 7.14(16) and 7.14(17) in the preceding expressions. Some of these are

$$\left.\begin{aligned} vI_\nu'(v) &= \nu I_\nu(v) + vI_{\nu+1}(v) \\ vI_\nu'(v) &= -\nu I_\nu(v) + vI_{\nu-1}(v) \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} vK_\nu'(v) &= \nu K_\nu(v) - vK_{\nu+1}(v) \\ vK_\nu'(v) &= -\nu K_\nu(v) - vK_{\nu-1}(v) \end{aligned}\right\} \tag{15}$$

Recurrence Formulas By recurrence formulas, it is possible to obtain the values for Bessel functions of any order, when the values of functions for any two other orders, differing from the first by integers, are known. For example, subtract (10) from (9). The result may be written

$$\frac{2\nu}{v} R_\nu(v) = R_{\nu+1}(v) + R_{\nu-1}(v) \tag{16}$$

As before, R_ν may denote J_ν, N_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$, but not I_ν or K_ν. For these, the recurrence formulas are

$$\frac{2\nu}{v} I_\nu(v) = I_{\nu-1}(v) - I_{\nu+1}(v) \tag{17}$$

$$\frac{2\nu}{v} K_\nu(v) = K_{\nu+1}(v) - K_{\nu-1}(v) \tag{18}$$

Integrals Integrals that will be useful in solving later problems are given below. R_ν denotes J_ν, N_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$:

$$\int v^{-\nu} R_{\nu+1}(v)\, dv = -v^{-\nu} R_\nu(v) \tag{19}$$

$$\int v^{\nu} R_{\nu-1}(v)\, dv = v^{\nu} R_\nu(v) \tag{20}$$

$$\begin{aligned}\int vR_\nu(\alpha v)R_\nu(\beta v)\, dv &= \frac{v}{\alpha^2 - \beta^2} \\ &\quad \times [\beta R_\nu(\alpha v)R_{\nu-1}(\beta v) - \alpha R_{\nu-1}(\alpha v)R_\nu(\beta v)], \ \alpha \neq \beta\end{aligned} \tag{21}$$

$$\begin{aligned}\int vR_\nu^2(\alpha v)\, dv &= \frac{v^2}{2}[R_\nu^2(\alpha v) - R_{\nu-1}(\alpha v)R_{\nu+1}(\alpha v)] \\ &= \frac{v^2}{2}\left[R_\nu'^2(\alpha v) + \left(1 - \frac{\nu^2}{\alpha^2 v^2}\right)R_\nu^2(\alpha v)\right]\end{aligned} \tag{22}$$

7.16 Expansion of a Function as a Series of Bessel Functions

In Sec. 7.11 a study was made of the method of Fourier series by which a function may be expressed over a given region as a series of sines or cosines. It is possible to evaluate the coefficients in such a case because of the orthogonality property of sinusoids. A study of the integrals, Eqs. 7.15(21) and 7.15(22), shows that there are similar orthogonality expressions for Bessel functions. For example, these integrals may be written for zero-order Bessel functions, and if α and β are taken as p_m/a and p_q/a, where p_m and p_q are the mth and qth roots of $J_0(v) = 0$, that is, $J_0(p_m) = 0$ and $J_0(p_q) = 0$, $p_m \neq p_q$, then Eq. 7.15(21) gives

$$\int_0^a rJ_0\left(\frac{p_m r}{a}\right)J_0\left(\frac{p_q r}{a}\right) dr = 0 \tag{1}$$

So, a function $f(r)$ may be expressed as an infinite sum of zero-order Bessel functions

$$f(r) = b_1 J_0\left(p_1 \frac{r}{a}\right) + b_2 J_0\left(p_2 \frac{r}{a}\right) + b_3 J_0\left(p_3 \frac{r}{a}\right) + \cdots$$

or

$$f(r) = \sum_{m=1}^{\infty} b_m J_0\left(\frac{p_m r}{a}\right) \tag{2}$$

The coefficients b_m may be evaluated in a manner similar to that used for Fourier coefficients by multiplying each term of (2) by $rJ_0(p_m r/a)$ and integrating from 0 to

a. Then by (1) all terms on the right disappear except the mth term:

$$\int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right) dr = \int_0^a b_m r\left[J_0\left(\frac{p_m r}{a}\right)\right]^2 dr$$

From Eq. 7.15(22),

$$\int_0^a b_m r J_0^2\left(\frac{p_m r}{a}\right) dr = \frac{a^2}{2} b_m J_1^2(p_m) \tag{3}$$

or

$$b_m = \frac{2}{a^2 J_1^2(p_m)} \int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right) dr \tag{4}$$

In the above, as in the Fourier series, the orthogonality relations enabled us to obtain coefficients of the series under the assumption that the series is a proper representation of the function to be expanded, but two additional points are required to show that the representation is valid. The series must of course converge, and the set of orthogonal functions must be *complete*, that is, sufficient to represent an arbitrary function over the interval of concern. These points have been shown for the Bessel series of (2) and for other orthogonal sets of functions to be used in this text.[17]

Expansions similar to (2) can be made with Bessel functions of other orders and types (Prob. 7.16a).

Example 7.16
BESSEL FUNCTION EXPANSION FOR CONSTANT IN RANGE $0 < r < A$

If the function $f(r)$ in (4) is a constant V_0 in the range $0 < r < a$, we have

$$b_m = \frac{2V_0}{a^2 J_1^2(p_m)} \int_0^a rJ_0\left(\frac{p_m r}{a}\right) dr \tag{5}$$

Using Eq. 7.15(20) with $R = J$, $\nu = 1$, and $v = p_m r/a$, the integral in (5) becomes

$$\begin{aligned}\left(\frac{a}{p_m}\right)^2 \int_0^a \left(\frac{p_m r}{a}\right) J_0\left(\frac{p_m r}{a}\right) d\left(\frac{p_m r}{a}\right) &= \left[\left(\frac{a}{p_m}\right)^2 \left(\frac{p_m r}{a}\right) J_1\left(\frac{p_m r}{a}\right)\right]_0^a \\ &= \frac{a^2}{p_m} J_1(p_m)\end{aligned} \tag{6}$$

[17] *See, for example, E. T. Whittaker and G. N. Watson,* A Course in Modern Analysis, *4th ed., pp. 374–378, University Press, Cambridge, 1927.*

and the series expansion (2) for the constant V_0 is

$$f(r) = \sum_{m=1}^{\infty} \frac{2V_0}{p_m J_1(p_m)} J_0\left(\frac{p_m r}{a}\right) \tag{7}$$

or, using the values of the zeros of J_0 in Table 7.15a,

$$\begin{aligned} f(r) &= \frac{0.832V_0}{J_1(2.405)} J_0\left(\frac{2.405r}{a}\right) + \frac{0.362V_0}{J_1(5.520)} J_0\left(\frac{5.520r}{a}\right) \\ &+ \frac{0.231V_0}{J_1(8.654)} J_0\left(\frac{8.654r}{a}\right) + \cdots \end{aligned} \tag{8}$$

Further evaluation of (8) requires reference to tables in the sources given in footnotes 14 and 15 or numerical evaluation of Eq. 7.13(11).

7.17 Fields Described by Cylindrical Harmonics

We will consider here the two basic types of boundary value problems which exist in axially symmetric cylindrical systems. These can be understood by reference to Fig. 7.17*a*. In one type both Φ_1 and Φ_2, the potentials on the ends, are zero and a nonzero potential Φ_3 is applied to the cylindrical surface. In the second type $\Phi_3 = 0$ and either (or both) Φ_1 or Φ_2 are nonzero. The gaps between ends and side are considered negligibly small. For simplicity, the nonzero potentials will be taken to be independent of the coordinate along the surface. In the first type, a Fourier series of sinusoids is used to expand the boundary potentials as was done in the rectangular problems. In the second situation, a series of Bessel functions is used to expand the boundary potential along the radial coordinate.

Nonzero Potential on Cylindrical Surface Since the boundary potentials are axially symmetric, zero-order Bessel functions should be used. The repeated zeros along the z coordinate dictate the use of sinusoidal functions of z. The potential in Eq. 7.13(20) is the appropriate form. Certain of the constants can be evaluated immediately. Since $K_0(\tau r)$ is singular on the axis, C_2' must be identically zero to give a finite potential there. The cos τz equals unity at $z = 0$ but the potential must be zero there so $C_4' = 0$. As in the problem discussed in Sec. 7.10 the repeated zeros at $z = l$ require that $\tau = m\pi/l$. Therefore the general harmonic which fits all boundary conditions except $\Phi = V_0$ at $r = a$ is

$$\Phi_m = A_m I_0\left(\frac{m\pi r}{l}\right) \sin\left(\frac{m\pi z}{l}\right) \tag{1}$$

Figure 7.17*b* shows a sketch of this harmonic for $m = 1$ and with the nonzero boundary potential on the cylinder. It is clear that we have here the problem of expanding the

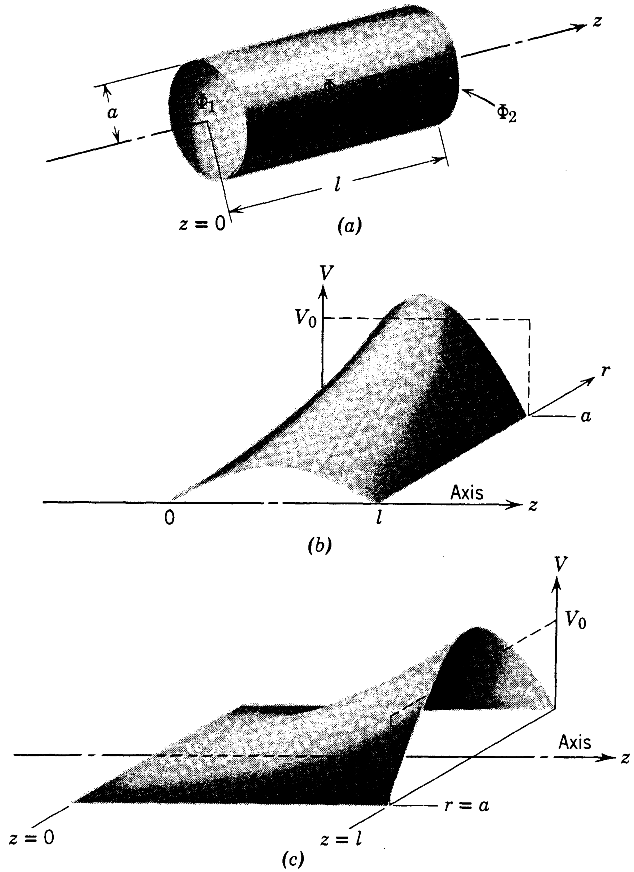

FIG. 7.17 (*a*) Cylinder with conducting boundaries. (*b*) One harmonic component for matching boundary conditions when nonzero potential is applied to cylindrical surface in (*a*). (*c*) One harmonic component for matching boundary conditions when nonzero potential is applied to end surface in (*a*).

boundary potential in sinusoids just as in the rectangular problem of Sec. 7.12. Following the procedure used there we obtain

$$\Phi(r, z) = \sum_{m \text{ odd}} \frac{4V_0}{m\pi} \frac{I_0(m\pi r/l)}{I_0(m\pi a/l)} \sin \frac{m\pi z}{l} \tag{2}$$

Nonzero Potential on End In this situation if we refer to Fig. 7.17*a*, we see that $\Phi_1 = \Phi_3 = 0$ and $\Phi_0 = V_0$. In selecting the proper form for the solution from Sec. 7.13, the boundary condition that $\Phi = 0$ at $r = a$ for all values of z indicates that the R function must become zero at $r = a$. Thus, we select the J_0 functions since the I_0's do not ever become zero. (The corresponding second solution, N_0, does not appear since potential must remain finite on the axis.) The value of T in Eq. 7.13(19) is determined from the condition that $\Phi = 0$ at $r = a$ for all values of z. Thus, if p_m is the mth root of $J_0(v) = 0$, T must be p_m/a. The corresponding solution for Z is in hyperbolic functions. The coefficient of the hyperbolic cosine term must be zero since Φ is zero at $z = 0$ for all values of r. Thus, a sum of all cylindrical harmonics with arbitrary amplitudes which satisfy the symmetry of the problem and the boundary conditions so far imposed may be written

$$\Phi(r, z) = \sum_{m=1}^{\infty} B_m J_0\left(\frac{p_m r}{a}\right) \sinh\left(\frac{p_m z}{a}\right) \tag{3}$$

One of the harmonics and the required boundary potentials are shown in Fig. 7.17*c*.

The remaining condition is that, at $z = l$, $\Phi = 0$ at $r = a$ and $\Phi = V_0$ at $r < a$. Here we can use the general technique of expanding the boundary potential in a series of the same form as that used for the potentials inside the region, as regards the dependence on the coordinate along the boundary. In Ex. 7.16 we expanded a constant over the range $0 < r < a$ in J_0 functions so that result can be used here to evaluate the constants in (3). Evaluating (3) at the boundary $z = l$, we have

$$\Phi(r, l) = \sum_{m=1}^{\infty} B_m \sinh\left(\frac{p_m l}{a}\right) J_0\left(\frac{p_m r}{a}\right) \tag{4}$$

Equations (4) and 7.16(7) must be equivalent for all values of r. Consequently, coefficients of corresponding terms of $J_0(p_m r/a)$ must be equal. The constant B_m is now completely determined, and the potential at any point inside the region is

$$\Phi(r, z) = \sum_{m=1}^{\infty} \frac{2V_0}{p_m J_1(p_m) \sinh(p_m l/a)} \sinh\left(\frac{p_m z}{a}\right) J_0\left(\frac{p_m r}{a}\right) \tag{5}$$

7.18 Spherical Harmonics

Consider next Laplace's equation in spherical coordinates for regions with symmetry about the axis so that variations with azimuthal angle ϕ may be neglected. Laplace's equation in the two remaining spherical coordinates r and θ then becomes (obtainable from form of inside front cover)

$$\frac{\partial^2(r\Phi)}{\partial r^2} + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) = 0 \tag{1}$$

or

$$r\frac{\partial^2\Phi}{\partial r^2} + 2\frac{\partial\Phi}{\partial r} + \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r\tan\theta}\frac{\partial\Phi}{\partial\theta} = 0 \tag{2}$$

Assume a product solution

$$\Phi = R\Theta$$

where R is a function of r alone, and Θ of θ alone:

$$rR''\Theta + 2R'\Theta + \frac{1}{r}R\Theta'' + \frac{1}{r \tan \theta}R\Theta' = 0$$

and

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = -\frac{\Theta''}{\Theta} - \frac{\Theta'}{\Theta \tan \theta} \tag{3}$$

From the previous logic, if the two sides of the equations are to be equal to each other for all values of r and θ, both sides can be equal only to a constant. Since the constant may be expressed in any nonrestrictive way, let it be $m(m + 1)$. The two resulting ordinary differential equations are then

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - m(m + 1)R = 0 \tag{4}$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{1}{\tan \theta}\frac{d\Theta}{d\theta} + m(m + 1)\Theta = 0 \tag{5}$$

Equation (4) has a solution which is easily verified to be

$$R = C_1r^m + C_2r^{-(m+1)} \tag{6}$$

A solution to (5) in terms of simple functions is not obvious, so, as with the Bessel equation, a series solution may be assumed. The coefficients of this series must be determined so that the differential equation (5) is satisfied and the resulting series made to define a new function. There is one departure here from an exact analog with the Bessel functions, for it turns out that a proper selection of the arbitrary constants will make the series for the new function terminate in a finite number of terms if m is an integer. Thus, for any integer m, the polynomial defined by

$$P_m(\cos \theta) = \frac{1}{2^m m!}\left[\frac{d}{d(\cos \theta)}\right]^m (\cos^2 \theta - 1)^m \tag{7}$$

is a solution to the differential equation (5). The equation is known as Legendre's equation; the solutions are called Legendre polynomials of order m. Their forms for the first few values of m are tabulated below and are shown in Fig. 7.18a. Since they are polynomials and not infinite series, their values can be calculated easily if desired, but values of the polynomials are also tabulated in many references.

$$\begin{aligned} P_0(\cos \theta) &= 1 \\ P_1(\cos \theta) &= \cos \theta \\ P_2(\cos \theta) &= \tfrac{1}{2}(3 \cos^2 \theta - 1) \\ P_3(\cos \theta) &= \tfrac{1}{2}(5 \cos^3 \theta - 3 \cos \theta) \\ P_4(\cos \theta) &= \tfrac{1}{8}(35 \cos^4 \theta - 30 \cos^2 \theta + 3) \\ P_5(\cos \theta) &= \tfrac{1}{8}(63 \cos^5 \theta - 70 \cos^3 \theta + 15 \cos \theta) \end{aligned} \tag{8}$$

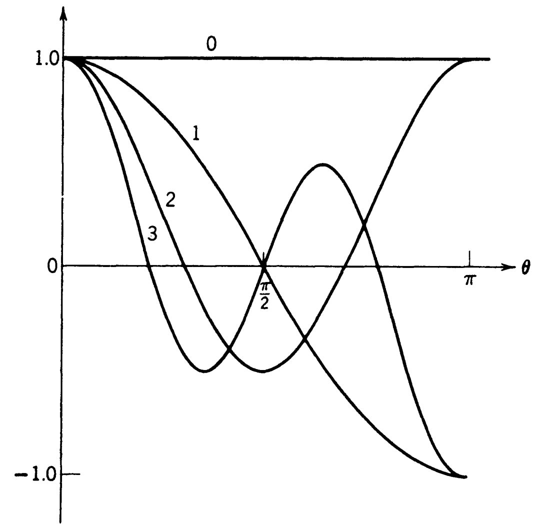

FIG. 7.18a Legendre polynomials.

It is recognized that $\Theta = C_1P_m(\cos\theta)$ is only one solution to the second-order differential equation (5). There must be a second independent solution, which may be obtained from the first in the same manner as for Bessel functions, but it turns out that this solution becomes infinite for $\theta = 0$. Consequently it is not needed when the axis of spherical coordinates is included in the region over which the solution applies. When the axis is excluded, the second solution must be included. It is typically denoted $Q_n(\cos\theta)$ and tabulated in the references.[18]

An orthogonality relation for Legendre polynomials is quite similar to those for sinusoids and Bessel functions which led to the Fourier series and expansion in Bessel functions, respectively.

$$\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta = 0, \qquad m \neq n \tag{9}$$

$$\int_0^\pi [P_m(\cos\theta)]^2 \sin\theta\,d\theta = \frac{2}{2m+1} \tag{10}$$

It follows that, if a function $f(\theta)$ defined between the limits of 0 to π is written as a series of Legendre polynomials,

$$f(\theta) = \sum_{m=0}^{\infty} \alpha_m P_m(\cos\theta), \qquad 0 < \theta < \pi \tag{11}$$

[18] *W. R. Smythe,* Static and Dynamic Electricity, *3rd ed., Hemisphere Publishing Co., Washington, DC, 1989.*

the coefficients must be given by the formula

$$\alpha_m = \frac{2m + 1}{2} \int_0^{\pi} f(\theta) P_m(\cos \theta) \sin \theta \, d\theta \tag{12}$$

Example 7.18a
HIGH-PERMEABILITY SPHERE IN UNIFORM FIELD

We will examine the field distribution in and around a sphere of permeability $\mu \neq \mu_0$ when it is placed in an otherwise uniform magnetic field in free space. The uniform field is disturbed by the sphere as indicated in Fig. 7.18*b*. The reason for choosing this example is threefold. It shows, first, an application of spherical harmonics. Second, it is an example of a situation in which the constants in series solutions for two regions are evaluated by matching across a boundary. Finally, it is an example of a magnetic boundary-value problem.

Since there are no currents in the region to be studied, we may use the scalar magnetic potential introduced in Sec. 2.13. The magnetic intensity is given by

$$\mathbf{H} = -\nabla \Phi_m \tag{13}$$

As the problem is axially symmetric and the axis is included in the region of interest, the solutions $P_m(\cos \theta)$ are applicable. The series solutions with these restrictions are

$$\Phi_m(r, \theta) = \sum_m P_m(\cos \theta)[C_{1m} r^m + C_{2m} r^{-(m+1)}] \tag{14}$$

The procedure is to write general forms for the potential inside and outside the sphere and match these across the boundary. Since the potential must remain finite at $r = 0$,

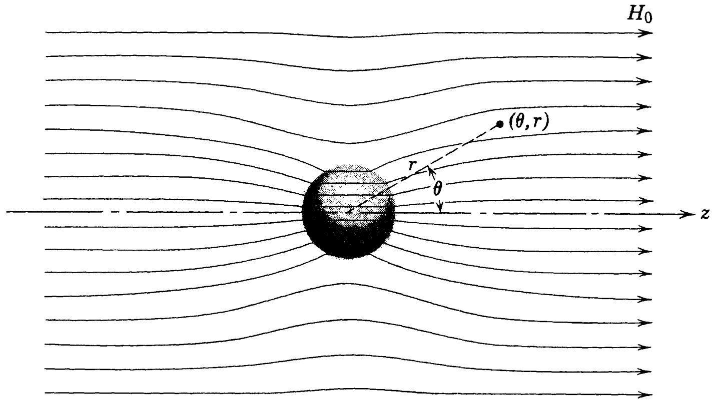

FIG. 7.18*b* Sphere of magnetic material in an otherwise uniform magnetic field.

the coefficients of the negative powers of r must vanish for the interior. The series becomes, for the inside region,

$$\Phi_m = \sum_m A_m r^m P_m(\cos\theta) \tag{15}$$

Outside, the potential must be such that it gives a uniform magnetic field H_0 at infinity. The potential form which satisfies this condition is

$$\Phi_m = -H_0 r \cos\theta \tag{16}$$

That this gives a uniform field may be seen by noting that $dz = dr\cos\theta$ so

$$H_z = -\frac{\partial\Phi_m}{\partial z} = -\frac{1}{\cos\theta}\frac{\partial\Phi_m}{\partial r} = H_0 \tag{17}$$

Terms of the series (14) having negative powers of r may be added to (16), since they all vanish at infinity. Then the form of the solution outside the sphere is

$$\Phi_m = -H_0 r\cos\theta + \sum_m B_m P_m(\cos\theta) r^{-(m+1)} \tag{18}$$

It was pointed out in Sec. 2.14 that Φ_m is continuous across boundaries without surface currents. Therefore, the terms in (15) and (18) having the same form of θ dependence are equated, giving

$$\begin{aligned} A_0 &= B_0 a^{-1} && m = 0 \\ A_1 a &= B_1 a^{-2} - H_0 a && m = 1 \\ &\vdots \\ A_m a^m &= B_m a^{-(m+1)} && m > 1 \end{aligned} \tag{19}$$

Furthermore, the normal flux density is continuous at the boundary so

$$\mu_0 \left.\frac{\partial\Phi_m}{\partial r}\right|_{r=a+} = \mu \left.\frac{\partial\Phi_m}{\partial r}\right|_{r=a-} \tag{20}$$

Substituting (15) and (18) in (20) and equating terms with the same θ dependence, we find

$$\begin{aligned} B_0 &= 0 && m = 0 \\ \mu A_1 &= -2\mu_0 B_1 a^{-3} - \mu_0 H_0 && m = 1 \\ &\vdots \\ \mu m A_m a^{m-1} &= -\mu_0(m+1) B_m a^{-(m+2)} && m > 1 \end{aligned} \tag{21}$$

From (19) and (21) we see that $A_0 = B_0 = 0$, and that for $m > 1$, all coefficients must be zero to satisfy the two sets of conditions. The only remaining terms are those with

$m = 1$. These two equations may be solved to give A_1 and B_1 in terms of H_0. Substituting the results in (18) gives, for $r > a$,

$$\Phi_m = \left[\left(\frac{\mu - \mu_0}{2\mu_0 + \mu}\right)\frac{a^3}{r^3} - 1\right]H_0 r \cos\theta \tag{22}$$

from which H can be found by using (13) for $r > a$. Substitution of A_1 into (15) gives, for $r < a$,

$$\Phi_m = -\left(\frac{3\mu_0}{2\mu_0 + \mu}\right)H_0 r \cos\theta \tag{23}$$

Applying (13), we find the field inside to be

$$\mathbf{H} = \hat{\mathbf{z}}\left(\frac{3\mu_0}{2\mu_0 + \mu}\right)H_0 \tag{24}$$

It is of interest to observe that the field inside the homogeneous sphere is uniform. Finally, multiplication of (24) by μ gives the flux density

$$\mathbf{B} = \hat{\mathbf{z}}\left(\frac{3\mu_0}{2(\mu_0/\mu) + 1}\right)H_0 \tag{25}$$

From (25) we see that for $\mu >> \mu_0$ the maximum possible value of the flux density is

$$\mathbf{B} = \hat{\mathbf{z}}3\mu_0 H_0 \tag{26}$$

Example 7.18b
EXPANSION IN SPHERICAL HARMONICS WHEN FIELD IS GIVEN ALONG AN AXIS

It is often relatively simple to obtain the field or potential along an axis of symmetry by direct application of fundamental laws, yet difficult to obtain it at any point off this axis by the same technique. Once the field is found along an axis of symmetry, expansions in spherical harmonics give its value at any other point. Suppose potential, or any component of field which satisfies Laplace's equation, is given for every point along an axis in such a form that it may be expanded in a power series in z, the distance along this axis:

$$\Phi\Big|_{\text{axis}} = \sum_{m=0}^{\infty} b_m z^m, \qquad 0 < z < a \tag{27}$$

If this axis is taken as the axis of spherical coordinates, $\theta = 0$, the potential off the axis may be written for $r < a$

$$\Phi(r, \theta) = \sum_{m=0}^{\infty} b_m r^m P_m(\cos\theta) \tag{28}$$

This is true since it is a solution of Laplace's equation and does reduce to the given potential (27) for $\theta = 0$ where all $P_m(\cos\theta)$ are unity.

If potential is desired outside of this region, the potential along the axis must be expanded in a power series good for $a < z < \infty$:

$$\Phi\Big|_{\theta=0} = \sum_{m=1}^{\infty} c_m z^{-(m+1)}, \qquad z > a \tag{29}$$

Then Φ at any point outside is given by comparison with the second series of (14):

$$\Phi = \sum_{m=0}^{\infty} c_m P_m(\cos\theta) r^{-(m+1)}, \qquad r > a \tag{30}$$

For example, the magnetic field H_z was found along the axis of a circular loop of wire carrying current I in Sec. 2.3 as

$$H_z = \frac{a^2 I}{2(a^2 + z^2)^{3/2}} = \frac{I}{2a[1 + (z^2/a^2)]^{3/2}} \tag{31}$$

The binomial expansion

$$(1 + u)^{-3/2} = 1 - \frac{3}{2}u + \frac{15}{8}u^2 - \frac{105}{48}u^3 + \cdots$$

is good for $0 < |u| < 1$. Applied to (31), this gives for $z < a$

$$H_z\Big|_{\text{axis}} = \frac{I}{2a}\left[1 - \frac{3}{2}\left(\frac{z^2}{a^2}\right) + \frac{15}{8}\left(\frac{z^2}{a^2}\right)^2 - \frac{105}{48}\left(\frac{z^2}{a^2}\right)^3 + \cdots\right]$$

Since H_z, axial component of magnetic field, satisfies Laplace's equation (Sec. 7.2), H_z at any point r, θ with $r < a$ is given by

$$H_z(r, \theta) = \frac{I}{2a}\left[1 - \frac{3}{2}\left(\frac{r^2}{a^2}\right)P_2(\cos\theta) + \frac{15}{8}\left(\frac{r^4}{a^4}\right)P_4(\cos\theta) + \cdots\right] \tag{32}$$

7.19 Product Solutions for the Helmholtz Equation in Rectangular Coordinates

The technique used in the preceding sections for finding product solutions to Laplace's equation will be applied here to the scalar Helmholtz equation. Whereas the single-product solution for static problems was seen in Sec. 7.10 to be of little value, such solutions will be seen in the next chapter to be of great importance as waveguide propagation modes and will be analyzed extensively there.

Let us consider the scalar Helmholtz equation. Here we make the assumption that the dependent variable depends on z in the manner of a wave, as $e^{-\gamma z}$. The variable ψ

remaining in the equation is, therefore, the coefficient of $e^{(j\omega t - \gamma z)}$. Written with the Laplacian explicitly in rectangular coordinates, we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -k_c^2 \psi \tag{1}$$

where $k_c^2 = \gamma^2 + \omega^2 \mu \varepsilon$. Let us assume that the solution can be written as the product solution $\psi = X(x)Y(y)$. Substituting this form in (1),

$$X''Y + XY'' = -k_c^2 XY$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = -k_c^2 \tag{2}$$

The primes indicate derivatives. If this equation is to hold for all values of x and y, since x and y may be changed independently of each other, each of the ratios X''/X and Y''/Y can be only a constant. There are then several forms for the solutions, depending upon whether these ratios are taken as negative constants, positive constants, or one negative constant and one positive constant. If both are taken as negative,

$$\frac{X''}{X} = -k_x^2$$

$$\frac{Y''}{Y} = -k_y^2$$

The solutions to these ordinary differential equations are sinusoids, and by (2) the sum of k_x^2 and k_y^2 is k_c^2. Thus

$$\psi = XY \tag{3}$$

where

$$\begin{aligned} X &= A \cos k_x x + B \sin k_x x \\ Y &= C \cos k_y y + D \sin k_y y \\ k_x^2 &+ k_y^2 = k_c^2 \end{aligned} \tag{4}$$

Either or both of k_x and k_y may be imaginary in which case the corresponding sinusoid becomes a hyperbolic function. Values of the constants k_x and k_y are determined by conditions on ψ at the boundaries in the x–y plane. Examples of the application of these general forms will be seen extensively in the following chapter where the dependent variable ψ is identified as E_z or H_z.

7.20 Product Solutions for the Helmholtz Equation in Cylindrical Coordinates

In cylindrical structures, such as coaxial lines or waveguides of circular cross section, the wave components are most conveniently expressed in terms of cylindrical coordinates. Assuming that the z dependence is in the waveform $e^{-\gamma z}$, the scalar Helmholtz equation becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} = -k_c^2\psi \tag{1}$$

where $k_c^2 = \gamma^2 + \omega^2\mu\varepsilon$. For this partial differential equation, we shall again substitute an assumed product solution and separate variables to obtain two ordinary differential equations.

Assume

$$\psi = RF_\phi$$

where R is a function of r alone and F_ϕ is a function of ϕ alone:

$$R''F_\phi + \frac{R'F_\phi}{r} + \frac{F''_\phi R}{r^2} = -k_c^2RF_\phi$$

Separating variables, we have

$$r^2\frac{R''}{R} + \frac{rR'}{R} + k_c^2r^2 = \frac{-F''_\phi}{F_\phi}$$

The left side of the equation is a function of r alone; the right of ϕ alone. If both sides are to be equal for all values of r and ϕ, both sides must equal a constant. Let this constant be ν^2. There are then the two ordinary differential equations:

$$\frac{-F''}{F} = \nu^2 \tag{2}$$

and

$$r^2\frac{R''}{R} + \frac{rR'}{R} + k_c^2r^2 = \nu^2$$

or

$$R'' + \frac{1}{r}R' + \left(k_c^2 - \frac{\nu^2}{r^2}\right)R = 0 \tag{3}$$

The solution to (2) is in sinusoids. By comparing with Eq. 7.14(3) we see that solutions to (3) may be written in terms of Bessel functions of order ν:

$$\psi = RF_\phi \tag{4}$$

where

$$R = AJ_\nu(k_cr) + BN_\nu(k_cr) \tag{5}$$

$$F_\phi = C\cos\nu\phi + D\sin\nu\phi$$

Either or both of the Bessel functions may be replaced by Hankel functions [Eqs. 7.14(11) and (12)] when one desires to look at waves as though propagation were in the radial direction. Thus, for example,

$$\begin{aligned} R &= A_1 H_\nu^{(1)}(k_c r) + B_1 H_\nu^{(2)}(k_c r) \\ F_\phi &= C \cos \nu\phi + D \sin \nu\phi \end{aligned} \tag{6}$$

If k_c is imaginary, the ordinary Bessel functions can be replaced by the modified Bessel functions, Eqs. 7.14(16) and (17). In the examples in the following chapter, the variable ψ will be identified with E_z or H_z.

PROBLEMS

7.2a Find the form of differential equation satisfied by E_r in cylindrical coordinates for a charge-free, homogeneous dielectric region. Repeat for E_ϕ. Note that these are not Laplace equations.

7.2b Show that none of the spherical components of electric field satisfy Laplace's equation for quasistatic problems in which $\nabla^2\mathbf{E} = 0$.

7.2c* Show that the rectangular component E_z of electrostatic field satisfies Laplace's equation expressed in spherical coordinates.

7.2d Derive Laplace's equation for **H**, **A**, and Φ_m in a current-free region with static fields and for **J** and **E** in a homogeneous conductor with dc currents.

7.2e Use superposition to find the potential on the axis of an infinite cylinder with a potential specified as $\Phi(\phi) = V_0 \sin \phi/2$, for $0 \le \phi \le 2\pi$ on the boundary.

7.2f A spherical surface is at zero potential except for a sector in the region $0 < \phi < \pi/3$, $0 < \theta < \pi/2$. Find the potential at the center of the sphere.

7.3a Calculate the capacitance of a parallel-plate capacitor with square plates having edge length a and spacing $d = a/2$ situated in free space using the method of moments. If you do the calculations by hand, divide each plate into four equal squares. If a computer program is written, run it for several subdivisions of the plates and plot the effect on capacitance.

7.3b Find a better approximation to the capacitance of the structure in Ex. 7.3a by subdividing each of the squares shown in Fig. 7.3*d* into four equal parts and repeating the method of moments calculation.

7.3c In applying the method of moments calculation to two-dimensional problems, the $\ln r_0$ term in Eq. 1.8(7) is neglected. As an illustration of the validity of this procedure, find the potential of two parallel line charges located as follows: $+q_1$ at $\phi = 0$, $r = \delta$ and $-q_1$ at $\phi = 0$, $r = 2\delta$; take the zero potential point to be $r = R$ on the $\phi = 0$ axis. Apply Eq. 1.8(7) and show that the $\ln r_0$ terms cancel to arbitrary accuracy as $R \to \infty$. How does this explain that the $\ln r_0$ terms can be neglected in the two-electrode two-dimensional method of moment problems in which the line charges have a variety of values?

7.3d* Write a computer program to find the stripline capacitance as in Ex. 7.3b. Extend the range included on the larger electrodes by one unit of the division in Fig. 7.3e and evaluate the effect on capacitance. Then use a subdivision of the electrodes one-half as fine as in the example. Compare the results to evaluate the importance of the grid size.

7.4a Check by the Cauchy–Riemann equations the analyticity of the general power term $W = C_n Z^n$ and a series of such terms,

$$W = \sum_{n=1}^{\infty} C_n Z^n$$

7.4b Check the following functions by the Cauchy–Riemann equations to determine if they are analytic:

$$W = \sin Z$$
$$W = e^Z$$
$$W = Z^* = x - jy$$
$$W = ZZ^*$$

7.4c Check the analyticity of the following, noting isolated points where the derivatives may not remain finite:

$$W = \ln Z$$
$$W = \tan Z$$

7.4d Take the change ΔZ in any general direction $\Delta x + j\,\Delta y$. Show that, if the Cauchy–Riemann conditions are satisfied, Eq. 7.4(3) yields the same result for the derivative as when the change is in the x direction or the y direction alone.

7.4e If by following a path around some point in the Z plane, the variable W takes on different values when the same Z is reached, the point around which the path is taken is called a *branch point*. Evaluate $W = Z^{1/2}$ and $W = Z^{4/3}$ along a path of constant radius around the origin to show that $Z = 0$ is a branch point for these functions. Discuss the analyticity of these functions at the branch point.

7.5a Plot the shape of the $u = \pm 0.5$ equipotentials for the $V = x^{4/3}$, $y = 0$ boundary condition used in Ex. 7.5.

7.5b A thin cylindrical shell of radius a has a potential described by $\Phi(a, \theta) = V_0 \cos 2\theta$. Use a method similar to that in Ex. 7.5 to find $\Phi(r,\theta)$.

7.5c Show that if u is the potential function, the field intensity E_y is equal to the imaginary part of dW/dZ and E_x equals the negative of the real part.

7.5d Use the results of Prob. 7.5c to find an expression for the slope of equipotential lines in terms of dW/dZ. Show that all equipotential lines except $u = 0$ are normal to the beam edge in the electron flow in Fig. 7.5*b*. ($W = Z^{4/3}$ is not analytic at $Z = 0$, as was shown in Prob. 7.4e, and the $W = 0$ line at $y = 0$ is a special case.) *Hint:* Write an expression for du in terms of partial derivatives and set $du = 0$ to get relations existing along an equipotential.

7.6a Plot a few equipotentials and flux lines in the vicinity of conducting corners of angles $\alpha = \pi/3$ and $3\pi/4$.

7.6b Evaluate the constant C_1 and C_2 in the logarithmic transformation so that u represents the potential function in volts about a line charge of strength q_l C/m. Let potential be zero at $r = a$.

7.6c Show that if v is taken as the potential function in the logarithmic transformation, it is applicable to the region between two semi-infinite conducting planes intersecting at an angle α, but separated by an infinitesimal gap at the origin so that the plane at $\theta = 0$ may be placed at potential zero and the plane at $\theta = \alpha$ at potential V_0. Evaluate the constants C_1 and C_2, taking the reference for zero flux at $r = a$. Write the flux function in coulombs per meter.

7.6d Find the form of the curves of constant u and constant v for the functions $\sin^{-1} Z$, $\cosh^{-1} Z$, and $\sinh^{-1} Z$. Do these permit one to solve problems in addition to those from the function $\cos^{-1} Z$?

7.6e Apply the results of the $\cos^{-1}$ transformation to item 4 in Ex. 7.6c. Take the right-hand semi-infinite plane extending from $x = a$ to $x = \infty$ at potential V_0. Take the left-hand semi-infinite plane extending from $x = -a$ to $x = -\infty$ at potential zero. Evaluate the scale factors and additive constant.

7.6f* Apply the results of the transformation to item 2 of Ex. 7.6c. Take the elliptic cylindrical conductor of semimajor axis a and semiminor axis b at potential V_0. The inner conductor is a strip conductor extending between the foci, $x = \pm c$, where

$$c = \sqrt{a^2 - b^2}$$

Evaluate all required scale factors and constants. Find the total charge per unit length induced upon the outer cylinder and the electrostatic capacitance of this two-conductor system.

7.6g* Modify the derivation in Ex. 7.6d to apply to the problem of parallel cylinders of unequal radius. Take the left-hand cylinder of radius R_1 with center at $x = -d_1$, the right-hand cylinder of radius R_2 with center at $x = d_2$, and a total difference of potential V_0 between cylinders. Find the electrostatic capacitance per unit length in terms of R_1, R_2, and $(d_1 + d_2)$.

7.6h The important bilinear transformation is of the form

$$Z = \frac{aZ' + b}{cZ' + d}$$

Take a, b, c, and d as real constants, and show that any circle in the Z' plane is transformed to a circle in the Z plane by this transformation. (Straight lines are considered circles of infinite radius.)

7.6i Consider the special case of Prob. 7.6h with $a = R$, $b = -R$, $c = 1$, and $d = 1$. Show that the imaginary axis of the Z' plane transforms to a circle of radius R, center at the origin, in the Z plane. Show that a line charge at $x' = d$ and its image at $x' = -d$ in the Z' plane transform to points in the Z plane at radii r_1 and r_2 with $r_1 r_2 = R^2$. Compare with the result for imaging line charges in a cylinder (Sec. 1.18).

7.7a Explain why a factor in the Schwarz transformation may be left out when it corresponds to a point transformed to infinity in the Z' plane.

7.7b In Eq. 7.7(2), separate Z into real and imaginary parts. Show that the boundary condition for potential is satisfied along the two conductors. Obtain the asymptotic equations for large positive u and for large negative u, and interpret the results in terms of the type of field approached in these limits.

7.7c* Work the example of Prob. 7.6e by the Schwarz technique and show that the same result is obtained. This is the problem of two coplanar semi-infinite plane conductors separated by a gap $2a$, with the left-hand conductor at potential zero and the right-hand conductor at potential V_0.

7.7d* For the first example of Table 7.7, find the electrostatic capacitance in excess of what would be obtained if a uniform field existed in both of the parallel-plane regions.

7.7e Plot the $V_0/2$ equipotential for Ex. 7.7.

7.8 Suppose that the wave-guiding structure in Fig. 7.8*a* is bounded on the outside by a dielectric $\varepsilon_3(r)$ which has the value ε_2 at R_0 and then decreases to an appreciably lower value as r is increased. As was seen in Sec. 6.12, waves incident on a plane boundary between two dielectrics from the higher ε side can be totally reflected. Find the limiting rate of decrease of ε_3 at R_0 which can permit total reflection of rays approaching the boundary, by studying the variation of the equivalent dielectric constant in the W plane.

7.9a The so-called circular harmonics are the product solutions to Laplace's equation in the two circular cylindrical coordinates r and ϕ. Apply the basic separation of variables technique to Laplace's equation in these coordinates to yield two ordinary differential equations. Show that the r and ϕ equations are satisfied respectively by the functions R and F_ϕ where

$$R = C_1 r^n + C_2 r^{-n}$$
$$F_\phi = C_3 \cos n\phi + C_4 \sin n\phi$$

7.9b An infinite rod of a magnetic material of relative permeability μ_r lies with its axis perpendicular to the direction of a uniform magnetic field in which it is immersed. Take the rod to be of circular cross section with radius a and use the expressions in Prob. 7.9a to find the fields inside and outside the rod. Note the uniformity of the field inside.

7.10a Plot the form of equipotentials for $\Phi = V_0/4, V_0/2$, and $3V_0/4$ for Fig. 7.10*a*.

7.10b Describe the electrode structure for which the single rectangular harmonic $C_1 \cosh kx \sin ky$ is a solution for potential. Take electrodes at potential V_0 passing through $|x| = a$ when $y = a/2$.

7.10c Describe the electrode structure and exciting potentials for which the single circular harmonic (Prob. 7.9a) $Cr^2 \cos 2\phi$ is a solution.

7.11a Obtain Fourier series in sines and cosines for the following periodic functions:

(i) A triangular wave defined by $f(x) = V_0(1 - 2x/L)$ from 0 to $L/2$ and $f(x) = V_0[(2x/L) - 1]$ from $L/2$ to L

(ii) A sawtooth wave defined by $f(x) = V_0 x/L$ for $0 < x < L$

(iii) A sinusoidal pulse given by $f(x) = (V_m \cos kx - V_0)$ for $-\alpha < kx < \alpha$, $f(x) = 0$, for $-\pi < kx < -\alpha$ and also for $\alpha < kx < \pi$.

7.11b Suppose that a function is given over the interval 0 to a as $f(x) = \sin \pi x/a$. What do the cosine and sine representations yield? Explain how this single sine term can be represented in terms of cosines.

7.11c Find sine and cosine representations for the function e^{kx} defined over the interval $0 < x < a$.

7.11d Plot $f(x)$ given by Eq. 7.11(14) in the neighborhood of the discontinuities using (i) five sine terms and (ii) ten sine terms and discuss differences from the rectangular function being represented.

7.11e A complex form of the Fourier series for a function $f(x)$ defined over the interval $0 < x < a$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nx/a}$$

Show that if this is valid, c_n must be given by

$$c_n = \frac{1}{a}\int_0^a f(x)e^{-j2\pi nx/a}\,dx$$

7.11f Find representations for the constant C over the interval $0 < x < a$ in the complex form of Prob. 7.11e, and compare the result with Eq. 7.11(14).

7.11g Find the Fourier integral representation for a decaying exponential, $f(x) = 0$, for $x < 0$ and $f(x) = ce^{-\alpha x}$ for $x > 0$.

7.12a Obtain a series solution for the two-dimensional box problem in which sides at $y = 0$ and $y = b$ are at potential zero, and end planes at $x = a$ and $x = -a$ are at potential V_0.

7.12b Find the potential distribution for the box of Prob. 7.12a with the same boundary conditions except that the potential on the side at $y = 0$ should be V_1 and that at $y = b$ should be $-V_1$.

7.12c In a two-dimensional problem, parallel planes at $y = 0$ and $y = b$ extend from $x = 0$ to $x = \infty$ and are at zero potential. The one end plane at $x = 0$ is at potential V_0. Obtain a series solution.

7.12d The fringing that occurs at the open ends of a pair of parallel plates as seen in Fig. 1.9*a* leads to a modification of the fields between the plates from the ideal uniform distribution. Consider $x = 0$ to be the ends of the plates, which are at $y = 0, b$. The analysis of Ex. 7.7 can show that the potential between the ends of the plates may be expressed approximately as $\Phi(0, y) = V_0[(y/b) + 0.06 \sin 2\pi y/b]$. Find the distance x at which the potential distribution between the plates is linear to within 1%, using the analysis of Prob. 7.12c.

7.12e A two-dimensional conducting rectangular solid is bounded on three sides by perfect conductors: at $y = 0, \Phi = 0$; at $x = 0, \Phi = 0$; at $y = b, \Phi = V_0$. It is bounded at $x = a$ by a dielectric with zero conductivity. Find an expression for the potential distribution inside the conducting solid.

7.12f Two concentric cylinders are located at $r = a$ and $r = b$. The inner $(r = a)$ cylinder is split along its length into two halves which are at different potentials. Potential is $-V_0$ for $-\pi < \phi < 0$ and V_0 for $0 < \phi < \pi$. The cylinder at $r = b$ is at zero potential. Find the potential between the two cylinders.

7.12g The potential along the plane boundary of a half-space is in strips of width a and alternates between $-V_0$ and V_0. Take the boundary to be at $y = 0$ and the strips to be invariant in the z direction. The origin of the x coordinate lies in the gap between strips so that the potential is $-V_0$ for $-a < x < 0$ and V_0 for $0 < x < a$. Find the potential distribution for $y \geq 0$ and determine the surface charge density along the $y = 0$ plane. Put the result in closed form (see Collin, footnote 3 of Chap. 8, p. 813) and plot for $-a < x < a$.

7.12h* Infinite parallel conducting plates are located at $y = 0$ and $y = a$. A conducting strip at $x = 0, a/2 \leq y \leq a, -\infty < z < \infty$, is connected to the plate at $y = a$, thus

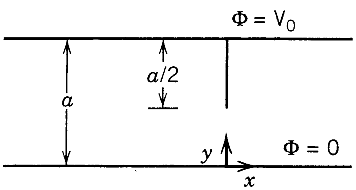

FIG. 7.12*h*

introducing additional capacitance between the plates. (See Fig. P7.12*h*.) Assume a linear potential variation for $0 \le y \le a/2$ at $x = 0$, and use superposition of boundary conditions to find an expression for the capacitance per meter in the z direction added by the strip at $x = 0$.

7.12i Consider a rectangular prism of width a in the x direction and b in the y direction with all four sides at zero potential extending from $z = 0$ to $z = \infty$. At $z = 0$ the prism has a cap with the following potential distribution:

$$V(x, y, 0) = \begin{cases} 0 & \text{for } 0 < x < a/2, \text{ all } y \\ V_0 & \text{for } a/2 < x < a, \text{ all } y \end{cases}$$

Find the potentials within the prism.

7.12j* For a box as in Ex. 7.12c, find the potential distribution if the box is filled with a homogeneous, isotropic dielectric with permittivity ε_1 in the bottom half of the box $0 \le z \le c/2$ and free space in the remainder.

7.13 Demonstrate that the series Eq. 7.13(10) does satisfy the differential equation 7.13(8).

7.16a Write a function $f(r)$ in terms of nth-order Bessel functions over the range 0 to a and determine the coefficients.

7.16b Determine coefficients for a function $f(r)$ expressed over the range 0 to a as a series of zero-order Bessel functions as follows:

$$f(r) = \sum_{m=1}^{\infty} c_m J_0\left(\frac{p'_m r}{a}\right)$$

where p'_m denotes the mth root of $J'_0(v) = 0$ [i.e., $J_1(v) = 0$].

7.17a A cylinder divided into a set of rings with appropriately applied voltages may be used to set up a nearly uniform electric field along the axis with advantageous focusing properties for electron beams. Suppose the field at the radius a of the cylinder is given approximately by $E_z(a, z) = E_0(1 + \cos \alpha z)$, where $\alpha = 2\pi/p$ and p is the period of the rings. Find the potential variation along the rings ($r = a$) and for $r < a$. Determine the field on the axis and the period required to have the periodic part of the field 1% of E_0.

7.17b Show that the function

$$\Phi(r, z) = A I_0(\tau r) \cos \tau z$$

satisfies the requirement of solutions of Laplace's equation that there should be no relative maxima or minima.

7.17c Find the series for potential inside the cylindrical region with end plates $z = 0$ and $z = l$ at potential zero and the cylinder of radius a in two parts. From $z = 0$ to $z = l/2$, it is at potential V_0; from $z = l/2$ to $z = l$, it is at potential $-V_0$.

7.17d The problem is as in Prob. 7.17c except that the cylinder is divided in three parts with potential zero from $z = 0$ to $z = b$ and also from $z = l - b$ to $z = l$. Potential is V_0 from $z = b$ to $z = l - b$.

7.17e Write the general formula for obtaining potential inside a cylindrical region of radius a, with zero-potential end plates at $z = 0$ and $z = l$, provided potential is given as $\Phi = f(z)$ at $r = a$.

7.17f Write the general formula for obtaining potential inside a cylinder of radius a which, with its plane base at $z = 0$, is at potential zero, provided that the potential is given across the plane surface at $z = l$, as

$$\Phi(r, l) = f(r)$$

7.17g Find the potential distribution inside a cylinder with zero potential on the cylindrical surface at $r = a$, on the end plate at $z = 0$ and where $a/2 < r < a$ on the end plate at $z = l$. It also has $\Phi(r, l) = V_0$ for $0 \leq r < a/2$.

7.18a Apply the separation of variables technique to Laplace's equation in the three spherical coordinates, r, θ, and ϕ, obtaining the three resulting ordinary differential equations. Write solutions to the r equation and the ϕ equation.

7.18b Assume a spherical surface split into two thin hemispherical shells with a small gap between them. Assume a potential V_0 on one hemisphere and zero on the other and find the potential distribution in the surrounding space.

7.18c Write the general formulas for obtaining potential for $r < a$ and for $r > a$, when potential is given as a general function $f(\theta)$ over a thin spherical shell at $r = a$.

7.18d For Ex. 7.18b, write the series for H_z at any point r, θ with $r > a$.

7.18e A Helmholtz coil is used to obtain very nearly uniform magnetic field over a region through the use of coils of large radius compared with coil cross sections. Consider two such coaxial coils, each of radius a, one lying in the plane $z = d$ and the other in the plane $z = -d$. Take the current for each coil (considered as a single turn) as I. Obtain the series for H_z applicable to a region containing the origin, writing specific forms for the first three coefficients. Show that if $a = 2d$, the first nonzero coefficient (other than the constant term) is the coefficient of r^4.

7.19 In Eqs. 7.19(3) and (4), let ψ be the axial electric field component E_z, and simplify by taking A and C zero in (4). Discuss the forms of solutions and the question of finding physical boundary conditions for (i) both k_x and k_y real, (ii) k_x real but k_y imaginary, and (iii) both k_x and k_y imaginary. For (ii) and (iii) would physical applicability of solutions be changed if either or both of A and C were nonzero?