$$W_e = \frac{1}{2}\int_{\Delta v} \rho_v V \, dv = \frac{1}{2}\int_{\Delta v} [\varepsilon_0 \nabla \cdot \mathbf{E}] V \, dv \tag{2.54}$$

In writing (2.54), we have included the first postulate of electrostatics (2.23) to replace the volume charge density ρ_v. Equation (2.54) can be rewritten as

$$W_e = \frac{\varepsilon_0}{2}\int_{\Delta v} [\nabla \cdot (V\mathbf{E}) - \mathbf{E} \cdot \nabla V] dv \tag{2.55}$$

using the vector identity

$$\nabla \cdot (a\mathbf{B}) = \mathbf{B} \cdot \nabla a + a \nabla \cdot \mathbf{B}$$

The first term on the right hand side of (2.55) can be converted to a closed surface integral using the divergence theorem

$$\int_{\Delta v} \nabla \cdot (V\mathbf{E}) dv = \oint V\mathbf{E} \cdot \mathbf{ds} \tag{2.56}$$

From (2.48) and (2.49), the electric field and the potential decay as R^{-2} and R^{-1} respectively, where R is the distance from a charge that is located at the origin of a spherical coordinate system. The surface area of a sphere increases as R^{+2}. Therefore, if the charge distribution is not infinite in extent, the surface integral in (2.56) approaches 0 as $R \Rightarrow \infty$. Therefore (2.55) can be written as

$$\boxed{W_e = \frac{\varepsilon_0}{2}\int_{\Delta v} [-\mathbf{E} \cdot \nabla V] dv = \frac{\varepsilon_0}{2}\int_{\Delta v} [\mathbf{E} \cdot \mathbf{E}] \, dv = \frac{\varepsilon_0}{2}\int_{\Delta v} E^2 \, dv} \tag{2.57}$$

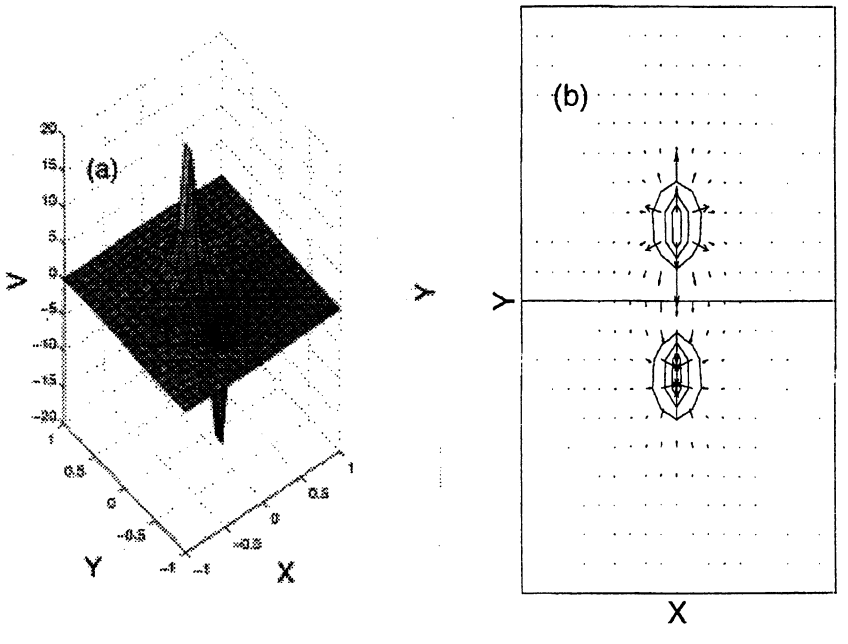
where we have incorporated (2.51). Note that the electrostatic energy depends upon the scalar quantity of the magnitude of the electric field squared. We will encounter (2.57) later.

Example 2-9. Find the potential V far from two charges of the opposite sign that are separated by a distance d. This configuration is known as an electric dipole. Using MATLAB, sketch the equipotential contours and the electric field surrounding the charges. The volume integral in (2.52) leads to $+Q/R_1$ and $-Q/R_2$. It also serves as a simple model for an atom.

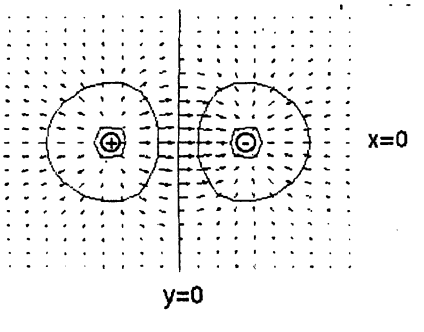
Answer: The potential at a distance R from the center of the dipole structure can be computed using the principle of superposition. It is given by

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} - \frac{Q}{4\pi\varepsilon_0 R_2}$$

The normalized potential profile resulting from dipole charges at $x = 0$ and $y = \pm 0.3$ is shown in (a). Note that it rapidly decays to zero.



Equipotential contours and the resulting electric fields are shown in (b).



Note that between the two charges, the electric field is directed from the positive charge to the negative charge.

With the assumption that $R >> d$, the three lines R_1, R_2 and R are almost parallel and the three angles are almost equal, that is $\theta_1 \approx \theta_2 \approx \theta$. Then we can approximate the distances as

$$R_1 \approx R - \frac{d}{2}\cos\theta$$

$$R_2 \approx R + \frac{d}{2}\cos\theta$$

Substituting these distances into the expression for the potential and using the ratio of (d/R) as a small expansion parameter, we obtain

$$V \approx \frac{Q}{4\pi\varepsilon_0\left(R - \frac{d}{2}\cos\theta\right)} - \frac{Q}{4\pi\varepsilon_0\left(R + \frac{d}{2}\cos\theta\right)} \approx \frac{Qd\cos\theta}{4\pi\varepsilon_0 R^2}$$

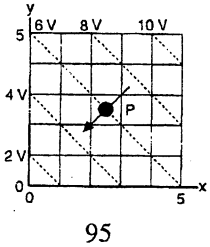
The term $Qd\cos\theta$ is frequently written as $\mathbf{p}\cdot\mathbf{u}_R$ where $\mathbf{p} = Qd\,\mathbf{u}_d$ is the dipole moment and $\mathbf{u}_R$ is the unit vector from the center of the dipole to the point of observation.

The MATLAB; program that was used to draw these figures is

```
clear; clf
[x, y] = meshgrid(-1:.1:1,-1:.1:1);
R1= (x.^2+(y-.25).^2).^.5;
R2 = (x.^2+(y+.25).^2).^.5;
V = 1./R1-1./R2;
subplot(1,2,1);
      surf(x,y,V)
      xlabel('X','fontsize',18);
      ylabel('Y','fontsize',18);
      zlabel('Z','fontsize',18);
      text(-.8,.9,15,'(a)', 'fontsize',18)
subplot(1,2,2);
      [px,py]=gradient(V,.1,.1);
      contour(V);
      hold on
      quiver(-px,-py);
      hold off
      xlabel('X','fontsize',18);
      ylabel('Y','fontsize',18);
      text(3,18,'(b)','fontsize',18)
```

Example 2-10. The potentials are measured at several locations in space. Equipotential contours are drawn on a graph. Estimate the electric field at the point *P*. The graph is 5 meters x 5 meters.

Answer: The electric field $\mathbf{E} = -\nabla V$. The measured equipotential contours are separated by a distance of $\sqrt{1^2+1^2} = \sqrt{2}$ meters. The electric field is



$$\mathbf{E} = -\nabla V \approx -\frac{2V}{\sqrt{2}}\left[\frac{\mathbf{u}_x + \mathbf{u}_y}{\sqrt{2}}\right] = -\left[\mathbf{u}_x + \mathbf{u}_y\right]\frac{\text{volts}}{\text{meter}}$$

In moving the charge from point *a* to point *b* in a region that contained an electric field, we found from equation (2.38) that work was required. If we move it back to point a along a slightly different path as shown in Figure 2-14 in an electrostatic field, we will find that the expended energy is returned to us. In this case, we write (2.38) as

$$\boxed{W_e = Q\oint \mathbf{E}\cdot\mathbf{dl} = 0} \tag{2.58}$$

where a closed path integral is indicated. Equation (2.58) states that no energy is either expended or created in this process. In this case, the electrostatic field belongs to a class of fields that are called *conservative* fields. Equation (2.58) is the second postulate of electrostatics. This equation can be converted into a surface integral via Stoke's theorem. We write

$$0 = \oint \mathbf{E}\cdot\mathbf{dl} = \int_{\Delta v} \nabla\mathbf{x}\mathbf{E}\cdot\mathbf{ds} \tag{2.59}$$

In order for this integral to be zero for any arbitrary surface, the integrand must be equal to zero This allows us to obtain the second postulate of electrostatics in differential form

$$\boxed{\nabla\mathbf{x}\mathbf{E} = 0} \tag{2.60}$$

We will encounter these postulates of electrostatics later when time-varying fields are described.

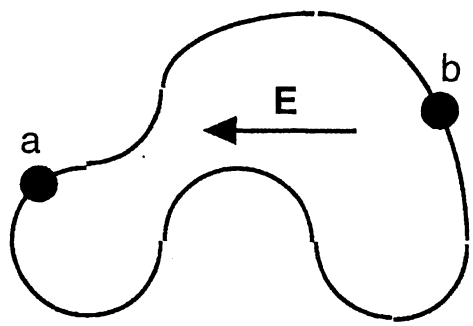


Figure 2-14. A closed path $a \Rightarrow b \Rightarrow a$.

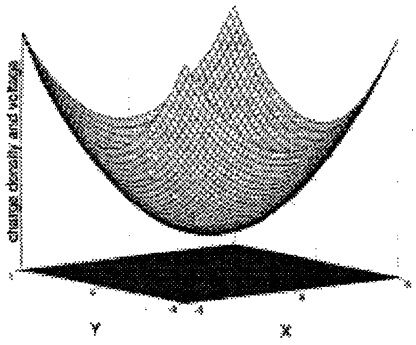
$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

The choice of which form of this operation to actually employ in a calculation is usually dictated by any possible symmetry considerations inherent in the problem. For example, the calculation of the potential within a spherical ball would suggest the application of $\nabla^2 V$ in spherical coordinates rather than other representations for $\nabla^2 V$. Definitions for the Laplacian operator exist for other coordinate systems such as in (2.72) and (2.73). A definition in a general orthogonal coordinate system can also be written. For complicated shapes and/or very difficult problems, a numerical solution may have to be attempted. This is typically the procedure that has to be followed in practice. We will encounter these procedures in the next chapter.

Example 2-13. The two-dimensional potential distribution can be approximated with the expression

$$V = \left(\frac{\rho_{vo}}{\varepsilon}\right)\left(x^2 + y^2\right)$$

Find and sketch the charge distribution that could create this potential.
Answer: From (2.69) and (2.71), we find $\rho_v = -4\,\rho_{vo}$.



The MATLAB program that was used to draw this figure is

```
clear; clf
[x,y]=meshgrid (-5:.2:5,-5:.2:5);
V=x.^2+y.^2;
mesh(x,y,V+10)
hold on
w=del2(V);
```

```
    mesh(x,y,w)
view(-37.6, 10)
xlabel('X','fontsize',18);
ylabel('Y','fontsize',18);
zlabel('charge density and voltage,'fontsize',18);
set(gca,'ZTickLabel',[])
set(gca,'ZTick',[])
```

Having derived Poisson's and Laplace's partial differential equations for three dimensional systems and having stated the definitions for ∇^2 in the three most useful coordinate systems, we will now obtain analytical solutions for these equations. Rather than first attempt a general three-dimensional solution, we will simplify the discussion by assuming that the potential depends on only one coordinate. The procedure that we will describe in this simpler problem will be followed in more difficult calculations. Several important results will, however, be obtained as we pass through this fairly difficult initial stage. Techniques that are germane to these more complicated problems will appear in the next chapter.

For example, the potential variation between two infinite parallel metal plates located in a vacuum, as shown in Figure 2-17a, would require solving Laplace's equation in one dimension. This problem would yield a result that approximates the potential distribution in a parallel plate capacitor where the separation between the plates is much less than any transverse dimension. Since the plates are assumed to be infinite and the conductivity of these metal plates is very high, they can be assumed to be equipotential surfaces. We will discuss conductivity later; we can think just that these metal plates have zero resistance. Hence, in the y and the z coordinates, we can postulate that

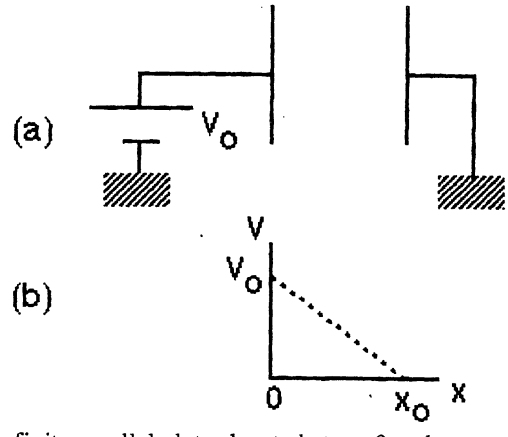


Figure 2-17. (a) Two infinite parallel plates located at $x = 0$ and at $x = x_0$. (b) Potential variation between the plates as determined from (2.76).