Fabry-Perot resonator

An examination of the standing wave depicted in Figure 5-14 leads us to conjecture that it should be possible to insert another high conductivity metal wall at any of the nodes where the tangential electric field is equal to zero without altering the remaining electric field structure. The applicable boundary condition is that the tangential electric field must be zero at a conducting surface. This is depicted in Figure 5-15 where plates have been inserted at two possible locations. For the moment, we will assume that the plates that are infinite in transverse extent are inserted at the nodes in zero time such that the electromagnetic energy is 'trapped' between the plates and nothing else is disturbed. This energy is actually 'coupled' between the plates with an antenna structure, a topic to be discussed later.

Let us now formally derive this result from the wave equation (5.18) which we rewrite as

$$\frac{d^2 E_y}{dz^2} + k^2 E_y = 0 \tag{5.60}$$

Recall that we have assumed a time-harmonic signal. The solution of this equation is given by

$$E_y = A \sin kz + B \cos kz \tag{5.61}$$

The constants of integration A and B are specified by the boundary condition that the tangential electric field must be equal to zero at a metal wall. These determine the constant $B = 0$ and $k = (n\pi/L)$ where n is an integer. If the maximum electric field has a magnitude E_{yo}, then the spatial distribution of the electric field is given by

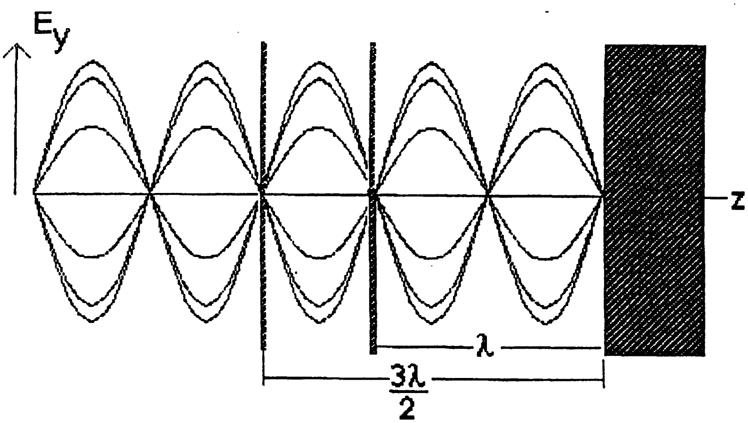


Figure 5-15. By inserting thin conducting plates separated by (nλ / 2) at the locations where the standing wave is zero, the electromagnetic field structure will not be altered. Two of many possible locations are indicated in the figure.

$$E_y = E_{yo} \sin\left(\frac{n\pi z}{L}\right) \tag{5.62}$$

The parallel plate cavity depicted in Figure 5-15 is called a *Fabry-Perot resonator.* This cavity has a very high 'Q' which could approach one million. Remember that the Q of an ordinary circuit is of the order of ten. Since it is so frequency selective, it has received wide application as the cavity that encloses various 'lasing' materials. The total lasing material – cavity entity carries the acronym *laser.* The term laser stands for 'Light amplification by stimulated emission and radiation.' At light frequencies, it is not a bad approximation to assume that the transverse dimension is a large number of wavelengths in extent. This very large number is approximated as being infinity.

We recall that the wave number k is a function of the frequency of oscillation ω and the velocity of light in the region between the two parallel plates $c = (1/\sqrt{\varepsilon\mu_0})$ where $\varepsilon = \varepsilon_r\,\varepsilon_0$. For the cavities depicted in Figure 5-15, this resonant frequency $\omega = \omega_r$ will be given by

$$\frac{\omega_r}{c} = k = \frac{n\pi}{L}$$

or

$$\omega_r = \frac{n\pi}{L}\frac{1}{\sqrt{\varepsilon\mu_0}} \tag{5.63}$$

For the two cavities depicted in Figure 5-16 which are either empty or filled with a dielectric, we find that the two Fabry-Perot cavities will resonate with slightly different frequencies. The difference of these two frequencies $\Delta\omega$ is given by

$$\Delta\omega = \omega_{r_a} - \omega_{r_b} = \frac{n\pi}{L}\frac{1}{\sqrt{\varepsilon_0\,\mu_0}} - \frac{n\pi}{L}\frac{1}{\sqrt{\varepsilon_r\,\varepsilon_0\,\mu_0}}$$

If the resonant frequency for the vacuum case (Figure 5-16(a)) can be computed or measured, this can be written as

$$\frac{\Delta\omega}{\omega_{r_a}} = \frac{\omega_{r_a} - \omega_{r_b}}{\omega_{r_a}} = \frac{\frac{n\pi}{L}\frac{1}{\sqrt{\varepsilon_0\,\mu_0}} - \frac{n\pi}{L}\frac{1}{\sqrt{\varepsilon_r\,\varepsilon_0\,\mu_0}}}{\frac{n\pi}{L}\frac{1}{\sqrt{\varepsilon_0\,\mu_0}}} = 1 - \frac{1}{\sqrt{\varepsilon_r}} \tag{5.64}$$

$Q = 2\pi$ (energy stored/power dissipated per cycle)

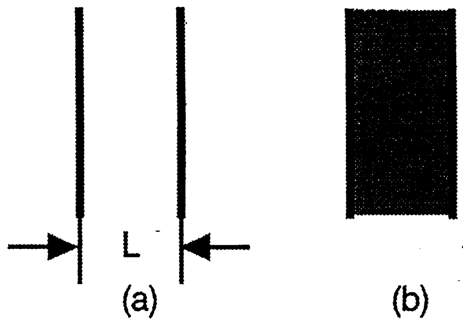
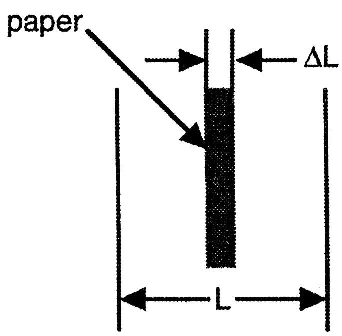


Figure 5-16. (a) An empty Fabry-Perot cavity. (b) A Fabry-Perot cavity filled with a dielectric $\varepsilon = \varepsilon_r \varepsilon_0$.

Example 5-16. An empty microwave Fabry-Perot cavity has a resonant frequency of 35 GHz. Determine the thickness ΔL of a sheet of paper that is then inserted between the plates if the resonant frequency changes to 34.99 GHz. The separation L between the parallel plates is 50 cm. Assume that the integer n that specifies the mode does not change. You may ignore any reflection at the paper interface.



Answer: The relative dielectric constant ε_{paper} of paper as determined from Appendix B is $\varepsilon_{paper} \approx 3$. The relative dielectric constant separating the plates with the paper inserted can be approximated as

$$\sqrt{\varepsilon_{eff}}\,L = \sqrt{\varepsilon_{paper}}\,\Delta L + 1(L - \Delta L) \approx \sqrt{\varepsilon_{paper}}\,\Delta L + L$$

Therefore, we write

$$\frac{\omega_{\text{vacuum}} - \omega_{\text{paper inserted}}}{\omega_{\text{vacuum}}} = \frac{\dfrac{n\pi}{L}\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} - \dfrac{n\pi}{L}\dfrac{1}{\sqrt{\mu_0 \varepsilon_{eff} \varepsilon_0}}}{\dfrac{n\pi}{L}\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}} \approx 1 - \frac{1}{\sqrt{1 + \dfrac{\Delta L}{L}\varepsilon_{\text{paper}}}} \approx \frac{\Delta L}{2L}\varepsilon_{\text{paper}}$$

Inserting the values, we compute

$$\frac{35-34.99}{35}=\frac{.01}{35}=\frac{\Delta L}{2\bullet 50}3$$

or ΔL = .01 cm.

From this example and the example mentioned earlier, we can discern that high frequency electromagnetic waves can be used in the diagnostics of various materials. This is a practical technique that has received wide currency in manufacturing paper where the ratio of less expensive water to the more costly wood pulp determines the ultimate grade of the paper. The relative dielectric constants of wood pulp and water are different.

Medical diagnostics for the determination of the ratio of diseased portion to the undiseased portion of a lung in an autopsy of a patient who died, of say pulmonary emphysema, can be performed (Figure 5-17(a)). Assuming that one of the lungs or a reasonable portion of one could be used to yield a value for relative dielectric constant for the lung, the percentage of the diseased lung could be ascertained. The disease has 'eaten' holes in the lung.

Since the resonant frequency depends on the distance L, two Fabry-Perot cavities could be set up as shown in Figure 5-17(b) and the thickness of the metal sheet could be controlled in a rolling mill. This can be written as

$$\frac{\Delta\omega}{\omega_{r_1}}=\frac{\frac{n\pi}{L_1}\frac{1}{\sqrt{\varepsilon_0\mu_0}}-\frac{n\pi}{L_2}\frac{1}{\sqrt{\varepsilon_0\mu_0}}}{\frac{n\pi}{L_1}\frac{1}{\sqrt{\varepsilon_0\mu_0}}}=\frac{L_2-L_1}{L_2} \tag{5.65}$$

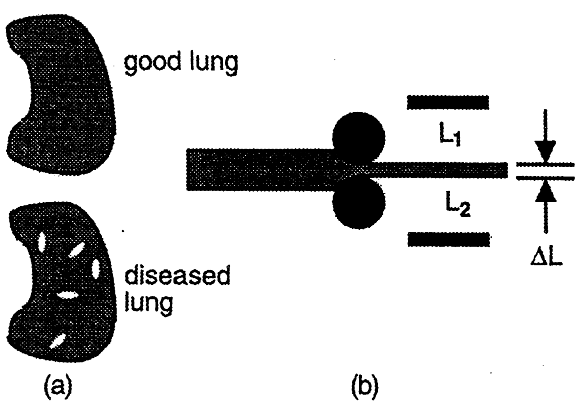


Figure 5-17. Two possible applications for the diagnostics of various objects. (a) Medical diagnostics to ascertain the ratio of diseased to good lung. (b) Controlling the thickness ΔL of a metal in a rolling mill.

If the separation L between the plates $L = L_1 + L_2 + \Delta L$ and the distance L_2 are known, this can be written as

$$\frac{\Delta\omega}{\omega_{r_1}} = \frac{L_2 - L_1}{L_2} = \frac{L_2 - (L - L_2 - \Delta L)}{L_2} = \frac{2L_2 - L + \Delta L}{L_2} \tag{5.66}$$

The assumption of knowing both distances is not unreasonable since the metal may be constrained to pass on rollers that are fixed at a certain distance above one of the plates.

Reflection of an obliquely incident electromagnetic wave

We might suspect that all bodies, be they dielectrics or conductors, do not always align themselves so that every incident electromagnetic wave has the wave vector **k** perpendicular to every surface. The wave vector **k** may also not even be coincident with the axis of a Cartesian coordinate system. This is shown in Figure 5-18. The vector **r** is a position vector from the origin of the coordinate system to *any point* on the plane. For simplicity, let us assume that the waves are propagating in a lossless space.

Intuition that we might have gained from sitting at a beach and noting that the waves that crashed with a thundering roar upon the shore usually had some 'velocity of crashing.' We could even define this velocity in several directions. One direction could be the velocity of this crashing directed along the water–sand interface. The study of the reflection and the transmission of electromagnetic waves in this case will be slightly

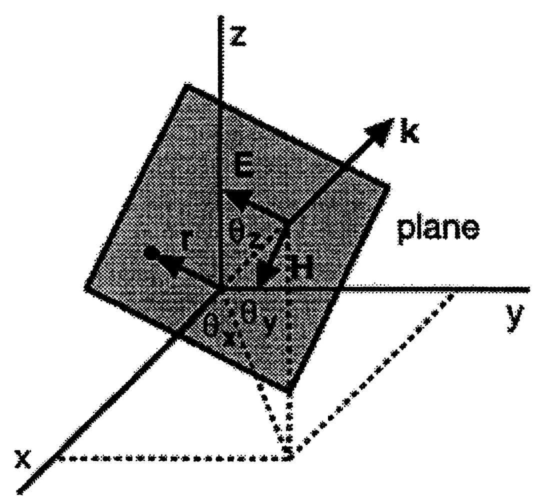


Figure 5-18. Plane wave propagating at an arbitrary angle with respect to the axes of a Cartesian coordinate system. Both the electric and magnetic fields have equiphase contours in the plane.