7

ANTENNAS

In the previous chapters, we learned that electromagnetic waves can propagate in infinite free space and that these same waves can also propagate along a common transmission line that can be held in our hands. A question that remains to be answered is whether the same electromagnetic wave can be excited in a finite region and then launched or radiated into infinite space. In this chapter, we will first examine the fundamentals of the radiation of electromagnetic waves. We will base our argument on material that has been uncovered in earlier chapters. This will naturally lead to an introduction into the important topic of antennas. Several of the important parameters and terms associated with antennas will be brought forth in this discussion.

Radiation fundamentals

Before we examine the radiation properties of an antenna, we should first understand a physical process that can actually cause the radiation of electromagnetic waves. This means that we have to examine possible radiation characteristics of an electric charge from a fundamental basis. There are certain requirements that an electric charge must meet in order to consider that it will actually radiate electromagnetic waves. These requirements will be argued from an intuitive point of view. If we understand this argument, the development of antenna radiation theory follows immediately since the principle of superposition applies in the linear medium that is being considered in this text and the antenna can be considered to consist of a large number of charges. The argument also illustrates the type of calculation that can be written on the backs of old envelopes.

We can understand radiation of electromagnetic waves using Poynting's theorem. This theorem states that the total power that is radiated from a source is given by the following closed surface integral

$$\text{total radiated power} = \oint \mathbf{E} \times \mathbf{H} \bullet \mathbf{ds} \tag{7.1}$$

Poynting's theorem tells us that the radiation of electromagnetic waves from a source that is located within a volume that is completely enclosed by a closed surface requires *both* an electric field and a magnetic field, the two fields being coupled together via Maxwell's equations.

A *stationary* charge that was discussed in Chapter 2 will *not* radiate electromagnetic waves. This can be easily understood since a stationary charge will cause no current to flow, hence there will be no magnetic field associated with a stationary charge. From (7.1), the total radiated power is therefore equal to zero. From this, we can conclude that there will be no radiation of electromagnetic waves from a stationary charge.

We can also come to this conclusion from another point of view. If the point of observation where the power is to be detected is far from the source and if there were a spherically radiating wave, it would appear to be almost a plane wave at large distances from the charge and we can make use of the fact that the electric and magnetic field intensities of propagating waves are related through the characteristic impedance of free space Z_c as given in Chapter 5. The magnitude of the magnetic field intensity H can be found from the electric field intensity E via $H = E/Z_c$.

Therefore, a source of electromagnetic power located at the center of a sphere whose radius is R shown in Figure 7-1 would radiate a total power whose value can be written as

$$\text{total radiated power} \approx \frac{E^2}{Z_c}(\text{surface area of sphere}) = \frac{E^2}{Z_c}4\pi R^2 \tag{7.2}$$

Let us assume at this stage that the antenna is an isotropic radiator and has no directional characteristics. The total radiated power is equal to that which is delivered from the source that we will assume to be a constant. Hence, the total radiated power is independent of the distance R. Therefore, we would conclude that the electric field E of an electromagnetic wave must change with distance as R^{-1}. However, we find that the electric field from a static charge varies as R^{-2}. Hence, we again come to the same conclusion that stationary charges cannot radiate electromagnetic waves.

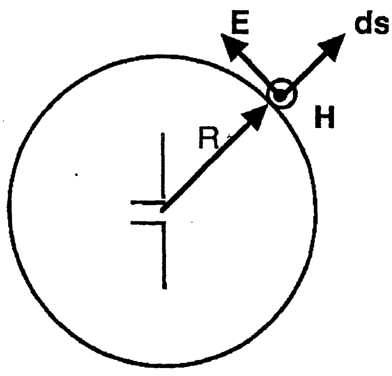


Figure 7-1. Antenna radiation of electromagnetic waves. For a stationary charge, **H** will be equal to zero.

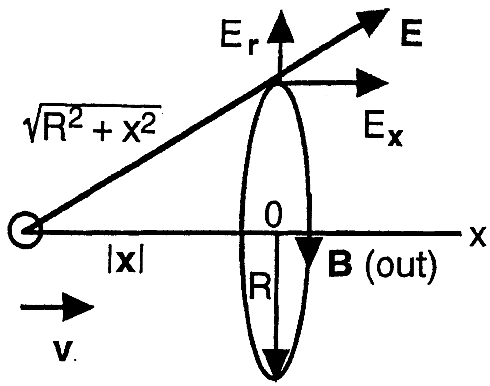


Figure 7-2. Electric and magnetic fields due to a moving charge. The velocity **v** is a constant and $v << c$.

The next question that should be posed is whether a charge that is in motion with a *constant velocity* $v <<$ the velocity of light c can radiate electromagnetic waves. We know that a charge in motion constitutes a current and currents cause magnetic fields. We are not able to invoke the simple argument based on the radiated power that we used for the lack of radiation from a static charge since both an electric field and a magnetic field will now be present. We will, however, use a slightly different argument that is still based on the Poynting vector.

Let us assume that a positive charge Q is moving in the positive $\mathbf{u_x}$ direction with a constant velocity **v** as shown in Figure 7-2. This velocity shall be chosen so that it is much less than the velocity of light c so it is *nonrelativistic*. We do not want to wade into the deep waters of relativity or advanced topics in physics at this time.

The static electric field **E** from the charge Q is computed to be

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R^2+x^2}\mathbf{u_r} \tag{7.3}$$

where $r = \sqrt{R^2+x^2}$. The magnetic field is computed from the Biot-Savart law. This leads to

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Q\,\mathbf{v}\,\mathbf{x}\,\mathbf{r}}{\left(R^2+x^2\right)^{3/2}} \tag{7.4}$$

Let us compute the direction of Poynting's vector associated with these two fields. This is facilitated from an examination of a sphere centered on the charge at a certain instant in time as shown in Figure 7-3. The electric field caused by a charge moving

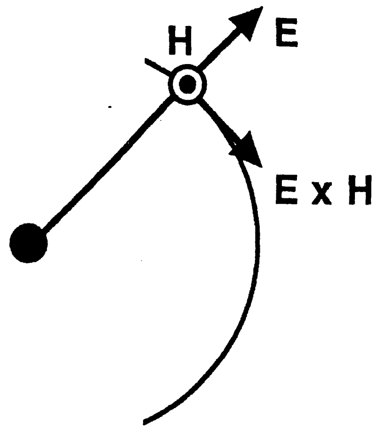


Figure 7-3. The Poynting vector associated with a charge moving with a constant velocity **v**.

with a uniform velocity is entirely normal to the spherical surface (the charge is slowly moving with a constant velocity) and the magnetic field is tangent to the surface. Hence, the Poynting vector **E x H** is completely confined within the spherical surface and it does *not* radiate in the radial direction away from the charge. Is there any hope for radiation?

Example 7-1. Calculate the component of the Poynting vector in the $\mathbf{u}_x$ direction in Figure 7-2 and the total energy flow rate through an infinitely large plane placed normal to the x axis. Discuss the meaning of this result.

Answer: The magnitude of the x component of the Poynting's vector is computed from $|(\mathbf{E}_r \times \mathbf{B})/\mu_0|$. This leads to

$$S_x = \left\{ \frac{Q}{4\pi\varepsilon_0} \frac{R}{\left(R^2+x^2\right)^{3/2}} \right\} \left\{ \frac{Qv}{4\pi} \frac{R}{\left(R^2+x^2\right)^{3/2}} \right\} = \frac{Q^2 v}{16\pi^2\varepsilon_0} \frac{R^2}{\left(R^2+x^2\right)^3}$$

The total energy flow rate becomes

$$\text{Power} = \int_0^\infty S_x 2\pi R \mathrm{dR} = \frac{Q^2 v}{8\pi\varepsilon_0} \int_0^\infty \frac{R^3}{\left(R^2+x^2\right)^3} \mathrm{dR}$$

The integral can be performed (see Problem 1) to yield

$$\text{Power} = \frac{Q^2 v}{32\pi\varepsilon_0}\frac{1}{x^2}W.$$

The distance $|x|$ is the instantaneous separation between the charge and the plane. In the one-dimensional system being considered here ($\mathbf{v} \rightarrow v\mathbf{u}_x$), magnitude of the velocity v can be written as $v = dx/dt$. The power can be rewritten in the form

$$\text{Power} = -\frac{d\left[\dfrac{Q^2}{32\pi\varepsilon_0 x}\right]}{dt}$$

The quantity

$$\frac{Q^2}{32\pi\varepsilon_0}\frac{1}{x_0} \quad (x_0 > 0)$$

is the electrostatic energy stored in the region $x > x_0$ (See problem 2). Therefore the power that is calculated using Poynting's theorem can be interpreted as the flow rate of electrostatic energy stored in space and has nothing to do with radiation. The magnetic energy will be of the order of $(v/c)^2$ times the electric energy and will be very small in nonrelativistic cases.

In order to answer the question whether there can be any radiation at all, let us consider a charge initially at rest at point A, which is accelerated in the x direction as shown in Figure 7-4. The acceleration lasts for a duration Δt seconds until it reaches a point B after which the charge moves with a constant velocity v << the velocity of light c to a point C and beyond. Remember that a signal cannot propagate faster than c.

We know that stationary charges and charges moving with a constant velocity do not radiate electromagnetic waves and have an electric field that is only radially outward (a Coulomb field). Thus the electric field lines when the charge is at A and at the point C are entirely radial. These electric field lines must be continuous since they are caused by the same charge. They are connected with 'kinked' lines. The kinks, that are disturbances in the electric field lines caused by the acceleration of the charge, propagate with the speed of light. It takes t seconds for the charge to move from the point A to the point B, therefore the separation between the two circles is approximately $c\,\Delta t$ = constant. In the kinks, there are components of electric field that are perpendicular to the Coulomb field. These transverse components are responsible for the radiation. Note that in this argument, there are directions where there are no radiated electric fields and only the static Coulomb field exists. The maximum radiated electric field will occur along the line that is perpendicular to the charge's acceleration.

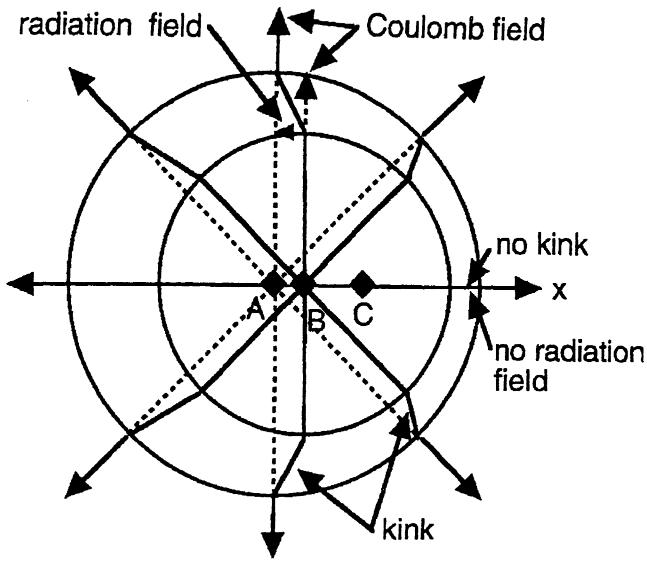


Figure 7-4. A charge that is accelerated does radiate electromagnetic waves. The dark lines are electric field lines **E**.

Consider a point L in Figure 7-5 that is normal to the direction of the charge's velocity at a certain instant. Let t be the time after the charge is accelerated from a stationary point A to point B where it has a velocity $v = a\,t$ where a is the acceleration. We will assume that $\Delta t << t$ so the distance $AB + BC = BC = vt$.

At point L, there will be two components of an electric field. The first is the radial Coulomb field that is given by

$$E_0 = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\rho^2} = \frac{Q}{\pi\varepsilon_0}\frac{1}{(ct)^2} \tag{7.5}$$

The radiation field E_t can be computed from the triangle JKL

$$\frac{JK}{KL} = \frac{JK}{AB + BC} \approx \frac{JK}{BC} = \frac{c\,\Delta t}{vt} = -\frac{E_0}{E_t} \tag{7.6}$$

Solving (7.6) for E_t, we obtain

$$E_t = -\frac{vt}{c\,\Delta t}E_0 = -\frac{vt}{c\,\Delta t}\frac{Q}{4\pi\varepsilon_0}\frac{1}{(ct)^2} = -\frac{Q}{4\pi\varepsilon_0 c^2}\frac{v}{\Delta t}\frac{1}{\rho} \tag{7.7}$$

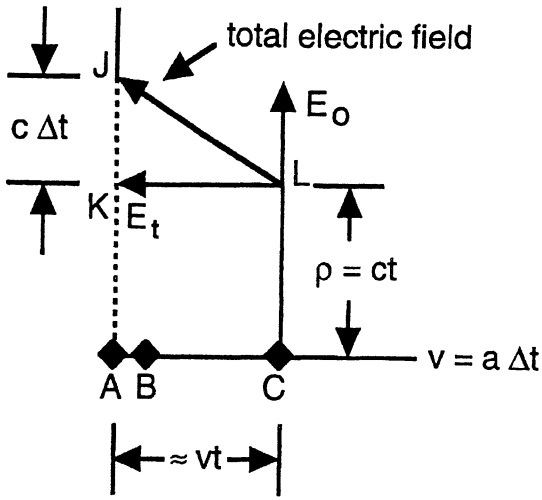


Figure 7-5. The components of the electric field caused by a charge Q that is accelerated during a time t from points A to B.

Eureka! This is what we were looking for! The transverse (or radiation) electric field is proportional to the acceleration $v/\Delta t$ and it has the proper spatial variation $1/\rho$ required in (7.2). The minus sign that appears here is due to the direction of E_t, that is opposite to the direction of the acceleration. From Figure 7-5, we note that there is a preferred direction for this radiation. If we define the angle θ as being the angle between the point of observation and the velocity of the accelerated charge, (7.7) can be written as

$$E_t = -\frac{Q}{4\pi\varepsilon_0 c^2}\frac{a\sin\theta}{\rho} \tag{7.8}$$

The magnetic field intensity H_t associated with E_t can be computed by just using the characteristic impedance of the material Z_c and the fact that the electric and magnetic field intensities are related by this characteristic impedance. We write

$$H_t = -\frac{1}{Z_c}\frac{Q}{4\pi\varepsilon_0 c^2}\frac{a\sin\theta}{\rho} \tag{7.9}$$

Poynting's vector is directed *radially outward* and it's magnitude S is given by

$$S = \frac{Q^2 a^2}{16\pi^2\varepsilon_0 c^3}\frac{\sin^2\theta}{\rho^2} \tag{7.10}$$

$$z = -z_0 \sin \omega t \tag{7.11b}$$

The acceleration for these two charges can be written respectively as

$$a_+ = \frac{d^2 z}{dt^2} = -z_0 \omega^2 \sin \omega t \tag{7.12a}$$

and

$$a_- = \frac{d^2 z}{dt^2} = z_0 \omega^2 \sin \omega t \tag{7.12b}$$

where the subscript indicates the sign of the charge.

From (7.8), the electric field due to the positive oscillating charge is written as

$$E_t\big|_{+Q} = -\frac{Q}{4\pi\varepsilon_0 c^2}\frac{a \sin\theta}{\rho} = -\frac{Q}{4\pi\varepsilon_0 c^2}\frac{\left(-z_0\omega^2 \sin \omega t\right)\sin\theta}{\rho} \tag{7.13a}$$

The radiation electric field for the negative charge is given by

$$E_t\big|_{-Q} = -\frac{Q}{4\pi\varepsilon_0 c^2}\frac{\left(z_0\omega^2 \sin \omega t\right)\sin\theta}{\rho} \tag{7.13b}$$

For a vacuum, we can use the *principle of superposition*. This states that the total electric field is given merely as the linear sum of the fields from the two individual oscillating charges. Therefore

$$E_t = 2\left\{\frac{Q}{4\pi\varepsilon_0 c^2}\frac{\left(z_0\omega^2 \sin \omega t\right)\sin\theta}{\rho}\right\} \tag{7.14}$$

The Poynting vector that is radiated from this dipole is found from (7.10) to be

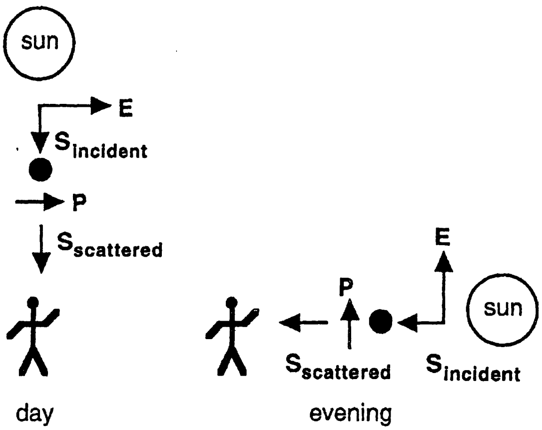
$$S = 4\frac{Q^2 z_0^2}{16\pi^2\varepsilon_0 c^3}\frac{\sin^2\theta}{\rho^2}\omega^4 \sin^2 \omega t \tag{7.15}$$

Again the numerical coefficient in (7.15) is small. The power is proportional to

$\omega^4 = (kc)^4 = \left(\frac{2\pi}{\lambda}c\right)^4$. The radiated power is inversely proportional to the wavelength to the power of 4.

Example 7-2. Explain why the sky appears blue during the day and red at night.

Answer: We consider atmospheric molecules excited by the sunlight to be dipole radiators that are polarized by the incident sunlight as shown in the figure. Sunlight is sometimes called white light since it contains all of the visible light and much more in its broad spectrum of radiation. The power reradiated by the atmospheric dipoles will be predominantly perpendicular to the polarizing electric field as shown in (7.15). From (7.15), the power in the longer wavelength red will be less than in the blue and the sky will appear blue during the day. At dusk, the light that reaches the observer lacks the blue[1] since it has passed through more air molecules and has been scattered away. Hence, the sky appears red.



In addition to the frequency dependence, we note that there is some *directivity* associated with the dipole radiation in that the field strength and the Poynting vector depend on the angle θ. In Figure 7-7, an equiamplitude contour $|\rho E_t|$ of the magnitude of the electric field as computed from (7.14) is displayed. The multiplication of the electric field by the distance ρ removes the distance from the radiation field. Hence, universal results can be obtained that can be applied in general to compare one type of antenna with another.

A contour of constant electric field multiplied by the distance ρ (the product ρE_t) is called the *radiation pattern* or *field pattern* for an antenna. The radiation pattern for a Hertzian dipole is symmetric about one axis and it does not depend on the coordinate φ.[2] We will calculate radiation patterns for other antennas later since this is one of the

[1] The wavelength of blue light is 5000 Å and the wavelength of red light is 6500 Å. The symbol Å stands for Ångstrom (1 Å = 10^{-10} meters).

[2] One might think of a doughnut or a bagel in order to understand this point.

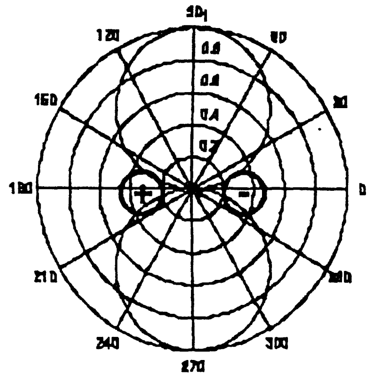


Fig.7-7 Radiation pattern for a dipole antenna.

fundamental properties that characterize an antenna. At least two reasons can be given for its importance. The first is concerned with possible interference. Listeners or viewers would not like to have two radio or television stations radiating a signal at the same frequency over the same region of the country unless one station is designed to 'jam' the other. The responsible government agency will order the stations to redirect their propagation or change their frequencies. The second is related to economics. An individual station located near the sea also has a strong interest in directing its signal to a potential audience of humans rather than sea gulls. Its radiation pattern will be appropriately selected based on this desire not to deliver the power in the wrong direction.

Antennas of this simple dipole type have been used by scientists to detect electromagnetic waves originating from the outer reaches of space.[1] This is shown in Figure 7-8. By measuring the potential difference between the two spheres separated by a dis-

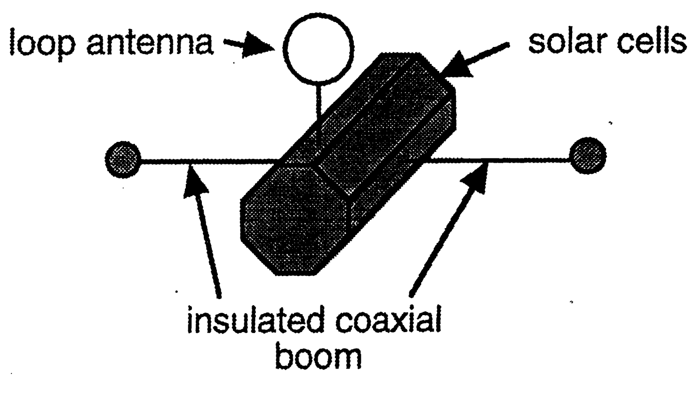


Figure 7-8. A satellite with its antenna arms and small loop extended to measure possible extra galactic electromagnetic radiation.

[1] It can be shown that the radiation pattern of an antenna used as a radiator or as a receiver are the same. This follows from a reciprocity relation.

tance L, the electric field intensity of the wave can be computed. A small loop is also employed in order to monitor the accompanying magnetic field intensity. As the satellite meanders about, the reception pattern that is equivalent to the radiation pattern, can be determined. By sampling the received signal at different frequencies, it is possible to postulate the mechanism that excites the wave in some far off region in space. The mechanism usually involves various plasma waves. From data obtained in these experiments, the existence or nonexistence of magnetic fields at various planets has been determined.

Formal treatment of a simple dipole antenna

In practice, we normally do not examine the radiation charac teristics of each individual oscillating charge element in order to predict and understand the radiation characteristics of an arbitrary antenna. Yes, we were able to obtain a 'feeling' for the radiation process for electromagnetic waves caused by the oscillating charge but such an effort for a realistic antenna would soon sap our patience. Here we will present a more general approach that is based on a computation of the vector potential **A** caused by an oscillating current. The components of the electromagnetic field **E** and **H** can be computed from the vector potential.

Recall from (2.108) that the vector potential for a current carrying wire is defined for a constant current $\mathbf{j}(\mathbf{r})$ via the integral

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\Delta v} \frac{\mathbf{j}(\mathbf{r}')}{R} dv' \tag{7.16}$$

where $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. The vector potential **A** is in the same direction as the current **j** as shown in Figure 7-9. From the vector potential, we were able to compute the magnetic field **B** via the definition $\mathbf{B} = \mu_0\mathbf{H} = \nabla \times \mathbf{A}$.

If the current varies in time, say as $e^{j\omega t}$, we will still be able to follow the same procedure with but one caveat. From our study in Chapter 5, we learned that it takes a nonzero time for an electromagnetic wave to propagate from the location where the current is located to the point of observation at a distance R from the current element. The smallest time delay is dictated by the velocity of light c. This causes the vector potential to be *retarded* in time by an amount equal to $\frac{R}{c} = \frac{\beta R}{\omega}$ seconds. This is sometimes given the descriptive name *retarded potential.* We regularly hear of recent detections or observations of effects such as the emergence or destruction of other galaxies occurring in the outer cosmos in distances measured in light years and times measured in eons. Hence (7.16) should be modified to read

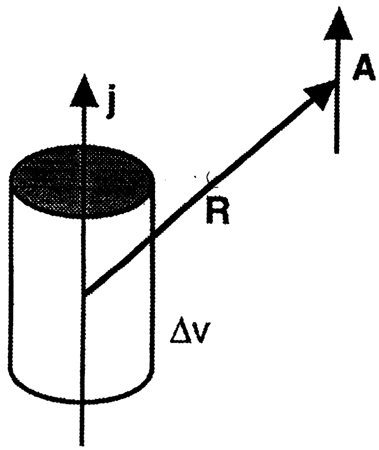


Figure 7-9. Vector potential **A** is created by the current density **j**. The volume Δv encloses the current.

$$A(\mathbf{r})e^{j\omega t} = \frac{\mu_0}{4\pi}\int_{\Delta v} \frac{j(\mathbf{r}')e^{j(\omega t-\beta R)}}{R}\mathrm{d}v' \tag{7.17}$$

The equation that we have just obtained using an argument can also be obtained directly from Maxwell's equations. The formal procedure will be left as an exercise in a later course due to our self imposed limitations. It is not a trivial calculation. We will, however, use it in a rigorous derivation of the radiated electromagnetic fields from a dipole antenna.

For the simple dipole that we are presently considering in Figure 7-9, the integral over the volume Δv can be easily performed. If the charge separation L is assumed to very small compared with the distance R and the current is given by $\mathbf{I} = \pm\, j\omega Q\mathbf{u}_z$, the vector potential $A(\mathbf{r})$ is then

$$A = \mathbf{u}_z \frac{\mu_0 I\,\Delta L}{4\pi}\left(\frac{e^{-j\beta\rho}}{\rho}\right) \tag{7.18}$$

The Hertzian dipole is very small and it is justified to assume that it is at the center of a *spherical coordinate* system. Hence, we have replaced R with ρ. In addition, we have to write the unit vector $\mathbf{u}_z$ in spherical coordinates as

$$\mathbf{u}_z = \mathbf{u}_\rho \cos\theta - \mathbf{u}_\theta \sin\theta \tag{7.19}$$

In spherical coordinates, the components of the vector potential $A = A_z \mathbf{u_z}$ are given by

$$A = A_\rho \mathbf{u}_\rho + A_\theta \mathbf{u}_\theta + A_\varphi \mathbf{u}_\varphi \quad (7.20$$

$$\left.\begin{aligned} A_\rho &= A_z \cos\theta = \frac{\mu_0 I\,\Delta L}{4\pi}\left(\frac{e^{-i\beta\rho}}{\rho}\right)\cos\theta && (a)\\ A_\theta &= -A_z \sin\theta = -\frac{\mu_0 I\,\Delta L}{4\pi}\left(\frac{e^{-i\beta\rho}}{\rho}\right)\sin\theta && (b)\\ A_\varphi &= 0 && (c)\end{aligned}\right\} \quad (7.21)$$

The magnetic field intensity is computed from the vector potential using the definition of the curl operation, again in spherical coordinates. We find that

$$H = \frac{1}{\mu_0}\nabla\times A = \mathbf{u}_\varphi \frac{1}{\mu_0\rho}\left[\frac{\partial(\rho A_\theta)}{\partial\rho} - \frac{\partial A_\rho}{\partial\theta}\right] = -\mathbf{u}_\varphi \frac{I\,\Delta L}{4\pi}\beta^2 \sin\theta\left[\frac{1}{j\beta\rho} + \frac{1}{(j\beta\rho)^2}\right]e^{-j\beta\rho} \quad (7.22)$$

The electric field is computed from Maxwell's equations to be

$$E = \frac{1}{j\omega\varepsilon_0}\nabla\times H = \frac{1}{j\omega\varepsilon_0}\left[\mathbf{u}_\rho \frac{1}{\rho\sin\theta}\frac{\partial(H_\varphi \sin\theta)}{\partial\theta} - \mathbf{u}_\theta \frac{\partial(\rho H_\varphi)}{\partial\rho}\right] \quad (7.23)$$

The components of the electric field after performing the appropriate differentiations are written as

$$\left.\begin{aligned} E_\rho &= -\frac{I\,\Delta L}{4\pi} Z_c \beta^2 2\cos\theta\left[\frac{1}{(j\beta\rho)^2} + \frac{1}{(j\beta\rho)^3}\right]e^{-j\beta\rho} && (a)\\ E_\theta &= -\frac{I\,\Delta L}{4\pi} Z_c \beta^2 \sin\theta\left[\frac{1}{j\beta\rho} + \frac{1}{(j\beta\rho)^2} + \frac{1}{(j\beta\rho)^3}\right]e^{-j\beta\rho} && (b)\\ E_\varphi &= 0 && (c)\end{aligned}\right\} \quad (7.24)$$

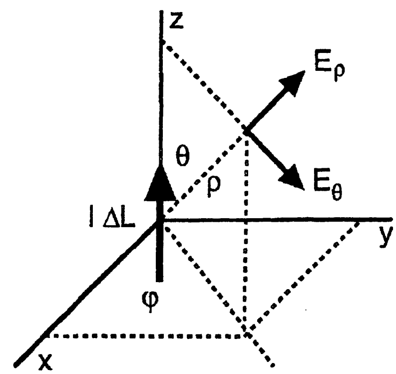


Figure 7-10. Electric field components caused by a simple dipole antenna.

where $Z_c = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \approx 120\pi\,\Omega$. The electric field components are shown in Figure 7-10.

We have found the electric and the magnetic fields that are excited by a small Hertzian dipole shown in Figure 7-6. Even in the case of this simplest of antennas, we find that the structure is very complicated and consists of several terms. Note that the time dependence $e^{j\omega t}$ has to be incorporated into these expressions at some time so the wave nature of the radiation becomes apparent. Since we wish to talk about the power that is radiated *from* the antenna, the wave variable $[j(\omega t - \beta\rho)]$ is used. If we had chosen the variable $[j(\omega t + \beta\rho)]$, the wave would have been excited at $\rho = \infty$ and would be converging toward the antenna and we could think of the antenna in terms of a receiver rather than a radiator.

We are able to find an expansion parameter $\left(\dfrac{1}{j\beta\rho}\right)$ that is common to all of the terms of the fields. In the regions very close to the antenna ($\beta\rho << 1$), the higher powers of this parameter will be larger than the lower powers of this parameter. Conversely, the lower powers of this parameter will dominate the higher powers at large distances from the antenna ($\beta\rho >> 1$). This fact will be used to separate and define the near field and the far field of an antenna.

In the *near field* ($\beta\rho << 1$), the product of $E_\rho H_\varphi{}^*$ and $E_\theta H_\varphi{}^*$ as computed from Poynting's vector will have terms that contain an additional factor of 'j'. This indicates that the energy is not radiated away from the antenna but is only stored in the region adjacent to the antenna. This is similar to storing energy in an inductor in a circuit. In fact, these terms of the electromagnetic fields are sometimes called the inductive or reactive terms of the radiation. In practice, this separation of the near field from the far field ($\beta\rho >> 1$) arises when accurate measurements of the phase of the wave are made. Errors in the phase can be usually accounted for by examining the properties of different near field regions. In addition, there will be real terms that have a higher even power of $\left(\frac{1}{j\beta\rho}\right)$ than 2. These terms will rapidly decay to zero as the distance from the antenna increases and are not important at large distances. They do not contribute to the far field radiation.

The electromagnetic field in the *far field* ($\beta\rho >> 1$) is of the most interest to us. Most antenna systems operate in regions that are many wavelengths in size. This is not only true for the Hertzian dipole that is being considered here, but also for most antenna structures. Keeping only the far field terms in (7.22) and (7.24), we write the electric and magnetic fields as

$$H_\varphi = j\frac{I\,\Delta L}{4\pi}\beta\sin\theta\frac{e^{-j\beta\rho}}{\rho} \tag{7.25}$$

$$E_\theta = j\frac{I\,\Delta L}{4\pi}Z_c\beta\sin\theta\frac{e^{-j\beta\rho}}{\rho} \tag{7.26}$$

We should stop at this moment and reflect on the meaning of the terms that we have just calculated. First the terms H_φ and E_θ are in space quadrature (perpendicular to each other) and in time phase. Second, the ratio of these two terms is equal to the characteristic impedance Z_c of free space. This means that the far field terms have the same properties as the plane waves that were studied in Chapter 5. This should not be too surprising since a spherical surface will approach a plane as its radius approaches infinity. Third, the total radiated complex power that is computed from Poynting's theorem can be written in phasor notation as

$$\frac{1}{2}\oint \boldsymbol{E\,x\,H^*} \bullet \mathbf{ds} = \frac{1}{2}\int_0^\pi \rho\sin\theta\, d\theta\int_0^{2\pi}\rho\left(\left|\boldsymbol{E\,x\,H^*}\right|\right)d\varphi \tag{7.27}$$

From (7.25) and (7.26), we find that both H_φ and E_θ decay with distance as $(1/\rho)$. Therefore, the dependence on the spatial variable ρ will cancel and the total real power that is radiated from the antenna will equal the power that is supplied to the antenna. This is a comforting conclusion after many lines of heavy mathematics and indicates that we have made no serious errors along the way. Finally, the radiation of the electromagnetic fields depends upon the angle θ in the same fashion as the Hertzian dipole (7.14). Hence the radiation pattern that is shown in Figure 7-7 can be employed here also.

In equations like (7.27), we encounter a factor of (1/2) that seems to mysteriously appear in front of the integral. The implication of this factor is that the peak amplitudes of the sinusoidally varying quantity are used in the integral and a time-average over one period of the oscillation has been taken.

Example 7-3. A small antenna 1 cm in length and 1 mm in diameter is designed to transmit a signal at 10 MHz inside the human body in a medical experiment. Assuming the dielectric constant of the body is similar to that of distilled water (ε_r = 80) and that the conductivity σ can be neglected, compute the maximum electric field at the surface of the body, 20 cm from the antenna. The maximum current that can be applied to the antenna is 10 μA.

Answer: The wavelength of the electromagnetic wave within the body is computed to be

$$\lambda = \frac{3\times 10^8}{\sqrt{80}}\times\frac{1}{10^7} = 3.3 \text{ m}$$

The characteristic impedance of the body is

$$Z_c = \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} = \frac{377}{\sqrt{80}} = 42\ \Omega$$

Since the dimensions of the antenna are much less than the wavelength, we can apply (7.26). Therefore

$$|E_\theta| = \frac{I\,\Delta L}{4\pi} Z_c \beta \sin\theta \frac{1}{\rho} = \frac{10^{-5}\times 10^{-2}}{4\pi}\times 42\times\frac{2}{3.3}\times\frac{1}{0.2} = 3\ \mu\text{V}/\text{m}$$

Example 7-4. The measured electric field at a distance of 1 km from a small dipole antenna is E_0. At what distance will the electric field decrease by 3 dB?

Answer: Since the decrease is expressed in dB, we must take the logarithm of the ratio to the base 10 and multiply by 20. This leads to

$$20\ \log_{10}\left|\frac{E(\rho)}{E(1000)}\right| = -3\text{ dB} = 20\ \log_{10}\left(\frac{1000}{\rho}\right)$$

Solving for ρ, we obtain ρ = 1.4 km.

Magnetic dipole

We will find the electromagnetic fields that are radiated from a *magnetic dipole* as a second example of a small antenna. This antenna is depicted in Figure 7-11. It consists of a small filamentary loop whose radius is a. The loop carries a harmonic current $i(t) = I\cos\omega t$ around its circumference. The vector potential caused by this current loop is determined from (7.16). Since the current is confined to the loop, this integral becomes (where the term $e^{j\omega t}$ is understood to be included)

$$A = \frac{\mu_0 I}{4\pi}\oint \frac{e^{-j\beta\rho'}}{\rho'}\,\mathrm{d}l' \tag{7.28}$$

This integral is not easy to evaluate since the terms within the integrand depend on the particular location where dl' is being evaluated. We can, however, obtain an approximate solution using the following procedure.[1] The exponential term can be written as

$$e^{-j\beta\rho'} \approx e^{-j\beta\rho} e^{-j\beta(\rho'-\rho)}\, e^{-j\beta\rho}\left[1 - j\beta(\rho'-\rho)\right] \tag{7.29}$$

In expanding the second exponential term, we have made the approximation that the loop is small with respect to the distance ρ between the center of the loop and the point of observation. Hence (7.28) can be written as

$$A = \frac{\mu_0 I}{4\pi} e^{-j\beta\rho}\left[(1 + j\beta\rho)\oint \frac{\mathrm{d}l'}{\rho'} - j\beta\oint \mathrm{d}l'\right] \tag{7.30}$$

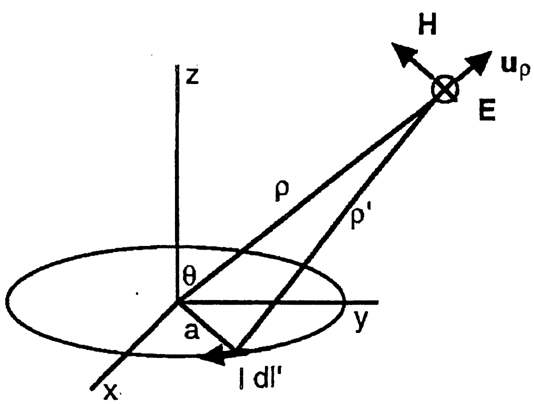


Figure 7-11. A magnetic dipole.

[1] In order to obtain analytical solutions in electromagnetics, we have to resort to many approximations. The ingenuity of the practitioner is tested when it comes to making sure that the approximations are reasonable. The success of the practitioner is tested when it comes to deciding what 'reasonable'means.

The second integral is equal to zero since this integral is akin to running around in a circle, we just return back to the same starting point and progress nowhere. The first integral is evaluated in Example 7-5. We find that the final evaluation of this integral leads to the vector potential being written as

$$A \approx \frac{\mu_0\left(\pi a^2 I\right)}{4\pi\rho^2}(1 + j\beta\rho)e^{-j\beta\rho}\sin\theta\,\mathbf{u}_\varphi \tag{7.31}$$

We recognize the term $(\pi a^2 I)$ from Example 2-24 as the magnitude m of the *magnetic dipole moment* $\mathbf{m} = (\pi a^2 I)\mathbf{u}_z$.

Having found the vector potential, we can find the magnetic and the electric fields from (7.22) and (7.23) to be

$$H_\rho = -\frac{j\omega\mu_0 m}{4\pi Z_c}\beta^2 2\cos\theta\left[\frac{1}{(j\beta\rho)^2} + \frac{1}{(j\beta\rho)^3}\right]e^{-j\beta\rho} \tag{7.32}$$

$$H_\theta = -\frac{j\omega\mu_0 m}{4\pi Z_c}\beta^2\sin\theta\left[\frac{1}{j\beta\rho} + \frac{1}{(j\beta\rho)^2} + \frac{1}{(j\beta\rho)^3}\right]e^{-j\beta\rho} \tag{7.33}$$

$$E_\varphi = \frac{j\omega\mu_0 m}{4\pi}\beta^2\sin\theta\left[\frac{1}{j\beta\rho} + \frac{1}{(j\beta\rho)^2}\right]e^{-j\beta\rho} \tag{7.34}$$

If we compare (7.32)–(7.34) with (7.22), (7.24a) and (7.24b), we note a similarity in that the various field components could be expressed in terms of the parameter $\left(\frac{1}{j\beta\rho}\right)$ to various powers. As we learned in our study of the electric dipole, we could express the radiation of the electromagnetic fields in terms of near fields and far fields. The far field radiation was of most interest to us since we could describe the power that actually left the antenna in terms of the far field terms only. The far field components of the magnetic dipole are given by

$$H_\theta = -\frac{\omega\mu_0 m}{4\pi Z_c}\beta\sin\theta\frac{e^{-j\beta\rho}}{\rho} \tag{7.35}$$

$$E_\varphi = \frac{\omega\mu_0 m}{4\pi}\beta\sin\theta\frac{e^{-j\beta\rho}}{\rho} \tag{7.36}$$

In the far field, the magnitude of the two fields each decay as ρ^{-1} and the ratio of the two fields is equal to the characteristic impedance Z_c of free space. This is similar to the behavior that we had already obtained for the electric dipole.

Example 7-5. Evaluate the integral $\oint \frac{dl'}{\rho'}$.

Answer: Use the vector identity (see Appendix A)

$$\oint_{\Delta l} a\, dl = \int_{\Delta s} \mathbf{u_n x} \nabla a \bullet \mathbf{ds}$$

to convert the closed line integral into a surface integral. The scalar quantity a is equal to $(1/\rho')$. With reference to Figure 7-10, we note that $\mathbf{u}_n = \mathbf{u}_z$ since the loop is in the *xy* plane. Therefore

$$\oint \frac{dl'}{\rho'} = \int_{\Delta s} \mathbf{u}_z \,\mathbf{x}\, \nabla'\left(\frac{1}{\rho'}\right) ds = \int_{\Delta s} \mathbf{u}_z \,\mathbf{x}\left(\frac{\mathbf{u}_\rho}{\rho'^2}\right) ds$$

where

$$\nabla'\left(\frac{1}{\rho'}\right) = \left(\frac{\mathbf{u}_\rho}{\rho'^2}\right)$$

For large distances from the current loop, we can let $\rho' = \rho$ and $\mathbf{u}_{\rho'} = \mathbf{u}_\rho$. With these approximations, the integral becomes

$$\int_{\Delta s} \mathbf{u}_z \,\mathbf{x}\left(\frac{\mathbf{u}_\rho}{\rho^2}\right) ds = \frac{\left(\mathbf{u}_z \,\mathbf{x}\, \mathbf{u}_\rho\right)}{\rho^2} \int_{\Delta s} ds$$

The surface integral yields a factor of πa^2. Finally, we make use of the vector relation

$$\mathbf{u}_z = \mathbf{u}_\rho \cos\theta - \mathbf{u}_\theta \sin\theta$$

to compute in spherical coordinates that

$$\mathbf{u}_z \,\mathbf{x}\, \mathbf{u}_\rho = \mathbf{u}_\varphi \sin\theta$$

Hence, the final result of the integrations yields

$$\oint \frac{\mathrm{d}\mathbf{l}'}{\rho'} = \frac{\mathbf{u}_\varphi}{\rho^2}\pi a^2 \sin\theta$$

Thin wire antenna

We have now covered the basic idea of the radiation of electromagnetic waves by two very small antennas, the electric and the magnetic dipole antennas. The mathematics has been straightforward although somewhat tedious at times. As we take a short trip away from this book and into the hinterlands, we may see some tall structures that reach into the heavens and have flashing red lights at the top to warn passing airplanes. These antennas certainly do not seem to fall into the class of being small. Herein we will describe a technique to generalize our treatment of antennas so that more realistic antennas can be studied.

Let us consider two thin metallic rods having a total length $2h$. This length may be of the order of the free space wavelength λ of the electromagnetic wave that is to be radiated. A sinusoidal voltage generator whose frequency of oscillation is ω is connected between the two rods as shown in Figure 7-12. This voltage generator will induce a current in the rods that can have a distribution $I(z)$ that is governed by the shape and length of the conductor.

It is reasonable to assume that the current distribution at the ends of the antenna ($z = \pm h$) is equal to zero and that the current distribution is symmetrical about the center ($z = 0$). The first assumption is predicated on the idea that no conduction current could extend beyond the metallic surface. Since the antenna is 'center-fed', symmetry arguments will follow. The assumption for the actual distribution $I(z)$ that is used in the integral requires some ingenuity. A typical requirement is that the current distribution

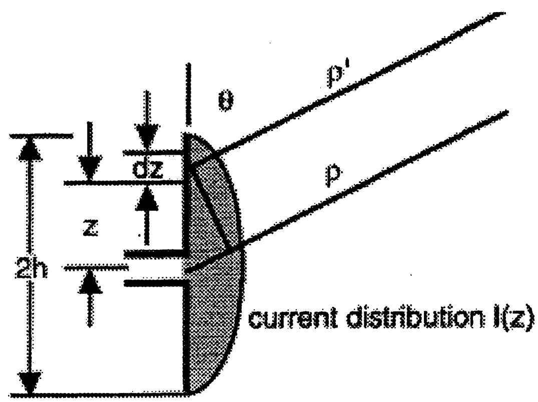


Figure 7-12. A center fed dipole with an arbitrary current distribution $I(z)$.

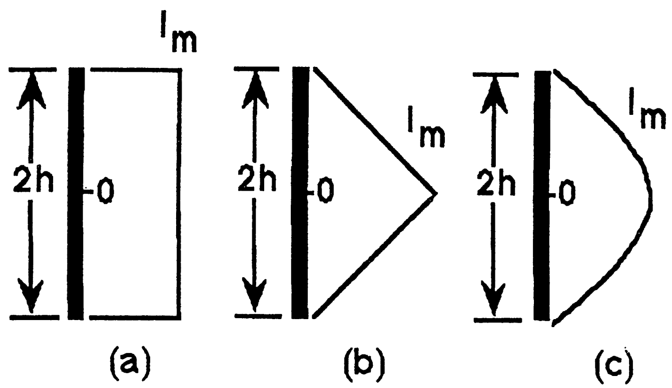


Figure 7-13. Three possible distributions of current on an antenna. (a) Uniform distribution. (b) Triangular distribution. (c) Sinusoidal distribution, $\beta = \pi/2$.

is selected so certain integrals can be actually be performed. Computers have now removed this restriction and more realistic distributions can be employed. The investigator can iterate the sequence: (1) choose a current distribution, (2) compute the radiated electromagnetic fields, (3) ascertain that the radiated fields satisfy all boundary conditions, (4) experimentally measure the radiated electromagnetic fields, and (5) reiterate the sequence with a modified current distribution. The computer, however, may introduce a degree of obfuscation that we would like to avoid at this level and we will stick with an analytical treatment.

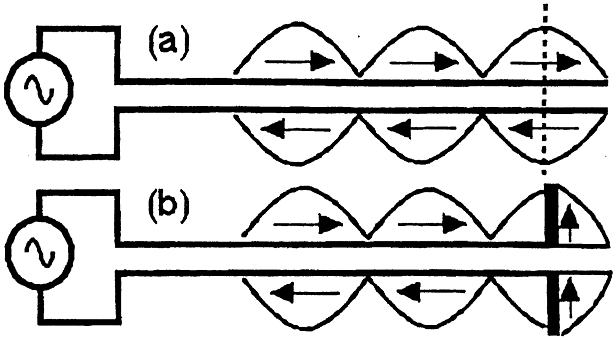
Three distributions for the current that have received considerable attention and lead to analytical solutions are

$$\left.\begin{array}{ll} I(z) = \begin{cases} I_m & \text{for } -h \le z \le +h \\ 0 & \text{elsewhere} \end{cases} & (a) \\ I(z) = I_m\left(1 - \dfrac{|z|}{h}\right) & (b) \\ I(z) = I_m \sin\left[\beta(h - |z|)\right] & (c) \end{array}\right\} \tag{7.37}$$

They all have the property that the current is confined to the wire. These are shown in Figure 7-13. The far field radiation properties are not extremely sensitive to the actual choice for the current distribution.

Example 7-6. Describe the excitation of a center fed dipole antenna using a transmission line model.

Answer: The current distribution of both the incident and the reflected components of



the current on an *open circuited* transmission line are depicted in the figure. Its spatial distribution is cosinusoidal as shown in (a). By bending the transmission line at $\lambda/4$ from the end, we form a half-wave dipole with the proper current distribution. This model has assumed that the last $\lambda/4$ of the transmission line is unaffected by the bending of the transmission line. The distribution of the current on the line will be altered since the load is not infinite due to fringing.

As we will see, once the current distribution is chosen, then the far field electromagnetic distribution can be calculated. There is no absolutely correct assumption that can be made. As just indicated, the procedure is iterative in that an assumption is first made for the current distribution the calculation performed; experimental measurements to check the predicted fields; modification of the assumed current distribution; and redoing the calculation. The initial calculation stage will be illustrated for the current distribution given in (7.37c).

In (7.25) and (7.26), we found the far field electric and magnetic field distributions caused by a small current element $I(z)dz$ to be

$$\mathrm{d}E_\theta = Z_c \mathrm{d}H_\varphi = j\frac{I(z)dz}{4\pi}\left(\frac{e^{-j\beta\rho'}}{\rho'}\right)Z_c\beta\sin\theta \tag{7.38}$$

The distance ρ' that appears in two terms, can be written in terms of the distance ρ between the point of observation and the center of the dipole as

$$\rho' = \left[\rho^2 + z^2 - 2\rho z\cos\theta\right]^{1/2} \approx \rho - z\cos\theta \tag{7.39}$$

See Figure 7-12. We are allowed to make this approximation since the field distribution in the far field is to be determined, that is $\rho >> z$. The difference in magnitude between $\frac{1}{\rho'}$ and $\frac{1}{\rho}$ is insignificant and can be neglected. However, it is important that we incorporate this difference in the phase term $e^{-j\beta\rho'}$. Small changes in distance may be a reasonable fraction of a wavelength λ that could cause this term to change sign from a "+" to a "−". This will have dramatic effects as will be shown below.

In order to actually compute the electromagnetic fields radiated from an antenna, we have to select the distribution for the current and perform an integration over the coordinate of the antenna z. This will be done with the current distribution given in (7.37c) for which we write that

$$E_\theta = Z_c H_\varphi = j\frac{I_m Z_c \beta \sin\theta}{4\pi\rho} e^{-j\beta\rho} \int_{-h}^{h} \sin\left[\beta(h - |z|)\right] e^{j\beta z \cos\theta} dz \qquad (7.40)$$

Before actually performing the integration required at this stage, let us comment on the terms within the integrand. The term $\sin[\beta(h - |z|)]$ is an even function in the variable of integration z as shown in Figure 7-13c. The product of this term and

$$e^{j\beta z\cos\theta} = \cos\left(\beta z \cos\theta\right) + j\sin\left(\beta z\cos\theta\right)$$

will yield two terms, one of which is odd in the variable z and one that is even in the variable z. Since the limits of the integral are symmetric about the origin, only the integrand that includes the even term will yield a nonzero result.[1] The integral (7.40) reduces to

$$E_\varphi = Z_c H_\varphi = j2\frac{I_m Z_c \beta \sin\theta}{4\pi\rho} e^{-j\beta\rho} \int_{0}^{h} \sin\left[\beta(h - |z|)\right] \cos(\beta z \cos\theta) dz \qquad (7.41)$$

Since we are considering vacuum or free space conditions, we can let Z_c = 120 Ω. After performing the integration, we finally obtain

$$E_\theta = Z_c H_\varphi = j60 I_m \frac{e^{-j\beta\rho}}{\rho} F(\theta) \qquad (7.42)$$

where

$$F(\theta) = \frac{\cos(\beta h \cos\theta) - \cos\beta h}{\sin\theta} \qquad (7.43)$$

The final solution is the product of two terms. The first is a term that corresponds to the radiation characteristics of an oscillating charge located at $\rho = 0$. The second term $F(\theta)$ is sometimes called the E-plane *pattern function* of the linear dipole antenna. It has the property that $F(\theta) = 0$ at $\theta = 0°$. This function will change if the length of the

[1] As an example, the integral of the integrands x and x^2, which are respectively odd and even, functions between $x = -1$ and $x = +1$ yields 0 and 2/3.

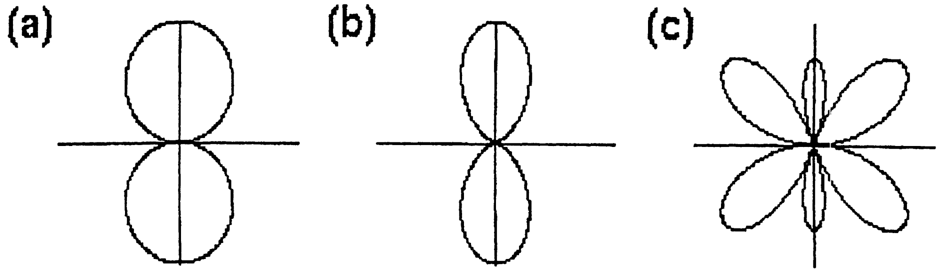


Figure 7-14. E-plane radiation patterns for center-fed dipole antennas of different lengths. (a) $h = \lambda/4$. (b) $h = \lambda/2$. (c) $h = 3\lambda/4$. The antenna with the dimension $h = \lambda/4$ is called a half-wave dipole.

antenna $\beta h = 2\pi \frac{h}{\lambda}$ is altered or if the current distribution is modified. In Figure 7-14, we illustrate the E-plane pattern function for three different dipole lengths.

The H-plane radiation patterns are azimuthally symmetric circles since $F(\theta)$ is independent of the angle φ. We note from Figure 7-14 that the maximum in the radiated power tends to shift away from $\theta = 90°$ as the length h is changed. If we set $h = \lambda$ in (7.43), we find that the radiation at $\theta = 90°$ is equal to zero. The contours depicted in Figure 7-14 are called *lobes*. The lobe at $\theta = 90°$ is called the *main lobe* and the others are called *side lobes*. If we were to traverse about the antenna at a constant radius and monitored the received signal with a meter sensitive to the phase, we would note a phase shift of 180° as we move from one lobe to the adjacent one. The lobe structure is another example of *phase mixing* that was discussed when the topics of dispersion and group velocity were presented.

Antenna parameters

In addition to the radiation pattern for the antenna that was discussed in the previous sections, there are other parameters that are used to characterize an antenna. If we connected the antenna to a transmission line, we could think of the antenna as being merely a load impedance. The radiation of electromagnetic power into the external environment removes the power from the circuit and hence acts like a resistor that just heats up. This is depicted in Figure 7-15.

In order to compute the value of the load impedance Z_L, we will have to return to *Poynting's vector*. Recall that this quantity is a measure of the power density at a point in space caused by the electromagnetic wave. The *total* power radiated from the antenna can be computed by surrounding the antenna with a large imaginary sphere whose radius is R (Figure 7-16). The radius R will be chosen so the sphere will be in the far field region. Then any power that is radiated from the antenna will have to pass through the

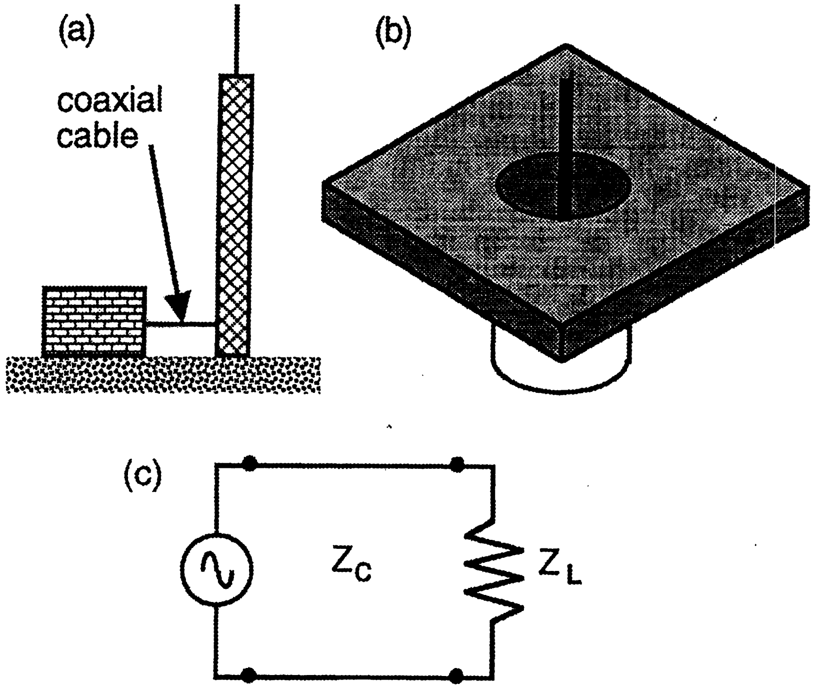


Figure 7-15. (a) An antenna that radiates electromagnetic energy is connected with a transmission line to a source of electromagnetic energy. (b) Coaxial cable connected to a ground plane. (c) Equivalent circuit of either structure.

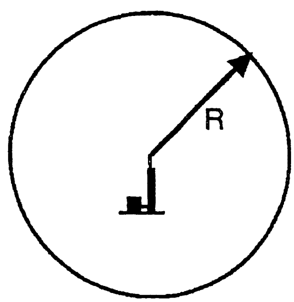


Figure 7-16. Electromagnetic power radiated from an antenna will pass through a sphere of radius R.

sphere in order to propagate to distances greater than this radius R.

The total average power that is radiated from the antenna is computed by integrating Poynting's vector over this entire closed spherical surface. From (7.1), this becomes

$$\text{average radiated power } P_r = \frac{1}{2}\oint \mathbf{E} \; x \; \mathbf{H}^* \bullet \mathbf{ds} = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} \left[E_\theta H_\varphi^*\right] R^2 \sin\theta \, d\theta \quad (7.44)$$

The factor of (1/2) arises since we are considering a time - average power over a temporal cycle of the oscillation. This average radiated power can be considered to be lost as far as the source is concerned it acts as if the antenna were a resistor that dissipated the power. This resistance is called the *radiation resistance* R_r and it is defined as

$$\boxed{R_r \equiv \frac{2P_r}{I_m^2}} \tag{7.45}$$

where I_m is the maximum amplitude of the current at the input terminals. We will calculate the radiation resistance for the Hertzian dipole and a half-wave dipole.

The latter antenna is of particular importance due to its desirable radiation pattern and as we will see, its impedance characteristics.

For the Hertzian dipole, we first calculate the power that is radiated from the antenna

$$\begin{aligned} P_r &= \frac{1}{2}\left\{\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\left[\left(\frac{I_m \Delta L}{4\pi R} Z_c \beta \sin\theta\right)\left(\frac{I_m \Delta L}{4\pi R}\beta \sin\theta\right)\right] R^2 \sin\theta\, d\theta\, d\varphi\right\} \\ &= \frac{I_m^2}{2}\frac{(\Delta L)^2 Z_c \beta^2}{8\pi^2}\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}(\sin\theta)^3 d\theta\, d\varphi \\ &= 80\pi^2 \frac{I_m^2}{2}\left(\frac{\Delta L}{\lambda}\right)^2 \end{aligned} \tag{7.46}$$

To obtain the last result, we have used the characteristic impedance for free space $Z_c = 120\,\pi\Omega$ and the definition that $\beta = \frac{2\pi}{\lambda}$. The radiation resistance for a Hertzian dipole follows from (7.45) to be

$$P_r = 80\pi^2 \left(\frac{\Delta L}{\lambda}\right)^2 \Omega \tag{7.47}$$

Let us insert numbers that satisfy our requirement for a Hertzian dipole that $L << \lambda$. Choosing $\Delta L = 0.01\,\lambda$ leads to a radiation resistance of only 0.08Ω, that is an extremely small value. This implies that the Hertzian dipole will be a very poor radiator of electromagnetic power (or it will 'dissipate' only a small amount of electrical power in the equivalent circuit shown in Figure 7-15c). An efficiency for an antenna can be defined as the ratio of the radiation resistance of the antenna to the input resistance of the antenna. Since the latter resistance includes ohmic losses in the antenna structure as well as losses to the ground, the efficiency of a Hertzian dipole is extremely small.

For the half-wave dipole whose length is $2h = \frac{\lambda}{2}$, we find that the *pattern function* $F(\theta)$ assuming a uniform current distribution becomes

$$F(\theta) = \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta} \tag{7.48}$$

This function has a maximum at $\theta = 90°$ with nulls at $\theta = 0°$ and at $\theta = 180°$. The radiation pattern for this antenna is shown in Figure 7-14. The far field electromagnetic field components follow from (7.42):

$$E_\theta = j60 I_m \left(\frac{e^{-j\beta\rho}}{\rho}\right) \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta} \tag{7.49}$$

and

$$H_\varphi = j\frac{I_m}{2\pi}\left(\frac{e^{-j\beta\rho}}{\rho}\right) \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta} \tag{7.50}$$

The total average power radiated from the antenna through the imaginary sphere whose radius $\rho = R$ is given from (7.44)

$$P_r = \frac{1}{2}\left\{\int_0^{2\pi} d\varphi \int_0^{\pi} \left(\frac{60 I_m}{R}\cos\left[\frac{\pi}{2}\cos\theta\right]\right)\left(\frac{I_m}{2\pi R}\cos\left[\frac{\pi}{2}\cos\theta\right]\right) R^2 \sin\theta \, d\theta\right\}$$

$$= 30 I_m^2 \int_0^{\pi} \left[\frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta}\right]^2 \sin\theta \, d\theta \tag{7.51}$$

The integral in (7.51) has to be evaluated numerically and it has a value of 1.217. Therefore, the total radiated power P_r has the value

$$P_r = 36.54 I_m^2 \ W \tag{7.52}$$

Using (7.45), we compute the radiation resistance R_r for a half-wave dipole to be R_r = 73.1. This value for a radiation resistance indicates that the half-wave dipole antenna can radiate significantly more power than the Hertzian dipole for the same value of input current and it is thus more efficient.

There are several other antenna parameters that should be de fined as they are found in practice. These terms are: (1) beam width, (2) directive gain, (3) directivity, and (4) effective area. We will discuss each of these terms below.

(1) The *beam width* is a parameter defining the sharpness of the radiation pattern of the main lobe. It is usually defined as the angular width of the pattern between the half-power points (or –3 dB points). For the electric field, this means the points where $E = \frac{E_{max}}{\sqrt{2}} \approx .707 E_{max}$. For the Hertzian dipole, the E-plane beam width is computed from (7.26) to be 90°. For the half-wave dipole, the E-plane beam width is computed from $F(\theta) = \frac{1}{\sqrt{2}}$ in (7.48) to be 78°.

From a comparison of the beam width of these two antennas, we can make a general statement that will apply to other antennas. If one desires to make an antenna with a very narrow beam width, then the physical dimension (normalized by the wavelength λ) should be as large as possible. Antennas with a physical size of the order of a football field have been used in some radar systems. Reception antennas with elements located at various sites throughout North America are used in astronomical applications to probe the mysteries of small areas in the heavens. Therefore, the total antenna size is of the order of the size of North America. The rotation of the earth about its axis and the earth's travel about the sun ensure that much of space can be scanned in a year.

(2) The second parameter that defines an antenna system is the *directive gain* or just the gain. The *gain* of an antenna is defined as the ratio of the power density radiated in a certain direction from the chosen antenna to the power density radiated in that direction from an isotropic source radiating the same total power. If we look for the direction where the maximum radiated power is found, we call this the *directive gain* of the antenna. The units of the time-average power per unit solid angle are watts per steradian. The directive gain $\mathcal{G}$ can be written as

$$\boxed{\mathcal{G} = \frac{P_{max}}{P_{av}} = \frac{P_{max}}{\left(\frac{P_r}{4\pi}\right)}} \tag{7.53}$$

since there are 4 steradians in a complete sphere. In terms of the electric field $E(\theta, \varphi)$, the gain of an antenna can be written as

$$\mathcal{G} = \frac{\left|E_{max}^2(\theta,\varphi)\right|}{\frac{1}{4\pi}\left\{\int_0^{2\pi} d\varphi \int_0^{\pi} (\sin\theta)^2 \sin\theta \, d\theta\right\}} \tag{7.54}$$

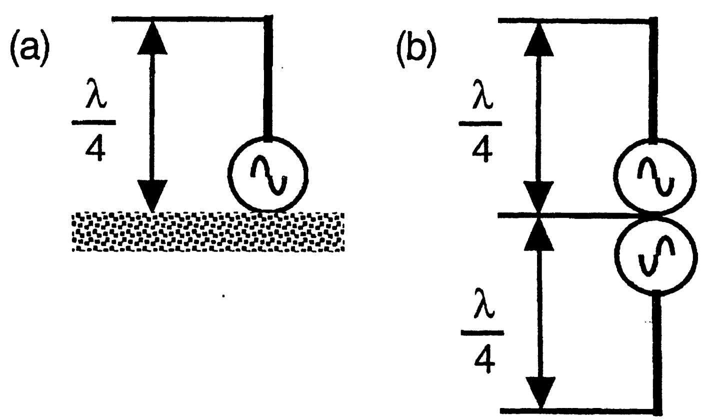
The gain of an antenna is frequently given in units of decibels (dB) with unity as a reference.

For the Hertzian dipole, we compute using (7.54) that the gain is

$$\mathcal{G} = \frac{\sin^2\theta}{\frac{1}{4\pi}\left\{\int_0^{2\pi} d\varphi \int_0^{\pi} (\sin\theta)^2 \sin\theta \, d\theta\right\}} = \frac{3}{2}\sin^2\theta \tag{7.55}$$

All of the other terms involving constants, current magnitudes and the radius have canceled since they are common factors in both the numerator and the denominator. The maximum value of the gain is found at an angle of $\theta = \frac{\pi}{2}$. Hence $\mathcal{G}$ = 1.5 or 10 $\log_{10}(1.5)$ = 1.76 dB. A similar calculation for the half-wave dipole leads to a directive gain $\mathcal{G}$ = 1.64 or 2.15 dB.

Example 7-7. A thin quarter-wavelength vertical antenna is located above a perfectly conducting ground plane as shown in (a). It is excited with a sinusoidal source at its base. Find the radiation pattern and the radiation resistance of the antenna.



Answer: Since the current consists of charge in motion and the charge is located above a perfectly conducting ground plane, we can replace the quarter-wavelength antenna depicted in (a) with the half-wavelength *dipole* antenna depicted in (b). The *image* antenna has the same length but the opposite phase from the real antenna above the ground.[1] The radiation pattern is therefore similar to that depicted in Figure 7-11 with the caveat that the antenna radiates only in the upper half plane ($0 \le \theta \le 90°$). Since it radiates only in the upper half plane, only one-half of the power computed in (7.51) can be radiated from the antenna. The radiation resistance is computed from

$$R_r = \frac{2P_r}{I_m^2} = \frac{2\left(18.27\, I_m^2\right)}{I_m^2} = 36.54\,\Omega$$

This value is equal to one-half of the radiation resistance of the half-wave antenna.

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(3) The *directivity* of an antenna provides us with some information about the entire radiation pattern of the antenna. The beam width describes the properties of the main lobe but it tells nothing about the side lobes and it is the directivity that includes these features. The angles where the maximum directive gain can be found leads to the directivity of the antenna.

(4) A parameter that is used to characterize a receiving antenna is the *effective area*.[2] The effective area $A_{eff}(\theta, \varphi)$ of a receiving antenna is defined as the ratio of the average power P_L delivered to a load impedance that is matched to the antenna receiving the incident time-average power density S_{av}

$$S_{av} = \frac{|E\, x\, H|}{2} = \frac{E^2}{2Z_c} = \frac{E^2}{240\pi} \tag{7.56}$$

The incident wave arrives in the direction (θ, φ). We write

$$P_L = S_{av} A_{eff}(\theta, \varphi) \tag{7.57}$$

The maximum effective area is attained if the load impedance is the complex conjugate of the antenna impedance. Assuming that the receiving antenna has an impedance Z_{ant}, this implies that the load impedance Z_L must have a value $Z_L = Z_{ant}$. Therefore, the power that is dissipated in the load is given by

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[1] This is smiliar to an image charge in static fields. A positive charge $+Q$ placed a distance d above an infinite grounded plane will induce a negative surface charge density on the plane. To calculate the electric field, we replace the surface charge with a charge $-Q$ at a distance d beneath the surface. The calculated electric field has a contribution from the real charge and the image charge.

[2] It is also referred to as the effective aperture or the receiving cross section.

$$P_L = \frac{1}{2} I_m^2 R_L = \frac{1}{2}\left(\frac{V_{oc}}{Z_{ant} + Z_L}\right)^2 R_L = \frac{1}{2}\left(\frac{V_{oc}}{Z_{ant} + Z_{ant}}\right)^2 R_{ant} = \frac{V_{oc}^2}{8R_{ant}} \tag{7.58}$$

where V_{oc} = [|**E**| sin θ *x* (effective length)] which is the parallel component of the incident wave. For the Hertzian dipole, the antenna resistance $R_{ant} = R_r$ that was given in (7.47) and we write

$$P_L = \frac{(E\sin\theta\Delta L)^2}{8\left(80\pi^2\left(\frac{\Delta L}{\lambda}\right)^2\right)} = \frac{E^2\sin^2\theta\lambda^2}{640\pi^2} \tag{7.59}$$

We find the effective area of the Hertzian dipole from (7.57) using (7.56) and (7.59) to be

$$A_{eff}(\theta, \varphi) = \frac{P_L}{S_{av}} = \frac{\frac{E^2\sin^2\theta\lambda^2}{640\pi^2}}{\frac{E^2}{240\pi}} = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\sin^2\theta\right) \tag{7.60}$$

The term within the parentheses is the gain $\mathcal{G}(\theta, \varphi)$ of a Hertzian dipole antenna as noted in (7.55). This is no accident and the effective area for any lossless antenna can be written as

$$\boxed{A_{eff}(\theta, \varphi) = \frac{\lambda^2}{4\pi}\mathcal{G}(\theta, \varphi)} \tag{7.61}$$

Let antenna *A* in Figure 7-17 transmit to antenna *B*. Both antennas must be in the far field. The gain of the transmitting antenna *A* in the direction of *B* is $\mathcal{G}_A(\theta_A, \varphi_A)$. Hence, the time average power density at *B* is

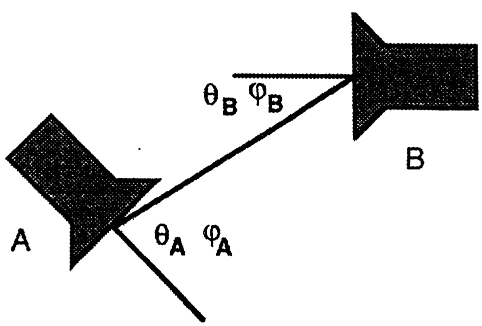


Figure 7-17. Two antennas separated by a distance *r*.

$$S_{av} = \frac{P_t}{4\pi r^2} \mathcal{G}_A(\theta_A, \varphi_A) \tag{7.62}$$

Write (7.57) and replace the effective area of antenna B A_{eff_B} using (7.61)

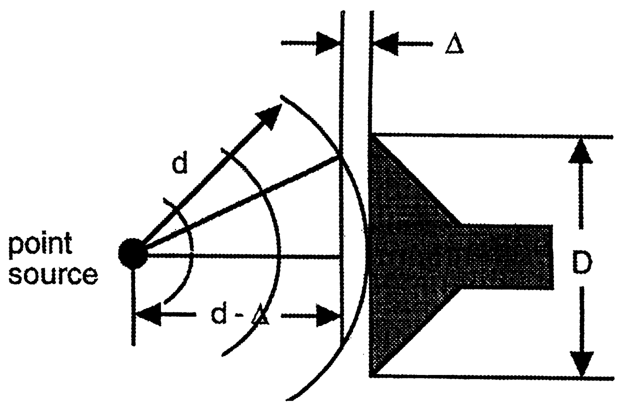
$$P_L = S_{av} A_{eff_B}(\theta_B, \varphi_B) = \left[\frac{P_t}{4\pi r^2} \mathcal{G}_A(\theta_A, \varphi_A)\right]\left[\frac{\lambda^2}{4\pi} \mathcal{G}_B(\theta_B, \varphi_B)\right] \tag{7.63}$$

We finally write

$$\boxed{\frac{P_L}{P_t} = \mathcal{G}_A(\theta_A, \varphi_A)\mathcal{G}_B(\theta_B, \varphi_B)\left(\frac{\lambda}{4\pi r}\right)^2} \tag{7.64}$$

Equation (7.64) is called the *Friis transmission equation.*

Example 7-7. Find a criterion that a receiving antenna is in the far field of a transmitting antenna.



Answer: A requirement that the Friis transmission equation be applicable is that both antennas be in the far field. The radiation from a point source is always in the far field. The receiving antenna will be in the far field if the incident spherical wave deviates from a plane wave by only a fraction of a wavelength. The largest dimension of the receiving antenna is D. This implies that the deviation Δ is approximately $\Delta = \lambda/k$, where $k = 2\pi/\lambda >> 1$. From the figure, we write

$$d^2 = (d-\Delta)^2 \approx d^2 - 2d\Delta + \frac{D^2}{4}$$

This implies that the receiving antenna will be in the far field if

$$d \approx \frac{D^2}{8\Delta} = \frac{kD^2}{8\lambda}$$

Antenna arrays

In many antenna applications, we will not find a single antenna tower located at the site where the transmitting antenna is located. There will be a number of towers that are situated at prescribed locations in order that the electromagnetic energy is radiated in the desired direction. We will examine one method of accomplishing this goal here. The technique to accomplish the desired radiation pattern that we will study is to set up an *antenna array*.

An antenna array is defined as a group of antennas that are arranged in various configurations (straight lines, squares, triangles, circles, etc.). We will initially assume that each antenna is similar in order to simplify the presentation. Each individual antenna that we will define as an *element* of the array is excited with the proper amplitude and proper phase so the desired radiation pattern is obtained. Since the elements are individually radiating into free space, we can find the radiation characteristics of the entire array using the principle of *superposition*. We will be concerned only with the far field radiation here. This principle we have employed before states merely that the field from the entire array can be computed from the linear vector addition of the fields from each individual element.

We will illustrate this for an antenna array where the elements are situated in a straight line. Such an array is called a *linear array*. To introduce the procedure, we will first examine an array that consists of two elements that are excited with the same amplitude but the phase in element b leads element a by an amount d. A linear array consisting of two elements is shown in Figure 7-18.

We should note at this point that we have not specified the radiation characteristics of an individual element. The individual element can be characterized by its pattern function $F(\theta, \varphi)$ that generalizes our previous definition to the dependence on both angles of spherical coordinates.

At point P in Figure 7-18, the total electric field consists of the sum of the contributions of the two individual elements

$$E = E_a + E_b = \Xi F(\theta, \varphi)\frac{e^{-j\beta\rho_a}}{\rho_a} + \Xi F(\theta, \varphi)\frac{e^{-j\beta\rho_b}e^{j\delta}}{\rho_b} = \Xi F(\theta, \varphi)\left[\frac{e^{-j\beta\rho_a}}{\rho_a} + \frac{e^{-j\beta\rho_b}e^{j\delta}}{\rho_b}\right] \tag{7.65}$$

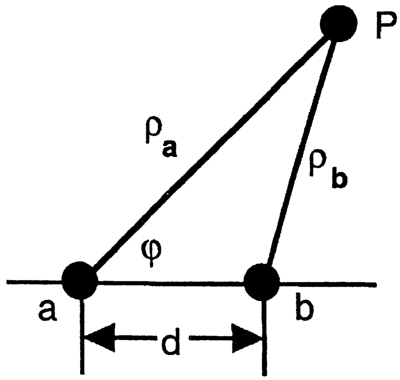


Figure 7-18. A two-element linear array.

In (7.65), the amplitude of the wave is Ξ. Note that the pattern function for an individual element $F(\theta, \varphi)$ has been factored out from the terms that represent the effects of the array. This can usually be done and we *multiply* the effects of the array by the pattern function for an individual element $F(\theta, \varphi)$. More about this later.

Remember that we are trying to find the total electric field in the *far field*. At this location, we can again make the approximation that $\rho_a = \rho_b$ in the amplitude terms of (7.65) but that a better approximation is required in the phase terms. With the two elements being almost parallel, we can write that

$$\rho_b \approx \rho_a - d \sin\theta \cos\varphi \tag{7.66}$$

in the phase terms. The angle θ is the angle between an axis parallel to the antenna element and a line to the point of observation. Substituting (7.66) into (7.65), we obtain

$$E \approx \Xi F(\theta, \varphi)\left(\frac{e^{-j\beta\rho_a}}{\rho_a}\right)\left[1 + e^{j(\beta d \sin\theta\cos\psi + \delta)}\right] \tag{7.67}$$

$$= \Xi F(\theta, \varphi)\left(\frac{e^{-j\beta\rho_a}}{\rho_a}\right)e^{j\psi/2}\left[2\cos\left(\frac{\psi}{2}\right)\right] \tag{7.68}$$

where

$$\psi = \beta d \sin\theta\cos\varphi + \delta$$

The magnitude of the electric field of the two-element array is given by

$$|E| = \frac{2\Xi}{\rho_a}|F(\theta, \varphi)|\left|\cos\frac{\psi}{2}\right| \tag{7.69}$$

The term $|F(\theta, \varphi)|$ is the magnitude of the pattern function of the element and the term $\left|\cos\frac{\psi}{2}\right|$ is the magnitude of the *array factor* of the antenna array. Note that this latter term depends on the array geometry and the amplitude and phase of the individual excitation applied to each element. In (7.65), the amplitudes were set equal, but this need not be a general requirement.

Let us illustrate the far-field radiation pattern for two isotropic radiating elements that are placed along the x axis. The two elements are separated by a distance d and the elements are excited with equal amplitude signals having a phase difference of δ. Both d and δ can be changed. For an isotropic radiating element, the pattern function for an individual element $F(\theta, \varphi) = 1$. From (7.69), we find that the magnitude of the electric field for this array is given by

$$|E| = \frac{2\Xi}{\rho_a}\left|F\left(\theta = \frac{\pi}{2}, \varphi\right)\right|\left|\cos\left(\frac{\beta d\cos\varphi + \delta}{2}\right)\right| = \frac{2\Xi}{\rho_a}\left|\cos\left(\frac{\beta d\cos\varphi + \delta}{2}\right)\right| \tag{7.70}$$

where we have specified that $\theta = \frac{\pi}{2}$.

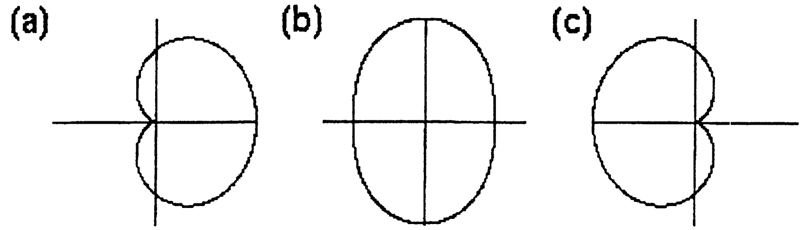
This suggests that this situation would correspond to two radio towers separated by a distance d located on the ground. Each tower isotropically radiates an electromagnetic wave having its electric field polarized normal to the ground. The station can adjust the phase between the signals fed to the two elements. We walk around the array at a constant distance from the array with a receiver that can pick up the radiated signal and plot the amplitude of the received signal.

Example 7-9. Find and sketch the array factor for two antennas separated by 25 centimeters. The frequency of the signal applied to each antenna is $f = 300$ MHz. The phase δ between the two antennas can be changed in units of $\pi/2$.

Answer: The value of βd is computed as

$$\beta d = \frac{2\pi}{\lambda} d = \frac{2\pi}{\left(\frac{3\times10^8 \text{ m/s}}{3\times10^8 \text{ Hz}}\right)} \times \frac{1}{4} = \pi/2$$

We find the radiation patterns change as shown in the figure with the change of phase. (a) $\delta = -\pi/2$. (b) $\delta = 0$. (c) $\delta = +\pi/2$.



We find that by electronically changing the phase δ of the applied signal, we can have the antenna 'sweep' through certain regions of space. Such a structure is called a *phased-array antenna.* Antennas of this type are of particular importance in large radar installations where it would be mechanically impossible to rotate an antenna that may be the size of a football field.

We can extend our investigation into these antenna array ideas in several ways. A technique of extending the analysis of antenna arrays is to consider more elements than two as shown in Figure 7-19. In this linear array, there is a progressive phase shift d in the current that feeds the identical N elements. In this case, (7.67) generalizes to

$$E \approx E_0 F(\theta, \varphi)\left(\frac{e^{-jk\rho_a}}{\rho_a}\right)\left[1 + e^{j(\beta d \sin\theta\cos\varphi+\delta)} + \ldots + e^{j(N-1)(\beta d \sin\theta\cos\varphi+\delta)}\right] \tag{7.71}$$

Fortunately, we do not have to carry along all of the terms within the square brackets since they can be summed using

$$\sum_{n=0}^{N-1} \xi^n = \frac{1-\xi^N}{1-\xi}$$

Therefore, the electric field in (7.62) becomes

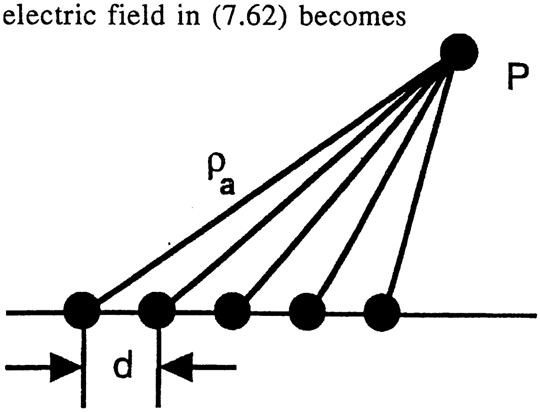


Figure 7-19. A uniform linear array. The current on the first element is $I(z)$, the current on the second element is $I(z)\ e^{j\delta}$, the current on the third element is $I(z)\ e^{j2\delta}$, etc.

$$E \approx E_0\, F(\theta,\varphi)\left(\frac{e^{-jk\rho_a}}{\rho_a}\right)\left[\frac{1-e^{jN(\beta d\sin\theta\cos\varphi+\delta)}}{1-e^{j(\beta d\sin\theta\cos\varphi+\delta)}}\right] \tag{7.72}$$

If we examine only the magnitude of the electric field $|E|$, we can simplify (7.72) with the relation

$$\left|1-e^{j\xi}\right| = \left|2j\sin\frac{\xi}{2}e^{j\xi/2}\right| = 2\sin\frac{\xi}{2}$$

Hence, the terms within the square brackets in (7.72) become

$$\left|G(\theta,\varphi)\right| \equiv \left|\frac{1-e^{jN(\beta d\sin\theta\cos\varphi+\delta)}}{1-e^{j(\beta d\sin\theta\cos\varphi+\delta)}}\right| = \left|\frac{\sin N\left(\dfrac{(\beta d\sin\theta\cos\varphi+\delta)}{2}\right)}{\sin\left(\dfrac{(\beta d\sin\theta\cos\varphi+\delta)}{2}\right)}\right| \tag{7.73}$$

The angles where the first null in the numerator of (7.73) occur define the main beam in the radiation pattern of the *linear array*. Similarly, zeroes in the denominator will yield maxima in the pattern.

In Figure 7-20, we show the variation of $|G(\theta,\varphi)|$ as the phase delay δ is changed in equal increments for a four element array. The separation of the elements $d = \lambda/2$. Hence, we observe that the antenna radiation pattern can be altered by changing the

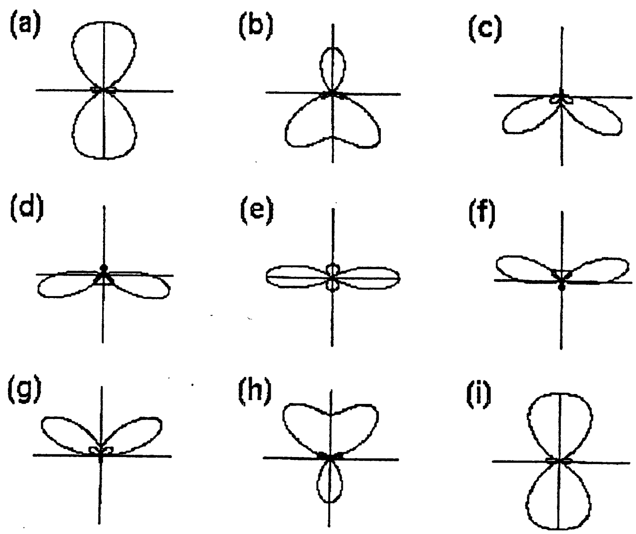


Figure 7-20. Field pattern of a four element phased array, $d = \lambda/2$. (a) $\delta = +4\pi/8$. (b) $\delta = +3\pi/8$. (c) $\delta = +2\pi/8$. (d) $\delta = +1\pi/8$. (e) $\delta = 0$. (f) $\delta = -1\pi/8$. (g) $\delta = -2\pi/8$. (h) $\delta = -3\pi/8$. (i) $\delta = -4\pi/8$.

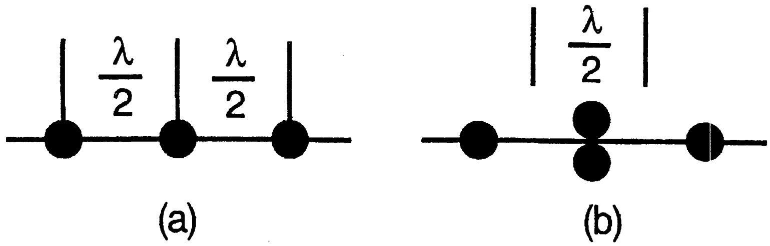


Figure 7-21. (a) A three-element array. (b) Equivalent displaced two-element arrays.

phase even though the physical elements are not changed.

A second way would be to examine the expected behavior if there is a prescribed nonuniform excitation of the elements. For example, let us assume that we have a linear array that consists of three elements that are displaced by a distance $d = \frac{\lambda}{2}$ and each element is excited in phase ($\delta = 0$). The excitation of the center element is twice as large as the outer two elements as shown in Figure 7-21a. The choice of this distribution of excitation amplitudes is based on the fact that 1:2:1 are the leading terms of a binomial series. The resulting array, that could be generalized to include more elements, is called a *binomial array*.

Because of the excitation of the center element being twice the outer two elements, we can consider that this three-element array is equivalent to two two-element arrays that are displaced by a distance $\frac{\lambda}{2}$ from each other. This allows us to make use of (7.69) where $F(\theta, \varphi)$ is interpreted to be the radiation pattern of this new element. We define

$$F(\theta, \varphi) = \cos\left(\frac{\pi}{2}\cos\varphi\right) \tag{7.74}$$

The array factor for these new elements is the same as the radiation pattern of one of the elements. Therefore, from (7.69) we write that the magnitude of the far field radiated electric field from this structure is given by

$$|E| = \frac{2E_0}{\rho}\left|\cos\left(\frac{\pi}{2}\cos\varphi\right)\right|^2 \tag{7.75}$$

The radiation pattern for this array is shown in Figure 7-22. It is contrasted with the two-element array and we note that the radiation pattern of the three-element array with a nonuniform excitation is narrower. We note that in this binomial array that there are no side lobes to absorb power. If more elements are included in the array, the beam

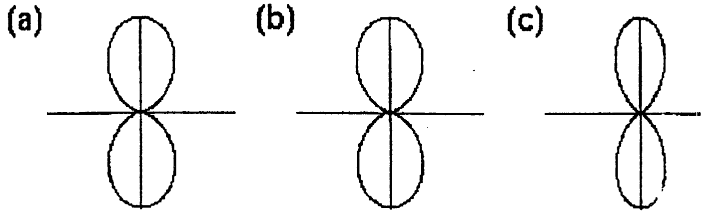
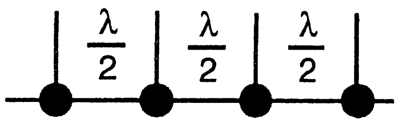


Figure 7-22. Radiation patterns of a two element dipole array and a three element binomial array. (a) Antenna pattern. (b) Array factor. (c) Antenna array.

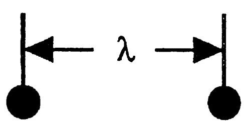
width will become narrower.

In drawing the composite figure for the antenna array that is comprised of two small dipoles that are separated by a half-wavelength, we have *multiplied* the radiation pattern of the individual antenna by the array factor. In this case, the array factor is the same since this is a binomial array. The multiplication is illustrated best by working through an example.

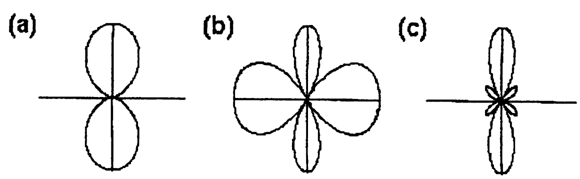
Example 7-10. Using the concept of the *multiplication of patterns*, find the radiation pattern of the array.



This array is to be replaced with two antennas



Answer: The radiation pattern of a pair of two nondirectional radiators separated by $\lambda/2$ has a radiation pattern depicted below in (a). This is the antenna pattern. The radiation pattern of two nondirectional radiators separated by λ and fed in phase is shown in (b). This is the array factor. The resultant pattern of the array is given in (c). We note that the final pattern has a null value at the same angles that *either* the individual antenna or the array factor has a null.



Conclusion

The fundamental radiation characteristics of electromagnetic waves and its relationship to accelerating and decelerating charges have been reviewed. Using these concepts, the small Hertzian dipole radiator was described. A radiation pattern for such an antenna was obtained.

A formal procedure that followed from the concept of the vector potential was introduced and applied to several antennas. We focused our attention to the examination of the far field properties of antennas. If the media in which the waves are propagating is linear, the principle of superposition applies. Constructive and destructive interference between fields radiated by displaced antenna elements with differing phases in the applied currents led to different radiation or reception characteristic of an antenna. Terms such as lobes, radiation resistance, beam width, directive gain, directivity, and effective area were defined and applied.

The solution that we have obtained for the radiation patterns was predicated on a valid approximation for the current distribution on the antenna. Several distributions were analyzed. The *method of moments* which was introduced in Chapter 3 can be equally well applied to antenna calculations. In this case, the field distribution is known and the current distribution becomes the unknown term that must be ascertained.

The subject of an antenna array that consisted of several identical antennas was introduced. By controlling either the phase or the amplitude of the signal that was applied to each individual antenna or its spatial separation, we found that the resulting radiation pattern could be changed. We found that predictions of the radiation pattern could be found by multiplying the radiation pattern of an individual antenna times an array factor in order to find the radiation pattern of the entire antenna array.

Problems

1. Perform the integration of the integral

$$\int_0^R \frac{\hat{R}^3}{\left(\hat{R}^2 + x^2\right)^3} d\hat{R}$$

that arises in Example 7-1.

2. Using dimensional arguments, show that the term

$$\frac{Q^2}{32\pi\varepsilon_0} \frac{1}{x_0} \quad (x_0 > 0)$$

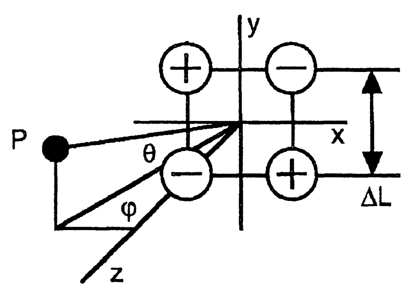
corresponds to the electrostatic energy stored in the region $x > x_0$ in Example 7-1.

3. Show that (7.9) follows from (7.8) using Maxwell's equations if a time dependence is assumed.

4. Show that (7.8) and (7.9) are related by the characteristic impedance of free space.

5. Let both oscillating charges depicted in Figure 7-6 have the same sign. Calculate the radiation pattern from this 'antenna.'

6. Find the far field potential from a 'Hertzian quadripole'. The charges are at the four corners of a square.

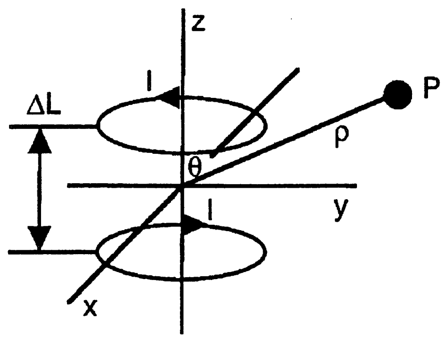


7. Verify that (7.22) and (7.24) are correct.

8. Sketch all of the separate terms of the field quantities in (7.22) and (7.24) as a function of distance ρ assuming each term has a value of 1 at ρ = 1. Identify the 'near' field and the 'far' field.

9. Explicitly evaluate all of the terms that are computed from Poynting's theorem from (7.22) and (7.24). Identify each of the terms.

10. Calculate the far fields at point P created by an antenna consisting of two magnetic dipoles in juxtaposition but separated by a distance L. The currents in the dipoles have a 180° phase difference.



11. An AM radio station of 1 MHz frequency uses an antenna 20 m long that is placed well above the ground.

(a). What is the radiation resistance of the antenna?

(b) If the station is to be operated at 50 kW power, what rms current should be applied to the antenna?

12. Find the far field radiation pattern from a thin rod of length $2h$ if the current on the rod can be described with the distribution

$$I(z)=\begin{cases} I_m & \text{for } -h\le z\le +h \\ 0 & \text{for } \quad z>|h| \end{cases}$$

Sketch this radiation pattern in the limit of $h/l << 1$.

13. Find the far field radiation pattern from a thin rod of length $2h$ if the current on the rod can be described with the distribution

$$I(z)=\begin{cases} I_m\left(1-\frac{|z|}{h}\right. & \left.\text{for } -h\le z\le +h\right) \\ 0 & \text{for } \quad z>|h| \end{cases}$$

Sketch this radiation pattern in the limit of $h/\lambda << 1$.

14. Find the far field radiation pattern from a thin rod of length $4h$ if the current on the rod can be described with the distribution

$$I(z)=\begin{cases} I_m\left(1-\frac{|z|}{h}\right) & \text{for } -2h\le z\le +2h \\ 0 & \text{for } \quad z>2|h| \end{cases}$$

Sketch this radiation pattern in the limit of $h/\lambda << 1$.

15. Calculate the radiation resistance of a magnetic dipole.

16. Calculate the radiation resistance for the antenna described in problem 12.

17. Calculate the radiation resistance for the antenna described in problem 13.

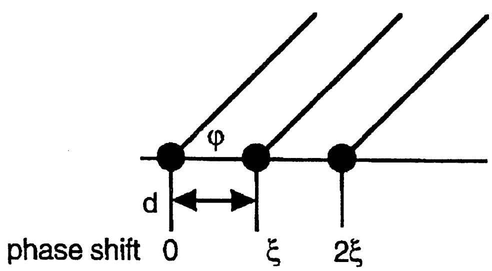
18. Calculate the directive gain $\mathcal{G}$ and the beam width of a magnetic dipole.

19. Calculate the directive gain $\mathcal{G}$ and the beam width of an antenna having the current distribution stated in problem 12.

20. Calculate the directive gain $\mathcal{G}$ and the beam width of an antenna having the current distribution stated in problem 13.

21. Find the effective area for a lossless half-wave dipole.

22. Assume that there are three identical antennas equally spaced along a straight line. Each antenna is fed with the same current but there is a uniform progressive phase shift along the line.



Find the array factor.

23. Show that the radiation pattern with $\delta = 0°$ in Figure 7-19 can be obtained using multiplication of patterns.

24. Show that an antenna with more elements that are equally spaced can produce a narrower main lobe. Assume that each element is fed in phase