3

MATHEMATICAL AND NUMERICAL TECHNIQUES

In the previous chapter, we learned that a static electric field would be created from a charge distribution. In addition, it was possible to determine this static electric field from a scalar potential. We also showed there that the potential V could also be obtained directly in terms of the charge distribution ρ_v via one of two partial differential equations: *Poisson's equation.*

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \tag{3.1}$$

or *Laplace's equation*

$$\nabla^2 V = 0 \tag{3.2}$$

if the charge density in the region of interest were equal to zero. In writing (3.1) or (3.2), we introduced the operator ∇^2 and called it a *Laplacian operator.* The Laplacian operator depends on the coordinate system that is chosen for a calculation and in Cartesian coordinates is written as

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{3.3}$$

The general procedure of solving (3.1) and (3.2) for the cases where the potential V depended on only one spatial coordinate was given there. In this chapter, we will introduce analytical and numerical techniques that will allow us to examine more complicated problems. For mathematical simplicity, however, only problems that can be written in terms of Cartesian coordinates will be examined.

We also introduce numerical techniques in this chapter and make extensive use of MATLAB in the process. Several MATLAB programs (dot m files) that have been used in this book are included in this chapter and in Appendix F for the reader's benefit. These programs can be easily altered and customized by the user. They can also be

translated into the reader's language of choice. They may also not be THE program for a particular task but they do work.

Introduction to separation of variables

The study of systems that could be modeled with a differential equation that had only one independent variable (say a distance x) has led to some important results that have practical significance. There are cases, in fact probably a majority of cases, where it is not justified to approximate a physical system with an equation that depends on only one independent variable. A simple example would be the calculation of the steady state temperature profile on a thin rectangular metal plate if a blowtorch were aimed at one corner of the plate. In this section, we will introduce the methodical procedure to effect the solution of this type of problem that is governed by Poisson's or Laplace's equation. The technique that will be employed to obtain this solution is the powerful *method of separation of variables.*

It is pedagogically convenient to introduce the technique with an example and then carefully work through the details. The problem that we will initially examine is to calculate the potential distribution within a charge-free region illustrated in Figure 3-1 where the potential is prescribed on all four edges. In our example, we will specify that the potential on two of the edges is equal to zero, approaches zero on the third edge, that is taken to be at $y \Rightarrow \infty$, and the potential has a particular distribution on the fourth edge.

Since there is no charge within the region, we should solve Laplace's equation (3.2). Since the object has rectangular symmetry and there is no dependence of the potential on the third coordinate z, the form of Laplace's equation in Cartesian coordinates (3.3) that we will use is written as

$$\nabla^2 V \equiv \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0 \tag{3.4}$$

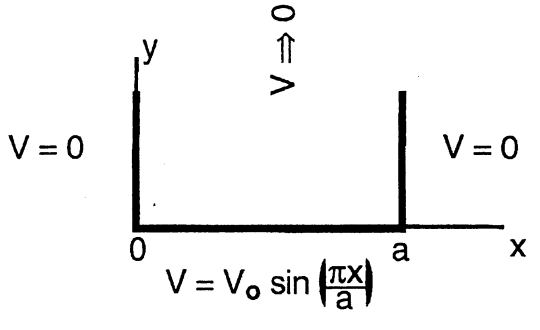


Figure 3-1. A rectangular region in which the potential V on two of the surfaces is specified to be equal to zero, $V \Rightarrow \infty$ as $y \Rightarrow \infty$ and $V = V_0 \sin\frac{\pi x}{a}$ on the fourth side.

In writing (3.4), we have explicitly stated that the potential $V = V(x,y)$ depends on the two independent variables x and y.

The philosophy of solving this equation by employing the method of separation of variables is to assert that the potential $V(x, y)$ is equal to the product of two terms $X(x)$ and $Y(y)$ that separately are functions of only one of the independent variables x and y each, that is

$$V(x, y) = X(x)\ Y(y) \tag{3.5}$$

This is a critical assertion and our solution depends on it being a correct assumption. We may wonder if other functional forms would work at this stage. They might or they might not. The resulting solutions that would be obtained using different combinations might physically not make any sense or they might not satisfy the boundary conditions. Therefore, we will follow in the footsteps of those pioneering giants who have led us through the dark forest containing problems of this genre in the past and just use (3.5) and not concern ourselves with these questions. "If it works, why fix it?" will be our motto.

Substitute (3.5) into (3.4) and write

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} = 0 \tag{3.6}$$

Note that the terms that are to be differentiated only involve one independent variable. Hence the partial derivatives can be replaced with ordinary derivatives and this will be done in the subsequent development.

The next step in this methodical procedure is to divide both sides of this equation by $V(x, y) = X(x)\ Y(y)$. Our friends in mathematics may stand up in horror at this suggestion! As we will later see, one of these terms could be zero at one or more points in space. Recall what a calculator or computer tells us when we do this 'evil' deed of dividing by zero. With this warning in hand and with a justified amount of trepidation, let us see what does result from this action. In our case, the end will justify the means. We find that

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0 \tag{3.7}$$

The first term on the left side of (3.7) is *independent* of the variable y. As far as the variable y, it can be considered to be a constant that we will take to be $-k_y^2$. Using a similar argument, the second term on the left side of (3.7) is *independent* of the variable

x and it also can be replaced with another constant[1] that will be written as $+k_x^2$. Therefore, (3.7) can be written as two *ordinary differential equations* and one algebraic equation.

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0 \tag{3.8}$$

$$\frac{d^2Y(y)}{dy^2} - k_y^2 Y(y) = 0 \tag{3.9}$$

$$k_x^2 - k_y^2 = 0 \tag{3.10}$$

A pure mathematician would have just written these equations down by inspection in order to avoid any problems with dividing by zero that we have so cavalierly glossed over.

The two second-order ordinary differential equations can be easily solved. We write that

$$X(x) = C_1 \sin k_x x + C_2 \cos k_x x \tag{3.11}$$

$$Y(y) = C_3 \exp(k_y y) + C_4 \exp(-k_y y) \tag{3.12}$$

where we include the constants of integration, C_1 to C_4. Let us now determine these constants of integration from the boundary conditions imposed in Figure 3-1. From (3.5), we note that the potential $V(x, y)$ is determined by multiplying the solution $X(x)$ with $Y(y)$. Therefore, we can specify the constants by examining each term separately. For any value of y at $x = 0$, the potential $V(0, y)$ is equal to zero. The only way that we can satisfy this requirement is to let the constant $C_2 = 0$ since cos 0 = 1. Nothing can be stated about the constant C_1 from this particular boundary condition since sin 0 = 0. For any value of x and in the limit of $y \Rightarrow \infty$, the potential $V(x, y \Rightarrow \infty) \Rightarrow 0$. This specifies that the constant $C_3 = 0$ since the term $\exp(k_y y) \Rightarrow \infty$ as $y \Rightarrow \infty$. The constant C_4 remains undetermined from the application of this boundary condition.The potential on the third surface $V(a, y)$ is also specified to be zero at $x = a$ from which we conclude that $k_x = \frac{n\pi}{a}$ since sin $n\pi = 0$. From (3.10), we also write that the constants $k_x = k_y$.

[1] Our choice for the signs of the constants is arbitrary. They have been selected by the author based on his knowledge of what was coming next. Another selection might have led to a later mechanical introduction of an additional factor of $j = \sqrt{-1}$ at some point in the development. It is also common to define $i = \sqrt{-1}$ in some mathematics or physics texts. We will follow the Electrical and Computer Engineering convention of using j.

With these values for the constants, our solution $V(x, y) = X(x)Y(y)$ becomes

$$V(x,y) = [C_1 C_4] \exp\left(-\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \tag{3.13}$$

For this example, the integer n will take the value of $n = 1$ in order to fit the fourth boundary condition at $y = 0$. Finally, the product of the two constants $[C_1 C_4]$ that is just another constant is set equal to V_0. The potential in this channel finally is given by

$$V(x,y) = V_0 \exp\left(-\frac{\pi y}{a}\right) \sin\left(\frac{\pi x}{a}\right) \tag{3.14}$$

The variation of this potential in space is shown in Figure 3-2.

An examination of Figure 3-2 will yield some important physical insight into the variation of the potential. First, the potential V only approaches zero as the coordinate $y \Rightarrow \infty$. Second, the boundary conditions at $x = 0$ and at $x = a$ were that the potential V equaled a constant that in this case was equal to zero. Recall from the previous chapter that $E_y = -\dfrac{\partial V}{\partial y}$. This implies that the component of electric field E_y must also be equal to zero along these two surfaces. We can make the general conclusion that the *tangential* component of electric field adjacent to an equipotential surface will be equal to zero. This conclusion will be of importance in several later calculations.

The procedure that we have conducted is the determination of the solution of a partial differential equation. Let us recapitulate the procedure before attacking a slightly more difficult problem.

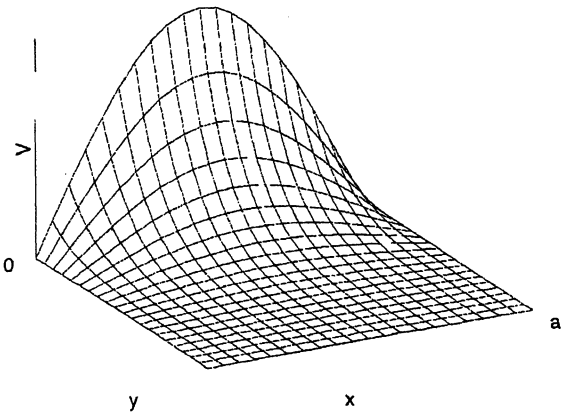


Figure 3-2. Variation of the potential within the region depicted in Figure 3-1.

(1) The proper form of the Laplacian operator ∇^2 for the coordinate system of interest was chosen. This choice was predicated on the symmetry and the boundary conditions of the problem.

(2) The potential $V(x, y)$ that depended on two independent variables was separated into two dependent variables that individually depended on only one of the independent variables. This allowed us to write the partial differential equation as a set of ordinary differential equations and an algebraic equation by assuming that the solution could be considered as a product of functions of the individual independent variables.

(3) Each of the ordinary differential equations was solved that led to several constants of integration. The solution of each ordinary differential equation was multiplied together to obtain the general solution of the partial differential equation.

(4) The arbitrary constants that appeared when the ordinary differential equations were solved were determined such that the boundary conditions would be satisfied. The solution for a particular problem has now been obtained.

Note that this technique is similar to the methodical procedure that we employed in the one dimensional case with the additional step unique to the separation of variables technique included to take care of the higher dimensions.

Let us examine the potential distribution in a bounded space as depicted in Figure 3-3. The procedure will be the same as for the unbounded case treated above. In this case, the potential is required to be equal to zero on three of the surfaces and it has the same potential as examined previously on the remaining edge.

In this case, the solution of Laplace's equation is again given by

$$X(x) = C_1 \sin(k_x x) + C_2 \cos(k_x x) \tag{3.15}$$

$$Y(y) = C_3 \exp(k_y y) + C_4 \exp(-k_y y) \tag{3.16}$$

We will later find that we will have to expand the potential in a Fourier series in order to match the boundary conditions at $x = 0$ and at $x = a$. However, we are able to predict the functional characteristics of the basic *eigenfunction*. This is a German word that means *characteristic* function. The values for k_x and k_y are called *eigenvalues* or characteristic values. In this case again, the eigenvalue $k_x = k_y$ as determined from (3.10). We may also find this function referred to as a proper function.

The constants are specified based on the following boundary conditions. The constant $C_2 = 0$ since the potential $V = 0$ at $x = 0$. The constant $k_x = \frac{n\pi}{a}$ since the potential $V = 0$ at $x = a$. From (3.16), we write

$$C_3 \exp(k_y b) + C_4 \exp(-k_y b) = 0$$

since the potential $V = 0$ at $y = b$. Therefore, the potential within an enclosed region as specified in Figure 3-3 can be written as

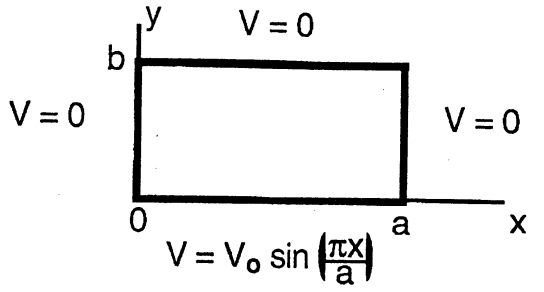


Figure 3-3. A boundary value problem for a bounded surface.

$$V = \left[C_1 C_3 \exp\left(\frac{n\pi b}{a}\right)\right]\left\{\exp\left(\frac{n\pi(y-b)}{a}\right) - \exp\left(-\frac{n\pi(y-b)}{a}\right)\right\}\sin\left(\frac{n\pi x}{a}\right)$$

$$V = \left[2C_1 C_3 \exp\left(\frac{n\pi b}{a}\right)\right]\sinh\left(\frac{n\pi(y-b)}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \tag{3.17}$$

The constants within the square brackets will be determined in the same manner using a Fourier series and the remaining boundary condition at $y = 0$ that we have just used.

The boundary condition at $y = 0$ states that $V = V_0 \sin\left(\frac{\pi x}{a}\right)$ for $0 \leq x \leq a$. Hence, the constant $n = 1$ and

$$C_1 C_3 = \frac{V_0}{2\exp\left(\frac{\pi b}{a}\right)\sinh\left(-\frac{\pi b}{a}\right)} \tag{3.18}$$

The potential is finally written as

$$V = \frac{V_0}{\sinh\left(\frac{-\pi b}{b}\right)}\sinh\left(\frac{\pi(y-b)}{a}\right)\sin\left(\frac{\pi x}{a}\right) \tag{3.19}$$

This is shown in Figure 3-4. Note that we do satisfy the imposed boundary condition that the potential equals zero on three edges.

Fourier series expansion

In the two examples that were treated above, we assumed that the boundary condition

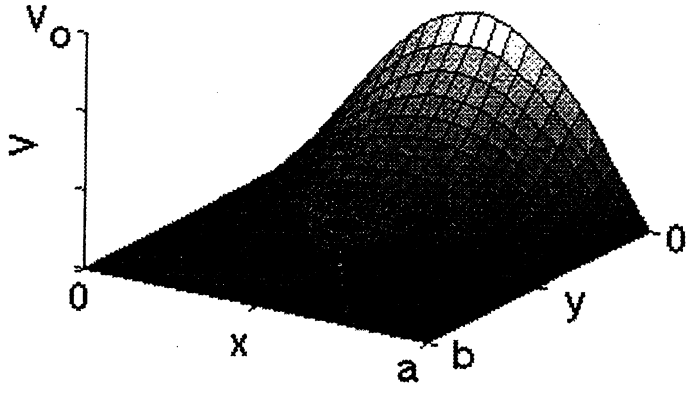


Figure 3-4. Normalized potential profile within the region described in Figure 3-3. Note that the potential is equal to zero on three edges.

at $y = 0$ had a nonuniform distribution (a sinusoidal variation). This was an academic type distribution rather than a realistic one and we were able to 'carry out the details' to the very end without having to introduce more complicated mathematics. However, we should look at the real world where we might expect that a more realistic distribution for the potential at $y = 0$ in Figure 3-1 would be to assume that the potential at $y = 0$ would be a constant, say $V = V_0$. The boundary conditions on the other three edges could remain the same in realistic situations. Let us carry through the details for this particular boundary condition.

Since the other boundary conditions have not been altered, the solution of Laplace's equation with several of the constants specified is given by (3.13) that we write as

$$V = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \tag{3.20}$$

where n is an integer. In writing this relation as a summation of an infinite series of sinusoidal functions, we are being guided by the fact that each term does satisfy the boundary condition that the potential $V = 0$ at $x = 0$ and at $x = a$, hence the infinite sum will also. The coefficients c_n will be determined to yield the best fit of the remaining boundary condition at $y = 0$ that has now been specified to be a constant potential $V = V_0$.

We may recognize (3.20) as the Fourier sine series and the constants c_n as the Fourier coefficients. The coefficients c_n and d_n are defined in general from the relations

$$c_n = \frac{2}{L}\int_0^L \Theta \sin\left(\frac{2n\pi\zeta}{L}\right)d\zeta \quad \text{and} \quad d_n = \frac{2}{L}\int_0^L \Theta\cos\left(\frac{2n\pi\zeta}{L}\right)d\zeta \tag{3.21}$$

where the general Fourier expansion for the periodic function Θ is given as

$$\Theta = \sum_{n=1}^{\infty} c_n \sin\left(\frac{2n\pi\zeta}{L}\right) + \sum_{n=0}^{\infty} d_n \cos\left(\frac{2n\pi\zeta}{L}\right) \tag{3.22}$$

In this general expression, L is defined as the period of the function Θ.

The potential V is specified only in the region $0 \leq x \leq a$ to be a constant. Outside of this region, it is not determined and can have any value that we specify in order to ease our mathematical difficulties.Let us choose it to be periodic in x as shown in Figure 3-5. In this case, the period of the wave is $L = 2a$. Our choice for the potential at the boundary is to assume that it is an *odd* function in the variable x. An odd function is defined as $\Phi(x) = -\Phi(-x)$. An even function is defined as $\Phi(x) = \Phi(-x)$. The sine function is an odd function and a cosine function is an even function.

In order to represent the periodic potential depicted in Figure 3-5, we should employ the series containing just odd functions, hence we will let the coefficients $d_n = 0$. Recall that the integral over a period or a symmetrical interval of an odd function times an even function is equal to zero. For example, the integral of the odd functions sin ξ or ξ

$$\int_{-\pi}^{\pi} \Phi_0 \sin\xi \, d\xi \quad \text{or} \quad \int_{-1}^{1} \Phi_0 \, \xi \, d\xi$$

are both equal to zero. In this example, Φ_0 which is a constant, is an even function and sin ξ and ξ are both odd functions.

Substitute the function defined in Figure 3-5 into (3.21) in order to determine the coefficients c_n

$$c_n = \frac{2}{2a}\left\{\int_0^a V_0 \sin\left(\frac{n\pi\zeta}{a}\right) d\zeta - \int_a^{2a} V_0 \sin\left(\frac{n\pi\zeta}{a}\right) d\zeta\right\}$$

$$c_n = \begin{cases} \dfrac{4V_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{3.23}$$

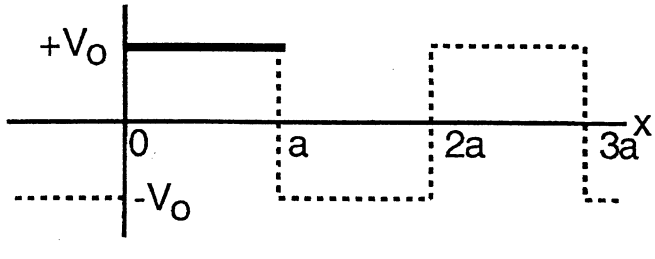


Figure 3-5. Periodic potential represents the constant potential $V = V_0$ within the region $0 \leq x \leq a$.

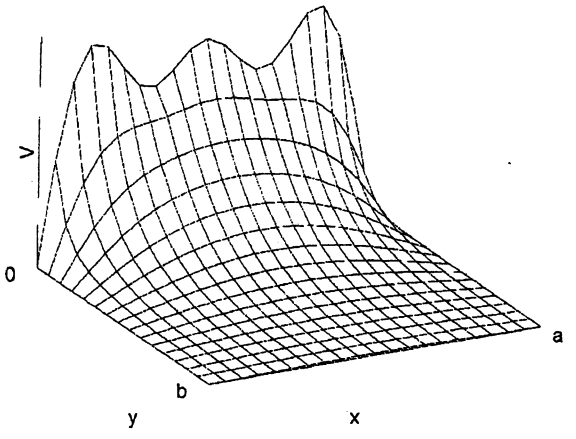


Figure 3-6. The potential variation within the region depicted in Figure 3-5 if the potential V at $y = 0$ is a constant. The sum of the first three Fourier modes is shown.

The potential is finally given by

$$V = \frac{4aV_0}{\pi}\left\{\sin\left(\frac{\pi x}{a}\right)\exp\left(-\frac{\pi y}{a}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{a}\right)\exp\left(-\frac{3\pi y}{a}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{a}\right)\exp\left(-\frac{5\pi y}{a}\right)+\ldots\right\} \tag{3.24}$$

The potential variation of just the first three terms is shown in Figure 3-6.

Certain general comments can be made about the potential variation that is shown in Figure 3-6, especially when it is compared with the potential profile in Figure 3-2 that is just the first Fourier term of the Fourier series. Again, the potential V only approaches zero as the coordinate $y \Rightarrow \infty$. The fit to a constant value at $y = 0$ is better if more modes are included in the expansion. We note that the higher order terms subtract from the lowest order mode at $x \approx \frac{a}{2}$ and they add to the lowest order mode in the regions $x = 0$ and $x = a$. The fit at $x \approx 0$ and at $x \approx a$ will not be possible as the function is double-valued there. For $y > 0$, the potential V is zero at these boundaries. However, for $y = 0$, $V = V_0$. If we had included more terms in the expansion, we would have observed a very rapid oscillation at either edge. This effect is given the name *Gibb's phenomenon* and it is a topic for further consideration in an advanced calculus course.

Example 3-1. Find the Fourier coefficients for the boundary condition specified in Figure 3-1.

Answer: From (3.21), we write

$$c_n = \frac{2}{2a}\int_0^{2a} V_0 \sin\left(\frac{\pi\zeta}{a}\right)\sin\left(\frac{2n\pi\zeta}{2a}\right)d\zeta$$

$$d_n = \frac{2}{2a}\int_0^{2a} V_0 \sin\left(\frac{\pi\zeta}{a}\right)\cos\left(\frac{2n\pi\zeta}{2a}\right)d\zeta$$

The coefficient $c_1 = V_0$ *for* $n = 1$. All other coefficients are equal to zero.

In this section, we found that it was desirable to sum all of the terms in the Fourier series in order to get a valid representation for the potential profile. As a general rule, we can say that the more terms that are included in the summation, the better the representation for the potential. The question then arises, "Is there something unique about each of the terms in the series?" We can answer this question by watching a gymnast jumping on a trampoline. If the gymnast lands in the middle of the trampoline, the perturbation in the canvas will be different from what it would be if the landing were at a point that is away from the center or if two gymnasts were doing their thing in tandem. There are different *modes* for the oscillation. The mathematical structure of the solution for an equation describing the motion of the canvas for all possible landing points is a solution that involves finding all of the *Fourier modes.*

Numerical integration

In order to illustrate the procedure, we will find the potential variation at various distances along the z axis from a finite sized charged sheet. This is shown in Figure 3-7. The potential from an arbitrary body of charge was obtained in the previous chapter and we rewrite it below.

$$V(x,y,z) = \int_{\Delta V} \frac{\rho_V(x',y',z')}{4\pi\varepsilon_0 R} dx'\,dy'\,dz' \tag{3.25}$$

where

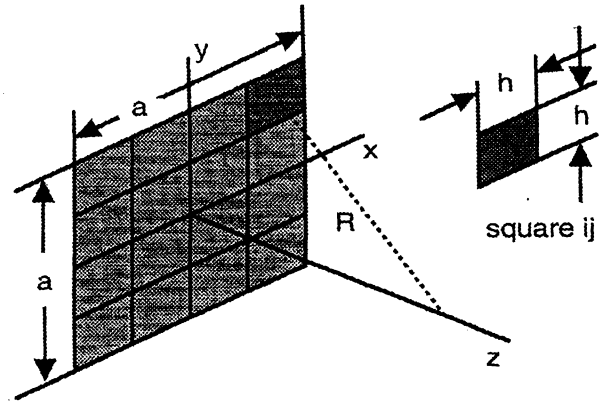


Figure 3-7. A finite size sheet of charge.

$$R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{3.26}$$

The primed variables refer to the location of the charge. We will let the charge be located in a thin sheet at $z'= 0$. We will also let the sheet of charge be square in shape and have it centered on the coordinate system. The charge density will be uniformly distributed and have a value ρ_s. With these assumptions, (3.25) becomes

$$V(x,y,z)=\frac{\rho_s}{4\pi\varepsilon_0}\int_{-a/2}^{a/2}dx'\int_{-a/2}^{a/2}dy'\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z)^2}} \tag{3.27}$$

Since the potential is to be determined along the z axis, this simplifies to

$$V(0,0,z)=\frac{\rho_s}{4\pi\varepsilon_0}\int_{-a/2}^{a/2}dx'\int_{-a/2}^{a/2}dy'\frac{1}{\sqrt{(x')^2+(y')^2+(z)^2}} \tag{3.28}$$

As defined by the charge distribution, there are two degrees of symmetry in this problem. They are in the x' and the y' coordinates. Hence we have to evaluate the integral over only 1/4 of the entire surface and the just multiply the result by 4. Hence

$$V(0,0,z)=4\frac{\rho_s}{4\pi\varepsilon_0}\int_{0}^{a/2}dx'\int_{0}^{a/2}dy'\frac{1}{\sqrt{(x')^2+(y')^2+(z)^2}} \tag{3.29}$$

In order to numerically evaluate the integral (3.29), we subdivide the entire plane in the region $\{0 < x' < a/2,\ 0 < y' < a/2\}$ into small squares whose dimensions are $h \times h$ as shown in the inset. The maximum size that the square can be is given by $h = a/2$. If there are N^2 squares in the quadrant, we have

$$h=\frac{a}{2N} \tag{3.30}$$

The charge ΔQ that is in each square is given by

$$\Delta Q = \rho_s h^2$$

since the charge is assumed to be uniformly distributed on the surface. The incremental potential ΔV_{ij} due to charge in the small square that is identified with the labels 'ij' is given by

$$\Delta V_{ij} = \frac{1}{4\pi\varepsilon_0} \frac{\Delta Q}{\sqrt{\left(ih - \frac{h}{2}\right)^2 + \left(jh - \frac{h}{2}\right)^2 + z^2}} \tag{3.31}$$

where $\left(ih - \frac{h}{2}, jh - \frac{h}{2}\right)$ defines the center point of a charge square.

All that we need do now is use superposition and sum the incremental potentials due to each incremental charge

$$V = 4\sum_{i=1}^{N}\sum_{j=1}^{N} \Delta V_{ij}$$

$$V = 4\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\rho_s h^2}{\sqrt{\left(ih - \frac{h}{2}\right)^2 + \left(jh - \frac{h}{2}\right)^2 + z^2}} \tag{3.32}$$

Hence, the double integral in (3.28) has been converted into a double summation The summation can be simplified by combining (3.30) and (3.32) to yield

$$V = 4\frac{a}{2N}\frac{\rho_s}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{\sqrt{\left(i - \frac{1}{2}\right)^2 + \left(j - \frac{1}{2}\right)^2 + \left(\frac{2Nz}{a}\right)^2}} \tag{3.33}$$

There is the integer N that specifies the number of small squares that are assumed to comprise the large plane. This value is determined by the user's compromising a desire of accuracy against computer time. Note the term $(2Nz/a)$ that appears in the denominator of (3.33). This states that the distance z from the plane of charge is scaled by the size of the plane.

Example 3-2. Evaluate the potential at $z = a$ as the number of divisions N is increased from 1 to 4. The accuracy should be written to four decimal places. The figure shows the quadrant that is to be evaluated using (3.33).

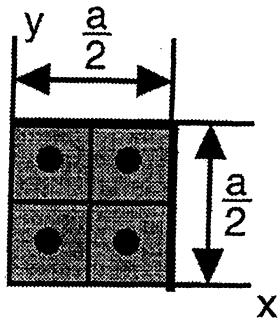
Answer: For $N = 1$, there is one square in the quadrant and only one term in the summation. We write (3.33) as

$$V = 4\frac{a}{2}\frac{\rho_s}{4\pi\varepsilon_0}\frac{1}{\sqrt{\left(1 - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{2}\right)^2 + (2)^2}} = \frac{\rho_s}{4\pi\varepsilon_0}(.9428)a$$

For $N = 4$, there will be 16 terms. We explicitly write out each of these terms in order to show the details that must actually be performed by the computer. From (3.33), the summation over j is performed first.

$$V = \frac{a}{2}\frac{\rho_s}{4\pi\varepsilon_0}\left\{\begin{array}{l}
\dfrac{1}{\sqrt{\left(1-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(1-\frac{1}{2}\right)^2+\left(2-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(1-\frac{1}{2}\right)^2+\left(3-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(1-\frac{1}{2}\right)^2+\left(4-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(2-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(2-\frac{1}{2}\right)^2+\left(2-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(2-\frac{1}{2}\right)^2+\left(3-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(2-\frac{1}{2}\right)^2+\left(4-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(3-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(3-\frac{1}{2}\right)^2+\left(2-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(3-\frac{1}{2}\right)^2+\left(3-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(3-\frac{1}{2}\right)^2+\left(4-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(4-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(4-\frac{1}{2}\right)^2+\left(2-\frac{1}{2}\right)^2+(8)^2}} \\
+\dfrac{1}{\sqrt{\left(4-\frac{1}{2}\right)^2+\left(3-\frac{1}{2}\right)^2+(8)^2}}+\dfrac{1}{\sqrt{\left(4-\frac{1}{2}\right)^2+\left(4-\frac{1}{2}\right)^2+(8)^2}}
\end{array}\right\}$$

$$V = \frac{\rho_s}{4\pi\varepsilon_0}(.9295)a$$



If we continue the process of subdividing the square, we compute the tabulated values for the coefficients. The MATLAB program is given in Example 3-3.

Example 3-3. Evaluate (3.33) at $z = a$ using MATLAB.
Answer: Since we may wish to evaluate this sum several times for different values for the maximum number of values of n [called 'nmax'], we write this as a .m file.

```
clear
s=square(nmax)
% This .mfile allows us to evaluate the potential in (3.33).
    nmax=input('What is the value of nmax?: ');
    for n=1:nmax
        s=0;
        for j=1:n
            for i=1:n
                s=s+(2/n)/((i-.5).^2+(j-.5).^2+(2*n).^2).^.5;
            end
        end
        n=4•n-1;
    end
```

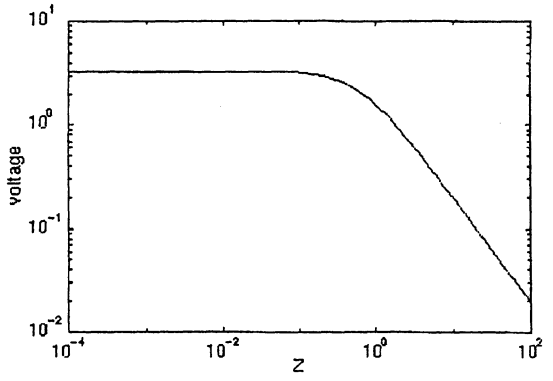
# of subsquares N	# of iterations	coefficient
1	1	0.9428
2	4	0.9320
4	16	0.9295
8	64	0.9288
16	256	0.9287
32	1024	0.9286

Example 3-4. Evaluate and sketch the potential along the z axis that is perpendicular to the center of a uniformly charged plane. Choose the value of $nmax = 4$.

Answer: Using MATLAB, we write a .m file. At distances close to the plane, the voltage

does not change with position. As the distance from the finite plane increases, the voltage decays linearly with distance. The finite plane acts as a point charge. This is in agreement with the results in Chapter 2.

```
clear
s=s(z,s);
clf
z=logspace(-4,2);
a=size(z);
for sz=1:a(2)
        nmax=4;
        for n=1:nmax
                s(sz)=0.0;
                for j=1:n
                        for i=1:n
                                s(sz)=s(sz)+(2/n)/((i-.5)^2+(j-.5)^2+(z(sz)*n)^2)^.5;
                        end
                end
                n=4*n-1;
        end
end
loglog(z,s)
xlabel('Z')
ylabel('voltage')
```



Finite difference equations

We have found in the previous chapter that the electric potential could be determined from a solution of differential equations called Laplace's or Poisson's equations. Analytical solutions can be obtained in several cases although not in full generality. Herein, we will show that these differential equations can be converted into finite difference equations that can be solved on the computer. There are software packages that are available that have been written to effect such calculations. These packages contain very efficient

algorithms. Our goal here is to obtain a flavor of what lies behind the program. With our choice of examples, we essentially will crack a walnut with a sledge hammer.

One dimensional problems

The first technique that will be examined is the numerical solution of Laplace's and Poisson's equation in one dimension. The first order derivatives can be derived with reference to Figure 3-8. We have three possibilities from which to choose. They are

$$\left.\frac{dV}{dx}\right|_{x_0} \approx \frac{V_3 - V_2}{h} \quad \text{(Forward difference method)} \tag{3.34}$$

$$\left.\frac{dV}{dx}\right|_{x_0} \approx \frac{V_2 - V_1}{h} \quad \text{(Backward difference method)} \tag{3.35}$$

$$\left.\frac{dV}{dx}\right|_{x_0} \approx \frac{V_3 - V_1}{2h} \quad \text{(Central difference method)} \tag{3.36}$$

We can interpret the Central difference method as an average between the other two methods. In order to ascertain which method is better, we must estimate the errors that might be expected to be found in each case.The errors can be estimated by expanding the voltage in a Taylor series about the point x_0.

$$V(x_0 + h) = V(h_0) + \frac{h}{1!}\left.\frac{dV}{dx}\right|_{x_0} + \frac{h^2}{2!}\left.\frac{d^2V}{dx^2}\right|_{x_0} + \frac{h^3}{3!}\left.\frac{d^3V}{dx^3}\right|_{x_0} + \text{higher order terms} \tag{3.37}$$

If we neglect the third derivative and higher order terms, we write

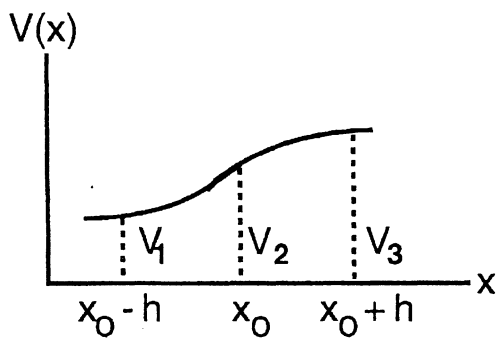


Figure 3-8. Voltage as a function of position. The finite difference equations will be derived with reference to this figure

$$\left.\frac{dV}{dx}\right|_{x_0} = \frac{V(x_0+h)-V(x_0)}{h} - \left.\frac{h}{2}\frac{d^2V}{dx^2}\right|_{x_0} \tag{3.38}$$

A comparison of (3.38) with (3.34) shows that this is equivalent to the Forward difference method with the exception of the error term

$$\left.\frac{h}{2}\frac{d^2V}{dx^2}\right|_{x_0} \tag{3.39}$$

There are, of course, additional higher order terms that must be included in this error term but they will be multiplied by h to a higher order power. If the parameter h can be made sufficiently small, (3.34) could be useful.

In a similar manner, we write the Taylor series expansion $V(x_0 - h)$ about the point

$$V(x_0-h) = V(x_0) - \left.\frac{h}{1!}\frac{dV}{dx}\right|_{x_0} + \left.\frac{h^2}{2!}\frac{d^2V}{dx^2}\right|_{x_0} - \left.\frac{h^3}{3!}\frac{d^3V}{dx^3}\right|_{x_0} + \text{higher order terms} \tag{3.40}$$

From (3.40), we compute

$$\left.\frac{dV}{dx}\right|_{x_0} = \frac{V(x_0)-V(x_0-h))}{h} + \left.\frac{h}{2}\frac{d^2V}{dx^2}\right|_{x_0} \tag{3.41}$$

for the Backward difference method. The error term is also given by (3.39) plus higher order terms.

Subtracting (3.40) from (3.37) yields

$$V(x_0+h)-V(x_0-h) \approx 2h\left.\frac{dV}{dx}\right|_{x_0} + 2\left.\frac{h^3}{6}\frac{d^3V}{dx^3}\right|_{x_0} \tag{3.42}$$

$$\left.\frac{dV}{dx}\right|_{x_0} = \frac{V(x_0+h)-V(x_0-h)}{2h} - \left.\frac{h^2}{6}\frac{d^3V}{dx^3}\right|_{x_0} \tag{3.43}$$

The error in the Central difference method is of the order of h^2. Hence, the error in using this method will be smaller than either of the other two methods and it will be the one employed throughout the rest of the chapter.

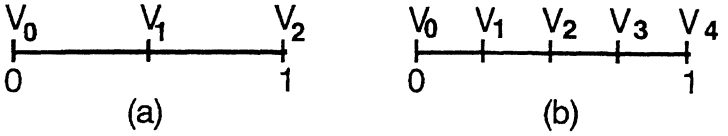
Using the Central difference method, we find the representation for the second derivative to be

$$\left.\frac{d^2V}{dx^2}\right|_{x_0} = \frac{\left.\frac{dV}{dx}\right|_{x_0+\frac{h}{2}} - \left.\frac{dV}{dx}\right|_{x_0-\frac{h}{2}}}{h} = \frac{\frac{V_3-V_2}{h} - \frac{V_2-V_1}{h}}{h}$$

or

$$\left.\frac{d^2V}{dx^2}\right|_{x_0} = \frac{V_3+V_1-2V_2}{h^2} \tag{3.44}$$

Example 3-5. Find the potential distribution between two surfaces if $V(x = 0) = 0$ and $V(x = 1) = 3$. There is no charge distribution in the space $0 \leq x \leq 1$.



Answer: Using the Central difference method, we write Laplace's equation as

$$\frac{d^2V_1}{dx^2} = \frac{V_0+V_2-2V_1}{\left(\frac{1}{2}\right)^2} = 0$$

for the first iteration depicted in (a) The boundary conditions imply $V_0 = V(x = 0) = 0$ and $V_2 = V(x = 1) = 3$. Hence

$$V_0+V_2-2V_1 = 0+3-2V_1 = 0 \Rightarrow V_1 = \frac{3}{2}$$

The second iteration depicted in (b) with the boundary conditions $V_0 = V(x = 0) = 0$ and $V_4 = V(x = 1) = 3$ leads to

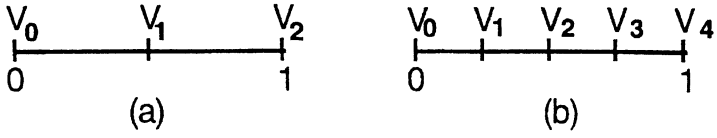
$$\frac{V_0+V_2-2V_1}{\left(\frac{1}{4}\right)^2} = 0;\ \frac{V_1+V_3-2V_2}{\left(\frac{1}{4}\right)^2} = 0;\ \frac{V_2+V_4-2V_3}{\left(\frac{1}{4}\right)^2} = 0$$

Inserting the boundary conditions, we are left with three simultaneous equations to solve. Note that V_2 is known from the previous iteration to be equal to 3/2.

The solution for the three intermediate points and the two end points are

$$V_0 = 0;\ V_1 = \frac{3}{4};\ V_2 = \frac{6}{4};\ V_3 = \frac{9}{4};\ V_4 = 3$$

Example 3-6. Repeat Example 3-5 with a uniform charge distribution $\rho_v(j) = -4\varepsilon_0$ in the space $0 \le x \le 1$. Find the potential distribution between two surfaces if $V(x = 0) = 0$ and $V(x = 1) = 3$.



Answer: Using the Central difference method, we write Poisson's equation as

$$\frac{d^2V_1}{dx^2} = \frac{V_0 + V_2 - 2V_1}{\left(\frac{1}{2}\right)^2} = 4$$

for the first iteration depicted in (a). The boundary conditions imply $V_0 = V(x = 0) = 0$ and $V_2 = V(x = 1) = 3$. Hence

$$\frac{V_0 + V_2 - 2V_1}{\left(\frac{1}{2}\right)^2} = \frac{0 + 3 - 2V_1}{\left(\frac{1}{2}\right)^2} = 4 \Rightarrow V_1 = 1$$

The second iteration depicted in (b) with the boundary conditions $V_0 = V(x = 0) = 0$ and $V_4 = V(x = 1) = 3$ leads to

$$\frac{V_0 + V_2 - 2V_1}{\left(\frac{1}{4}\right)^2} = 4;\ \frac{V_1 + V_3 - 2V_2}{\left(\frac{1}{4}\right)^2} = 4;\ \frac{V_2 + V_4 - 2V_3}{\left(\frac{1}{4}\right)^2} = 4$$

The solution for the three intermediate points and the two end points are

$$V_0 = 0;\ V_1 = \frac{3}{8};\ V_2 = 1;\ V_3 = \frac{15}{8};\ V_4 = 3$$

The term V_2 equals 1 from the previous iteration.

Example 3-7. Using MATLAB, write a program that will yield the results that were obtained in Example 3-6.

Answer: We write a .m file.

```
clear
%specify array size and boundary conditions
      sz=input('ARRAY SIZE...');
      md=(sz+1)/2;
%specify charge density
      rho=input('Charge density ...');
```

```
%specify boundary conditions
      lb=input('Left boundary ...');
      rb=input('Right boundary ...');
      v(1)=lb;
      v(sz+2)=rb;
%set all cells to zero
      for i=2:sz+1
            v(i)=NaN;
      end
      w=0;
      while w==0
            for i=2:sz+1
            if isnan(v(i))
                  w=0;
            end
      end
      for h=md:-1:1
            for i=2:sz+1
                  if (i+h)<=(sz+2) & (i-h)>=1
                        if isnan(v(i)) & ~isnan(v(i-h)) &~isnan(v(i+h))
                              v(i)=(v(i-h)+v(i+h)+rho*(h/(sz+1))^2)/2.0
                        end
                  end
            end
      end
end
%-----------------------
% If the results are to be plotted on a graph, a semicolon must follow the line
%                              v(i)=(v(i-h)+v(i+h)+rho*(h/(sz+1))^2)/2.0
% and the following line must be added to the program
%-----------------------
%clf
%plot(1:sz+2,v(1:sz+2))
%xlabel('Z')
%ylabel('voltage')
%-----------------------
      v =
            0          NaN        1          NaN        3
      v =
            0          0.3750     1.0000     NaN        3.0000
      v =
            0          0.3750     1.0000     1.8750     3.0000
      >>
```

A critical restriction on the mesh size is that the first point must be in the center. This point will be evaluated from the two boundaries. This will restrict the number of internal

points N to contain the following number of points.

$$1; 3; 7; 15; 31; 63; ...[2^N - 1]$$

Let us call this the array size.

We may have noted that the length 'h' that appears in our application of (3.44) has changed from 1/2 to 1/4. In the next step, it will be reduced to 1/8 and then 1/16 and so on. We can also evaluate (3.44) and keep h as a prescribed value but as we will see, the calculation will have to be repeated several times. The numbers will converge, hopefully in a reasonable time, to the correct answer.

In order to introduce the procedure, we will redo the calculation of the one-dimensional Laplace's equation that we have just performed but now make the *a priori* assumption that $h = 1/4$. We set the three values internal to the fixed boundaries as initially being equal to zero.Hence, we write

$$V_1(0) = 0, V_2(0) = 0, V_3(0) = 0$$

The boundary values will remain fixed at all iterations, namely $V_0 = 0$ and $V_4 = 3$.

In our first iteration denoted with the '(1)' using (3.44), we write

$$\frac{V_0 + V_2(0) - 2V_1(1)}{\left(\frac{1}{4}\right)^2} = 0; \quad \frac{V_1(1) + V_3(0) - 2V_2(1)}{\left(\frac{1}{4}\right)^2} = 0; \quad \frac{V_2(1) + V_4 - 2V_3(1)}{\left(\frac{1}{4}\right)^2} = 0$$

In the second equation, we include the value for $V_1(1)$ that had just been obtained from the previous equation since it is now known. A similar argument holds for the third equation. In fact, this will be a general pattern. The simultaneous solution of this set of equations leads to

$$V_1(1) = 0, V_2(1) = 0, V_3(1) = 1.5$$

In order to compute the values at the second iteration, we use the values from the first iteration and sequentially write

$$\frac{V_0 + V_2(1) - 2V_1(2)}{\left(\frac{1}{4}\right)^2} = 0; \quad \frac{V_1(2) + V_3(1) - 2V_2(2)}{\left(\frac{1}{4}\right)^2} = 0; \quad \frac{V_2(2) + V_4 - 2V_3(2)}{\left(\frac{1}{4}\right)^2} = 0$$

From this set, we compute

$$V_1(2) = 0,\ V_2(2) = 0.75,\ V_3(2) = 1.875$$

The third iteration is

$$\frac{V_0 + V_2(2) - 2V_1(3)}{\left(\frac{1}{4}\right)^2} = 0;\quad \frac{V_1(3) + V_3(2) - 2V_2(3)}{\left(\frac{1}{4}\right)^2} = 0;\quad \frac{V_2(3) + V_4 - 2V_3(3)}{\left(\frac{1}{4}\right)^2} = 0$$

We obtain

$$V_1(3) = 0.375,\ V_2(3) = 1.125,\ V_3(3) = 2.0625$$

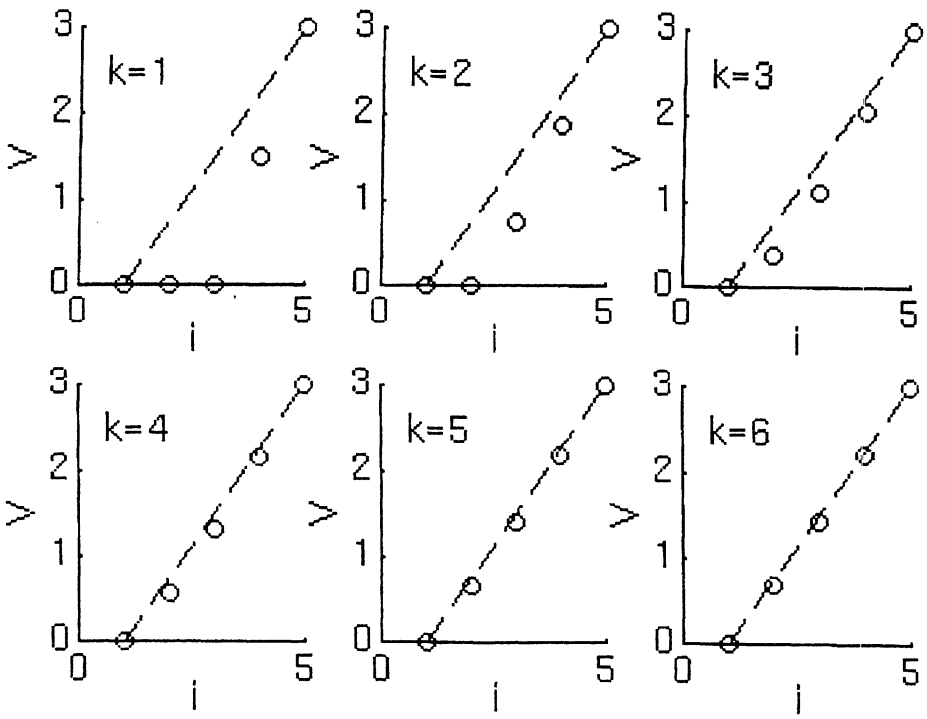
We could keep going using our calculator, but let us stop here.The numbers seem to be approaching an asymptotic limit. We will write a MATLAB program to do this work below. A more interesting question arises at this point. Is the answer correct? We can easily check this by dividing the parameter 'h' by two and redoing the calculation. If the numbers are the same, we are finished.

Example 3-8. Write a MATLAB program to evaluate and plot the first six iterations of the solution of the one dimensional Laplace's equation.The boundary conditions are: $V(1) = 0$ *and* $V(5) = 3$.

Answer: We write the MATLAB program. The iterations are indicated with the integer k. The analytical solution $V(x) = [(3/4) \bullet (x - 1)]$. is shown with a dashed line. Note that we include the iteration number k in the final figures.

```
clear
clf
lb=0;rb=3;km=6;nx=4;
% specify boundary conditions
      v(1)=lb;
      v(nx+1)=rb;
% set all internal cells to zero
      zeros(size(2:nx));
s='k=k';
for k=1:km
      subplot(2,3,k)
      for i=2:nx
            v(i)=(v(i+1)+v(i-1))/2;
      end
      for i=1:nx+1
            hold on
            plot(i,v(i),'o')'hold off'
      end
```

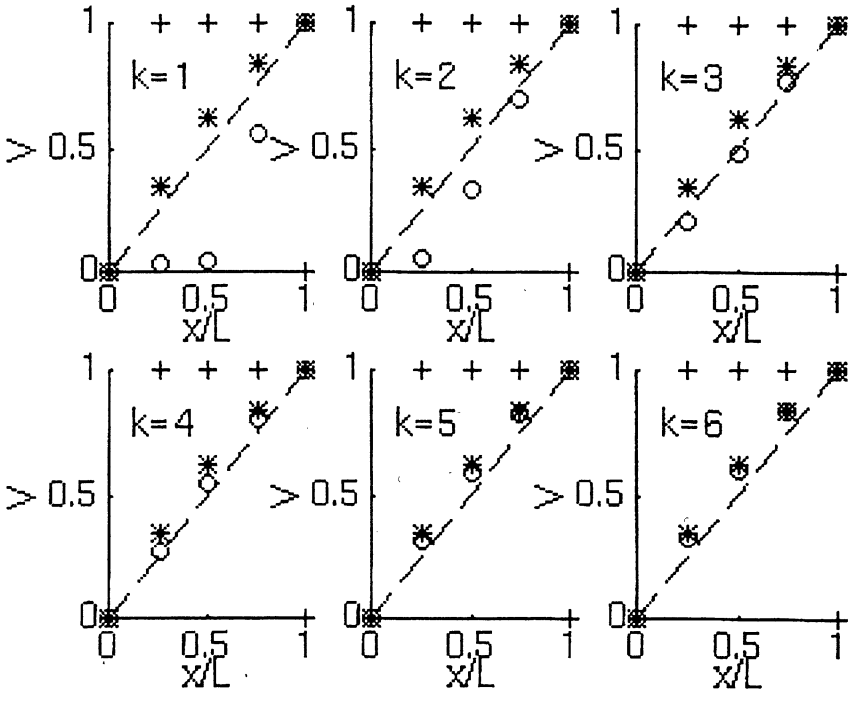
```
        %label graphs
            s(3)=num2str(k);
            text(.5,2.5,s)
            xlabel('i')
            ylabel('V')
        %draw analytical solution
            a=[1 1+nx];
            b=[0 3];
            plot(a,b,'linestyle','--')
            hold off
end
```



Example 3-9. Write a MATLAB program to evaluate and plot the first six iterations of the solution of the one dimensional Poisson's equation.The charge density $\rho_v = -\varepsilon_0$.The boundary conditions are: $V(1) = 0$ and $V(5) = 1$.

Answer: We write the MATLAB program. The iterations are indicated with the integer k and the computed solutions are shown with a o. The charge is indicated with $a+$. Points from the analytical solution $V(x) = [(3/4)\cdot(x-1)]$ are indicated with $a*$. The potential profile in a vacuum is shown with a dashed line. Note that we include the iteration number k in the final figures.

```
clear
clf
L=1;lb=0;rb=1; nx=4; km=6;
LL=L/nx;
%charge density distribution
      for i=1:nx+1
            rho(i)=+1;
      end
      rho(1)=0;
      rho(nx+1)=0;
% specify boundary conditions
      v(1)=lb;
      v(nx+1)=rb;

% set all internal cells to zero
      zeros(size(2:nx));
s='k=k';
for k=1:km
      subplot(2,3,k)
      for i=2:nx
          v(i)=(v(i+1)+v(i-1))/2+((LL^2)*rho(i))/2;
      end
      for i=1:nx+1
            hold on
            j=(i-1)/(nx);
            plot(j,v(i),'o',j,rho(i),'+')
            hold off
      end
      s(3)=num2str(k);
      text(.1,.8,s)
      xlabel('x/L')
      ylabel('V')
      hold on
for i=1:nx+1
            j=(i-1)/nx;
            u(i)=-((j.^2)/2)+3*j/2;
            plot(j,u(i),'*')
         end
a=[0 1];
b=[0 1];
plot(a,b,'linestyle','--')
      hold off
end
```

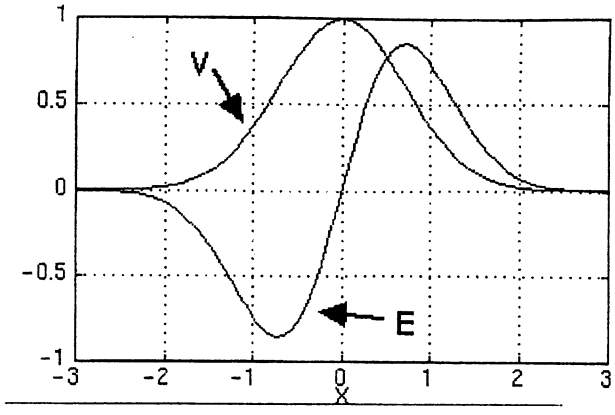


Example 3-10. Using MATLAB, plot the voltage $V = \exp(-x^2)$ and the electric field $E = -dV/dx$ as a function of x in the range $-3 \leq x \leq 3$.

The difference operation 'diff(y)./diff(x)' sequentially performs and stores the values $\{[(V(2)-V(1)]/h,\ [V(3)-V(2)]/h,...[V(n)-V(n-1)]/h\}$. There are only $(n-1)$ values of the derivatives. Therefore, if we wish to plot the results, we plot y in the range x_a to x_b at increments of h and dy/dx in the range $[x_a + h/2]$ to $[x_b - h/2]$ with the same increment h. The grid is added with the 'grid' command.

Answer: Using the 'diff' operator, we write:

```
clf
x=-3:dx:3
V=exp(-x.^2);
hold on
plot(x,V);
dV=diff(V)/dx;
xx=-2.95:dx:2.95
plot(xx,- dV)
grid
xlabel('X')
```



Higher dimensional problems

The finite difference scheme outlined above can be extended to two and to three dimensions as shown in Figure 3-9 to find V_0. The finite difference representation of Laplace's equation in two dimensions is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h^2} = 0 \tag{3.45}$$

We write from (3.45) that

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4) \tag{3.46}$$

The procedure that is to be followed is indicated in Figure 3-9 for two dimensions. The potential V_0 is the *average* of the four adjacent points. For Laplace's equation, the parameter 'h' factors out. However, in Poisson's equation, it will modify the charge

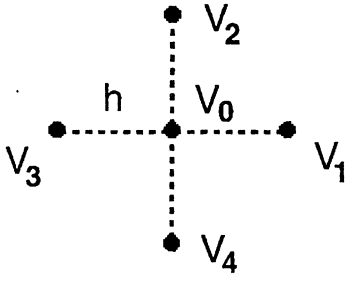


Figure 3-9. A five-point star used in two dimensional difference equations to determine the voltage V_0.

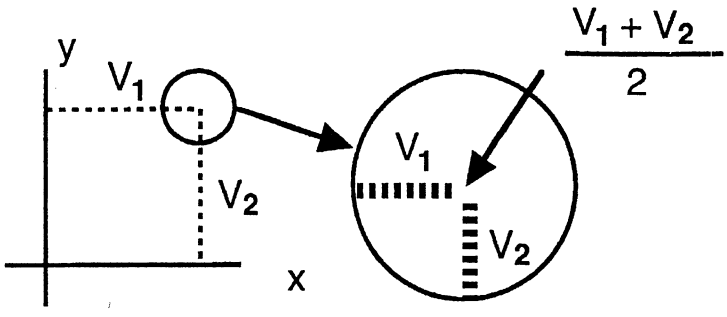


Figure 3-10. The value of the potential at the corners is chosen to be equal to the average of the adjacent sides.

density that is to be evaluated at the point located at the central point. If the potentials at the two surfaces at a corner are different, a convention is to assume that the potential equals the average of the potentials of the adjacent sides as shown in Figure 3-10.

The extension to three dimensions is straight forward. In three dimensions, we would use the four values of potential given in (3.46) plus two additional ones. In addition, the factor (4) that appears in (3.46) would be replaced by a factor (6). The square has been replaced by a cube.

After computing the first iteration, let us determine the potential at other points within the nine point mesh. This will involve two more iterations as shown in Figure 3-11. In the second iteration, all of the potentials at the locations indicated by a solid circle ● in Figure 3-11a are now known. The values indicated by a square ■ are to be computed in this iteration using (3.46). In the third iteration, the values of the potential indicated by the solid circles ● and squares ■ are known from the previous two iterations or initially in the calculation. Again employing (3.46), the values of the potential at the locations

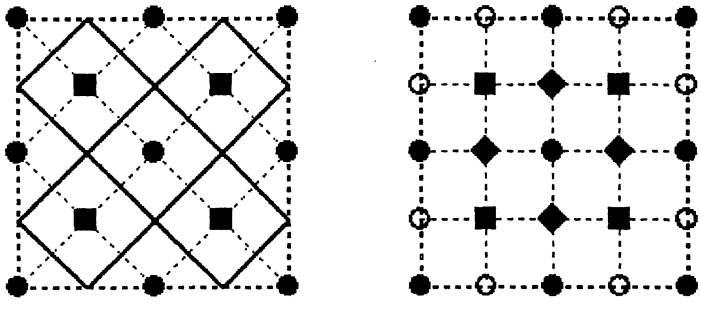


Figure 3-11. The second and third iterations. (a) The values of the potential indicated by the solid circles ● are known. The values at the locations of the solid squares ■ are computed in the second iteration. (b) The potentials at the boundaries indicated by the hollow circles ○ are assumed to be known. The potentials at the locations indicated by the diamonds ◆ are computed in the third iteration.

indicated by the diamonds ◆ can be computed. In this mesh, it is assumed that the potentials at the boundaries can be stated, hence the potentials at the locations indicated by the hollow circles ○ are also known.

This iterative procedure can continue until the adjacent points in the decreasing meshes become close to each other. Accuracy of the calculation can be insured by repeating the calculation with a different initial mesh size. Other meshes other than the mesh that has been used here could be employed. In addition, the tedious reiteration procedure outlined here can be performed with a computer.

A critical restriction is also found on the square mesh size in that the first point must be in the center of the square. This point will be evaluated from the four boundaries of the square. This will restrict the number of internal points N of the square to contain the following number of points.

$$1^2; 3^2; 7^2; 15^2; 31^2; 63^2; ...[2^N-1]^2$$

Let us call this the array size.

At this point, we shall assume that the region that we are to study has the shape of a square. We could have scaled various dimensions to create this square. Later, we will examine shapes that are more complicated. We will follow the reiterative procedure that we have previously encountered in numerically solving the one dimensional Laplace's equation.

Example 3-11. Given that the potential at the four sides of the square trough are as indicated, compute the values at the indicated points.

Answer: Employing the convention that the potential at the corner is one-half of the two adjacent sides, the potential at the top two corners is 8. Due to the symmetry of this example, the potential only has to be calculated at a limited number of points. (first iteration – (a))

$$V_E = \frac{0+0+0+16}{4} = 4$$

Then (second iteration - (b))

$$V_A = V_C = \frac{0+8+16+4}{4} = 7$$

and (c)

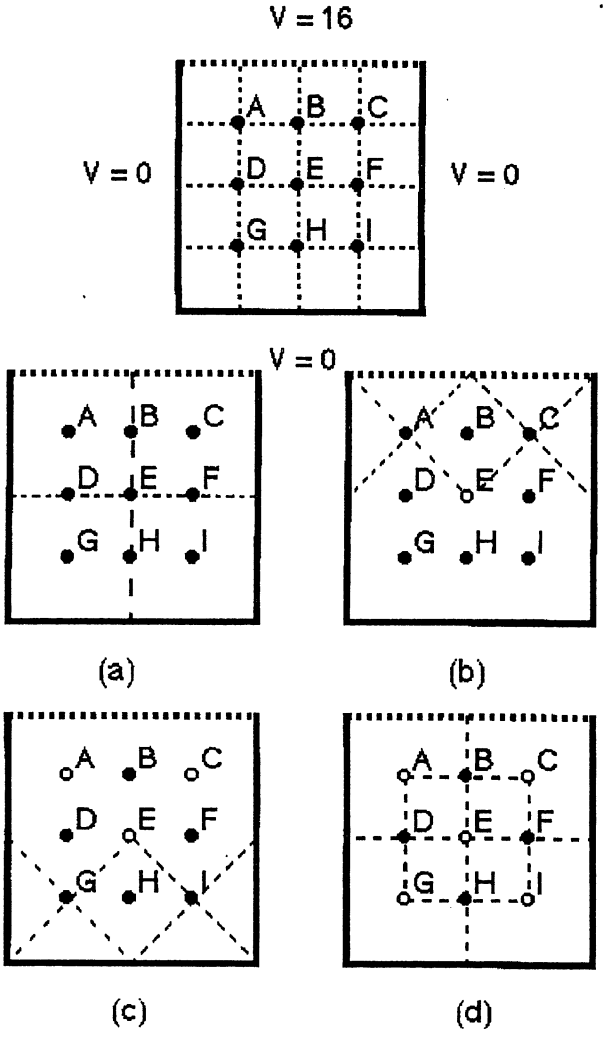
$$V_G = V_I = \frac{0+4+0+0}{4} = 1$$

The remaining potentials (third iteration - (d)) are

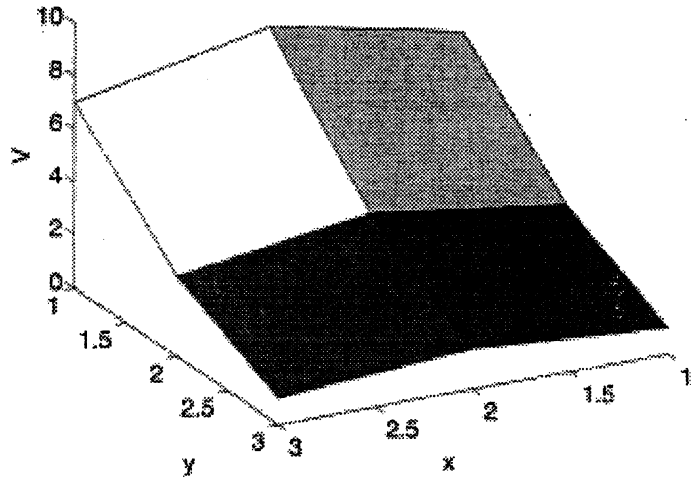
$$V_B = \frac{7+16+7+4}{4} = 8.5$$

$$V_H = \frac{1+4+1+0}{4} = 1.5$$

$$V_D = V_F = \frac{0+7+4+1}{4} = 3$$



The potential within the square region is shown below.

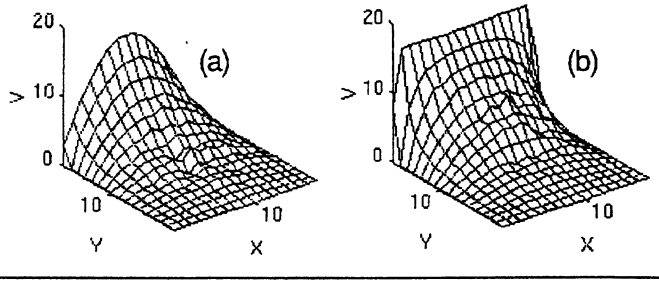


Example 3-12. Write a MATLAB program to evaluate and plot the potential V in the square region by numerically solving Laplace's equation. The maximum potential on one wall is 16 V, and the other three edges are grounded. Use an array size of 15 × 15. Assume that nonzero wall potentials are (a) $V = 16 \sin(\pi x/a)$ and (b) $V = 16$. The voltages at the corners are (a) [(16 sin (π/16) + 0)/2] and (b) [(16 + 0)/2]. Compare the numerical solution with Figure 3-4 and Figure 3-6.

Answer: The MATLAB program is written as.

```
clear
clf
%specify array size and boundary conditions
sz=input('ARRAY SIZE ... ');
md=(sz+1)/2;
bndt=input('Top boundary ... ');
bndl=input('Left boundary ... ');
bndr=input('Right boundary ... ');
bndb=input('Bottom boundary ... ');
%specify boundary conditions
for i=2:sz+1
        v(1,i)=bndt;
        v(i,1)=bndl;
        v(i,sz+2)=bndr;
        v(sz+2,i)=bndb;
end
v(1,1)=(bndt+bndl)/2;
v(1,sz+2)=(bndt+bndr)/2;
v(sz+2,1)=(bndl+bndb)/2;
```

```
v(sz+2,sz+2)=(bndr+bndb)/2;
%set all cells to zero
for i=2:sz+1
        for j=2:sz+1
              v(i,j)=NaN;
        end
end
w=0;
while w==0
        w=1;
        for i=1:sz+2
                for j=1:sz+2
                        if isnan(v(i,j))
                                w=0;
                        end
                end
        end
        for h=md:-1:1
                for i=2:sz+1
                        for j=2:sz+1
                                if (i+h)<=sz+2 & (j+h)<=sz+2 &(i-h)>=1  & (j-h)>=1
                                        if isnan(v(i,j)) & ~isnan(v(i-h,j)) & ~isnan(v(i,j+h)) &
                                                        ~isnan(v(i+h,j)) & ~isnan(v(i,jh))
                                                v(i,j)=(v(i-h,j)+v(i,j+h)+v(i+h,j)+v(i,j-h))/4.0;
                                            elseif isnan(v(i,j)) & ~isnan(v(i-h,j-n)) &
                                          ~isnan(v(i+h,j+h))& ~isnan(v(i+h,j-h)) & ~isnan(v(i+h,j+h))
                                                v(i,j)=(v(i-h,j-h)+v(i-h,j+h)+v(i+h,j-h)+v(i+h,j+h))/4.0;
                                        end
                                    end
                        end
                end
        end
end
%plot the results
clg;surf(v(2:sz+1,2:sz+1));axis('equal');axis('square')
shading interp
colormap(jet)
```



In the discussion above, we assumed that the region to which the star given in (3.46) and depicted in Figure 3-9 had the shape of a square.This technique seemed to work well. We will, however, suggest another technique here following our discussion of one dimensional equations.

As an example to lead us into the two dimensional iterative procedure, we consider a rectangular shape that is one and one half times as long as it is wide that is depicted in Figure 3-12. The length could be greater than this but we will use this example to illustrate a point. For simplicity, we let the charge density equal zero. Superimposed on the rectangle and centered on the points a and b, are the two stars depicted in Figure 3-9. The potentials at these two interior points are *a priori* unknown. We will, however, assume that they initially have the value of zero and then find their values

The values at the interior points will be found using an iterative procedure. In the first iteration using (3.46), we obtain

$$V_a = (0 + 0 + 0 + 0) / 4 = 0 \text{ and } V_b = (16 + 0 + 0 + 0) / 4 = 4.$$

In the next iteration, we obtain

$$V_a = (0 + 0 + 0 + 4) / 4 = 1 \text{ and } V_b = (16 + 0 + 0 + 1) / 4 = 4.25.$$

Proceeding to the next iteration, we write to two place accuracy

$$V_a = (0 + 0 + 0 + 4.25) / 4 = 1.06 \text{ and } V_b = (16 + 0 + 0 + 1.06) / 4 = 4.27$$

We see that the potentials at these two interior points seem to approach two asymptotic values. There are questions of convergence that may have to be investigated. In particular, can we converge to the final answer faster using one scheme instead of another?

Example 3-13. Write a MATLAB program that will calculate the first ten iterations of the potential at the points a and b in Figure 3-12.

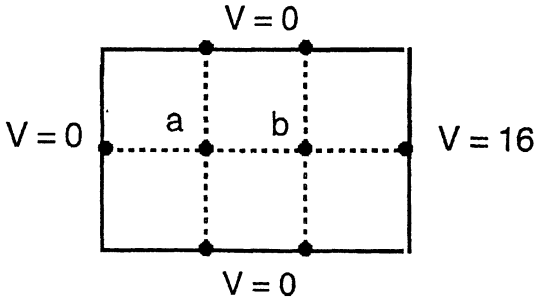


Figure 3-12. A rectangular region whose size is $3h$ times $2h$.

```
>> VA(1) = 0;
>> VB(1) = 0;
>> for j = 1:10
      VA(j + 1) = (VB(j)) / 4;
      VB(j + 1) = (16 + VA(j)) / 4;
   end
>> [(1:11)', VA', VB']
ans =

      1.0000            0            0
      2.0000            0       4.0000
      3.0000       1.0000       4.0000
      4.0000       1.0000       4.2500
      5.0000       1.0625       4.2500
      6.0000       1.0625       4.2656
      7.0000       1.0664       4.2656
      8.0000       1.0664       4.2666
      9.0000       1.0667       4.2666
      10.0000      1.0667       4.2667
      11.0000      1.0667       4.2667
>>
```

In this example, the columns are the step, VA and VB respectively and we have used the transpose operation (') to form vertical columns.

Example 3-14. Write a MATLAB program that will calculate and show the first, the ninth, and the seventeenth iteration of a two dimensional potential profile. The potential at one edge is nonzero and it is zero elsewhere.

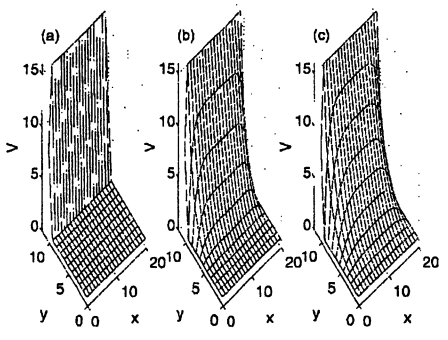
Answer: The program is

```
clear
clf
lb=0; bb=0; tb=0; rb=16; km=17; ny=20; nx=10;
s='(a)';

%set boundary conditions
      for i=2:nx
            v(i,1)=tb;
            v(i,ny+1)=bb;
      end
      for j=2:ny
      v(1,j)=lb;
            %constant boundary condition
            v(nx+1,j)=rb;
            %sinusoidal boundary condition-not used
```

```
            %v(nx+1,j)=rb*sin((j*pi)/((ny+1)));
        end

%corners
        v(1,1)=(lb+tb)/2;
        v(1,ny+1)=(lb+bb)/2;
        %constant boundary condition
            v(nx+1,1)=(rb+tb)/2;
            v(nx+1,ny+1)=(rb+bb)/2;

        %sinusoidal boundary condition-not used
            %v(nx+1,1)=0;
            %v(nx+1,ny+1)=0;

for k=1:km
    if k==1
        %set all cells to zero initially
            %show boundary conditions
        for i=2:nx
            for j=2:ny
                v(i,j)=0;
            end
        end
            subplot(1,3,k)
                surf(v(1:nx+1,1:ny+1));
                xlabel('x')
                ylabel('y')
                zlabel('V')
                axis([0 20 0 12 0 16])
                text(2,14,17,s)
                s(2)=setstr(s(2)+1);
   else
        %calculate the interior voltages
        for i=2:nx
            for j=2:ny
                v(i,j)=(v(i+1,j)+v(i-1,j)+v(i,j+1)+v(i,j-1))/4;
            end
        end
        if k==8*(fix((k-1)/8))+1
            subplot(1,3,((k-1)/8)+1)
                surf(v(1:nx+1,1:ny+1));
                xlabel('x')
                ylabel('y')
                zlabel('V')
                axis([0 20 0 12 0 16])
```

```
      text(2,14,17,s)
      s(2)=setstr(s(2)+1);
      end
   end
end
```

The results are shown below.



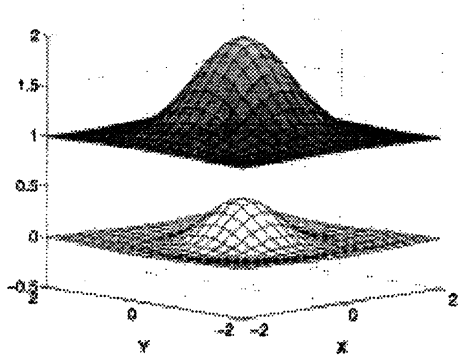
We note that the graph 'fills in' at each iteration.

The examples that are treated here are particularly simple. The convergence of the numbers to a final value was also rapid. There are also techniques that can be employed to enhance this rate of convergence. If the boundary of the object had a slope to it as in a trapezoid, we may have to replace the star in Figure 3-9 that is based upon a square to one based on a right triangle. It is not a large step to get into examples that are 'beyond the scope of this text.' We will let others tread in those waters.

Example 3-15. Write a MATLAB program to evaluate and plot both the potential $V = V_0 \exp(-x^2 - y^2)$ and the charge density from Poisson's equation.

Answer: The MATLAB program is:

```
clear;
clf
>>  [x,y]=meshgrid(-2:.2:2,-2:.2:2);
>>  z=exp(-x.^2-y.^2);
>>  w= - del2(z);
>>  mesh(x,y,10*w)
>>  hold on
>>  surf(x,y,z+1)
>>  view(-45,10)
>>  whitebg
>>  colormap(gray)
>>  xlabel('X')
>>  ylabel('Y')
```



Dielectric interfaces

In the material described so far, we have assumed that the potential was specified at the boundaries of a uniform region for which the potential was to be numerically determined. If the region contains two dielectrics as shown in Figure 3-13, we have to obtain an algorithm that will allow us to evaluate the potential on both sides of the dielectric interface.

In order to calculate the boundary condition for the interface of the two dielectrics, we make use of Gauss's law. This is written as

$$\oint \varepsilon \mathbf{E} \cdot \mathbf{ds} = Q_{\text{enc}} = 0 \tag{3.47}$$

where we have assumed that there is no surface charge density at the interface. With reference to Figure 3-13, (3.47) can be written as

$$\oint \varepsilon \mathbf{E} \cdot \mathbf{ds} = \Delta z \oint \varepsilon \mathbf{E} \cdot \mathbf{dl} = -\Delta z \oint \varepsilon \frac{\partial V}{\partial n} dl = 0 \tag{3.48}$$

where we have replaced the electric field with $(\partial V/\partial n)$, the derivative of the potential that is normal to the surface. The term Δz is the distance in the third coordinate. The surface integral has become a contour integral times this distance Δz that is directed out of the page. In terms of Figure 3-13, we write

$$\begin{aligned} \oint \varepsilon \frac{\partial V}{\partial n} dl = {} & \frac{V_1 - V_0}{h}\left(\varepsilon_2 \frac{h}{2} + \varepsilon_1 \frac{h}{2}\right) + \frac{V_2 - V_0}{h}(\varepsilon_1 h) \\ & + \frac{V_3 - V_0}{h}\left(\varepsilon_2 \frac{h}{2} + \varepsilon_1 \frac{h}{2}\right) + \frac{V_4 - V_0}{h}(\varepsilon_2 h) \end{aligned} \tag{3.49}$$

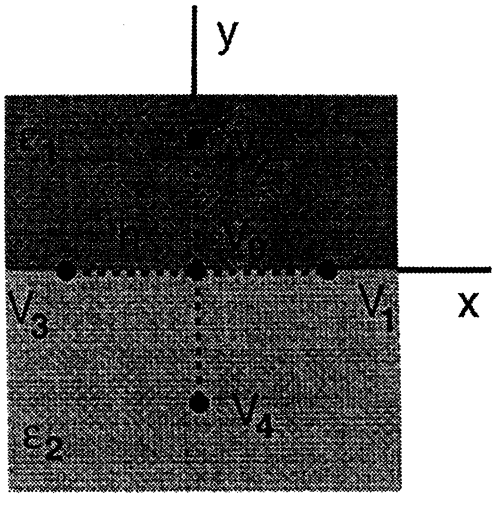


Figure 3-13. Interface of two dielectrics. Points of the five-point star are in the two dielectrics.

Rearranging terms, we rewrite (3.49) as

$$2\varepsilon_1 V_2 + 2\varepsilon_2 V_4 - 4(\varepsilon_1 + \varepsilon_2)V_0 + (\varepsilon_1 + \varepsilon_2)V_1 + (\varepsilon_1 + \varepsilon_2)V_3 = 0 \tag{3.50}$$

or

$$V_0 = \frac{1}{4(\varepsilon_1 + \varepsilon_2)}\{2\varepsilon_1 V_2 + 2\varepsilon_2 V_4 + (\varepsilon_1 + \varepsilon_2)V_1 + (\varepsilon_1 + \varepsilon_2)V_3\} \tag{3.51}$$

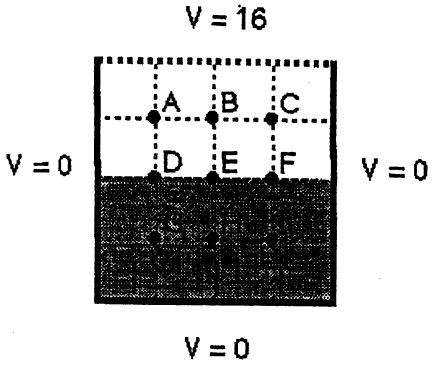
Using the algorithm developed in (3.51), we can relate the potentials on one side of a dielectric to the other.

Example 3-16. In Example 3-11, the bottom half of the region is filled with a dielectric that has a relative dielectric constant $\varepsilon_r = 3$. The top half is a vacuum where $\varepsilon_r = 1$. Find the potential at all points within the region. The potential at the top two corners is equal to 8 as noted previously.

Answer: We follow the sequence of computations outlined in Example 3-11. The potential at point *E* requires that we use (3.51). Hence

$$V_E = \frac{1}{4(1+3)}\{32+0+0+0\} = 2$$

The second iteration allows us to compute the potentials at *A* and *C* and at *G* and *I*



$$V_A = V_C = \frac{(8+16+2+0)}{4} = 6.5$$

$$V_G = V_I = \frac{(0+2+0+0)}{4} = 0.5$$

The potentials at *D* and *F* require (3.25)

$$V_D = V_F = \frac{1}{4(1+3)}\{13+3+0+8\} = 1.5$$

The potentials at *B* and *H* are

$$V_B = \frac{(6.5+16+6.5+2)}{4} = 7.75$$

$$V_H = \frac{(0.5+2+0.5+0)}{4} = .75$$

Note that the potentials differ from Example 3-11.

Method of moments

In the previous chapter, we found that the electric potential V could be computed from a charge distribution via an integral that we rewrite here

$$V(x,y,z) = \int_{\Delta v} \frac{\rho_v(x',y',z')}{4\pi\varepsilon_0 R}\, dx'\, dy' dz' \qquad (3.52)$$

If the charge distribution is known, then the potential can be easily computed. We have just noted that (3.52) can be converted into a summation and hence the integral can be numerically calculated.

There are cases, however, where the potential may actually be known and the charge distribution may be unknown. Static fields abounds with such problems. An example could be the determination of the surface charge distribution on a conductor that would result in the wire's potential being a specified voltage.[1]

In order to introduce the procedure, consider that two charges are located on the x axis as shown in Figure 3-14. The absolute potential at the point P_1 is known to be equal to $V = 1$ volt.

We write the potential at P_1 due to the two point charges as

$$1 = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1}{4} + \frac{Q_2}{5}\right\} \tag{3.53}$$

Various combinations of charge values will satisfy this relation. For example: $Q_1 = 0$ and $Q_2 = 20\ \pi\varepsilon_0\ C.$ or $Q_1 = 8\ \pi\varepsilon_0\ C.$ and $Q_2 = 10\ \pi\varepsilon_0\ C.$ could be used. There is no unique solution in this case. This is because there are two charges and only one condition (3.53).

Let us impose a second condition and require that the potential be equal to 1 V at two points as indicated in Figure 3-15. Hence, we write

$$\left.\begin{aligned} 1 &= \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1}{4} + \frac{Q_2}{5}\right\} \\ 1 &= \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1}{5} + \frac{Q_2}{4}\right\} \end{aligned}\right\} \tag{3.54}$$

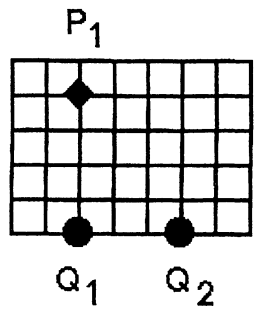


Figure 3-14. Two charges yield the potential at P to be equal to one volt. The grid separation is in one meter increments

[1] Several examples and references appear in L.T. Tsai and C.E. Smith, 'Moment Methods in Electromagnetics for Undergraduates,' IEEE Transactions on Education, Vol. E-21, pp. 14-22 (1978).

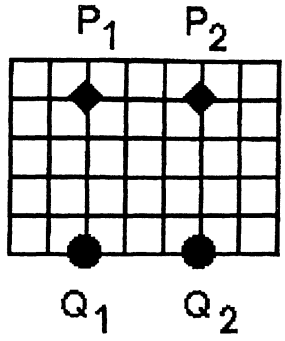


Figure 3-15. Two charges yield the potential at the two points P_1 and P_2 to be equal to one volt. The grid separation are in one meter increments.

This set of linear equations can be written in compact matrix notation as

$$\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{4\pi\varepsilon_0}\begin{bmatrix}\frac{1}{4} & \frac{1}{5}\\ \frac{1}{5} & \frac{1}{4}\end{bmatrix}\cdot\begin{bmatrix}Q_1\\Q_2\end{bmatrix} \tag{3.55}$$

If we invert the matrix, we obtain

$$Q_1=Q_2=\frac{80\pi\varepsilon_0}{9}\,C \tag{3.56}$$

This solution is unique.

Example 3-17. Using MATLAB, solve for the charges in (3.56).

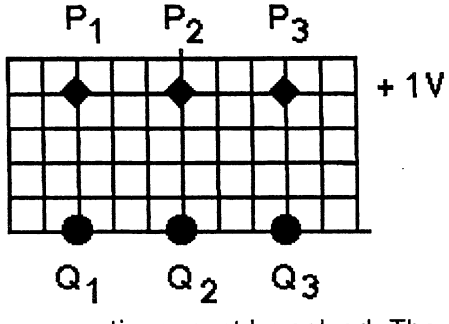
Answer: Type in the matrix elements

```
>> a(1, 1) = 1/4;
>> a(1, 2) = 1/5;
>> a(2, 1) = 1/5;
>> a(2, 2) = 1/4;
>> a
a =
      0.2500      0.2000
      0.2000      0.2500

>> V(1, 1) = 1;
```

```
>> V(2, 1) = 1;
>> V
V =
      1
      1
Q = V'/a'
Q =
    2.2222    2.2222
```

This is the result given in (3.56) if the numbers are multiplied by $4\pi\varepsilon_0$. The term $Q = V'/a'$ is Q = [V transpose] / [a transpose].

Example 3-18. Three charges are located at the indicated points on the graph. Determine their values if the potential at the indicated points are all equal to 1 V.



Answer: Three simultaneous equations must be solved. They are

$$1 = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1}{4} + \frac{Q_2}{5} + \frac{Q_3}{\sqrt{4^2+6^2}}\right\}$$

$$1 = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1}{5} + \frac{Q_2}{4} + \frac{Q_3}{5}\right\}$$

$$1 = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_1}{\sqrt{4^2+6^2}} + \frac{Q_2}{5} + \frac{Q_3}{4}\right\}$$

This set can be written in matrix notation as

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{4\pi\varepsilon_0}\begin{bmatrix}\frac{1}{4} & \frac{1}{5} & \frac{1}{\sqrt{4^2+6^2}}\\ \frac{1}{5} & \frac{1}{4} & \frac{1}{5}\\ \frac{1}{\sqrt{4^2+6^2}} & \frac{1}{5} & \frac{1}{4}\end{bmatrix} \cdot \begin{bmatrix}Q_1\\Q_2\\Q_3\end{bmatrix}$$

The inversion of the matrix leads to

$$Q_1 = Q_3 = 2.91\ (4\pi\varepsilon_0) \text{ and } Q_2 = -.66\ (4\pi e_0).$$

Example 3-19. Since MATLAB was originally written to manipulate matrix operations, explicitly state the sequence of commands that will allow you to solve Example 3-16 using it. Use the more compact notation to represent vectors and matrices, i.e. *A* = [1/4 1/5 1/sqrt (16+36)].

Answer: The '>>' is the MATLAB prompt for us to input data. The other numbers are the MATLAB reply.

```
>>  A = [1/4 1/5 1/(sqrt(16+36)); 1/5 1/4 1/5; 1/sqrt(16+36) 1/5 1/4]
A =
    0.2500    0.2000    0.1387
    0.2000    0.2500    0.2000
    0.1387    0.2000    0.2500
>>  B = [1; 1; 1]
B =
    1
    1
    1
>>  Q = B' / A'
Q =
    2.9123   -0.6596    2.9123
```

Consider the configuration shown in Figure 3-16. Four charges are located in space. A coordinate system is also introduced and the location of the centers of the four charges are specified with reference to this coordinate system. The potential at two of the charges (Q_1 and Q_2) is specified to be $V = -1$ and at the other two (Q_3 and Q_4), it is specified to be $V = +1$. Let us find the value of the charges. We write four linear equations for the potentials at the four points.

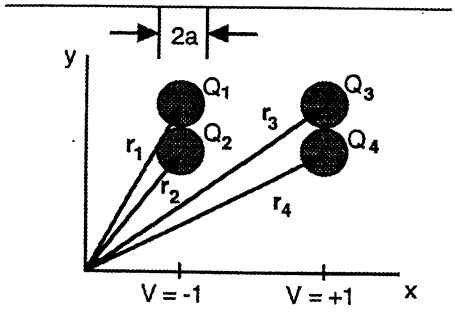


Figure 3-16. Four charges distributed in space. The potential at the indicated points are $V = -1$ and $V = +1$.

$$\left.\begin{aligned}
-1 &= \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}_1-\mathbf{r}_1|}+\frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}_1-\mathbf{r}_2|}+\frac{Q_3}{4\pi\varepsilon_0|\mathbf{r}_1-\mathbf{r}_3|}+\frac{Q_4}{4\pi\varepsilon_0|\mathbf{r}_1-\mathbf{r}_4|}\\
-1 &= \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}_2-\mathbf{r}_1|}+\frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}_2-\mathbf{r}_2|}+\frac{Q_3}{4\pi\varepsilon_0|\mathbf{r}_2-\mathbf{r}_3|}+\frac{Q_4}{4\pi\varepsilon_0|\mathbf{r}_2-\mathbf{r}_4|}\\
+1 &= \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}_3-\mathbf{r}_1|}+\frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}_3-\mathbf{r}_2|}+\frac{Q_3}{4\pi\varepsilon_0|\mathbf{r}_3-\mathbf{r}_3|}+\frac{Q_4}{4\pi\varepsilon_0|\mathbf{r}_3-\mathbf{r}_4|}\\
+1 &= \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}_4-\mathbf{r}_1|}+\frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}_4-\mathbf{r}_2|}+\frac{Q_3}{4\pi\varepsilon_0|\mathbf{r}_4-\mathbf{r}_3|}+\frac{Q_4}{4\pi\varepsilon_0|\mathbf{r}_4-\mathbf{r}_4|}
\end{aligned}\right\} \tag{3.57}$$

In matrix notation, we write this set of equations as

$$\begin{bmatrix}-1\\-1\\+1\\+1\end{bmatrix} = \frac{1}{4\pi\varepsilon_0}\begin{bmatrix}
\frac{1}{|\mathbf{r}_1-\mathbf{r}_1|} & \frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} & \frac{1}{|\mathbf{r}_1-\mathbf{r}_3|} & \frac{1}{|\mathbf{r}_1-\mathbf{r}_4|}\\
\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|} & \frac{1}{|\mathbf{r}_2-\mathbf{r}_2|} & \frac{1}{|\mathbf{r}_2-\mathbf{r}_3|} & \frac{1}{|\mathbf{r}_2-\mathbf{r}_4|}\\
\frac{1}{|\mathbf{r}_3-\mathbf{r}_1|} & \frac{1}{|\mathbf{r}_3-\mathbf{r}_2|} & \frac{1}{|\mathbf{r}_3-\mathbf{r}_3|} & \frac{1}{|\mathbf{r}_3-\mathbf{r}_4|}\\
\frac{1}{|\mathbf{r}_4-\mathbf{r}_1|} & \frac{1}{|\mathbf{r}_4-\mathbf{r}_2|} & \frac{1}{|\mathbf{r}_4-\mathbf{r}_3|} & \frac{1}{|\mathbf{r}_4-\mathbf{r}_4|}
\end{bmatrix}\cdot\begin{bmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{bmatrix} \tag{3.58}$$

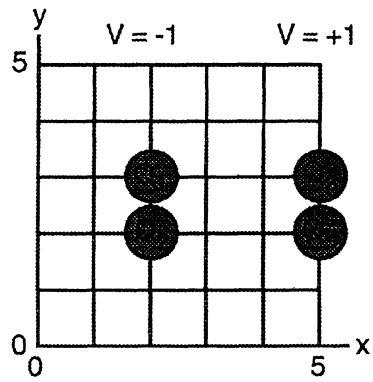
The main diagonal terms of this matrix appear to give us problems since they become very large. These terms are called *singular.* We can get away from this problem if the potential at these singular points is evaluated at the edge of the charge at a radius $r = a$ rather than at the center and assume that the entire sphere has this potential. The potential at the edge of the spherical volume of radius $r = a$ is given by

$$V_j = \frac{Q_j}{4\pi\varepsilon_0 a} \tag{3.59}$$

Hence (3.58) becomes

$$\begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \end{bmatrix} = \frac{1}{4\pi\varepsilon_0} \begin{bmatrix} \frac{1}{a} & \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} & \frac{1}{|\mathbf{r}_1 - \mathbf{r}_3|} & \frac{1}{|\mathbf{r}_1 - \mathbf{r}_4|} \\ \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} & \frac{1}{a} & \frac{1}{|\mathbf{r}_2 - \mathbf{r}_3|} & \frac{1}{|\mathbf{r}_2 - \mathbf{r}_4|} \\ \frac{1}{|\mathbf{r}_3 - \mathbf{r}_1|} & \frac{1}{|\mathbf{r}_3 - \mathbf{r}_2|} & \frac{1}{a} & \frac{1}{|\mathbf{r}_3 - \mathbf{r}_4|} \\ \frac{1}{|\mathbf{r}_4 - \mathbf{r}_1|} & \frac{1}{|\mathbf{r}_4 - \mathbf{r}_2|} & \frac{1}{|\mathbf{r}_4 - \mathbf{r}_3|} & \frac{1}{a} \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \tag{3.60}$$

Example 3-20. Find the values of the charges that will cause the potentials as shown in the figure. The grid separation is 1 meter and the diameter of the charged region is 1 meter.



Answer: The matrix (3.60) has the terms

$$\begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \end{bmatrix} = \frac{1}{4\pi\varepsilon_0} \begin{bmatrix} \frac{1}{(1/2)} & \frac{1}{1} & \frac{1}{3} & \frac{1}{\sqrt{1^2+3^2}} \\ \frac{1}{1} & \frac{1}{(1/2)} & \frac{1}{\sqrt{1^2+3^2}} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{\sqrt{1^2+3^2}} & \frac{1}{(1/2)} & \frac{1}{1} \\ \frac{1}{\sqrt{1^2+3^2}} & \frac{1}{3} & \frac{1}{1} & \frac{1}{(1/2)} \end{bmatrix} \cdot \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

Inverting the matrix leads to $Q_1 = Q_2 = -Q_3 = -Q_4 = -.43\ (4\pi\varepsilon_0)$.

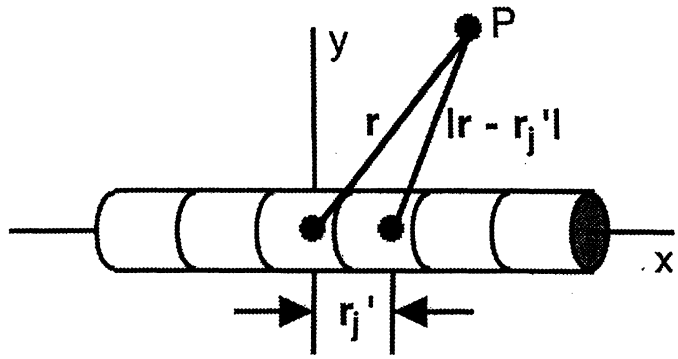


Figure 3-17. The potential at point *P* results from charges $\rho_L \Delta x_j$ located at the centers of the *j*-th section.

We could continue on with individual charges as has been presented up to this point. However, it is advantageous to examine cases where the charge is distributed in various surfaces. If the charge were distributed on a line as shown in Figure 3-17, then there would be a linear charge density ρ_L Coulombs / meter. The charges $\Delta Q_j = 2\pi a\, \Delta x\, \Delta_{Lj}$ would be located at the center of the j^{th} section and we would proceed as has just been described. The same problem with the singularity that was discussed above with individual charges will also be encountered in cylindrical coordinates.

The singularity in the matrix is removed in the same way that we removed it for the spherical charges. We evaluate the potential at the surface of the cylinder and assert that it is also equal to the potential at the 'singular point'. The evaluation of this potential V_j at the surface of this cylindrical section shown in Figure 3-18 is found from the integral.

$$V_j = \frac{1}{4\pi\varepsilon_0}\int_{-\frac{\Delta l_j}{2}}^{\frac{\Delta l_j}{2}}\int_0^{2\pi}\frac{\rho_L\, a\, d\varphi\, dx'}{\sqrt{a^2 + x'^2}} \tag{3.61}$$

The integral of (3.61) can be performed and we find

$$V_j = \frac{2\pi a\rho_L}{4\pi\varepsilon_0}\ln\left[x' + \sqrt{a^2 + x'^2}\right]\Bigg|_{-\frac{\Delta l_j}{2}}^{\frac{\Delta l_j}{2}} = \frac{a\rho_L}{2\varepsilon_0}\ln\left[\frac{\frac{\Delta l_j}{2} + \sqrt{a^2 + \left(\frac{\Delta l_j}{2}\right)^2}}{-\frac{\Delta l_j}{2} + \sqrt{a^2 + \left(\frac{\Delta l_j}{2}\right)^2}}\right] \approx$$

$$\approx \frac{a\rho_L}{2\varepsilon_0} \ln \left[\frac{\Delta l_j}{-\frac{\Delta l_j}{2} + \frac{\Delta l_j}{2}\left(1 + \frac{1}{2}\left(\frac{a^2}{\left(\frac{\Delta l_j}{2}\right)^2}\right)\right)} \right] \tag{3.62}$$

where we have employed the approximation that $a \ll \Delta l_j$. Equation (3.62) is written finally as

$$V_j = \frac{a\rho_L}{\varepsilon_0} \ln \left[\frac{\Delta l_j}{a} \right] \tag{3.63}$$

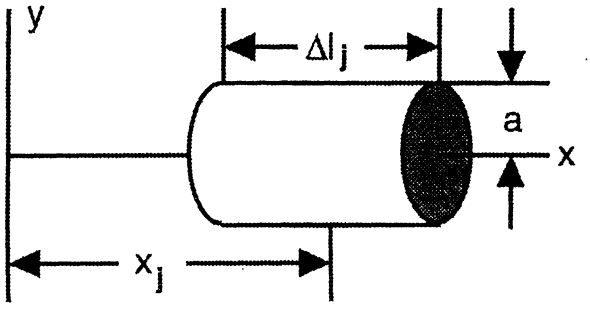
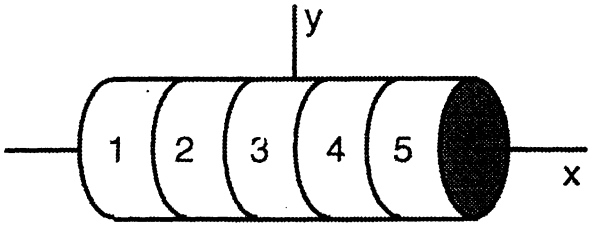


Figure 3-18. The j^{th} section of a linearly charged line.

Example 3-21. Find the charge distribution on the cylindrical conductor whose radius is 1 m and whose length is 100 m. The potential on the surface is $V = 1$ V. You may assume that the charge in each section is uniformly distributed. The term $2\pi a\, \Delta x = 40\pi$.



Answer: The matrix equation relating the potentials to the charges is

$$\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}=\frac{1}{4\pi\varepsilon_0}\begin{bmatrix}4\pi\ln(20) & \frac{40\pi}{20} & \frac{40\pi}{40} & \frac{40\pi}{60} & \frac{40\pi}{80}\\ \frac{40\pi}{20} & 4\pi\ln(20) & \frac{40\pi}{20} & \frac{40\pi}{40} & \frac{40\pi}{60}\\ \frac{40\pi}{40} & \frac{40\pi}{20} & 4\pi\ln(20) & \frac{40\pi}{20} & \frac{40\pi}{40}\\ \frac{40\pi}{60} & \frac{40\pi}{40} & \frac{40\pi}{20} & 4\pi\ln(20) & \frac{40\pi}{20}\\ \frac{40\pi}{80} & \frac{40\pi}{60} & \frac{40\pi}{40} & \frac{40\pi}{20} & 4\pi\ln(20)\end{bmatrix}\cdot\begin{bmatrix}\rho_{L1}\\ \rho_{L2}\\ \rho_{L3}\\ \rho_{L4}\\ \rho_{L5}\end{bmatrix}$$

The inverse of the matrix leads to $\rho_{L1} = .2556\ \varepsilon_0$, $\rho_{L2} = .2222\ \varepsilon_0$, $\rho_{L3} = .2170\ \rho_0$, $\rho_{L4} = .2222\ \varepsilon_0$, $\rho_{L5} = .2556\ \rho_0$.

Similarly, the charge could be distributed upon a surface resulting in a surface charge density of ρ_s Coulomb / meter^2. We subdivide the surface into small rectangular areas Δs_{ij}. This is shown in Figure 3-19. As the reader should expect at this stage, we will again encounter a singularity. The singular rectangular region whose area is Δs_{ii} will be replaced with a circular region containing the same incremental charge. We assume that this charge is distributed within the perimeter of the circle and compute the potential at the perimeter. The potential at the center is set equal to this value and is given by

$$V_{ii}=\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{a}\frac{\rho_s\, r\, dr\, d\varphi}{r}=\frac{\rho_s}{4\pi\varepsilon_0}2\pi a=\frac{\rho_s}{2\varepsilon_0}a=\frac{\rho_s}{2\varepsilon_0}\sqrt{\frac{\Delta s_{ii}}{\pi}} \qquad (3.64)$$

where the radius of the circle is written in terms of the area of the grid element.

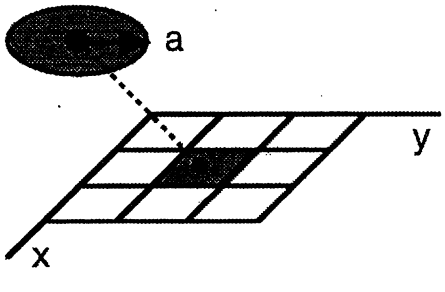


Figure 3-19. Charge is distributed on a surface and has a density ρ_s Coulomb / meter^2. Singular elements in the structure are replaced with discs.

Typical problems that are encountered in this case would be the calculation of the capacitance of a parallel plate capacitor. In such a calculation, the two plates are each subdivided into N^2 sections or subareas; as shown in Figure 3-20.

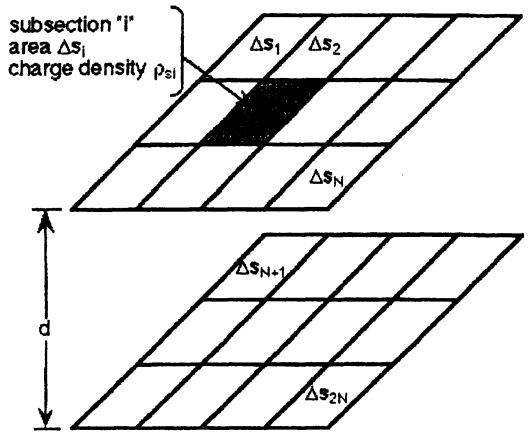
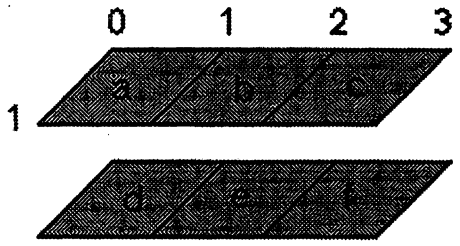


Figure 3-20. A parallel plate capacitor.

Example 3-22. Set up the matrix that describes two parallel plates (1 m × 3 m) separated by a distance of 1 meter. Each plate is subdivided into three equal sections. The potential of the top plate is +1 V and the bottom plate is −1 V.



Answer: The matrix is written in general as

$$\begin{bmatrix} +1 \\ +1 \\ +1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \frac{\Delta s}{4\pi\varepsilon_0} \begin{bmatrix} \frac{1}{|a-a|} & \frac{1}{|b-a|} & \frac{1}{|c-a|} & \frac{1}{|d-a|} & \frac{1}{|e-a|} & \frac{1}{|f-a|} \\ \frac{1}{|a-b|} & \frac{1}{|b-b|} & \frac{1}{|c-b|} & \frac{1}{|d-b|} & \frac{1}{|e-b|} & \frac{1}{|f-b|} \\ \frac{1}{|a-c|} & \frac{1}{|b-c|} & \frac{1}{|c-c|} & \frac{1}{|d-c|} & \frac{1}{|e-c|} & \frac{1}{|f-c|} \\ \frac{1}{|a-d|} & \frac{1}{|b-d|} & \frac{1}{|c-d|} & \frac{1}{|d-d|} & \frac{1}{|e-d|} & \frac{1}{|f-d|} \\ \frac{1}{|a-e|} & \frac{1}{|b-e|} & \frac{1}{|c-e|} & \frac{1}{|d-e|} & \frac{1}{|e-e|} & \frac{1}{|f-e|} \\ \frac{1}{|a-f|} & \frac{1}{|b-f|} & \frac{1}{|c-f|} & \frac{1}{|d-f|} & \frac{1}{|e-f|} & \frac{1}{|f-f|} \end{bmatrix} \cdot \begin{bmatrix} \rho_{sa} \\ \rho_{sb} \\ \rho_{sc} \\ \rho_{sd} \\ \rho_{se} \\ \rho_{sf} \end{bmatrix}$$

where |a–b| is the distance between the center of subsection *a* to subsection *b*, |a–c| is the distance between the center of subsection *a* to subsection *c*, etc. The singular terms that are found in the main diagonal are removed using (3.64). These terms are equal to $2\sqrt{\pi}$ and $\Delta s = 1$. The matrix becomes

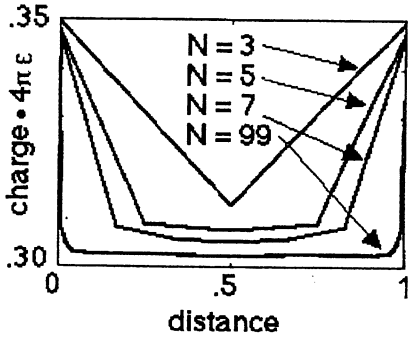
$$\begin{bmatrix} +1 \\ +1 \\ +1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \frac{\Delta s}{4\pi\varepsilon_0} \begin{bmatrix} 2\sqrt{\pi} & \frac{1}{1} & \frac{1}{2} & \frac{1}{2\sqrt{\pi}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{1} & 2\sqrt{\pi} & \frac{1}{2\sqrt{\pi}} & \frac{1}{\sqrt{2}} & \frac{1}{1} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{1} & 2\sqrt{\pi} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} & \frac{1}{1} \\ \frac{1}{1} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} & 2\sqrt{\pi} & \frac{1}{1} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{1} & 2\sqrt{\pi} & \frac{1}{1} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} & \frac{1}{1} & \frac{1}{2} & \frac{1}{1} & 2\sqrt{\pi} \end{bmatrix} \cdot \begin{bmatrix} \rho_{sa} \\ \rho_{sb} \\ \rho_{sc} \\ \rho_{sd} \\ \rho_{se} \\ \rho_{sf} \end{bmatrix}$$

Inverting the matrix leads to $\rho_{sa} = (0.3497)\ 4\pi\varepsilon_0$; $\rho_{sb} = (0.3124)\ 4\pi\varepsilon_0$; $\rho_{sc} = (0.3497)\ 4\pi\varepsilon_0$; $\rho_{sd} = -(0.3497)\ 4\pi\varepsilon_0$; $\rho_{se} = -(0.3124)\ 4\pi\varepsilon_0$; $\rho_{sf} = -(0.3497)\ 4\pi\varepsilon_0$.

Example 3-23. Solve Example 3-22 using MATLAB. Plot the charge distribution. Be able to change the number of elements *N*.

Answer: Using MATLAB, we write the program.

```
clear; clg;
N=input('Number of elements = ')
%set voltages on plates
for i=1:N
        A(i,1)=1;
        A(i+N,1)=-1;
end
%identify cells and locations
for i=1:N
        dx(i)=i;
        dx(i+N)=i;
        dy(i)=1;
        dy(i+N)=0;
end
%calculate matrix elements (4*eo)
for i=1:2*N
         for j=1:2*N
              if i==j
                    B(i,j)=2*sqrt(pi);
              else
                    B(i,j)=1/sqrt((dx(j)-dx(i))^2 +(dy(j)-dy(i))^2);
              end
        end
end
Q=A'/B';
plot(0:1/(N-1):1,Q(1:N))
hold on
axis([0 1 0.3 0.35])
hold off
```



The values of N are: N = 3, 5, 7, and 99.

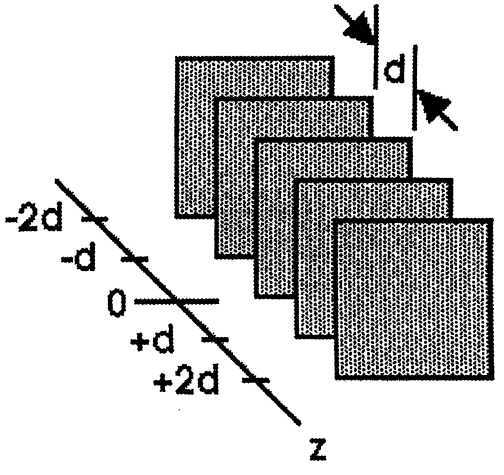


Figure 3-21. Charged sheet model to represent the *pn* junction.

Let us now apply the method of moments to a slightly different topic, that of ascertaining the expected one dimensional charge distribution from an assumed potential profile. The potential profile and the charge distribution could be very nonuniform as in, for example, the depletion layer of a *pn* junction;. In this case, we assume that there are distributed sheets of charge as shown in Figure 3-21. The charge on each sheet '*j*' is uniformly distributed and has a value σ_j. The separation between each sheet is also assumed to be uniform and the separation is *d*.

The electric field surrounding an infinite plane of charge 'σ_j' is given by

$$E_j = \frac{\sigma_j}{2\varepsilon} \tag{3.65}$$

The electric potential at a distance *z* from the charged sheet *j* is found from the integral of (3.65).

$$V_j = -\frac{\sigma_j}{2\varepsilon} z \tag{3.66}$$

The constant of integration in (3.66) is set equal to zero due to symmetry. The potentials at the two extremities ($z = -2d$ and $z = +2d$ in Figure 3-21) have a value that is equal to 1/2 of the value that is given in (3.66). This additional factor of 1/2 arises since the entire electric field is constrained to be directed into the junction. The singularity that was previously encountered is automatically removed since a factor of '0' is introduced in (3.66).

Finally, the potential profile in the junction must be specified. There are several possibilities that could be employed. The simplest is to assume that the potential changes linearly in position z. For the case depicted in Figure 3-21 where there are five charged sheets, we write

$$\left.\begin{aligned}
V(-2d) &= \frac{-1}{2\varepsilon}\left\{\frac{\sigma_1}{2}(0)+\sigma_2(d)+\sigma_3(2d)+\sigma_4(3d)+\frac{\sigma_5}{2}(4d)\right\} \quad (a)\\
V(-d) &= \frac{-1}{2\varepsilon}\left\{\frac{\sigma_1}{2}(d)+\sigma_2(0)+\sigma_3(d)+\sigma_4(2d)+\frac{\sigma_5}{2}(3d)\right\} \quad (b)\\
V(0) &= \frac{-1}{2\varepsilon}\left\{\frac{\sigma_1}{2}(2d)+\sigma_2(d)+\sigma_3(0)+\sigma_4(d)+\frac{\sigma_5}{2}(2d)\right\} \quad (c)\\
V(+d) &= \frac{-1}{2\varepsilon}\left\{\frac{\sigma_1}{2}(3d)+\sigma_2(2d)+\sigma_3(d)+\sigma_4(0)+\frac{\sigma_5}{2}(d)\right\} \quad (d)\\
V(+2d) &= \frac{-1}{2\varepsilon}\left\{\frac{\sigma_1}{2}(4d)+\sigma_2(3d)+\sigma_3(2d)+\sigma_4(d)+\frac{\sigma_5}{2}(0)\right\} \quad (e)
\end{aligned}\right\} \tag{3.67}$$

The factors of 1/2 that occur at either edge are explicitly stated for clarity. The MATLAB program that can be used to generate the elements of the following matrix is in Appendix F.

$$\begin{bmatrix} V(-2d)\\ V(-d)\\ V(0)\\ V(+d)\\ V(+2d) \end{bmatrix} = \frac{-d}{2\varepsilon}\begin{bmatrix} 0 & 1 & 2 & 3 & \frac{4}{2}\\ \frac{1}{2} & 0 & 1 & 2 & \frac{3}{2}\\ \frac{2}{2} & 1 & 0 & 1 & \frac{2}{2}\\ \frac{3}{2} & 2 & 1 & 0 & \frac{1}{2}\\ \frac{4}{2} & 3 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} \sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5 \end{bmatrix} \tag{3.68}$$

A MATLAB program to generate (3.68) of arbitrary size is given in Appendix F.

Let us now specify the values of the potential as

$$\begin{bmatrix} V(-2d) \\ V(-d) \\ V(0) \\ V(+d) \\ V(+2d) \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ +1 \\ +2 \end{bmatrix} \tag{3.69}$$

This implies a linear change of the potential or an electric field that is a constant in the depletion layer. Inverting the matrix, we find that

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \frac{2\varepsilon}{d} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ +1 \end{bmatrix} \tag{3.70}$$

This result expresses the fact that the electric field that is proportional to the negative gradient of the linearly varying potential is a constant that is determined by charge densities at either edge.

Other potential distributions can be employed. A different choice will, of course, yield a different charge distribution. It is also applicable in other fields.[1]

Uniqueness

We have just encountered some heavy mathematics, but we have been able to obtain a solution for the partial differential equation, in this case Laplace's equation. If we had a charge distribution within the region where the potential was to be examined, we would then have to solve Poisson's equation. For a given charge distribution within the region, there can be only one potential distribution that satisfies the same boundary conditions. This physical statement has its mathematical basis in that there is one unique solution; for the problem. Let us substantiate this statement. By carrying out the verification, we will gain a little manipulative skill with some vector identities.

Assume that there would exist two potentials φ and ψ that were caused by the same charge distribution ρ_v. Each potential must satisfy the same Poisson equation

[1] K.E. Lonngren, P.V. Schwartz, E.W. Bai, W.C. Theisen, R.L. Merlino, and R.T. Carpenter, "Extracting Double Layer Charge Density Distributions using the Method of Moments," IEEE Transactions on Plasma Science, Vol.24, pp. 278-280 (1994).

$$\nabla^2\varphi = -\frac{\rho_v}{\varepsilon} \tag{3.71}$$

$$\nabla^2\psi = -\frac{\rho_v}{\varepsilon} \tag{3.72}$$

To show that the potentials φ and ψ are the same, let us multiply both sides of (3.72) by φ and subtract it from (3.71) that has been multiplied by ψ. We find that

$$\psi\nabla^2\varphi - \varphi\nabla^2\psi = -\frac{\rho_v}{\varepsilon}(\psi - \varphi) \tag{3.73}$$

The term on the left hand side of this equation can be written as

$$\begin{aligned}\psi\nabla^2\varphi - \varphi\nabla^2\psi &= \psi\nabla\cdot\nabla\varphi - \varphi\nabla\cdot\nabla\psi \\ &= \left[\nabla\cdot(\psi\nabla\varphi) - \nabla\psi\cdot\nabla\varphi\right] - \left[\nabla\cdot(\varphi\nabla\psi) - \nabla\varphi\cdot\nabla\psi\right] \\ &= \nabla\cdot(\psi\nabla\varphi - \varphi\nabla\psi)\end{aligned} \tag{3.74}$$

Hence, we write that

$$\nabla\cdot(\psi\nabla\varphi - \varphi\nabla\psi) = -\frac{\rho_v}{\varepsilon}(\psi - \varphi) \tag{3.75}$$

The potential that arises from the charge distribution can be computed for an unbounded or a bounded volume. Let us initially assume that the volume is a large spherical volume Δv as shown in Figure 3-22. We can make the sphere large enough such that it is possible to assume that the charge distribution is localized at the origin of the sphere. Let us now integrate both sides of equation (3.75) over the volume Δv of the sphere.

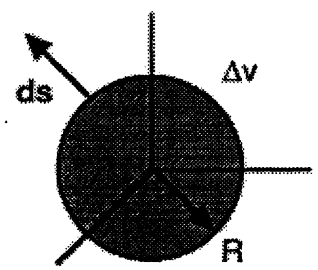


Figure 3-22. The charge density is localized within the spherical volume Δv.

$$\int_{\Delta v} \nabla \cdot (\psi \nabla \varphi - \varphi \nabla \psi)\, dv = -\int_{\Delta v} \frac{\rho}{\varepsilon}(\psi - \varphi)\, dv \tag{3.76}$$

Using Gauss's law, this expression becomes

$$\oint (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathbf{ds} = -\int_{\Delta v} \frac{\rho}{\varepsilon}(\psi - \varphi) dv \tag{3.77}$$

Let us examine the surface integral for two cases: an unbounded region where the surface is taken to be infinite in extent and a bounded region where conducting surfaces surround the region of interest. The charge density ρ is enclosed within the surrounding surface.

For an unbounded region, it is possible to estimate the magnitude of the various terms in the closed surface integral that we will later allow to become infinite is size. This is an electrostatic field, we can therefore identify the electric field as $\mathbf{E} = -\nabla\varphi$ (or $-\nabla\psi$). From Chapter 2, we found that both the electric field and the potential will decrease as we move farther from the source. The potential decays as $\varphi \approx (1/R)$ and the electric field decays as $E \approx (1/R^2)$. The spherical surface area will increase with radius as $4\pi R^2$. Therefore, we write that

$$\lim_{R \Rightarrow \infty} \oint (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathbf{ds} \approx \lim_{R \Rightarrow \infty} \left(\frac{1}{R} \frac{1}{R^2} R^2 \right) \Rightarrow 0 \tag{3.78}$$

For the case of the bounded region, we employ the following argument to show that the surface integral on the conducting surfaces is equal to zero. The surface integral will be decomposed into two surface integrals. The first surface is taken to be the one that approaches infinity and we have just shown that this integral approaches zero. The second surface integral is over the finite surfaces where the potential is already known. Hence, we write

$$\begin{aligned} &\oint (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathbf{ds} \\ &\qquad = \oint (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathbf{ds}\Big|_{\text{infinite}} + \oint (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathbf{ds}\Big|_{\text{finite}} \\ &\qquad = 0 + \oint (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathbf{ds}\Big|_{\text{finite}} \end{aligned} \tag{3.79}$$

The conducting surfaces are equipotential surfaces which states that the potentials are constants. Therefore, the integral becomes

$$\oint (\psi \nabla\varphi - \varphi\nabla\psi)\cdot \mathbf{ds}\Big|_{\text{finite}} = (\psi - \varphi)\oint (\nabla\varphi - \nabla\psi)\cdot \mathbf{ds} \tag{3.80}$$

We note that the terms within the integral can be identified as the normal components of the electric field since the vector direction of **ds** is normal to the surface. Since this surface is a conductor, the normal component of the electric field must terminate on a surface charge density. We write

$$(\psi - \varphi)\oint (\nabla\varphi - \nabla\psi)\cdot \mathbf{ds} = -\frac{\psi - \varphi}{\varepsilon}\oint \left(\rho_{s\varphi} - \rho_{s\psi}\right) ds \tag{3.81}$$

The two closed surface integrals of the surface charge density implied in (3.81) are equal to the same total charge that is distributed on the surfaces. Hence, the contribution given in (3.81) is equal to zero.

Therefore, we have shown that the left hand side of (3.77) for either bounded or unbounded regions is equal to zero. For an arbitrary volume Δv, the only way that the right hand side will also equal zero is if the integrand is equal to zero.[1] Therefore $\varnothing = \psi$ that states that there is only one correct solution. It is the *unique solution*! This means that students who solve a problem using a computer will obtain the same solution as those who use a piece of paper, assuming both students make neither numerical nor algebraic mistakes. Hopefully they will both agree with measurements performed in a laboratory.

Conclusion

Solving boundary value problems for potentials has led us to certain general conclusions concerning the methodical procedure. First, nature has given us certain physical phenomena that can be described by partial or ordinary differential equations. These equations can be solved, in many cases analytically, as has been done here. Other cases may require numerical solutions. The solutions so obtained contain constants of integration. Nature also tells us enough information that will allow us to evaluate these constants and thus obtain the solution for the problem of interest. Assuming that neither mathematical nor numerical mistakes were made, we can rest assured that this is the solution.

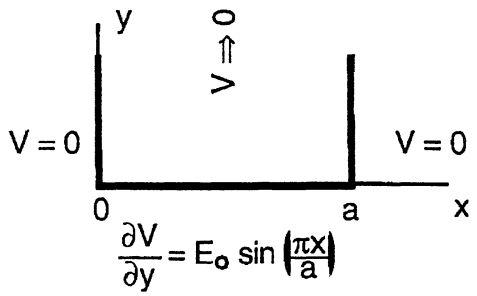
Four techniques that are frequently encountered when we attempt a solution of more complicated problems in electromagnetic theory were introduced in this chapter. The latter three techniques involve the application of numerical methods. Given a known charge distribution, it is possible to numerically integrate over this charge distribution to find the electric potential or the electric field. Poisson's and Laplace's equations, which

[1] Mathematicians also require that the terms within the brackets be 'positive definite'. Subtleties of this magnitude are beyond the scope of this text.

are differential equations, can also be numerically integrated to yield the potential variation in a bounded region. Finally, the method of moments which allowed us to find an unknown charge distribution from given voltage requirements was outlined. The numerical techniques that were introduced here are applicable to difficult problems that may be later encountered. We applied the readily available MATLAB. Numerical questions concerning other specific programming languages, convergence requirements, numerical errors, aliasing, etc. are better left to later courses. As problems arise, solutions can hopefully be found.

Problems

1. For the indicated boundary conditions that are specified in the figure, find the potential distribution within the enclosed region by solving Laplace's equation.



2. Find the Fourier series expansion for a potential $V = V_0 (x/a)$ that describes a potential variation in the region $0 \leq x \leq a$. Sketch the sum of the first three terms of your series and compare it with the actual function.

3. Find the potential within the channel given in problem 1 using the boundary conditions at $x = 0$, a and $y = \infty$ as stated there. The boundary condition at $y = 0$ is given as $\mathrm{E} = E_0 \mathbf{u}_y$ where E_0 is a constant.

4. Find the potential within the channel given in problem 1 using the boundary conditions at $x = 0$, a and $y = \infty$ as stated there. The boundary condition at $y = 0$ is: $V = +V_0$ for $0 < x < (a/2)$ and $V = -V_0$ for $(a/2) < x < a$.

5. Using a product solution in cylindrical coordinates $V(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, show that the term $R(r)$ satisfies the ordinary differential equation

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - \left(\lambda^2 - \frac{n^2}{r^2}\right) R(r) = 0$$

The solution for this equation can be found in terms of an infinite series. This series is given the name – 'Bessel series'. Find the corresponding ordinary differential equation for $\Theta(\theta)$.

6. Using a product solution in spherical coordinates $V(r, \theta, \varphi) = R(r)\,\Theta(\theta)\,\vartheta(\varphi)$, show that the term $R(r)$ satisfies the ordinary differential equation

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} - \frac{\alpha^2}{r} R(r) = 0$$

if the term $\vartheta(\varphi)$ is a constant. The solution for this equation can be found in terms of an infinite series. This series is given the name – spherical Bessel series. Find the corresponding

ordinary differential equation for $\Theta(\theta)$.

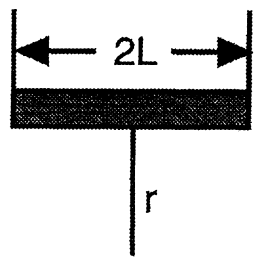
7. If the separation constant $\alpha = 0$ in the Bessel equation given in problem 5, show that $R(r) \approx r^{-1}$.

8. Using a power series expansion for $R(r)$ for the equation given in problem 5. For $n = 0$, the leading two terms of the Bessel series are given by

$$R(r) = 1 - \left(\frac{r}{2}\right)^2$$

Compare this approximate value with the cosine function that is found in solving Laplace's equation in Cartesian coordinates.

9. Compute the potential surrounding at the midpoint of a finite length of uniformly distributed charge. The total charge $Q = 1$ C. Compare your numerical result with analytical predictions.

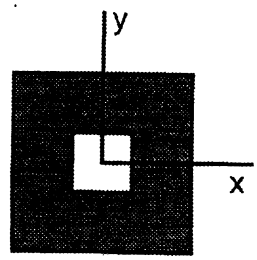


10. Repeat problem 9 with a nonuniform charge distribution $\rho_v = \rho_0 [1 - (z/L)]$ where $z = 0$ is at the midpoint of the line. Show that the value changes as you make finer approximations for the charge distribution.

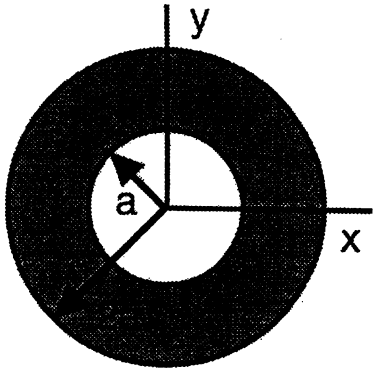
11. Repeat problem 9 but compute the electric field instead of the potential.

12. Repeat problem 10 but compute the electric field instead of the potential.

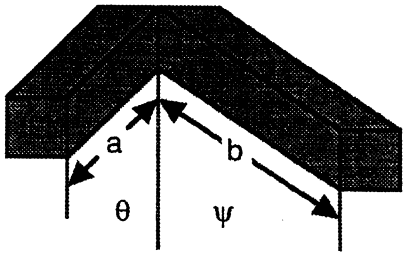
13. Compute the potential along the z axis from the square washer whose size is $3a \times 3a$. The dimensions of the hole are $a \times a$ Charge $Q = 1$ C. is distributed uniformly upon the shaded region.



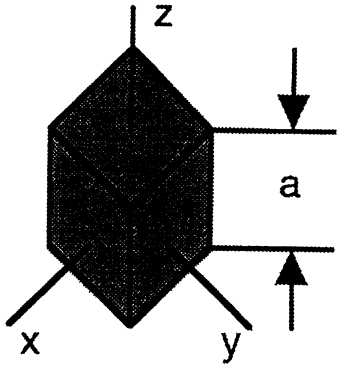
14. Compute the potential along the z axis from the circular washer. Charge Q = 1 C. is distributed uniformly upon the shaded region.



15. Compute the potential along the z axis that is perpendicular to the plane containing the rods as a function of two parameters. The linear charge density is 1 C./ m. Two parameters that can be varied are the ratios (a/b) and (θ/ψ).



16. Compute the potential along the z axis from a cube. Charge Q = 1 C. is distributed uniformly within the cube.



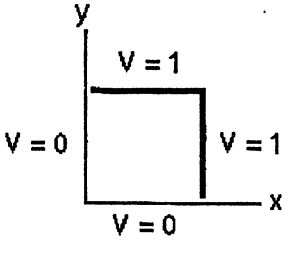
17. Find the potential distribution between two surfaces if $V\ (x = 0) = 3$ and $V\ (x = 1) = 0$. There is no charge distribution in the space $0 \leq x \leq 1$. Find at least three values in the range $0 < x < 1$.

18. Find the potential distribution between two surfaces if $V\ (x = 0) = 3$ and $V\ (x = 1) = 0$. There is a uniform charge distribution $\rho_v\ (j) = -4\ \varepsilon_0$ in the space $0 \leq x \leq 1$. Find at least three values in the range $0 < x < 1$.

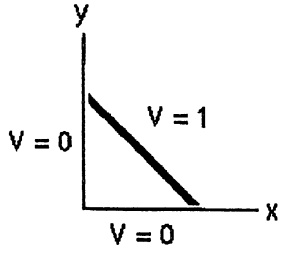
19. Solve problem 17 by writing a MATLAB program.

20. Solve problem 18 by writing a MATLAB program.

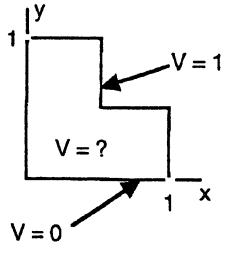
21. Find the potential distribution in the region between the two surfaces. Display the potential profile.



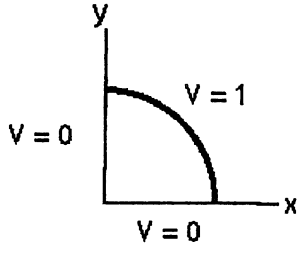
22. Find the potential distribution in the region between the two surfaces. Display the potential profile.



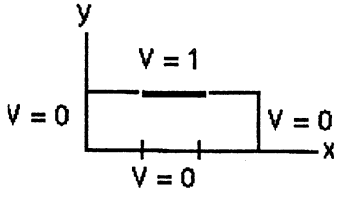
23. Find the potential distribution in the region between the two surfaces. Display the potential profile.



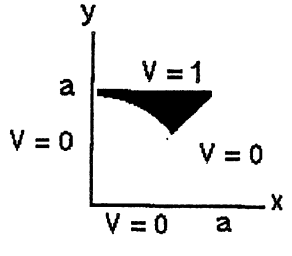
24. Find the potential distribution in the region between the two surfaces. Display the potential profile.



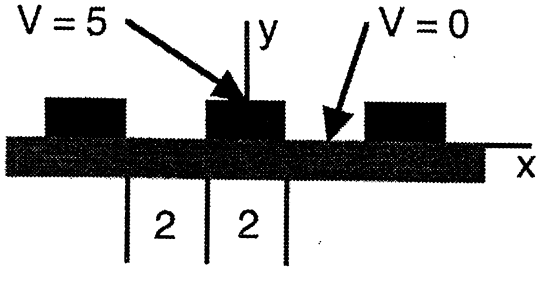
25. Find the potential distribution in the region between the two surfaces. Display the potential profile.



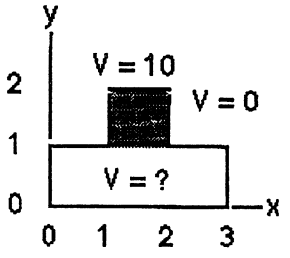
26. Find the potential distribution in the region between the two surfaces. Display the potential profile.



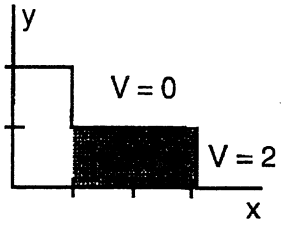
27. Find the potential distribution in the region $y > 0$. This problem models a VLSI circuit where conductors lay on an insulating surface.



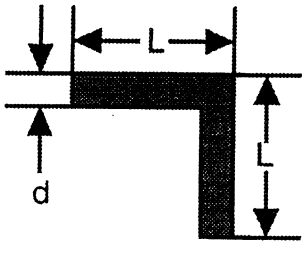
28. Find the potential distribution in the enclosed region. The shaded region is a dielectric with $\varepsilon_r = 4$.



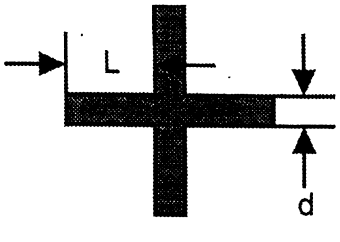
29. Find the potential distribution in the enclosed region. The voltage on all boundaries $V = 0$ except one where $V\ (x = 3) = 2$. The shaded region is a dielectric with $\varepsilon_r = 4$.



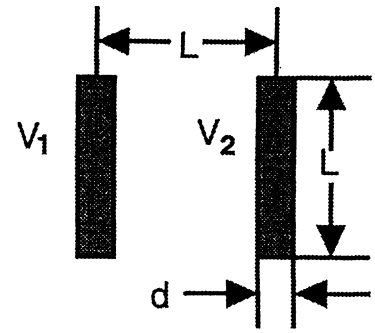
30. Use the method of moments to find the charge distribution on the wire. The potential of the wire is 1 V. The wire dimensions are $d = 1$ cm and $L = 10$ cm.



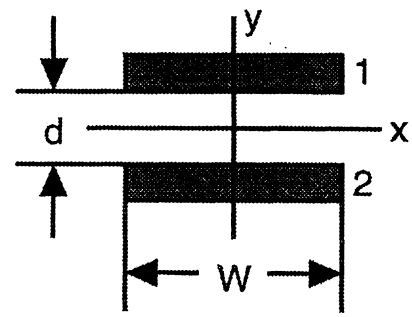
31. Use the method of moments to find the charge distribution on the wire. The potential of the wire is 1 V. The wire dimensions are $d = 1$ cm and $L = 10$ cm.



32. Use the method of moments to find the charge distribution on the wires. The wire dimensions are d = 1 cm and L = 10 cm. The voltages V_1 = 1 V and V_2 = 2V.



33. Use the method of moments to find the charge distribution on the strip lines. This distribution will be ρ_L where the third direction is out of the paper. The dimensions are d = 1 cm and W = 10 cm. The voltages V_1 = +1 V and V_2 = –1 V.



34. Use the method of moments to find the charge distribution on the strip lines. This distribution will be ρ_L where the third direction is out of the paper. The dimensions are d_1 = 1 cm, d_2 = 1.25 cm and W = 10 cm. The voltages V_1 = +1 V and V_2 = –1 V.