CHAPTER 9

TIME-DEPENDENT FIELDS

In all the previous chapters we have been primarily concerned with static fields, i.e., fields that are independent of time. In this chapter this restriction will be removed, and we shall then find that the electric and magnetic fields are intimately related to each other. An example of this interrelationship was provided by Faraday's law, which showed that a time-varying magnetic field induced an electric field. To complete the picture we shall show that a reciprocal effect, namely, that a time-varying electric field induces a magnetic field, also exists. This mutual support of each other, i.e., a magnetic field producing an electric field and an electric field producing a magnetic field, results in the phenomenon of wave propagation. The prediction of electromagnetic waves and the subsequent successful use of these waves in communication systems were an outstanding climax to the centuries of exploration and experimentation that preceded it.

9.1. Modification of Static Field Equations under Time-varying Conditions

Before presenting the general equations for the time-varying electromagnetic field we shall summarize the basic equations that govern the static electric and magnetic fields and the stationary current flow field. A number of equivalent choices are possible, but the following equations are chosen because they clearly show the irrotational property of the electrostatic field and the divergenceless property of the magnetostatic and stationary current flow fields.

For the electrostatic field we have

$$\nabla \times \mathbf{E} = 0 \tag{9.1a}$$
$$\nabla \cdot \mathbf{D} = \rho \tag{9.1b}$$

while for the magnetostatic field

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{9.1c}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{9.1d}$$

and for stationary currents

$$\nabla \cdot \mathbf{J} = 0 \tag{9.1e}$$

We already know that some of the above equations must be modified when the fields vary with time. In particular, Faraday's law of induction shows that (9.1a) must be replaced by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9.2}$$

when the field **B** varies with time. Also we know that when **J** and ρ vary with time, the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{9.3}$$

must hold since current is a flow of charge, and hence the divergence of the current at a point must always equal the time rate of decrease of charge density at that point.

At this time we might very well ask whether there is any need to modify any of the other equations. The answer is, "Yes, there is," for we may easily show that the set of equations (9.1b), (9.1c), (9.1d), (9.2), and (9.3) do not form a self-consistent set. The divergence of the curl of any vector is identically zero, and hence from (9.1c) we obtain

$$\nabla \cdot \nabla \times \mathbf{H} = 0 = \nabla \cdot \mathbf{J} \tag{9.4}$$

This result is in contradiction with (9.3) when **J** varies with time. If the basic form of (9.1c) is to be retained under time-varying conditions, then the right-hand side must be solenoidal and reduce to **J** under time-stationary conditions. The necessary form of the right-hand side of (9.1c) can be deduced from (9.3) if we make use of (9.1b) in the expression for ρ. With this substitution we may write

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \tag{9.5}$$

The vector quantity in the parentheses of (9.5) is solenoidal and reduces to **J** if $\partial/\partial t = 0$. Consequently, the previous equations will become consistent if this quantity is substituted for the right-hand side of (9.1c); that is,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{9.6}$$

The term $\partial \mathbf{D}/\partial t$ was originally introduced into the curl equation for **H** by Maxwell and is called the displacement current density because it has the dimensions of a current density. Although the way in which we introduced this term above in no way proves the correctness of (9.6), it has been found experimentally that the conclusions drawn from (9.6) are in accord with all known experimental facts; so there is no reason to doubt its validity. The sum of the terms on the right-hand side of (9.6) is in the form of a total current, which, as we are aware, is solenoidal.

As for the remaining equations, they are self-consistent and no experimental evidence for requiring any further modifications has been found. The above equations governing the time-varying electromagnetic field are known collectively as Maxwell's equations and will be discussed in greater detail in the next section.

9.2. Maxwell's Equations

From the previous section we have the following set of equations which govern the behavior of the time-varying electromagnetic field

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9.7a}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{9.7b}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{9.7c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{9.7d}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{9.7e}$$

In (9.7c) and (9.7e) the current $\mathbf{J}$ will consist, in general, of a conduction current $\sigma\mathbf{E}$, caused by the presence of an electric field $\mathbf{E}$ in a material with finite conductivity σ, and a convection current $\rho\mathbf{v}$, consisting of a free-charge distribution ρ flowing with a velocity $\mathbf{v}$. The convection current is of importance in many practical devices such as electron tubes, cathode-ray tubes, etc. In the majority of situations that we deal with in this book, however, the convection current is zero.

In place of the above equations, which are in derivative form, we may write an equivalent set of equations in integral form. Integration of (9.7a) over an open surface S and application of Stokes' law give

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{9.8a}$$

where C is the boundary of S. For (9.7b) we integrate throughout a volume V and use the divergence theorem to obtain

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho\, dV \tag{9.8b}$$

Similarly, the remaining three equations give

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{\partial}{\partial t}\int_S \mathbf{D} \cdot d\mathbf{S} + \int_S \mathbf{J} \cdot d\mathbf{S} \tag{9.8c}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{9.8d}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \rho\, dV \tag{9.8e}$$

The integral form of Maxwell's equations is easier to interpret physically and is also useful in deriving the boundary conditions that the field vectors must satisfy. However, in the solution of a physical problem, the derivative form is invariably used.

The first equation in the set (9.8) is just Faraday's law of induction and states that the total voltage induced around a contour C is equal to the negative time rate of change of magnetic flux through this contour. Equation (9.8*b*) simply states that the total displacement flux through a closed surface S is equal to the enclosed charge (Gauss' law).

Equation (9.8*c*) is a generalization of Ampère's circuital law by the addition of the displacement current term. Without this term electromagnetic waves would not exist. It is not surprising that this term was not discovered experimentally, since it is only at radio frequencies that the displacement current becomes comparable to the conduction current in its effects. At the time of Maxwell, means for generating high-frequency currents and fields were virtually nonexistent and certainly, at best, very poorly understood. After introducing the displacement current Maxwell was able to show theoretically the existence of electromagnetic waves having a velocity of propagation equal to that of light. This prediction obviously led to the conclusion that light was electromagnetic in nature. It was only years later that the brilliant experimental work of Hertz, in generating electromagnetic waves by means of spark gaps and demonstrating that their properties were similar to that of light, verified the correctness of Maxwell's assumption.

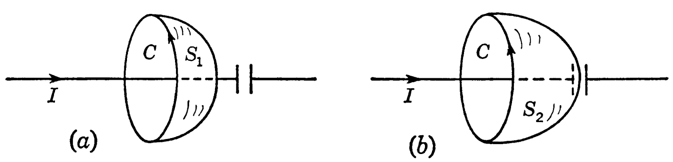

FIG. 9.1. Illustration of need for displacement current term in Ampère's circuital law.

As an example to illustrate the need for the displacement current, consider a parallel-plate capacitor connected to an a-c generator by means of two conducting wires. If we draw an arbitrary closed contour C through which the circuit passes and construct a surface S_1 that intersects the conductor, as in Fig. 9.1*a*, Ampère's circuital law gives

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I$$

where I is the total current flowing in the conductor. We could equally well draw our surface as a surface S_2 that passes between the plates of the capacitor, as in Fig. 9.1*b*. If we use the unmodified form of Ampère's circuital law, we should be led to the conclusion that the line integral of

H around C was equal to zero since no conduction current flows through the surface S_2. This would be an embarrassing situation, since the contour C is still the same contour as used in Fig. 9.1*a*. If we include the displacement current in Ampère's circuital law, we are able to resolve this difficulty. Let A be the area of the capacitor plate, and let d_0 be the separation. The capacitance C_0 is given by

$$C_0 = \frac{\epsilon_0 A}{d_0}$$

if fringing effects are neglected. When a current I is flowing into a capacitor, the voltage V across it is given by

$$I = C_0 \frac{dV}{dt}$$

But the voltage V is equal to Ed_0, where E is the electric field between the plates. Hence

$$I = C_0 d_0 \frac{dE}{dt} = \frac{C_0 d_0}{\epsilon_0 A} \frac{d\epsilon_0 AE}{dt} = \frac{d\epsilon_0 AE}{dt}$$

The latter term is the total displacement current flowing between the capacitor plates and is also given by

$$\frac{d}{dt}\left(\int_{S_2} \mathbf{D} \cdot d\mathbf{S}\right) = \frac{d\epsilon_0 AE}{dt} = I$$

Hence

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{d}{dt}\left(\int_{S_2} \mathbf{D} \cdot d\mathbf{S}\right) = I$$

which is the same result as obtained by choosing the surface as S_1 in Fig. 9.1*a*. Of course, this example only confirms that consistent results are obtained through the inclusion of a displacement current term in Ampère's circuital law. It does not verify that the law can be extended to the time-varying case in the way stated. Only an appeal to experiment can confirm this, as it does.

9.3. Source-free Wave Equation

In a source-free region of free space, $\mathbf{J} = \rho = 0$, and Maxwell's equations reduce to the following:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{9.9a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{9.9b}$$

$$\nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = 0 \tag{9.9c}$$

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0 \tag{9.9d}$$

If we eliminate either **E** or **H**, we obtain a three-dimensional wave equation for the remaining quantity. For example, take the curl of (9.9*a*) and use (9.9*b*) to eliminate $\nabla \times \mathbf{H}$ on the right-hand side and thus obtain

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

We may expand $\nabla \times \nabla \times \mathbf{E}$ to get $\nabla\nabla \cdot \mathbf{E} - \nabla^2\mathbf{E}$, and since $\nabla \cdot \mathbf{E} = 0$ from (9.9*c*), we have

$$\nabla^2\mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

or

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{9.10}$$

where $c = (\mu_0\epsilon_0)^{-1/2}$. The parameter c has the dimensions of velocity and is numerically equal to 3×10^8 meters per second, i.e., the velocity of light in free space or vacuum. Equation (9.10) is the standard form of a three-dimensional vector wave equation. The field **H** satisfies the same equation, as may be readily shown by eliminating **E** from (9.9*b*). In practice, if we know **E**, we can obtain **H** by using (9.9*a*).

In order to examine the nature of (9.10) more closely, note that each rectangular component of **E** satisfies the scalar wave equation; e.g.,

$$\nabla^2 E_x - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0$$

If we now assume that E_x is a function of the z coordinate and the time coordinate t only, a further simplification results. We obtain

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0 \tag{9.11}$$

for which any function $f(z - ct)$ is a solution. This latter statement is readily verified. If we let $z - ct = u$, then

$$\frac{\partial f(z - ct)}{\partial z} = \frac{\partial f(u)\,\partial u}{\partial u\,\partial z} = \frac{\partial f(u)}{\partial u} = f'$$

and

$$\frac{\partial^2 f(z - ct)}{\partial z^2} = f''$$

Similarly,

$$\frac{\partial f(z - ct)}{\partial t} = \frac{\partial f(u)}{\partial u}\frac{\partial u}{\partial t} = -cf'$$

and

$$\frac{\partial^2 f(z - ct)}{\partial t^2} = c^2 f''$$

Consequently, we obtain

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = f'' - \frac{c^2}{c^2}f'' = 0$$

which verifies that $f(z - ct)$ is a solution of (9.11). A function such as $f(z - ct)$ represents a disturbance that propagates along the z axis with a velocity c. A typical plot of $f(z - ct)$ as a function of z for various values of t is given in Fig. 9.2 and clearly shows that the disturbance propagates in the z direction with a velocity c.

Another solution to (9.11) is any arbitrary function $f(z + ct)$. This solution is similar to the previous one, except that it represents a disturbance propagating in the negative z direction.

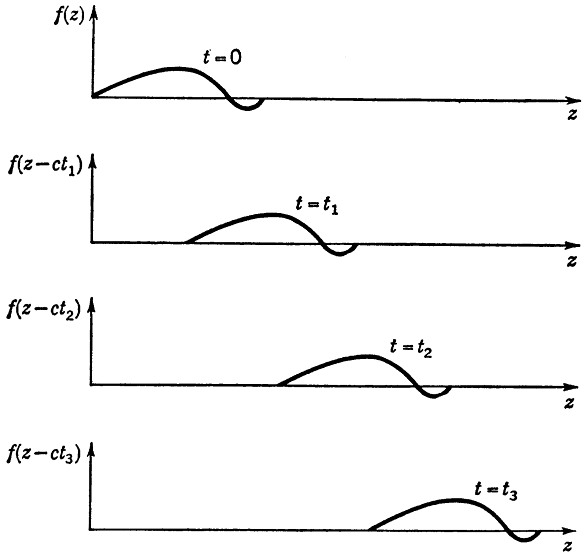

FIG. 9.2. Propagation of a disturbance $f(z)$.

In a homogeneous, isotropic, source-free material body with a permittivity ϵ and a permeability μ, a similar derivation shows that the wave equation satisfied by $\mathbf{E}$ is

$$\nabla^2\mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{9.12}$$

This equation is similar to (9.10), with $\mu\epsilon$ replacing $\mu_0\epsilon_0$. The solution is also similar, with the exception that the velocity of propagation is now $v = (\mu\epsilon)^{-1/2}$ instead of c.

In a conducting body with parameters μ, ϵ, and σ, where σ is the conductivity, we expect to obtain a wave equation with a damping term present. The presence of an electric field $\mathbf{E}$ will cause a conduction current $\mathbf{J} = \sigma\mathbf{E}$ to flow, and this will result in a loss of energy because of joulean heat loss. In a conducting medium we must replace (9.9b) by

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma\mathbf{E} \tag{9.13}$$

The current that occurs here is not an impressed source current but rather the conduction current that flows as a result of the presence of the field **E**. The free charge in the conductor may be assumed to be zero, and hence $\nabla \cdot \mathbf{E}$ is still zero. Any free charge initially present decays to zero in an extremely short time interval since the relaxation time for a good conductor is very small. If we now use (9.13) to eliminate $\nabla \times \mathbf{H}$ in (9.9*a*), we find that

$$\nabla \times \nabla \times \mathbf{E} = \nabla\nabla \cdot \mathbf{E} - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = -\mu\epsilon \frac{\partial^2\mathbf{E}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

or

$$\nabla^2\mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\epsilon \frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{9.14}$$

As anticipated, we obtain a damping term $-\mu\sigma(\partial\mathbf{E}/\partial t)$ which is directly proportional to the conductivity σ. The presence of this term results in an exponential decay of the wave as it propagates away from the source. A fuller appreciation of (9.14) will be obtained in a later section when sinusoidal time-varying fields are analyzed.

9.4. Power Flow and Energy

Energy may be transported through space by means of electromagnetic waves. In addition, energy may be stored in the electromagnetic field, a result we could well anticipate in view of our earlier results in connection with energy storage in the static electric and magnetic fields. In this section relations will be derived that permit the evaluation of the energy stored in a given volume of space and the flow of energy in the electromagnetic field. In ordinary circuit theory, power flow is related to the product of voltage and current. For the electromagnetic field we shall find that the power flow across an element of area $d\mathbf{S}$ is given by $\mathbf{E} \times \mathbf{H} \cdot d\mathbf{S}$. In this expression **E** is analogous to voltage and has the dimensions of volts per meter, while **H** is analogous to current and has the dimensions of amperes per meter. Power flow is a vector quantity, and hence it is not surprising to find that it is given by a vector relation such as $\mathbf{E} \times \mathbf{H}$ watts per unit area.

To derive the relations that we wish to obtain, consider a volume V bounded by a closed surface S. Let the material inside S be isotropic, homogeneous, and characterized by electrical parameters μ, ϵ and conductivity σ. Consider the expression $\nabla \cdot \mathbf{E} \times \mathbf{H}$, which we may expand to obtain

$$\nabla \cdot \mathbf{E} \times \mathbf{H} = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}$$

On the right-hand side we now replace $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ by $-\partial\mathbf{B}/\partial t$ and $\partial\mathbf{D}/\partial t + \sigma\mathbf{E}$, as obtained from Maxwell's equations, so that

$$\nabla \cdot \mathbf{E} \times \mathbf{H} = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \sigma\mathbf{E} \cdot \mathbf{E}$$

When μ and ϵ are constant we can write

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = \mu\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} = \frac{\mu}{2}\frac{\partial(\mathbf{H}\cdot\mathbf{H})}{\partial t} = \frac{\mu}{2}\frac{\partial H^2}{\partial t}$$

and similarly $$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} = \frac{\epsilon}{2}\frac{\partial E^2}{\partial t}$$

where H and E represent the magnitudes of $\mathbf{H}$ and $\mathbf{E}$. We now obtain the basic result

$$\nabla\cdot\mathbf{E}\times\mathbf{H} = -\frac{\partial}{\partial t}\left(\frac{\mu}{2}H^2 + \frac{\epsilon}{2}E^2\right) - \sigma E^2 \tag{9.15}$$

In this expression we interpret $(\mu/2)H^2$ and $(\epsilon/2)E^2$ as the density of energy stored in the magnetic and electric fields. This interpretation is carried over directly from the similar results that were derived earlier for the static fields. The term $-\sigma E^2$ is interpreted as the power loss per unit volume due to joulean heating brought about by the flow of conduction current $\sigma\mathbf{E}$. Equation (9.15) is thus understood to relate the divergence of power (which is a rate of flow of energy) from a unit element of volume to the sum of the time rate of decrease of the energy stored in the magnetic and electric fields per unit volume minus the power loss per unit volume.

A macroscopic form of (9.15) is obtained by integrating throughout the volume V and converting the volume integral of the divergence to a surface integral by means of the divergence theorem. We obtain

$$\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V\left(\frac{\mu}{2}H^2 + \frac{\epsilon}{2}E^2\right)dV - \int_V \sigma E^2\,dV \tag{9.16}$$

a result which states that the instantaneous flow of power across a closed surface S is equal to the time rate of decrease of the energy stored in the field in the interior of S minus the power loss due to joulean heating within the volume V. The vector

$$\mathbf{P} = \mathbf{E}\times\mathbf{H} \tag{9.17}$$

is called the Poynting vector and gives the instantaneous flow of power (both magnitude and direction) per unit area. The total power flow across a given surface is obtained by integrating the normal component of $\mathbf{E}\times\mathbf{H}$ over the surface in question. Energy flows in a direction that is perpendicular to both the $\mathbf{E}$ and $\mathbf{H}$ field vectors. While the interpretation of $\mathbf{E}\times\mathbf{H}$ as representing power density at a point is ordinarily a useful one, it should be noted that (9.16) states only that the total surface integral of $\mathbf{P}$ gives a net power flow across a closed surface.

Example 9.1. Plane Waves. Let us assume that the only electric field component present is E_x and that this component is a function of

z and t only, as in Sec. 9.3. If the field varies sinusoidally in time with a radian frequency ω, a possible solution to the wave equation is

$$E_x = f(z - ct) = E_0 \sin \frac{\omega}{c} (z - ct) = E_0 \sin \left(\frac{\omega}{c} z - \omega t \right) \quad (9.18a)$$

where E_0 is an amplitude constant. From the equation

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

we obtain

$$-\mu_0 \frac{\partial \mathbf{H}}{\partial t} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \mathbf{a}_y \frac{\partial E_x}{\partial z}$$

and hence
$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{\omega E_0}{\mu_0 c} \mathbf{a}_y \cos \frac{\omega}{c} (z - ct)$$

or by integrating with respect to time,

$$H_y = \frac{E_0}{\mu_0 c} \sin \left(\frac{\omega}{c} z - \omega t \right) = E_0 \left(\frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}} \sin \left(\frac{\omega}{c} z - \omega t \right) \quad (9.18b)$$

The parameter $(\epsilon_0/\mu_0)^{\frac{1}{2}}$ has the dimensions of admittance and is called the intrinsic admittance of free space. The reciprocal quantity $(\mu_0/\epsilon_0)^{\frac{1}{2}}$ is called the intrinsic impedance of free space and will be denoted by Z_0; that is,

$$Z_0 = \left(\frac{\mu_0}{\epsilon_0} \right)^{\frac{1}{2}} \quad (9.19)$$

Numerically, $Z_0 = 120\pi = 377$ ohms.

The particular solution of Maxwell's equations given by (9.18) is called a uniform-plane electromagnetic wave since both **E** and **H** lie in a plane (xy plane in this case) perpendicular to the direction of propagation (z direction in this example). The plane wave is uniform since neither E_x nor H_y varies with the transverse coordinates x and y. The space relationship between **E** and **H** is shown in Fig. 9.3.

The power flow is given by

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = E_x H_y \mathbf{a}_x \times \mathbf{a}_y = E_x H_y \mathbf{a}_z$$

and is seen to be in the direction of propagation. Substituting for E_x and H_y, we obtain

$$\mathbf{P} \cdot \mathbf{a}_z = \frac{E_0^2}{Z_0} \sin^2 \frac{\omega}{c} (z - \omega t)$$

as the instantaneous power flow across a unit area of the xy plane. The

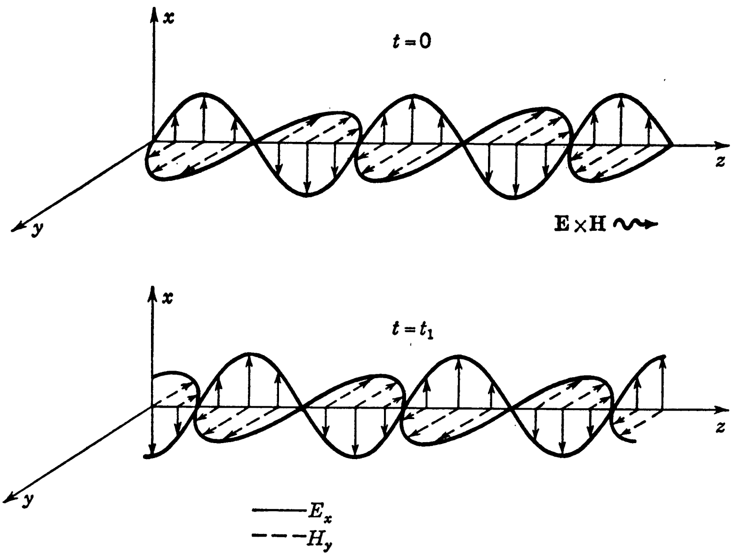

FIG. 9.3. Space relationship between **E** and **H** in a plane TEM wave.

time-average power flow per unit area is given by

$$P_{av} = \frac{1}{2}\frac{E_0^2}{Z_0}$$

The instantaneous energy stored in the electric field per unit volume is

$$\frac{\epsilon_0}{2} E_0^2 \int_0^1 \int_0^1 \int_0^1 \sin^2 \frac{\omega}{c}(z - ct)\, dx\, dy\, dz$$
$$= \frac{\epsilon_0}{2} E_0^2 \int_0^1 \tfrac{1}{2}\left[1 - \cos\frac{2\omega}{c}(z - ct)\right] dz$$
$$= \frac{\epsilon_0}{4} E_0^2 \left[z - \frac{c}{2\omega}\sin\frac{2\omega}{c}(z - ct)\right]_0^1$$

If we average over one period in time, we obtain

$$U_e = \frac{\epsilon_0 E_0^2}{4} \tag{9.20}$$

as the time-average energy stored in the electric field per unit volume. A similar derivation shows that the time-average energy stored in the magnetic field per unit volume is given by

$$U_m = \frac{\mu_0}{4}\left(\frac{E_0}{Z_0}\right)^2 = \frac{\epsilon_0}{4} E_0^2 = U_e \tag{9.21}$$

and is equal to the time-average energy stored in the electric field. Power is a rate of flow of energy, and hence if we multiply the total energy stored in the field per unit volume by the velocity of energy transport, we should obtain the expression for power flow. For the present example,

$$c(U_e + U_m) = \frac{c\epsilon_0}{2} E_0^2 = \frac{\epsilon_0}{2(\mu_0\epsilon_0)^{1/2}} E_0^2$$
$$= \frac{E_0^2}{2Z_0} = P_{av} \tag{9.22}$$

which checks with our earlier result. We are thus able to say that for a plane electromagnetic wave, energy is transported with the velocity c in free space; that is, the velocity of propagation of energy is the same as the phase velocity of the wave as given in (9.18a). Later we shall discover circumstances where the two velocities are not the same.

9.5. Sinusoidal Time-varying Fields

In practice, we generally deal with steady-state sinusoidal time-varying fields. Just as in circuit theory, it is convenient to introduce an abbreviated notation and to represent each field vector as a complex phasor. If the angular radian frequency is ω, we write, for the electric field,

$$\mathbf{E}'(x,y,z,t) = \text{Re}\,[\mathbf{E}(x,y,z)e^{j\omega t}] \tag{9.23}$$

where the prime is used to signify the real physical field. For brevity, we represent the electric field simply by the complex phasor $\mathbf{E}(x,y,z)$, where $\mathbf{E}(x,y,z)$ is a complex space vector and a function of x, y, z only. Logically, we should adopt a different notation for the phasor quantity, but the use of standard boldface type should provide very little confusion since we shall be dealing almost entirely with sinusoidal time-varying fields in the remainder of the book. Each space component of the phasor $\mathbf{E}$ is a complex quantity, for example, $E_x(x,y,z) = E_{xr}(x,y,z) + jE_{xi}(x,y,z)$, where E_{xr} is the real part and E_{xi} the imaginary part. Particular care must be used to avoid thinking of E_{xr} and E_{xi} as components of a space vector, as is sometimes done in circuit theory. The quantity $E_{xr} + jE_{xi}$ forms one component, the x component, of the complex phasor space vector $\mathbf{E}$. The physical field is always obtained by multiplying by $e^{j\omega t}$ and taking the real part (or the imaginary part).

When using complex phasor notation the time-average electric- and magnetic-energy densities are given by

$$U_e = \frac{\epsilon}{4}\mathbf{E}\cdot\mathbf{E}^* \tag{9.24a}$$

$$U_m = \frac{\mu}{4}\mathbf{H}\cdot\mathbf{H}^* \tag{9.24b}$$

where the asterisk denotes the complex conjugate phasor. The additional factor $\frac{1}{2}$ arises because of averaging over one period in time. In (9.24) it is assumed that ϵ and μ are real; the case when ϵ and μ are complex will be considered later. The proof of (9.24) may be developed along lines similar to the following. If **E** has only an x component, then the physical field $E'_x(x,y,z,t)$ is given by

$$\begin{aligned} E'_x &= \text{Re}\,[E_{xr}(x,y,z) + jE_{xi}(x,y,z)]e^{j\omega t} \\ &= E_{xr}\cos\omega t - E_{xi}\sin\omega t \end{aligned}$$

where E_{xr} and E_{xi} are real. The time average of E'^2_x is obviously equal to $\frac{1}{2}(E^2_{xr} + E^2_{xi})$, a result which is seen to be equal to $\frac{1}{2}E_xE^*_x$ as well.

When we deal with steady-state sinusoidal time-varying fields, all time derivatives $\partial/\partial t$ may be replaced by $j\omega$. Thus Maxwell's equations reduce to the following form:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{9.25a}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J} \tag{9.25b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{9.25c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{9.25d}$$

$$\nabla \cdot \mathbf{J} = -j\omega\rho \tag{9.25e}$$

When time-varying fields are applied to material bodies, the polarization vectors **P** and **M** vary with time at the same frequency as the applied fields. Because of damping forces which are always present to some extent, the polarization vectors **P** and **M** will usually lag behind the fields **E** and **H**. This means that, in general, ϵ and μ must be complex. The complex nature of ϵ and μ is a manifestation of power loss that will occur in the material because of the work that must be done in overcoming the frictional damping forces. As an example, let **E** be the complex phasor representing the field acting to polarize a dielectric material. The polarization **P** per unit volume is given by

$$\mathbf{P} = \epsilon_0\alpha e^{-j\phi}\mathbf{E}$$

where α is a positive real constant and ϕ is the phase angle by which **P** lags **E**. We now see that the susceptibility is given by

$$\chi_e = \alpha e^{-j\phi}$$

and hence the dielectric permittivity ϵ is given by

$$\epsilon = \epsilon_0(1 + \chi_e) = \epsilon_0(1 + \alpha\cos\phi - j\alpha\sin\phi) = \epsilon' - j\epsilon'' \tag{9.26}$$

and is a complex quantity.

If in (9.25b) we let ϵ be complex and let **J** be a conduction current, we obtain

$$\nabla \times \mathbf{H} = j\omega\epsilon'\mathbf{E} + (\omega\epsilon'' + \sigma)\mathbf{E} \tag{9.27}$$

This result shows that the imaginary part of ϵ is equivalent to an increase in the conductivity of the medium. At high frequencies σ for a good dielectric is very small and most of the energy loss is caused by polarization damping forces that bring in the term ϵ''. For convenience when dealing with dielectric materials that have a finite conductivity σ, a single complex dielectric permittivity

$$\epsilon = \epsilon' - j\epsilon'' - \frac{j\sigma}{\omega}$$

is usually introduced so as to include the effect of both damping losses and conduction losses in a single term. The properties of the dielectric material are usually specified by giving its dielectric constant κ and its loss tangent tan δ_l. The value of ϵ is then

$$\epsilon = \kappa\epsilon_0(1 - j \tan \delta_l)$$

In this equation tan δ_l includes the effects of finite conductivity as well as the effects of polarization damping forces.

Remarks similar to the above may be made about the permeability μ. When it is necessary to consider complex μ, we shall use the notation $\mu = \mu' - j\mu''$. In passive materials the imaginary parts of ϵ and μ are negative since these terms must correspond to a loss in the material. If the imaginary parts were positive, they would indicate a generation of energy by the material body, which is a violation of the condition that the material is passive in nature. The proof of these remarks will be given in Sec. 9.7.

9.6. Helmholtz's Equation

For sinusoidal time-varying fields the wave equation for waves in free space may be obtained from (9.10) simply by replacing $\partial^2/\partial t^2$ by $-\omega^2$; thus

$$\nabla^2\mathbf{E} + \omega^2\mu_0\epsilon_0\mathbf{E} = 0 \tag{9.28}$$

or

$$\nabla^2\mathbf{E} + k_0^2\mathbf{E} = 0 \tag{9.29}$$

where $k_0 = \omega(\mu_0\epsilon_0)^{1/2} = \omega/c$ is called the free-space wave number and $\mathbf{E}$ in (9.29) is now a complex phasor space vector that is independent of time. Equation (9.29) is commonly referred to as the vector Helmholtz equation. A simple application of this equation is given in the following example.

Example 9.2. Sinusoidal Time-varying Plane Wave. As a simple example of a solution to (9.29), consider the case where $\mathbf{E}$ is a function of z only and, furthermore, has only an x component. We now have

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x = 0$$

for which a general solution is

$$E_x(z) = A_1 e^{-jk_0 z} + A_2 e^{jk_0 z} \tag{9.30}$$

where A_1 and A_2 are amplitude constants. The physically real field $E'_x(z,t)$ is given by

$$\begin{aligned} E'_x(z,t) &= \text{Re}\,(A_1 e^{-jk_0 z + j\omega t} + A_2 e^{jk_0 z + j\omega t}) \\ &= A_1 \cos\,(k_0 z - \omega t) + A_2 \cos\,(k_0 z + \omega t) \end{aligned} \tag{9.31}$$

provided A_1 and A_2 are real. We thus see that $A_1 e^{-jk_0 z}$ represents a plane wave propagating in the positive z direction while $A_2 e^{+jk_0 z}$ represents a plane wave propagating in the negative z direction.

The distance a wave must propagate in order for its phase angle to change by an amount 2π is called the wavelength. In free space we shall denote the wavelength by the symbol λ_0. By definition we now have $k_0 \lambda_0 = 2\pi$, or

$$\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi}{\omega} c = \frac{c}{f} \tag{9.32a}$$

and also

$$k_0 = \frac{2\pi}{\lambda_0} \tag{9.32b}$$

The relationship between wavelength, velocity, and frequency obtained here is undoubtedly familiar to the reader from earlier courses in physics.

The magnetic field **H** corresponding to the electric field **E** given by (9.30) is readily found from (9.25*a*) and is

$$\mathbf{H} = \mathbf{a}_y (Y_0 A_1 e^{-jk_0 z} - Y_0 A_2 e^{jk_0 z}) \tag{9.33}$$

where $Y_0 = (\epsilon_0/\mu_0)^{1/2}$ and is the intrinsic admittance of free space. It is seen that the direction of **H** is reversed for the wave propagating in the negative z direction. This is a necessary requirement in order to obtain a reversal in the direction of power flow. Note that the electric and magnetic fields are in time phase but space quadrature.

The time-average power flow is given by one-half of the real part of the complex Poynting vector (details of the complex Poynting theorem are developed in the next section)

$$\mathbf{P} = \tfrac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{9.34}$$

If we consider the wave propagating in the positive z direction only ($A_2 = 0$), the power flow across a unit area in the xy plane is found to be

$$P_{\text{av}} = \tfrac{1}{2} Y_0 |A_1|^2 \tag{9.35}$$

Helmholtz's Equation in Dielectric and Conducting Media

In a dielectric medium with a permittivity ϵ (relative dielectric constant $\kappa = \epsilon/\epsilon_0$) and negligible losses, we have, in place of (9.29),

$$\nabla^2\mathbf{E} + \omega^2\mu_0\epsilon\mathbf{E} = 0$$

or

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{9.36}$$

where $k = \kappa^{1/2}k_0 = (\kappa\omega^2\mu_0\epsilon_0)^{1/2}$. The solution to this equation is similar to that for (9.29) with k_0 replaced by k. The velocity of propagation is $v = \kappa^{-1/2}c$ instead of c. The parameter $\kappa^{1/2}$ is called the index of refraction and will be denoted by the symbol η. In a dielectric the wavelength of plane waves is λ, where

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\eta k_0} = \frac{\lambda_0}{\eta} \tag{9.37}$$

and is less than the free-space wavelength.

In a medium with finite conductivity σ, the required Helmholtz equation is found by replacing $\partial/\partial t$ by $j\omega$ in the general time-varying wave equation (9.14). We obtain

$$\nabla^2\mathbf{E} - j\omega\mu\sigma\mathbf{E} + \omega^2\mu\epsilon\mathbf{E} = 0 \tag{9.38}$$

This may be rewritten as

$$\nabla^2\mathbf{E} - j\omega\mu(j\omega\epsilon + \sigma)\mathbf{E} = 0$$

The term $j\omega\epsilon\mathbf{E}$ is the displacement current density, while $\sigma\mathbf{E}$ is the conduction current density. For metals, σ is of the order of 10^7 mhos per meter (for copper $\sigma = 5.8 \times 10^7$ mhos per meter) and ϵ is approximately equal to $\epsilon_0 = (36\pi)^{-1} \times 10^{-9}$ farad per meter. Consequently, $\sigma/\omega\epsilon \approx 10^{18}\omega^{-1}$, and hence for all frequencies up to the optical range we can neglect $\omega\epsilon$ in comparison with σ; for example, for $f = 10{,}000$ megacycles we have $\omega\epsilon/\sigma \approx 5 \times 10^{-8}$, so that $\omega\epsilon$ is certainly negligible in comparison with σ. This means that in metals the displacement current is entirely negligible compared with the conduction current.

In place of (9.38), we can now write

$$\nabla^2\mathbf{E} - j\omega\mu\sigma\mathbf{E} = 0 \tag{9.39}$$

as the equation satisfied by **E** (and **H** also) in a metal. This equation is a diffusion equation and not a wave equation, since for the general time-varying case it would be of the form

$$\nabla^2\mathbf{E}' - \mu\sigma\frac{\partial\mathbf{E}'}{\partial t} = 0 \tag{9.40}$$

In metals the fields may be thought of as diffusing into the material and will undergo both attenuation and phase retardation in the process. It is only when the displacement current term is predominant that we obtain true wave propagation. On the other hand, (9.39) is similar to the free-space Helmholtz equation, and consequently its solutions are also formally similar. In fact, any solution of (9.29) is a solution of (9.39) if we replace k_0 by $(-j\omega\mu\sigma)^{1/2} = (1 - j)(\omega\mu\sigma/2)^{1/2}$. For a plane wave depending on z only and having only an E_x component of electric field, a solution is

$$E_x = A \exp\left[-j\left(\frac{\omega\mu\sigma}{2}\right)^{1/2} z - \left(\frac{\omega\mu\sigma}{2}\right)^{1/2} z\right] \qquad (9.41)$$

The rate of attenuation is $(\omega\mu\sigma/2)^{1/2}$ nepers per meter. The skin depth δ is defined as

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2} \qquad (9.42)$$

and is the distance the wave must propagate in order to decay by an amount e^{-1}. By using (9.42), we may rewrite (9.41) as

$$E_x = Ae^{-(1+j)z/\delta} \qquad (9.43)$$

9.7. Complex Poynting Vector

The basic relation between power flow and energy storage in the sinusoidal time-varying electromagnetic field may be derived in a manner similar to that used in Sec. 9.4. The curl equations for **E** and **H*** are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad \nabla \times \mathbf{H}^* = -j\omega\epsilon^*\mathbf{E}^* + \sigma\mathbf{E}^*$$

where we have assumed that μ and ϵ may be complex. If we expand $\nabla \cdot \mathbf{E} \times \mathbf{H}^*$, we obtain

$$\begin{aligned}\nabla \cdot \mathbf{E} \times \mathbf{H}^* &= \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* \\ &= -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* + j\omega\epsilon^*\mathbf{E} \cdot \mathbf{E}^* - \sigma\mathbf{E} \cdot \mathbf{E}^* \qquad (9.44)\end{aligned}$$

after substituting for the curl of **E** and **H***. We now integrate (9.44) throughout a volume V bounded by a surface S and use the divergence theorem, with the result that

$$\begin{aligned}\oint_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} = -j\omega \int_V (\epsilon^*\mathbf{E} \cdot \mathbf{E}^* - \mu\mathbf{H} \cdot \mathbf{H}^*)\, dV \\ + \int_V \sigma\mathbf{E} \cdot \mathbf{E}^*\, dV \qquad (9.45)\end{aligned}$$

by taking the vector area $d\mathbf{S}$ directed *into* the volume V. The time-average electric and magnetic energy stored in the field per unit volume is given by

$$U_e = \tfrac{1}{4}\,\mathrm{Re}\, \epsilon\mathbf{E} \cdot \mathbf{E}^* = \tfrac{1}{4}\epsilon'\mathbf{E} \cdot \mathbf{E}^* \qquad (9.46a)$$
$$U_m = \tfrac{1}{4}\,\mathrm{Re}\, \mu\mathbf{H} \cdot \mathbf{H}^* = \tfrac{1}{4}\mu'\mathbf{H} \cdot \mathbf{H}^* \qquad (9.46b)$$

The sum of the following terms will be shown to represent the time-average power loss per unit volume:

$$\tfrac{1}{2}\sigma \mathbf{E}\cdot\mathbf{E}^* + \frac{\omega}{2}\operatorname{Im}(\epsilon\mathbf{E}\cdot\mathbf{E}^* + \mu\mathbf{H}\cdot\mathbf{H}^*)$$
$$= \tfrac{1}{2}\sigma \mathbf{E}\cdot\mathbf{E}^* + \frac{\omega}{2}(\epsilon''\mathbf{E}\cdot\mathbf{E}^* + \mu''\mathbf{H}\cdot\mathbf{H}^*) \quad (9.47)$$

In (9.46) ϵ', μ' are the real parts of ϵ and μ, while in (9.47) ϵ'', μ'' are the imaginary parts that represent loss due to polarization damping.

If we separate (9.45) into its real and imaginary parts we have

$$\tfrac{1}{2}\operatorname{Re}\oint_S \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} = \int_V \frac{\omega}{2}(\epsilon''\mathbf{E}\cdot\mathbf{E}^* + \mu''\mathbf{H}\cdot\mathbf{H}^*)\,dV + \int_V \frac{\sigma}{2}\mathbf{E}\cdot\mathbf{E}^*\,dV \quad (9.48a)$$

and

$$\operatorname{Im}\oint_S \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} = 4\omega\int_V (U_m - U_e)\,dV \quad (9.48b)$$

since $\epsilon = \epsilon' - j\epsilon''$ and $\mu = \mu' - j\mu''$. Equation (9.48a) states that the average power flow into the volume V is given by the integral of one-half the real part of the complex Poynting vector $\mathbf{E}\times\mathbf{H}^*$ over the surface S bounding V and that this is equal to the time-average power loss in V due to conduction current losses and polarization damping losses. The expressions for polarization damping losses given in (9.47) are identified by analogy with the expression for conduction current losses. For example, Maxwell's curl equation for $\mathbf{H}$ is

$$\nabla\times\mathbf{H} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E} = j\omega\epsilon'\mathbf{E} + (\omega\epsilon'' + \sigma)\mathbf{E}$$

and hence if $(\sigma/2)\mathbf{E}\cdot\mathbf{E}^*$ is the average power loss per unit volume due to joulean heating, then $(\omega\epsilon''/2)\mathbf{E}\cdot\mathbf{E}^*$ is the average power loss due to electric polarization damping forces. From (9.48a) it is readily seen that the imaginary parts of ϵ and μ must be negative, since these terms represent energy dissipation and not energy generation in passive materials.

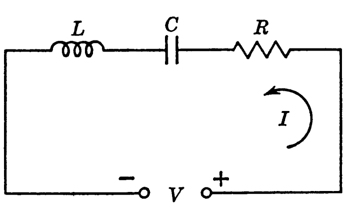

FIG. 9.4. A series RLC circuit.

Equation (9.48b) states that the integral of the imaginary part of the complex Poynting vector over the surface S bounding V is equal to 4ω times the difference in the average energy stored in the magnetic and electric field. This result is reminiscent of that obtained for low-frequency circuits, as is shown below.

Consider a simple RLC series circuit, as in Fig. 9.4. With an applied voltage V, a current $I = V/(R + j\omega L + 1/j\omega C)$ flows. For this circuit we have

$$\tfrac{1}{2}VI^* = \tfrac{1}{2}II^*Z_{in} = \tfrac{1}{2}II^*R + 2j\omega\left(\frac{II^*L}{4} - \frac{II^*}{4\omega^2C^2}C\right) \quad (9.49)$$

But $\tfrac{1}{2}II^*R$ is the average power loss in the resistor, $II^*L/4$ is the average energy stored in the inductor, and $II^*C/4(\omega C)^2$ is the average energy stored in the capacitor since $I/\omega C$ is the voltage across the capacitor. Thus (9.49) is the low-frequency circuit equivalent of (9.48). The result is not an unexpected one, since, after all, low-frequency circuit theory is based on Maxwell's equations. The relation between circuit theory and field theory is examined in greater detail in Sec. 9.10.

9.8. Boundary Conditions

In an infinite unbounded homogeneous medium the solutions to the field equations are relatively easy to obtain. In most practical situations, however, we require a solution for the fields in the presence of conducting bodies and boundaries separating material media with differing electrical parameters ϵ and μ. In order to obtain a solution, a knowledge of the boundary conditions to be applied to the field vectors is needed. The time-varying field satisfies boundary conditions similar to those obeyed by the static fields, as we show in the following analysis.

Boundary between Two Dielectric Media

Consider two dielectric media with electrical permittivity ϵ_1 and ϵ_2 and permeability μ_1 and μ_2 and having a common boundary, as in Fig. 9.5. Construct an infinitesimal "coin"-shaped box with end faces of area ΔS in adjacent media and the end surfaces parallel to the common boundary surface. Since $\nabla \cdot \mathbf{D} = 0$ in the present case, it follows by applying Gauss' law to the volume enclosed by the coin-shaped box that

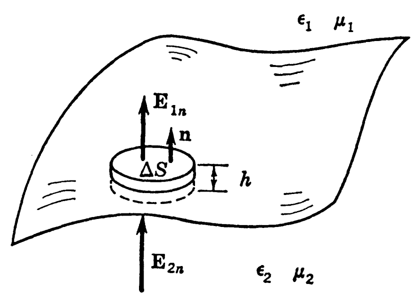

FIG. 9.5. Boundary between two dielectric media.

$$\int_V \nabla \cdot \mathbf{D}\, dV = 0 = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

and hence $$\lim_{h \to 0} \oint_S \mathbf{D} \cdot d\mathbf{S} = (D_{1n} - D_{2n})\,\Delta S = 0$$

where the subscript n denotes the component normal to the surface. The limit $h \to 0$ is taken to ensure that there will be no flux passing out through the sides of the box. We now have

$$D_{1n} = D_{2n} \qquad \text{or} \qquad \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

which may be written in vector form as

$$\mathbf{n} \cdot \mathbf{D}_1 = \mathbf{n} \cdot \mathbf{D}_2 \tag{9.50}$$

where $\mathbf{n}$ is the unit normal to the boundary surface. This relation is the same as that derived for the static field.

If we apply Gauss' law to the field $\mathbf{B}$, we obtain in a similar way the result

$$\mathbf{n} \cdot \mathbf{B}_1 = \mathbf{n} \cdot \mathbf{B}_2 \tag{9.51}$$

since $\nabla \cdot \mathbf{B} = 0$. Thus the normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous across a surface separating two dielectric media.

To derive the boundary relations that apply to tangential field components, consider a small contour C of length Δl with sides lying on

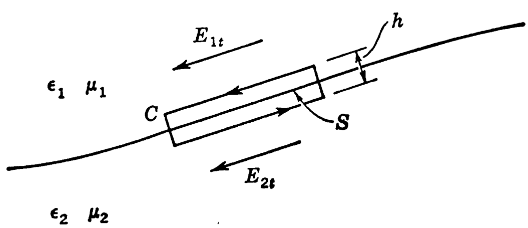

FIG. 9.6. Contour C for deriving boundary conditions for the tangential components of the field.

adjacent sides of the surface, as in Fig. 9.6. Application of Stokes' law to the equation $\nabla \times \mathbf{E} = -j\omega\mathbf{B}$ gives

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -j\omega \int_S \mathbf{B} \cdot d\mathbf{S}$$

Now

$$\lim_{h\to 0} \oint_C \mathbf{E} \cdot d\mathbf{l} = (E_{1t} - E_{2t})\,\Delta l$$

and

$$\lim_{h\to 0} - j\omega \int_S \mathbf{B} \cdot d\mathbf{S} = \lim_{h\to 0} - j\omega B_t h\,\Delta l = 0$$

so that we obtain

$$E_{1t} = E_{2t} \tag{9.52}$$

A similar application of Stokes' law to the equation $\nabla \times \mathbf{H} = j\omega\mathbf{D}$ gives

$$H_{1t} = H_{2t} \tag{9.53}$$

In (9.52) and (9.53) the subscript t denotes the component of the field that is tangential to the common surface separating the two media. These equations state that the tangential fields are continuous across a boundary between two different dielectric media.

Boundary of a Perfect Conductor

In the interior of a perfect conductor ($\sigma = \infty$), the electromagnetic time-varying field is zero. This may be seen from the expression (9.42) for the skin depth δ. As σ tends to infinity, $\delta = (2/\omega\mu\sigma)^{1/2}$ approaches zero. Thus the field decays infinitely fast (by an amount e^{-1} in a distance δ) and cannot penetrate into the conductor. Actually, we never have perfect conductors, but in most cases σ is so large that at high frequencies negligible error is made in assuming that the field in the interior of the conductor is zero. For example, for copper at a frequency of 1,000 megacycles, $\delta \approx 2 \times 10^{-3}$ millimeter; so we could consider the depth of penetration as zero without appreciable error. On the other hand, for a frequency of 1,000 cycles, we have $\delta \approx 2$ millimeters, which in many cases would not be negligible.

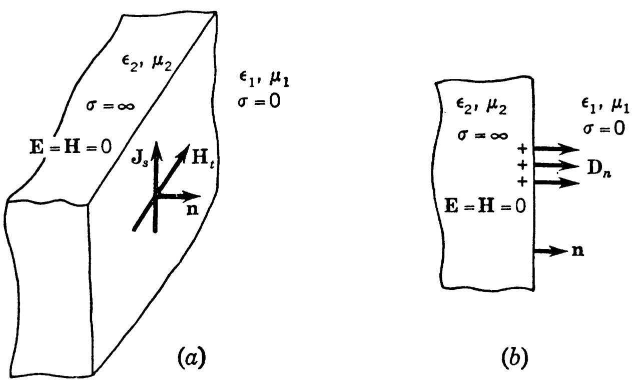

FIG. 9.7. Boundary conditions at a perfect conductor surface.

Because of the phenomenon described above, which is called "skin effect," as σ approaches infinity, the current flows in a narrower and narrower layer, until in the limit a true surface current exists on the surface (this problem is examined in detail in the next chapter). With reference to Fig. 9.7a, let $\mathbf{J}_s$ be the surface current density in amperes per meter. Since the displacement current in the conductor, as well as the field $\mathbf{H}$, is zero, Ampère's circuital law shows that $\mathbf{H}_t$ is perpendicular to $\mathbf{J}_s$ and equal to $\mathbf{J}_s$ in magnitude; thus

$$H_t = J_s$$

or in vector notation,

$$\mathbf{n} \times \mathbf{H} = \mathbf{J}_s \tag{9.54}$$

Similarly, Gauss' law shows that

$$\mathbf{n} \cdot \mathbf{D} = \rho_s \tag{9.55}$$

where ρ_s is the surface charge density, as in Fig. 9.7*b*. While these results are rigorously true only for $\sigma \to \infty$, they are excellent approximations for practical conductors at high frequencies, where by high frequency we mean one that yields a value of skin depth that is small compared with all conductor dimensions.

Since the field in the interior of the conductor is zero and the tangential electric field E_t and normal magnetic field B_n are continuous across a boundary, it follows that

$$E_t = 0 \qquad \text{or} \qquad \mathbf{n} \times \mathbf{E} = 0 \tag{9.56}$$

$$\mathbf{n} \cdot \mathbf{B} = 0 \tag{9.57}$$

at the surface of a perfect conductor.

In any practical problem it is sufficient to ensure that the tangential components of the field satisfy the proper boundary conditions since this will automatically ensure that the normal components of **D** and **B** satisfy their respective boundary conditions. We may prove this statement as follows. Let $\nabla = \nabla_t + \nabla_n$, where ∇_t is the part of the del operator which represents differentiation with respect to the coordinates along the boundary surface separating two different media, and $\nabla_n = \mathbf{n}(\partial/\partial n)$ represents differentiation with respect to the coordinate normal to the boundary surface. The equation $\nabla \times \mathbf{E} = -j\omega\mathbf{B}$ separates into two parts,

$$\nabla_n \times \mathbf{E}_t + \nabla_t \times \mathbf{E}_n = -j\omega\mathbf{B}_t \tag{9.58a}$$

$$\nabla_t \times \mathbf{E}_t = -j\omega\mathbf{B}_n \tag{9.58b}$$

when the tangential and normal components are equated. This result is arrived at by noting that

$$\begin{aligned}\nabla \times \mathbf{E} &= \nabla_t \times (\mathbf{E}_t + \mathbf{E}_n) + \nabla_n \times (\mathbf{E}_t + \mathbf{E}_n) \\ &= \nabla_t \times \mathbf{E}_t + \nabla_n \times \mathbf{E}_t + \nabla_t \times \mathbf{E}_n\end{aligned}$$

since $\nabla_n \times \mathbf{E}_n = \partial(\mathbf{n} \times \mathbf{E}_n)/\partial n = 0$. The term $\nabla_t \times \mathbf{E}_t$ is a vector directed along the normal **n**, while $\nabla_n \times \Gamma_t + \nabla_t \times \mathbf{E}_n$ is a vector in the boundary surface. If we make $\mathbf{E}_t$ continuous across the boundary surface, then the derivatives of $\mathbf{E}_t$ with respect to the coordinates along the boundary surface are also continuous. Therefore $\nabla_t \times \mathbf{E}_t$ is continuous, and likewise $\mathbf{B}_n$ must be continuous across the surface since

$$-j\omega\mathbf{B}_n = \nabla_t \times \mathbf{E}_t$$

For $\mathbf{D}_n$ and $\mathbf{H}_t$ we have the equation $\nabla_t \times \mathbf{H}_t = j\omega\mathbf{D}_n$, and a similar argument shows that $\mathbf{D}_n$ is continuous if $\mathbf{H}_t$ is continuous across the boundary. In the case when $\mathbf{H}_t$ is discontinuous across the boundary because of a surface current, $\mathbf{D}_n$ is also discontinuous. The discontinuity in $\mathbf{D}_n$ is equal to the surface charge density ρ_s. Furthermore, the surface current $\mathbf{J}_s$ and surface charge ρ_s satisfy a continuity equation

$$\nabla_t \cdot \mathbf{J}_s = -j\omega\rho_s \tag{9.59}$$

on the surface. It turns out that if $\mathbf{H}_t$ is made discontinuous by an amount equal to the surface current density, this automatically makes $\mathbf{D}_n$ discontinuous by an amount equal to the surface charge density.

The above results are of great importance in practice, since they make it necessary only to match the tangential field components at a discontinuity surface. This simplifies the analytical details of constructing a solution of Maxwell's field equations.

For the time-varying field a uniqueness theorem exists† which states that if a solution to the field equations has been found such that all boundary conditions are satisfied and also such that the fields have the proper behavior (singularity) at the position of the impressed sources, then this solution is unique. The proof may be constructed along lines similar to those employed in the proof of the uniqueness theorem for electrostatic boundary-value problems in Chap. 2. The details of the proof are not too important; so we omit them. The important fact is that such a theorem exists and thus guarantees the uniqueness of the solution once it has been found.

9.9. Scalar and Vector Potentials

The existence of an electromagnetic field implies a source of impressed currents and charges. If the impressed currents and charges are known, then the field may be determined by means of the equations to be derived in this section. We shall assume sinusoidal time variation, and hence all quantities we deal with are phasor quantities. For time-varying currents and charges the continuity equation $\nabla \cdot \mathbf{J} = -j\omega\rho$ serves to link the current $\mathbf{J}$ and the charge density ρ. As a consequence, we may not specify ρ and $\mathbf{J}$ independently.

The reader may readily verify that when $\mathbf{J}$ and ρ are not zero, the separation of Maxwell's equations into an equation for $\mathbf{E}$ alone or $\mathbf{H}$ alone gives

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = j\omega\mu\mathbf{J} + \nabla\left(\frac{\rho}{\epsilon}\right) = j\omega\mu\mathbf{J} - \frac{1}{j\omega\epsilon}\nabla\nabla \cdot \mathbf{J} \tag{9.60}$$

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = -\nabla \times \mathbf{J} \tag{9.61}$$

These equations are referred to as inhomogeneous Helmholtz equations. As seen, the impressed current density $\mathbf{J}$ enters into these equations in a relatively complicated way. For this reason we generally do not find the fields $\mathbf{E}$ and $\mathbf{H}$ directly, but rather first compute a scalar and a vector

† J. Stratton, "Electromagnetic Theory," sec. 9.2, McGraw-Hill Book Company, Inc., New York, 1941.

potential from which the fields may subsequently be found. The advantage of doing this is analogous to the similar procedure that was used for the static fields.

The field **B** always has zero divergence, and hence we may take

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{9.62}$$

since $\nabla \cdot \nabla \times \mathbf{A}$ is identically zero. From Maxwell's equation

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

so we can now write

$$\nabla \times \mathbf{E} = -j\omega\nabla \times \mathbf{A}$$

or

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0$$

The vector quantity $(\mathbf{E} + j\omega\mathbf{A})$ is irrotational; so it may be derived from the gradient of a scalar potential Φ; that is, the general integral of the above equation is

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi \tag{9.63}$$

where Φ is, as yet, an arbitrary scalar function. If we now substitute (9.62) and (9.63) in the curl equation for **H**, we obtain

$$\nabla \times \mathbf{H} = \frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = j\omega\epsilon\mathbf{E} + \mathbf{J} = j\omega\epsilon(-\nabla\Phi - j\omega\mathbf{A}) + \mathbf{J}$$

Expanding $\nabla \times \nabla \times \mathbf{A}$ to give $\nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}$, we get

$$\nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} = -j\omega\epsilon\mu\,\nabla\Phi + k^2\mathbf{A} + \mu\mathbf{J} \tag{9.64}$$

According to the Helmholtz theorem, a vector function is completely specified by its divergence and curl. Since (9.62) gives only the curl of **A**, we are at liberty to specify the divergence of **A** in any way we choose. If we examine (9.64), it is clear that this equation simplifies considerably if we choose

$$\nabla \cdot \mathbf{A} = -j\omega\epsilon\mu\Phi \tag{9.65}$$

so that

$$\nabla\nabla \cdot \mathbf{A} = -j\omega\epsilon\mu\,\nabla\Phi$$

This particular choice is known as the Lorentz condition. Making use of (9.65) reduces (9.64) to

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} \tag{9.66}$$

which is simpler than either (9.60) or (9.61).

Up to this point all Maxwell's equations except the equation $\nabla \cdot \mathbf{D} = \rho$ have been made use of and are therefore satisfied. To ensure that $\nabla \cdot \mathbf{D} = \rho$, we replace **E** by (9.63) and use the Lorentz condition (9.65) to obtain

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-j\omega \mathbf{A} - \nabla\Phi) = -\nabla^2\Phi - k^2\Phi = \frac{\rho}{\epsilon}$$

or

$$\nabla^2\Phi + k^2\Phi = -\frac{\rho}{\epsilon} \tag{9.67}$$

This equation determines Φ, and then $\nabla \cdot \mathbf{A}$ is obtained from (9.65). The divergence equation $\nabla \cdot \mathbf{D} = \rho$ is thus satisfied provided Φ is a solution of (9.67), and the divergence of **A** is determined from the Lorentz condition.

In practice, we do not need to solve for the scalar potential Φ. If we make use of the Lorentz condition, we can express both **B** and **E** in terms of the vector potential **A** alone. We have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{9.68a}$$

$$\mathbf{E} = -j\omega\mathbf{A} + (j\omega\epsilon\mu)^{-1} \nabla\nabla \cdot \mathbf{A} \tag{9.68b}$$

This result may seem rather strange at first, since normally we should expect to need both the scalar potential Φ and vector potential **A** in order to completely determine the field. The explanation lies in the fact that for time-varying sources the charge density ρ is determined by the current density **J** through the continuity equation. Thus specification of **J** alone is sufficient to completely determine all sources, and hence a solution for **A** in terms of the current density **J** contains all the necessary information to completely specify the time-varying field. In actual fact, the Lorentz condition is merely the continuity equation in disguise, as the following discussion shows.

If we take the Laplacian of (9.65), we find that

$$\nabla \cdot \nabla^2\mathbf{A} = -j\omega\epsilon\mu\nabla^2\Phi$$

since

$$\nabla^2(\nabla \cdot \mathbf{A}) = \nabla \cdot (\nabla^2\mathbf{A})$$

Replacing $\nabla^2\mathbf{A}$ from (9.66) and $\nabla^2\Phi$ from (9.67) gives

$$-k^2\nabla \cdot \mathbf{A} - \mu\nabla \cdot \mathbf{J} = -j\omega\epsilon\mu\left(-k^2\Phi - \frac{\rho}{\epsilon}\right)$$

or

$$-k^2(\nabla \cdot \mathbf{A} + j\omega\epsilon\mu\Phi) = \mu(\nabla \cdot \mathbf{J} + j\omega\rho) \tag{9.69}$$

The left-hand side vanishes by virtue of the Lorentz condition, and hence the right-hand side must also vanish. This means that

$$\nabla \cdot \mathbf{J} = -j\omega\rho$$

The Lorentz condition is seen to be a condition that ensures that the current and charge satisfy the continuity equation; that is, it provides a function Φ, corresponding to an **A** that satisfies (9.66), so that Φ is a solution of (9.67) for a source ρ that is related to **J** in (9.66) by the continuity equation. It is for this reason that Φ may be eliminated and the field determined in terms of the vector potential **A** alone.

When ω equals zero, the equations for **A** and Φ reduce to Poisson's equation, which is the appropriate result for static fields. In the static case ρ and **J** are no longer related, and hence Φ and **A** are now independent. Thus for static fields we require a scalar potential Φ to determine **E** and a vector potential **A** to determine **B**. In this respect the determination of the time-varying field is simpler.

The integration of (9.66) and (9.67) is very similar to the integration of Poisson's equation. If we let (x',y',z') be the source point and (x,y,z) be the field point, the solutions are

$$\mathbf{A}(x,y,z) = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(x',y',z')}{R}\, e^{-jkR}\, dV' \tag{9.70a}$$

$$\Phi(x,y,z) = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(x',y',z')}{R}\, e^{-jkR}\, dV' \tag{9.70b}$$

where $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$ and $k^2 = \omega^2\mu\epsilon$. These solutions represent waves propagating radially outward from the source point; that is, $e^{-jkR+j\omega t}$ is a radially outward propagating wave. As the wave propagates outward its amplitude falls off as $1/R$.

To verify the above solutions consider (9.70b) and note that

$$(\nabla^2 + k^2)\int_V \frac{\rho(x',y',z')}{4\pi\epsilon R}\, e^{-jkR}\, dV' = \frac{1}{4\pi\epsilon}\int_V \rho(x',y',z')(\nabla^2 + k^2)\frac{e^{-jkR}}{R}\, dV' \tag{9.71}$$

We may treat R as the radial coordinate in a spherical coordinate system. Since there is no θ or ϕ variation,

$$\nabla^2 \equiv \frac{1}{R^2}\frac{\partial}{\partial R} R^2 \frac{\partial}{\partial R}$$

By direct differentiation it is now found that

$$\nabla^2\left(\frac{e^{-jkR}}{R}\right) = -\frac{k^2}{R}\, e^{-jkR} \qquad R \neq 0$$

Hence the integrand in (9.71) vanishes at all points except $R = 0$, where it has a singularity. We now surround the singularity point $R = 0$, that is, the point (x,y,z), by a small sphere of radius δ and volume V_0. For all values of x', y', z' within this sphere we can replace $\rho(x',y',z')$ by $\rho(x,y,z)$ and e^{-jkR} by unity, provided we choose δ small enough. Note that the maximum value R can have is δ, so that e^{-jkR} can be made to approach unity with a vanishingly small error. The right-hand side of (9.71) thus reduces to

$$\frac{\rho(x,y,z)}{4\pi\epsilon}\int_{V_0}(\nabla^2 + k^2)\,\frac{1}{R}\,dV'$$

Now
$$\int_{V_0}\frac{k^2}{R}\,dV' = k^2\int_{V_0} R\sin\theta\,d\theta\,d\phi\,dR = 2\pi k^2\delta^2$$

and vanishes as δ tends to zero. We are therefore left with

$$\frac{\rho(x,y,z)}{4\pi\epsilon}\int_{V_0}\nabla^2\left(\frac{1}{R}\right)dV' = -\frac{\rho(x,y,z)}{\epsilon}$$

since
$$\int_{V_0}\nabla^2\left(\frac{1}{R}\right)dV' = -4\pi$$

as was demonstrated in Chap. 1, in connection with the integration of Poisson's equation. Therefore (9.70*b*) is a solution of (9.67).

Equation (9.66) for the vector potential **A** may be written as the sum of three scalar equations. For each component the above proof may be applied to show that (9.70*a*) is a solution. Application of the above solutions for the potentials will be made in Chap. 11, in connection with radiation from antennas.

Quasi-static Potentials

Let us assume that we have impressed sources located in free space where $k = k_0 = 2\pi/\lambda_0 = 2\pi f/c$. If we are interested in the fields in the immediate vicinity of the sources, and if the extent of the source region is small compared with a wavelength, then $k_0R = 2\pi R/\lambda_0$ is very small. We may now replace e^{-jk_0R} by unity, and the solutions for the potentials reduce to the static solutions

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(x',y',z')}{R}\,dV' \tag{9.72a}$$

$$\Phi = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(x',y',z')}{R}\,dV' \tag{9.72b}$$

with the exception that both **J** and ρ have a time variation according to $e^{j\omega t}$. The fields derived from these potentials are called quasi-static fields, since the fields vary with time but the frequency is sufficiently low so that propagation effects are not important for the range of R of interest. In other words, for a region containing the sources that are small compared with the wavelength, the fields are quasi-static and similar in character to the static field.

For somewhat greater values of k_0R, the approximation

$$e^{-jk_0R} = 1 - jk_0R$$

may be used, and we obtain

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \mathbf{J} \frac{1 - jk_0 R}{R} dV' \tag{9.73a}$$

$$\Phi = \frac{1}{4\pi\epsilon} \int_V \rho \frac{1 - jk_0 R}{R} dV' \tag{9.73b}$$

The presence of the term $-jk_0R$ is an indication that propagation effects are becoming important and the contributions to the potentials from the various source elements no longer add in phase. Higher-order approximations are obtained if more terms in the expansion of e^{-jk_0R} are retained.

Retarded Potentials

If we replace k by $k_0 = \omega/c$ and restore the time function $e^{j\omega t}$, the solutions for the potentials may be written as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}}{R} e^{j\omega(t-R/c)}\, dV' \tag{9.74a}$$

$$\Phi = \frac{1}{4\pi\epsilon} \int_V \frac{\rho}{R} e^{j\omega(t-R/c)}\, dV' \tag{9.74b}$$

In this form the potentials are referred to as retarded potentials. The factor $e^{j\omega(t-R/c)}$ shows that at any point a distance R away from the source, the effects caused by changes in the source are not felt until a time interval R/c, the propagation time, has elapsed; that is, contributions to the potential at a point from current or charge sources must include the finite propagation time from each source element to the field point. This means that the potentials are related to source distributions in effect at an earlier time; i.e., they are retarded potentials.

The concept of a retarded potential, although introduced above for sinusoidal time-varying sources, is valid for arbitrary time variation as well. The retarded-potential concept is similar to the action-at-a-distance concept embodied in Coulomb's and Ampère's force laws as contrasted with the field concept.

9.10. Relation between Field Theory and Circuit Theory

Maxwell's equations provide a rigorous and detailed description of the electric and magnetic fields arising from arbitrary current sources in the presence of material bodies. Because of the complexities, rigorous solutions to time-varying electromagnetic problems are obtainable only under very special circumstances, usually where the geometry is particularly simple. In other cases simplifying approximations must be sought.

The reader may be familiar with the well-established techniques for discussing the properties of electrical networks. These are usually characterized by constant lumped-parameter elements such as resistors, capacitors, and inductors. Under steady-state conditions the properties of such networks may be established by setting up and solving a system of algebraic equations. The latter equations arise from an application of Kirchhoff's loop and node equations to the given network, with an assumed current-voltage relationship for each element.

It may seem very surprising that such a relatively simple procedure is available for circuit analysis. Viewed as an electromagnetic boundary-value problem, it is almost hopeless to find a field solution that satisfies the boundary conditions over the connecting leads, the coiled wires of the inductors, and the various shaped conductors that make up the variety of types of capacitors, and for which the primary source is, say, an electron stream within a high-vacuum tube. Circuit theory is obviously an approximation, and if we are to understand the nature and limitations of this technique, it is necessary to determine the assumptions that are required to deduce circuit theory from Maxwell's equations.

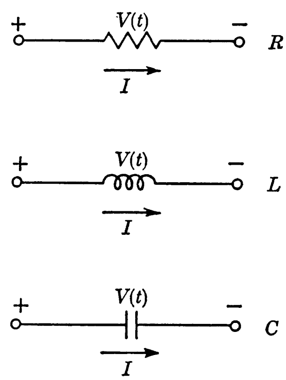

FIG. 9.8. Circuit elements.

As a start, let us briefly review the underlying structure of circuit theory. We assume the existence of four network parameters, the resistance R, the capacitance C, the inductance L, and mutual inductance M. The properties of these parameters are defined in terms of their voltage-current relationships as follows, where for simplicity the harmonic time variation $e^{j\omega t}$ is deleted:

$$V = RI \tag{9.75a}$$

$$V = j\omega LI \tag{9.75b}$$

$$V = \frac{I}{j\omega C} \tag{9.75c}$$

The above may also be specified in terms of the following inverse relations:

$$I = GV \tag{9.76a}$$

$$I = j\omega CV \tag{9.76b}$$

$$I = \frac{V}{j\omega L} \tag{9.76c}$$

where $G = 1/R$ is the conductance. Figure 9.8 illustrates schematically the three circuit elements described above. In each case the voltage V is considered to be the difference in potential between the terminals of the element. For an arbitrary network it is, furthermore, supposed that a unique potential may be assigned each circuit node. Consequently, the sum of the voltages taken around any closed path is zero, which is the statement of the Kirchhoff voltage law.

The current I is assumed to pass continuously through each element from one terminal to the other. Only conduction currents of this type are assumed to exist; consequently, their algebraic sum at any junction is zero, in order to conserve charge. This is the statement of Kirchhoff's current law.

When two coils are coupled so that some of the magnetic flux is common to both, a mutual-inductance term must be added to the circuit equations. Figure 9.9 illustrates schematically a circuit element which requires the use of a mutual inductance M. Since this is a four-terminal device, two equations must be written to describe its voltage-current characteristics. These equations are

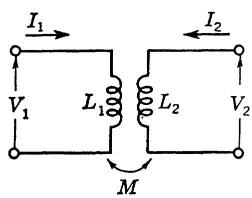

FIG. 9.9. Transformer-type circuit element.

$$V_1 = j\omega L_1 I_1 + j\omega M I_2 \qquad (9.77a)$$
$$V_2 = j\omega M I_1 + j\omega L_2 I_2 \qquad (9.77b)$$

In the absence of mutual coupling, (9.77) gives the voltage-current relations of the two separate inductors and is of the form (9.75b).

With these assumptions concerning the nature of V and I it is possible to establish Kirchhoff's laws. Using the latter and the properties of the circuit elements as described in (9.75) and (9.77), the entire steady-state theory of linear circuit analysis can be developed.† If we are to establish a justification of this theory, we must confirm that (9.75), (9.77), and unique relations for V and I can be derived from Maxwell's equations under suitable conditions.

In order to establish the desired result, we shall assume, first of all, that the maximum circuit dimensions are small compared with wavelength. If we consider any electronic device for which we intuitively feel that circuit theory should be applicable, we shall find the above assumption well justified. For example, an ordinary radio receiver has dimensions of much less than 1 meter, which is quite small compared with the smallest signal wavelength, which is around 200 meters. As a consequence of the above assumption, the fields will be quasi-static in nature; that is, when we expand e^{-jk_0R} we obtain the following expansions for the potentials:

$$\mathbf{A}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(x',y',z')}{R}\,dV' - \frac{jk_0\mu_0}{4\pi}\int_V \mathbf{J}(x',y',z')\,dV' - \frac{k_0^2\mu_0}{4\pi}\int_V \mathbf{J}(x',y',z')R\,dV' + \cdots$$

† It is not difficult to extend this work to transient conditions as well, but it will be easier to emphasize the fundamentals if we assume steady-state harmonic time variations.

$$\Phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(x',y',z')}{R}\,dV' - \frac{jk_0}{4\pi\epsilon_0}\int_V \rho(x',y',z')\,dV' - \frac{k_0^2}{4\pi\epsilon_0}\int_V \rho(x',y',z')R\,dV' + \cdots$$

where $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$. The first term in the expansion for **A** and Φ is the same as for stationary sources, except that here both **J** and ρ are permitted to vary with time. These terms define the quasi-static potentials. The second term in each expansion integrates to a constant and drops out when the expressions $\nabla \times \mathbf{A}$ and $\nabla\Phi$ are formed. This term does contribute to the evaluation of the electric field through the quantity $-j\omega\mathbf{A}$, but the entire expression is proportional to k_0^2. This is the same order in k_0 as the third terms for Φ and **A** given above. These terms, along with the higher-order terms in k_0, are negligible for circuit elements that are small compared with the wavelength. Thus we may compute the instantaneous electric and magnetic fields from the charges and currents that exist at that instant as if the sources were time-stationary.

The characteristics of the R, L, and C elements in a circuit may be specified from the field-theory point of view on an energy basis. Thus we consider the ideal capacitor as a lossless element which stores electric energy. A practical capacitor would then be one for which the magnetic stored energy and the losses were negligible compared with the stored electric energy. The ideal inductor would, conversely, store only magnetic energy, and in the practical case it would be assumed that the stored electric energy and losses were negligible. The resistor, however, ideally dissipates energy; practically, some energy storage is unavoidable, but usually negligible. Let us consider each element in greater detail.

The Inductor

Figure 9.10 illustrates an inductor L. Physically, it usually consists of a solenoidal winding of good conducting wire. Because of its construction it sets up a magnetic field which tends to be localized in the region of the coil. This is easily confirmed for a solenoid whose length-to-diameter ratio is large. Because of the quasi-static nature of the field, the results of Chap. 6 apply and reveal (see Prob. 6.8) that the magnetic field is negligible, except within the solenoid, where it is essentially uniform. For moderate length-to-diameter ratios the magnetic field may still be assumed localized in the vicinity of the inductor. This characteristic is emphasized in Fig. 9.10 by crosshatching the assumed local region of the magnetic field.

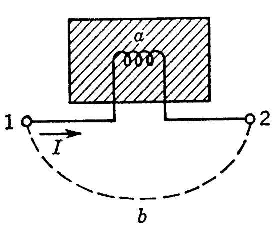

FIG. 9.10. An inductor.

If Faraday's law is applied to the closed contour $1a2b$ in Fig. 9.10, we obtain

$$\oint_{1a2b} \mathbf{E} \cdot d\mathbf{l} = \int_{1a2} \mathbf{E} \cdot d\mathbf{l} + \int_{2b1} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} \qquad (9.78)$$

where the path $1a2$ is along the coil wires, while $2b1$ is any return path in air. If the conductivity of the wire forming the inductor coil is sufficiently great, then within the wires $\mathbf{E} = \mathbf{J}/\sigma \approx 0$, so that (9.78) becomes

$$\int_{2b1} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} \qquad (9.79)$$

The assumption of quasi-static conditions means that we may use time-stationary concepts to evaluate $\int \mathbf{B} \cdot d\mathbf{S}$, which is the total flux linking the closed circuit indicated. By definition of inductance,

$$L = \frac{\int \mathbf{B} \cdot d\mathbf{S}}{I}$$

and consequently,

$$\int_{1b2} \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt}(LI) = j\omega LI \qquad (9.80)$$

after replacing d/dt by $j\omega$ and changing the sense of the path $2b1$ to $1b2$. We may note that the value of L depends somewhat on the external path $1b2$, but to the extent that most of the flux is localized as shown, moderate changes in $1b2$ will not particularly affect the net flux linkage. As a consequence, L may be thought of as a property of the inductor rather than the circuit; that is, we anticipate that the inductor will be connected to other circuit elements, and we assume that nearby resistors and capacitors contribute negligible flux in the evaluation of the right-hand side of (9.80). Furthermore, nearby coils are assumed also to provide negligible flux. When the latter condition is not fulfilled, then a mutual-inductance element is involved, for which a separate discussion is necessary. Thus whether as an individual element or an element in a circuit, the right-hand side of (9.80) depends on the self-inductance of the coil by itself and is essentially independent of the path $1b2$. This also means that $\int_{1b2} \mathbf{E} \cdot d\mathbf{l}$ is a unique quantity under steady-state conditions, and we may use it as a definition of difference in potential V across the inductor terminals; that is,

$$V = \int_{1b2} \mathbf{E} \cdot d\mathbf{l} = j\omega LI \qquad (9.81)$$

where the path $1b2$ is any path external to the inductor but in its general vicinity. This result confirms (9.75b).

For the case of mutual inductance, the procedure outlined above needs to be modified only slightly. In this case,

$$\int_{1b2} \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}$$

as before, except that **B** arises from currents in the coil itself and from currents in some other coil. Since quasi-static conditions are involved, we know from Sec. 8.3 that the total flux linkage $\int \mathbf{B} \cdot d\mathbf{S}$ is given by

$$\int \mathbf{B} \cdot d\mathbf{S} = LI_1 + MI_2 \tag{9.82}$$

where L is the self-inductance of the coil being considered and M is the mutual inductance to another coil carrying the current I_2. Consequently, we finally get

$$\int_{1b2} \mathbf{E} \cdot d\mathbf{l} = V = j\omega LI_1 + j\omega MI_2$$

Of course, the given coil can be magnetically coupled to more than one coil. The modification in that case should be fairly obvious from both the field and the circuit standpoint.

At this time it will be useful to consider how practical inductors depart from the above ideal inductor in their behavior. First of all, we know that the wire has a finite conductivity; so there will always be a potential drop across the coil resistance. At very low frequencies the current is uniform over the wire cross section, and hence if **J** is the current density, the electric field in the wire is $\mathbf{E} = \mathbf{J}/\sigma$. In (9.78) the first integral on the right now becomes

$$\int_{1a2} \mathbf{E} \cdot d\mathbf{l} = \int_{1a2} \frac{I}{S_0\sigma}\, dl = \frac{lI}{\sigma S_0}$$

where l is the total length of wire, S_0 is its cross-sectional area, and the total current $I = S_0J$. The low-frequency resistance is $R = l/\sigma S_0$, so that in place of (9.81), we have

$$V = j\omega LI + RI \tag{9.83}$$

In many applications $\omega L \gg R$, so that R may be neglected. At higher frequencies this approximation, however, becomes poorer, because the current now flows in a thin layer at the surface of the wire (skin effect), so that the resistance is much greater than the low-frequency value.

A second effect that exists in practical inductors is stray capacitance between turns. The association of this "stray capacitance" with the inductor is merely the recognition of the fact that the electric field and resultant accumulation of charge along the wire are not negligible. When there is a net accumulation of charge, the conduction current flowing in at one terminal does not equal that flowing at various other points in the inductor at each instant of time. The difference is equal to the rate of accumulation or depletion of charge between the two points under consideration; or in other words, the conduction current at the terminals of the inductor is not continuous through the coil winding at each instant of time. Only the total current, conduction plus displacement current, will be continuous. The displacement current, of course, accounts for the charging and discharging of the stray capacitance associated with the inductor.

The effect of stray capacitance is particularly noticeable at high frequencies since an increase in frequency is accompanied by an increase in the displacement current density $j\omega\mathbf{D}$. In fact, all practical inductors behave as capacitors at sufficiently high frequencies. In view of the above discussion, it is not surprising to find that most practical inductors must be characterized by all three ideal parameters, that is, inductance, resistance, and capacitance, particularly at the higher frequencies. An analysis for the equivalent circuit under these conditions is given in a subsequent section.

The Capacitor

Figure 9.11 is a schematic description of a capacitor. From a field standpoint the practical capacitor is characterized by the storage of electric energy, with a negligible accompanying magnetic energy or power loss. If we consider that a conduction current I flows in the leads, then from the continuity equation a charge $Q = I/j\omega$ accumulates on the plates. The electric field between the plates can be written in terms of the vector and scalar potentials as

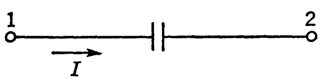

FIG. 9.11. A capacitor.

$$\mathbf{E} = -\nabla\Phi - j\omega\mathbf{A}$$

The scalar potential Φ arises essentially from charge stored on the capacitor plates, while the vector potential $\mathbf{A}$ is due mainly to current in the leads. For a given current magnitude, considering frequency as a variable, $j\omega\mathbf{A}$ is proportional to frequency while $\nabla\Phi$ is inversely proportional to frequency (that is, $\Phi \propto Q \propto 1/\omega$). Consequently, a low enough frequency exists for any capacitor so that $|\nabla\Phi| \gg |j\omega\mathbf{A}|$. The exact frequency below which this approximation is satisfactory can be specified only by a detailed consideration of the particular capacitor configuration involved.

When the vector potential contribution is negligible, the electric field may be derived from the negative gradient of Φ only. However, it is important to note that Φ may arise not only from the charge accumulated on the capacitor plates, but also from charge accumulations at other locations in the circuit. But this is nothing more than additional capacitive coupling between the capacitor under consideration and other nearby bodies and can be taken care of by introducing additional capacitors to describe the over-all circuit (see the discussion in Sec. 3.5 on multibody capacitors).

In order to arrive at the desired voltage-current relationship for the ideal capacitor, it is necessary to assume that the contribution to Φ from charges other than those on the capacitor plates is negligible in the region between the plates. Consequently, we may write, for the field between the capacitor plates,

$$\mathbf{E} = -\nabla\Phi$$

If we define the voltage drop across the capacitor terminals as

$$V = \int_1^2 \mathbf{E} \cdot d\mathbf{l} = \Phi_1 - \Phi_2 \tag{9.84}$$

where the path is through the leads and arbitrarily across the plate spacing, then the result depends only on the value of the scalar potential at each plate. (We again assume good conducting leads and neglect $\int \mathbf{E} \cdot d\mathbf{l}$ along them.) But the scalar potential Φ is derived from the charge on the capacitor according to the static formula, and so all the consequences of the work in Chap. 3 must hold. In particular, we may define the capacitance as $C = Q/(\Phi_1 - \Phi_2) = Q/V$, and in view of the continuity condition which requires $Q = I/j\omega$, we get

$$V = \frac{I}{j\omega C} \tag{9.85}$$

thus confirming (9.75c).

The Resistor

We consider, now, the properties of the resistor. As an ideal circuit element, it should set up negligible stored electric and magnetic energy, but is responsible for the dissipation of energy. We illustrate a resistor, schematically, in Fig. 9.12. At any point within the resistor we have

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma}$$

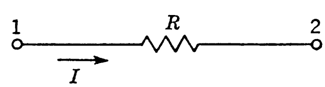

FIG. 9.12. A resistor.

If, for simplicity, we assume uniformity over the length of the resistor, then

$$E = \frac{I}{\sigma S}$$

where S is the effective cross-sectional area. The value of I is the total current and equals the terminal current provided that no substantial displacement current between the terminals exists. Since **E** is a constant, then

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} = \frac{Il}{\sigma S}$$

where l is the length of the resistor and the integration path is taken through the resistor. There is no ambiguity in this result; so we define it to be the potential drop V across the resistor; that is,

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} = V = IR \tag{9.86}$$

and $R = l/\sigma S$ is the usual definition. We note that (9.86) is in the form of (9.75*a*).

The Circuit

In order to connect the elements together to make a network, leads are necessary. From a field standpoint these should dissipate no energy nor store any energy. In this case we treat them like resistances with zero resistance. If the energy dissipated is not negligible, they will be treated like true resistances.

We have at this point established the validity of the assumed voltage-current relationships for the fundamental circuit elements, provided certain given conditions are met. We should like to illustrate the application of these ideas to a simple circuit which includes an applied emf and an R, L, and C in series. The circuit is illustrated in Fig. 9.13.

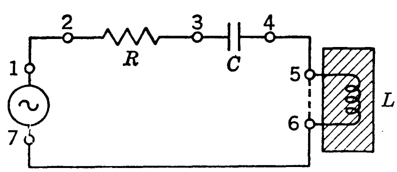

FIG. 9.13. *RLC* circuit with source of emf.

If we integrate the electric field around the circuit, but follow the dashed path 5-6 across the inductance, and integrate through the source of the emf, then

$$\oint \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} - \frac{d\psi}{dt} \tag{9.87}$$

which is a generalization of Faraday's law when the path of integration includes a source of emf. Because the flux associated with the inductance is excluded from the surface bounded by the chosen contour,

$$\frac{d\psi}{dt} = j\omega\psi = 0$$

Thus

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = \int_1^2 \mathbf{E} \cdot d\mathbf{l} + \int_2^3 \mathbf{E} \cdot d\mathbf{l} + \int_3^4 \mathbf{E} \cdot d\mathbf{l} + \int_5^6 \mathbf{E} \cdot d\mathbf{l} + \int_6^7 \mathbf{E} \cdot d\mathbf{l} + \int_7^1 \mathbf{E} \cdot d\mathbf{l}$$

If the source of emf is, say, a rotating machine, then $\int_7^1 \mathbf{E} \cdot d\mathbf{l}$ taken through the high-conductivity armature winding is negligible, as is true for the high-conductivity leads in the case of $\int_1^2 \mathbf{E} \cdot d\mathbf{l}$ and $\int_6^7 \mathbf{E} \cdot d\mathbf{l}$. Consequently,

$$\mathcal{E} = IR + \frac{I}{j\omega C} + j\omega LI \tag{9.88}$$

which is the generalization of Kirchhoff's voltage law to include sources of emf.

In summary we may say that the usual circuit concepts hold provided:

1. Circuit dimensions are small compared with wavelength, so that quasi-static conditions prevail. This ensures, among other things, negligible radiated energy.
2. Inductors and capacitors dissipate negligible energy.
3. The magnetic field associated with resistances and capacitors is negligible.
4. The displacement current associated with all circuit elements, except that between capacitor plates, is negligible. Otherwise the entire terminal current may not flow through the element. Furthermore, the Kirchhoff current law can be violated if displacement current flows into or out of a node as a result of the physical arrangement of a network.

The expert in working with circuits is, of course, aware of the above limitations and often employs techniques to extend the realm of network analysis. For example, if condition 2 is not satisfied, we have seen that the lossy inductor may usually be satisfactorily represented by an ideal inductor in series with a resistor, while the lossy capacitor may be represented by a series or shunt resistor with an ideal capacitor. Condition 3 is recognized in the ultra-high-frequency range as requiring the use of short lead lengths to minimize lead inductance, as already noted. Finally, displacement current between windings of a coil is accounted for by the "stray capacitance," which can often be approximated by a circuit with an ideal inductor paralleled by an ideal capacitor. How well these approximate circuit techniques will represent the actual conditions cannot be completely decided in advance, since what is involved is the detailed account of construction, frequency, and dimensions of the actual device.

When the frequency is such that circuit dimensions become comparable to wavelength, field techniques begin to take over as it becomes more difficult to continue to separate electric and magnetic stored-energy regions. Even under these conditions it is still possible to specify an equivalent two-terminal lumped-parameter network that correctly describes the behavior of the physical device at any given frequency.

The following section provides an introduction to this general-equivalent-circuit analysis.

General Equivalent Circuits

Consider an arbitrary physical structure made up from conductors, dielectric material, and magnetizable material in general. Let the device have two terminals, as in Fig. 9.14. Furthermore, let a total current $Ie^{j\omega t}$ flow into the device at the upper terminal and flow out at the lower terminal. Also, let the voltage between the two terminals be $Ve^{j\omega t}$. If we could solve the boundary-value problem to determine the electric and magnetic fields around the structure, we could evaluate the energy W_e stored in the electric field, the energy W_m stored in the magnetic field, and the power loss P_l due both to energy dissipation within the structure and to power radiated. The integrals to be evaluated to obtain the time-average quantities are (note that $\epsilon = \epsilon' - j\epsilon''$, $\mu = \mu' - j\mu''$, and are assumed to be constant)

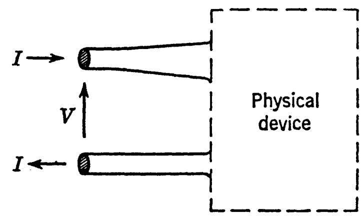

FIG. 9.14. Terminal voltage and current for a physical device.

$$W_e = \frac{\epsilon'}{4}\int_V \mathbf{E}\cdot\mathbf{E}^*\,dV \tag{9.89a}$$

$$W_m = \frac{\mu'}{4}\int_V \mathbf{H}\cdot\mathbf{H}^*\,dV \tag{9.89b}$$

$$P_l = \frac{\sigma}{2}\int_V \mathbf{E}\cdot\mathbf{E}^*\,dV + \frac{\omega}{2}\int_V (\epsilon''\mathbf{E}\cdot\mathbf{E}^* + \mu''\mathbf{H}\cdot\mathbf{H}^*)\,dV + \tfrac{1}{2}\oint_S \operatorname{Re}\mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} \tag{9.89c}$$

In the expression for P_l, the integral of the real part of the Poynting vector $\mathbf{E}\times\mathbf{H}^*$ is to be taken over the surface of an infinite sphere surrounding the structure. This integral gives the total time-average power radiated. The terms involving ϵ'' and μ'' give the losses due to polarization damping forces present in the material.

We shall now define the capacitance, inductance, and resistance of the structure by means of the following relations:

$$C = \frac{II^*}{4\omega^2 W_e} \tag{9.90a}$$

$$L = \frac{4W_m}{II^*} \tag{9.90b}$$

$$R = \frac{2P_l}{II^*} \tag{9.90c}$$

In (9.90*a*) and (9.90*b*) the numerical factor is 4, since we are considering time-average quantities. The above definitions are consistent with those which arise under time-stationary conditions, and their justification will be discussed below.

The values of W_e, W_m, and P_l are unique under a given set of terminal conditions. If the voltage V and current I can also be specified uniquely, then the circuit parameters—capacitance C, inductance L, and resistance R—are uniquely defined by the above formulas. Under static conditions it has been shown in earlier chapters that the above definitions for R, L, and C are equivalent to the geometrical definitions. However, the above definitions are more general in that they recognize the fact that ideal circuit elements do not exist physically; e.g., a parallel-plate capacitor has some inductance and resistance associated with it, as we have noted earlier. In other words, by defining capacitance in terms of electric energy storage, account is taken of all portions of the physical structure that contribute to the capacitance of the over-all device, and similarly for the inductance and resistance. It might be noted that these definitions for R, L, and C are equally applicable to distributed circuits, lumped circuits, or a combination of both. Again we emphasize that it is necessary to be able to define unique terminal currents and voltages in order for these parameters to have unique values.

In order to establish a relationship between the terminal current and voltage, we shall make use of the complex Poynting vector theorem established in Sec. 9.7. At the same time conditions for the unique specification of the terminal voltage will be obtained. If the physical device has two conducting leads, the terminal current is clearly unique, it being simply the total conduction current flowing into the structure in one lead and out at the other lead.

Let us construct a closed surface, consisting of an infinite plane (xy plane for convenience) and the surface of a hemisphere at infinity, that completely encloses the physical device except for the terminal leads which protrude through the plane surface, as in Fig. 9.15. The electric and magnetic fields are uniquely determined by the scalar and vector potentials Φ, $\mathbf{A}$, where Φ and $\mathbf{A}$ are determined from the charge density ρ and current density $\mathbf{J}$ distributed throughout all space, both inside and outside S. The electric field $\mathbf{E}$ is given by $\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$. In terms

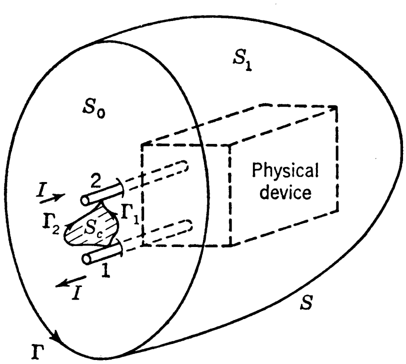

FIG. 9.15. Closed surface $S = S_0 + S_1$ surrounding a physical device.

of a line integral of **E**, the voltage between terminals 1 and 2 in Fig. 9.15 is

$$V = V_{12} = -\int_1^2 \mathbf{E}\cdot d\mathbf{l} = \int_1^2 \nabla\Phi\cdot d\mathbf{l} + j\omega\int_1^2 \mathbf{A}\cdot d\mathbf{l}$$
$$= \Phi_2 - \Phi_1 + j\omega\int_1^2 \mathbf{A}\cdot d\mathbf{l} \tag{9.91}$$

The integral of $\nabla\Phi$ from 1 to 2 may be taken over any arbitrary path without changing its value. However, this is not true for the line integral of **A**. We therefore see that the voltage V will be unique only if the contribution to its value from the vector potential **A** is zero. In general, this latter contribution is not zero, but as discussed earlier, it is negligible at sufficiently low frequencies. Physically, it is easy to see why there is a contribution from the vector potential **A** in general. If we integrate $\mathbf{A}\cdot d\mathbf{l}$ from 1 to 2 along the path Γ_1 and from 2 to 1 along the path Γ_2, we have

$$\int_{\Gamma_1} \mathbf{A}\cdot d\mathbf{l} + \int_{\Gamma_2} \mathbf{A}\cdot d\mathbf{l} = \oint_{\Gamma_1+\Gamma_2} \mathbf{A}\cdot d\mathbf{l} = \int_{S_c} \nabla\times\mathbf{A}\cdot d\mathbf{S} = \int_{S_c} \mathbf{B}\cdot d\mathbf{S} \tag{9.92}$$

where S_c is the area bounded by $\Gamma_1 + \Gamma_2$ and Stokes' law together with the relation $\mathbf{B} = \nabla\times\mathbf{A}$ has been used. The integral of **A** around the closed path gives the total magnetic flux through the area S_c. Since this flux induces an electric field when it changes with time, it follows that the condition for the vector potential not to contribute to the voltage V is that there be no magnetic lines of flux cutting through the boundary plane S_0 on which the line integral of **E** is taken. This requires that the vector potential **A** have no x or y (tangential) components on the surface S_0. When this is true a unique value for the terminal voltage V in terms of the line integral of **E** may be specified.

According to (9.48) the integral of the inward normal component of the complex Poynting vector over a closed surface S gives

$$\tfrac{1}{2}\oint_S \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} = -j\frac{\omega}{2}\int_V (\epsilon'\mathbf{E}\cdot\mathbf{E}^* - \mu'\mathbf{H}\cdot\mathbf{H}^*)\,dV$$
$$+ \tfrac{1}{2}\int_V (\omega\epsilon''\mathbf{E}\cdot\mathbf{E}^* + \omega\mu''\mathbf{H}\cdot\mathbf{H}^* + \sigma\mathbf{E}\cdot\mathbf{E}^*)\,dV$$
$$= 2j\omega(W_m - W_e) + P_{l1} \tag{9.93}$$

where W_m and W_e are the time-average magnetic and electric energy stored inside V, and P_{l1} is the time-average power loss inside V. We may split the integral over S into two parts:

$$\tfrac{1}{2}\oint_S \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} = \tfrac{1}{2}\int_{S_0} \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} + \tfrac{1}{2}\int_{S_1} \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S}$$

where $-\tfrac{1}{2}\int_{S_1} \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S}$ is the radiated power flowing out through the surface of the hemisphere. We may now rewrite (9.93) as

$$\tfrac{1}{2} \int_{S_0} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} = 2j\omega(W_m - W_e) + P_l \tag{9.94}$$

where $P_l = P_{l1} - \tfrac{1}{2} \int_{S_1} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S}$.

We shall demonstrate shortly that under certain conditions $\mathbf{E}$ and $\mathbf{H}^*$ can be uniquely related to V and I^*, respectively, so that (9.94) becomes

$$\tfrac{1}{2}VI^* = 2j\omega(W_m - W_e) + P_l \tag{9.95}$$

Since I is linearly related to V, we may now define an input impedance Z_{in} for the physical device so that $V = IZ_{\text{in}}$. Hence

$$\tfrac{1}{2}Z_{\text{in}}II^* = 2j\omega(W_m - W_e) + P_l \tag{9.96}$$

From (9.90) we have

$$W_e = \frac{II^*}{4\omega^2 C}$$
$$W_m = \frac{L}{4} II^*$$
$$P_l = \tfrac{1}{2}RII^*$$

When we substitute the appropriate terms into (9.96) and solve for Z_{in}, we obtain

$$\begin{aligned} Z_{\text{in}} &= \frac{2}{II^*}\left[2j\omega\left(\frac{L}{4} II^* - \frac{II^*}{4\omega^2 C}\right) + \tfrac{1}{2}RII^*\right] \\ &= R + j\omega L - \frac{j}{\omega C} \end{aligned} \tag{9.97}$$

An analysis similar to the above may be carried out when R, L, and C are defined on a voltage basis as

$$R = \frac{VV^*}{2P_l} \tag{9.98a}$$

$$L = \frac{VV^*}{4\omega^2 W_m} \tag{9.98b}$$

$$C = \frac{4W_e}{VV^*} \tag{9.98c}$$

An admittance Y_{in} may be introduced so that $I = Y_{\text{in}}V$, and (9.95) then gives

$$\tfrac{1}{2}Y_{\text{in}}VV^* = -2j\omega(W_m - W_e) + P_l \tag{9.99}$$

This leads to the result

$$Y_{\text{in}} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C \tag{9.100}$$

and specifies an equally valid equivalent circuit for the physical device. However, it is important to note that the circuit parameters R, L, and C in (9.100) are not, in general, the same as those occurring in (9.97) since in the present case the normalization factor is VV^* in place of II^*. The equivalent circuit specified by (9.97) is a series RLC circuit, whereas (9.100) specifies an equivalent circuit consisting of R, L, and C in a parallel connection.

The use of the complex-Poynting-vector theorem and appropriate definitions for R, L, and C thus leads to the specification of an equivalent circuit for the physical device. It is important to note that since we have considered a general device, the parameters R, L, and C do not specify ideal elements, but rather only equivalent parameters for the device. They will therefore, in general, be functions of the applied frequency. As a matter of fact, this is also true of all practical circuit elements.

We must now return to an earlier point and demonstrate that under certain conditions

$$\tfrac{1}{2}VI^* = \tfrac{1}{2}\int_{S_0} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S}$$

Since we require a unique voltage V between terminals, we assume that $\mathbf{E} = -\nabla\Phi$; that is, the vector potential contribution $-j\omega\mathbf{A}$ is negligible. Consequently, we can write

$$\int_{S_0} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} = -\int_{S_0} (\nabla\Phi) \times \mathbf{H}^* \cdot d\mathbf{S}$$

Now

$$\nabla \times (\Phi\mathbf{H}^*) = (\nabla\Phi) \times \mathbf{H}^* + \Phi\nabla \times \mathbf{H}^*$$

and hence

$$\int_{S_0} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} = \int_{S_0} \Phi\nabla \times \mathbf{H}^* \cdot d\mathbf{S} - \int_{S_0} \nabla \times \Phi\mathbf{H}^* \cdot d\mathbf{S}$$

The last integral may be transformed to a contour integral around the boundary of S_0 by means of Stokes' law; thus

$$\int_{S_0} \nabla \times \Phi\mathbf{H}^* \cdot d\mathbf{S} = \oint_{\Gamma} \Phi\mathbf{H}^* \cdot d\mathbf{l}$$

Since Φ and $\mathbf{H}^*$ vanish at infinity to at least an order $1/R^2$ ($R \to \infty$), while Γ increases only as R, this integral vanishes, and we are left with

$$\int_{S_0} \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} = \int_{S_0} \Phi(\mathbf{J}^* - j\omega\mathbf{D}^*) \cdot d\mathbf{S} \qquad (9.101)$$

upon replacing $\nabla \times \mathbf{H}^*$ by $\mathbf{J}^* - j\omega\mathbf{D}^*$ from Maxwell's equations. Provided the displacement current $j\omega\mathbf{D}^*$ can be neglected, (9.101) reduces simply to an integral over the cross section of the leads of the physical

Chapter 9

9.1. A cylindrical capacitor has an inner radius of a, an outer radius of b, and a length L. A sinusoidal voltage $V \sin \omega t$ is applied and $2\pi c/\omega \gg L$, so that the electric field distribution is that for static conditions. Calculate the displacement current density in the dielectric and also the total displacement current crossing a cylindrical surface of radius r ($a < r < b$). Show that the latter equals the conduction current in the leads to the capacitor.

9.2. Repeat Prob. 9.1, but for a spherical capacitor with inner radius a and outer radius b, and find the total displacement current crossing a spherical surface of radius r ($a < r < b$).

9.3. (a) Confirm that the one-dimensional wave equation of (9.11) possesses a general solution $E_x = f(z - ct) + f'(z + ct)$, where f and f' are arbitrary functions.

(b) For the specific wave

$$\mathbf{E} = E_0 \cos k_0(z - ct)\, \mathbf{a}_x + E_0 \sin k_0(z - ct)\, \mathbf{a}_y$$

where $k_0 = 2\pi/\lambda_0 = \omega/c$, compute $\mathbf{H}$ and the Poynting vector.

(c) For a given z, determine the locus of $\mathbf{E}$ as a function of time. This wave is called circular-polarized.

9.4. The plates of the accompanying parallel-plate capacitor are circular and have a radius R_0, and the medium between the plates is conductive with conductivity σ. A battery with emf V_0 is connected to the capacitor plates (assumed perfectly conducting) as shown.

(a) Compute $\mathbf{E}$ and $\mathbf{H}$ in the conducting media.

(b) Calculate the Poynting vector, and verify that it correctly evaluates the power flow from the battery.

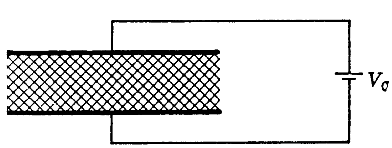

FIG. P 9.4

9.5. A shorted coaxial line has a resistive inner conductor and a perfectly conducting outer conductor. A d-c battery with emf V_0 is connected at the input end. The inner conductor has a radius of a, and the inside of the outer conductor a radius of b. The length is L, and the total current is I_0.

(a) Derive an expression for E and H in the dielectric region of the coaxial line. Assume that the battery is arranged to set up a potential that varies as $\ln r$ on the input surface. (If this is not done, is the problem uniquely specified?)

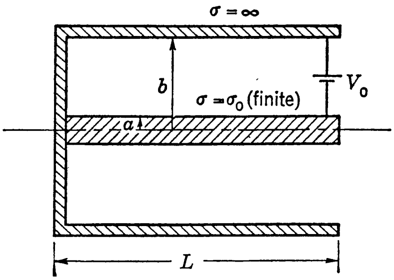

FIG. P 9.5

(b) Evaluate the Poynting vector, and compute the net power flow into the center conductor. Compare with a conventional circuit analysis.

9.6. An electrostatic field due to static charges and a magnetic field due to a permanent magnet are set up in the same region. In this way, a finite $\mathbf{E} \times \mathbf{H} = \mathbf{P}$ can exist, but no net power flow is taking place. Confirm that $\oint \mathbf{P} \cdot d\mathbf{S} = 0$ for any surface.

9.7. For a plane wave normally incident on an infinitely conducting infinite-plane reflector, the **E** and **H** fields as a function of the distance z from the reflector (with the given polarization) turn out to be

$$\mathbf{E} = jE_0 \sin k_0 z\, \mathbf{a}_x$$
$$\mathbf{H} = \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} E_0 \cos k_0 z\, \mathbf{a}_y$$

where $k_0 = 2\pi/\lambda_0 = \omega/c$.

(a) What is the instantaneous Poynting vector at $z = 0$, $\lambda_0/8$, $\lambda_0/4$?

(b) Compute the time-average Poynting vector at the above positions.

9.8. For Prob. 9.7, calculate the currents and charges set up on the conducting screen.

9.9. Given two uniform plane waves with electric fields as follows:

$$\mathbf{E}_1 = E_1 e^{-j\omega_1 z/c} \mathbf{a}_x$$
$$\mathbf{E}_2 = E_2 e^{-j\omega_2 z/c} \mathbf{a}_x$$

Both propagate in the same medium simultaneously. If $\omega_1 \neq \omega_2$, prove that the net time-average power flow equals the sum of the individual time-average power flows.

9.10. Find the skin depth for the following common materials, whose conductivity (mhos per meter) is given, at $f = 1$, 10^4, 10^6, 10^{10} cycles per second.

Silver...............	6.17×10^7
Copper.............	5.80×10^7
Aluminum..........	3.72×10^7
Sea water...........	4.5

9.11. At a frequency $f = 10^8$ cycles per second fused quartz has a relative permittivity of 3.8 and a loss tangent equal to 10^{-4}. Calculate the phase velocity, intrinsic impedance, and attenuation constant for a uniform plane wave propagating in this medium.

9.12. For the following media plot the attenuation of a uniform plane wave vs. frequency over the range 0 to 10^{10} cycles per second (σ in mhos per meter):

Sea water.............	$\sigma = 4.5$,	$\kappa = 80$
Good ground..........	$\sigma = 10^{-2}$,	$\kappa = 15$
Poor ground...........	$\sigma = 10^{-3}$,	$\kappa = 5$

9.13. A long cylindrical conductor of radius a is uniformly excited by an electromagnetic field so that a current flows in the axial direction and has no circumferential variations. We may take the current density at the surface to be equal to J, and for the purposes of this problem it is possible to neglect the axial variations. If the conductivity is σ and $\sigma \gg \omega\epsilon$,

(*a*) Show that

$$\nabla_t^2 J_z = \frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_z}{dr}\right) = j\omega\mu\sigma J_z$$

where the axial current density is in the z direction and a function of r only.

(*b*) Solve (*a*) for J_z.

Answer. Solutions to (*b*) come out in the form of a Bessel function $J_0(j^{-1/2}x)$, for which tables are available. Often new functions are defined as

$$\begin{aligned}\text{Ber } x &= \text{Re } J_0(j^{-1/2}x)\\ \text{Bei } x &= \text{Im } J_0(j^{-1/2}x)\end{aligned}$$

in terms of which the solution can be written

$$J_z(r) = J\frac{\text{Ber }(\sqrt{2}\,r/\delta) + j\text{ Bei }(\sqrt{2}\,r/\delta)}{\text{Ber }(\sqrt{2}\,a/\delta) + j\text{ Bei }(\sqrt{2}\,a/\delta)}$$

where $\delta = 1/\sqrt{\pi f\mu_0\sigma}$ is the skin depth and J is the current density at $r = a$.

9.14. The current distribution in the conductor of Prob. 9.13 may be found approximately if a/δ is large. In this case we may think of the conductor as if it were flat with a uniform plane wave incident. The diffusion of current into the conductor should be in the form of (9.43), except that the radial variable replaces the coordinate normal to the surface; that is, since **J** and **E** are related by the constant σ, we have

$$J_z = Je^{-(1+j)(a-r)/\delta}$$

By using the asymptotic expressions for Bessel functions of large argument, show that the result of Prob. 9.13 reduces to the above expression.

9.15. Show that in a source-free region of space where $\nabla \cdot \mathbf{E} = 0$, the electric and magnetic fields may be found from a magnetic-type vector potential $\mathbf{A}_m$ by means of the equations

$$\begin{aligned}\mathbf{E} &= \nabla \times \mathbf{A}_m\\ \mathbf{H} &= j\omega\epsilon\mathbf{A}_m - \frac{\nabla\nabla \cdot \mathbf{A}_m}{j\omega\mu}\end{aligned}$$

and $\mathbf{A}_m$ is a solution of

$$\nabla^2\mathbf{A}_m + \omega^2\mu\epsilon\mathbf{A}_m = 0$$

The derivation is similar to that for the electric-type vector potential **A**.

9.16. In a region of space where the only source for the electromagnetic field is a volume dielectric polarization of density **P**, show that the electric and magnetic fields may be found from a vector potential $\mathbf{\Pi}_e$ by means of the equations

$$\begin{aligned}\mathbf{H} &= j\omega\epsilon_0 \nabla \times \mathbf{\Pi}_e\\ \mathbf{E} &= k_0^2\mathbf{\Pi}_e + \nabla\nabla \cdot \mathbf{\Pi}_e = \nabla \times \nabla \times \mathbf{\Pi}_e - \frac{\mathbf{P}}{\epsilon_0}\end{aligned}$$

and $\mathbf{\Pi}_e$ satisfies the equation

$$\nabla^2\mathbf{\Pi}_e + k_0^2\mathbf{\Pi}_e = -\frac{\mathbf{P}}{\epsilon_0}$$

where $k_0^2 = \omega^2\mu_0\epsilon_0$. The derivation is similar to that for the vector potential **A**, and a Lorentz-type condition must be invoked. The polarization **P** is introduced by replacing **D** by $\epsilon_0\mathbf{E} + \mathbf{P}$ in Maxwell's equations. The potential function $\mathbf{\Pi}_e$ is known as the electric-type Hertzian potential.

The above result may be extended to arbitrary source distributions through the relations

$$\rho = -\nabla \cdot \mathbf{P}$$
$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}$$

9.17. If the only source for the electromagnetic field is the magnetic polarization **M** per unit volume, show that the fields found from a magnetic Hertzian vector potential $\mathbf{\Pi}_m$ by means of the equations

$$\mathbf{E} = -j\omega\mu_0 \nabla \times \mathbf{\Pi}_m$$
$$\mathbf{H} = k_0^2\mathbf{\Pi}_m + \nabla\nabla \cdot \mathbf{\Pi}_m$$
$$= \nabla \times \nabla \times \mathbf{\Pi}_m - \mathbf{M}$$

satisfy Maxwell's equations, provided $\mathbf{\Pi}_m$ is a solution of

$$\nabla^2\mathbf{\Pi}_m + k_0^2\mathbf{\Pi}_m = -\mathbf{M}$$

9.18. In an idealized velocity-modulated electron tube, the electron stream can be taken to consist of d-c convection current of amplitude i_0 and an a-c current of the form

$$\mathbf{i}_{\text{a-c}} = i_1 e^{j\omega t - \gamma z}\mathbf{a}_z$$

where i_0 and i_1 are in amperes per unit area of cross section. Note that the a-c current is in the form of a wave and that a geometry is implied where there are no variations with x or y.

The a-c convection current sets up an electromagnetic field most easily found by first computing the vector potential **A**. If, as we assume, $i_1 \ll i_0$, then all a-c quantities have the space-time dependence $e^{j\omega t - \gamma z}$. (The d-c current can be ignored in calculating time-varying fields.)

(*a*) Find the a-c charge density.

(*b*) Write and solve the differential equation for vector potential **A**.

(*c*) From (*b*) find the electric field **E**.

9.19. It is virtually impossible to obtain solutions to the vector wave equation if the fields are written in terms of their spherical components and if spherical coordinates are used. Yet for boundary conditions imposed on spherical boundaries, it is equally difficult to utilize rectangular coordinates since the boundary is not a natural one. It turns out, however, that the vector

$$\mathbf{M} = \mathbf{r} \times \nabla\psi$$

where **r** is the radius vector, satisfies the vector wave equation provided that ψ satisfies the scalar wave equation

$$\nabla^2\psi + \omega^2\mu\epsilon\psi = 0$$

Another solution is

$$\mathbf{N} = \frac{1}{\omega\sqrt{\mu\epsilon}} \nabla \times \mathbf{M}$$

Note that **M** and **N** may be identified with the **E** and **H** field, or vice versa. In view of the fact that **M** is transverse to spherical surfaces, spherical boundary-value problems may be readily formulated.

Confirm that **N** and **M** do indeed satisfy the vector wave equations

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = -\nabla \times \nabla \times \mathbf{A} + \nabla\nabla \cdot \mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = 0$$

provided that

$$\nabla^2\psi + \omega^2\mu\epsilon\psi = 0$$

9.20. An isotropic dielectric medium is nonuniform, so that ϵ is a function of position. Show that **E** satisfies

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = -\nabla\left(\mathbf{E} \cdot \frac{\nabla\epsilon}{\epsilon}\right)$$

9.21. If the gauge is chosen so that $\nabla \cdot \mathbf{A} = 0$, confirm that the following equations for Φ and **A** result:

$$\nabla^2\Phi = \frac{-\rho}{\epsilon_0}$$

$$\nabla \times \nabla \times \mathbf{A} - \omega^2\mu\epsilon\mathbf{A} = \mu\mathbf{J} - j\omega\epsilon\mu\,\nabla\Phi$$

where **E** and **H** remain related to Φ and **A** as in (9.62) and (9.63).

What is the relationship between the Φ and **A** in this gauge as compared with that for which the Lorentz condition is satisfied?

9.22. Confirm that $\psi = (1/4\pi R)e^{-jk_0R}$ satisfies the scalar Helmholtz equation $\nabla^2\psi + k_0^2\psi = 0$, provided $R \neq 0$. Show further that for a spherical (ΔV) volume of vanishing radius surrounding the origin, $\int_{\Delta V} (\nabla^2\psi + k_0^2\psi)\,dV$ is finite and actually equals -1.

HINT: Write $\nabla^2\psi = \nabla \cdot \nabla\psi$, and use the divergence theorem.

As a consequence of the above, $\psi = (1/4\pi R)e^{-jk_0R}$ may be said to be a solution of the following inhomogeneous Helmholtz equation:

$$\nabla^2\psi + k_0^2\psi = -\delta(\mathbf{r} - \mathbf{r}')$$

where $R\mathbf{a}_R = \mathbf{r} - \mathbf{r}'$ and $\mathbf{r}'$ is the location of a unit (delta) source. The delta function has the property that

$$\int_V \delta(\mathbf{r} - \mathbf{r}')\,dV = \begin{cases} 0 & \text{if } \mathbf{r}' \text{ is not contained in } V \\ 1 & \text{if } \mathbf{r}' \text{ is contained in } V \end{cases}$$

9.23. Confirm the statement that appears in the text that if, at a perfect conductor, we satisfy $\mathbf{J}_s = \mathbf{n} \times \mathbf{H}$, then $\rho_s = \mathbf{n} \cdot \mathbf{D}$.