CHAPTER 8

QUASI-STATIONARY MAGNETIC FIELD

In order to complete a discussion of magnetostatics, along lines analogous to that in electrostatics, it will be necessary to derive an expression for the stored magnetic energy. This will then make possible a full discussion of inductance (the counterpart of capacitance in electrostatics) and also an analysis of forces between current-carrying circuits. It turns out, however, that in order to determine the formula for stored magnetic energy due to time-stationary currents, it is necessary to know something about time-varying currents and time-varying magnetic fields. Consequently, this chapter starts out with a statement and discussion of Faraday's law of induction. Following this, inductance, energy, and force are considered, thereby completing the analysis of magnetostatics and preparing for the subject of general time-varying fields.

The procedure used to derive the field expression for electric stored energy involved evaluating the work done in assembling the charges that established the electric field. For the static magnetic field we might expect that a similar procedure could be used, except that in this case it would be necessary to evaluate the work done in assembling a system of current loops. This is true; however, the forces acting on the current loops multiplied by their respective displacements are not alone equal to the energy of assembly. In the process of moving the loops relative to each other, the magnetic flux linking each loop continually changes. It turns out that this changing flux results in an induced voltage in each loop and the battery must do work (or have work done on it) in order to keep the currents constant. This additional work must be taken into account in evaluating the net energy of assembly of the loops, which by the above definition equals the net stored magnetic energy.

Although we begin with a general formulation for time-varying magnetic effects, we shall be mostly concerned in this chapter with a quasi-static field. By a quasi-stationary magnetic field we mean a field that varies so slowly with time that all radiation effects are negligible. In a following chapter we shall discover that a system of conductors carrying sinusoidal currents must have dimensions of the order of a wavelength in order to radiate efficiently. For a frequency f, the wavelength in free space is given by $\lambda_0 = c/f$, where c is the velocity of light (3×10^8 meters per second). Thus, if the frequency is 100 kilocycles per second, $\lambda_0 = 3{,}000$ meters. A typical system of coils and loops used in the laboratory might have dimensions of around one meter; consequently, for frequencies less than 100 kilocycles, radiation effects would certainly be entirely negligible. The quasi-static application is thus widely applicable.

8.1. Faraday's Law

The discovery of electric induction by a changing magnetic field is credited to Michael Faraday. On Aug. 29, 1831, the classic experiment on induction was carried out. Faraday wound two separate coils on an iron ring and found that whenever the current in one coil was changed, an induced current would flow in the other coil. He also found that a similar induced current would be produced when a magnet was moved in the vicinity of the coil. At about the same time similar effects were being studied by Joseph Henry in America. However, Faraday was more fortunate in that he worked at the Royal Institution in London and his work was published and made known to the scientific world earlier than the work of Henry. As a consequence, the law of electric induction is known as Faraday's law.

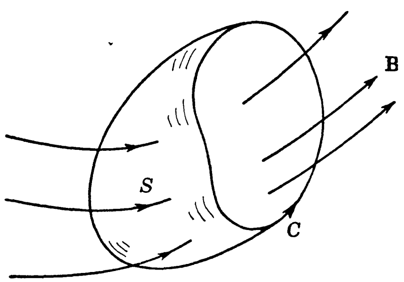

FIG. 8.1. Illustration of Faraday's law.

If we consider any closed stationary path in space which is linked by a changing magnetic field, it is found that the induced voltage around this path is equal to the negative time rate of change of the total magnetic flux through the closed path. Let C denote a closed path, as in Fig. 8.1. The induced voltage around this path is given by the line integral of the induced electric field around C and is

$$V_{ind} = \oint_C \mathbf{E} \cdot d\mathbf{l}$$

The magnetic flux through C is given by

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

where S is any surface with C as its boundary. Thus the mathematical statement of Faraday's law is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{8.1}$$

Basically, the law states that a changing magnetic field will induce an electric field. The induced electric field exists in space regardless of whether a conducting wire is present or not. When a conducting wire is present, a current will flow, and we refer to this current as an induced current. Faraday's law is the principle on which most electric generators operate. Note that the electric field set up by a changing magnetic field is nonconservative, as (8.1) clearly indicates. The changing magnetic field becomes a source for an electric field.

In addition to (8.1) there are several other equivalent statements of Faraday's law. Since **B** may be obtained from the curl of a vector potential **A**, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \oint_C \mathbf{A} \cdot d\mathbf{l} \tag{8.2}$$

by using Stokes' law to convert the surface integral to a line integral. Equation (8.2) permits the induced voltage to be evaluated directly from the vector potential **A**.

The differential form of (8.1) is obtained by using Stokes' law to replace $\oint_C \mathbf{E} \cdot d\mathbf{l}$ by a surface integral, so that

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$$

or

$$\int_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} = 0$$

Since S can be an arbitrary surface, the integrand must be equal to zero, and we obtain

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8.3}$$

This result again shows that the electric field induced by **B** is not of the same nature as the electrostatic field for which the curl or rotation is zero. Our concept of the curl or rotation as being a measure of the line integral of the field around an infinitesimal contour per unit area makes (8.3) a natural consequence of (8.1).

Example 8.1. Induced Voltage in a Coil. Figure 8.2*a* illustrates a single-turn coil of wire of radius d. The coil is located in a uniform magnetic field $\mathbf{B} = \mathbf{B}_0 \sin \omega t$ and with the normal to the plane of the coil at an angle θ with respect to the lines of magnetic flux. The induced voltage measured between the two open ends of the coil is given by (8.1) as

$$V = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} (\pi d^2 B_0 \cos \theta \sin \omega t)$$
$$= -\omega \pi d^2 B_0 \cos \theta \cos \omega t$$

since the total magnetic flux linking the coil is $\pi d^2 B_0 \cos\theta \sin\omega t$. In Fig. 8.2b a coil with N turns is illustrated. To evaluate $\int_S \mathbf{B} \cdot d\mathbf{S}$, a surface must be constructed so that the coil forms the periphery and the total flux crossing the surface is evaluated. This surface resembles a spiral staircase. The net result is roughly equivalent to the notion that each turn is separately linked by the magnetic flux, a notion that is quite good for tightly wound coils. With this point of view, then, in each turn the induced voltage is given by the above expression. These voltages

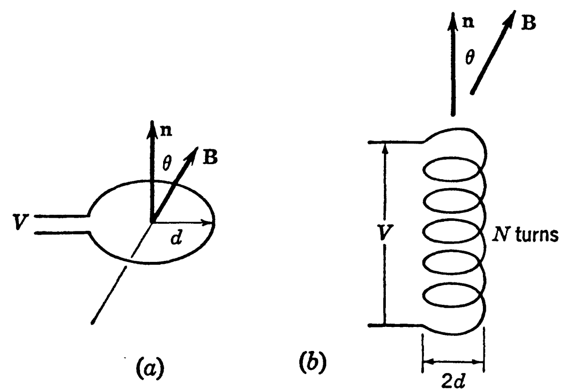

FIG. 8.2. Electric induction in a coil.

add in series, so that the total voltage across the complete coil is N times greater and hence given by

$$V = -N\omega\pi d^2 B_0 \cos\theta \cos\omega t$$

The induced voltage is proportional to the rate of change of the magnetic field, the number of turns, and the magnitude of the magnetic flux linking each turn.

8.2. Induced Electric Field Due to Motion

When conductors are moving through a static magnetic field, an induced voltage (we shall define this more precisely later) is produced in the conductor. This voltage is in addition to that calculated by (8.1). The magnitude of this voltage may be found from the Lorentz force equation. This states that a particle of charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force $\mathbf{F}$ given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{8.4}$$

This force, known as the Lorentz force, is similar to the analogous relation $\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$. (Note that $q\mathbf{v}$ can be interpreted as a current element.)

The force is seen to act in a direction perpendicular to both **v** and **B**. The interpretation of the Lorentz force gives rise to the concept that an observer moving through a static magnetic field sees, in addition to the magnetic field, an electric field also. A unit of charge moving with the observer appears to be stationary, and any force experienced by that charge is ascribed to the existence of an electrostatic field. But a force is experienced and is given by (8.4). Consequently, in the moving reference frame, this fact is interpreted as revealing the existence of an electric field **E** given by

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \tag{8.5}$$

Equation (8.5) gives an alternative and more general method of evaluating the induced voltage in a moving conductor. This equation is the mathematical formulation of Faraday's second observation of induction by moving magnets.

As an example, consider a conducting wire moving with a velocity **v** through a uniform field **B**, as in Fig. 8.3, where **B** is orthogonal to **v**. Each electron in the conductor experiences a force $F = -evB$, which tends to displace the electron along the wire in the direction indicated. As a result of this force electrons move toward the end marked P_1, leaving a net positive charge in the vicinity of the end marked P_2. When equilibrium has been reached, there is no further movement of the electrons along the wire, and this requires that there be no net force. What happens is that the displaced charges set up an electrostatic field which opposes the displacement of the charges due to the Lorentz force. When sufficient charge has been built up so that the electrostatic field produces a force equal and opposite to the Lorentz force, equilibrium is established. In this case $E = -vB$.

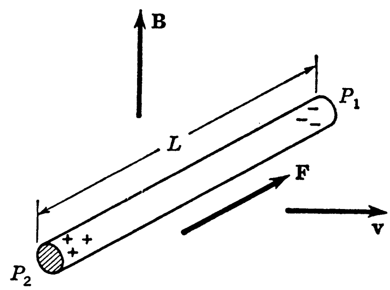

FIG. 8.3. Induced voltage in a moving conductor.

The induced voltage between the ends of the conductor is defined by

$$V = \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

and in this example

$$V = vB \int_{P_1}^{P_2} dl = vBL$$

a result that is true when **v** and **B** are orthogonal. The induced voltage caused by motion of a conductor through a magnetic field is called motional emf (electromotive force).† The electrostatic field set up by the displaced charges may be observed in both the stationary frame of reference and the moving frame attached to the conductor.

Moving Conductor in a Time-varying Magnetic Field

When a closed conducting loop C, as in Fig. 8.4*a*, is moving with a constant velocity $\mathbf{v}$ through a nonuniform time-varying magnetic field $\mathbf{B}$, the

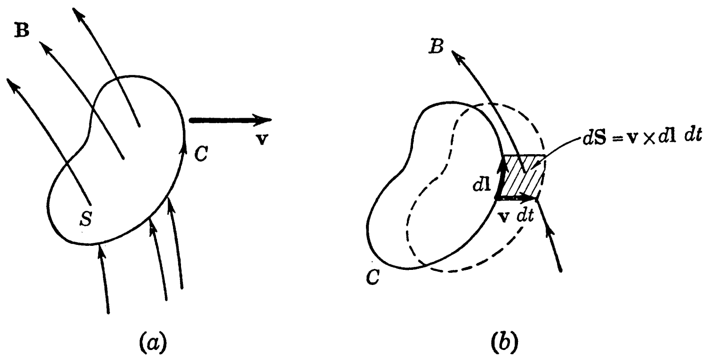

FIG. 8.4. Conductor C moving in a time-varying field $\mathbf{B}$.

induced voltage is given by

$$V_{ind} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} \tag{8.6}$$

In this expression the first term represents the contribution due to the time variation of $\mathbf{B}$ while the second term is the contribution representing the motional induced voltage.

The velocity $\mathbf{v}$ of different portions of the loop need not be the same, so that the loop C may be changing in shape as well as undergoing translation and rotation. However, in (8.6), the integral of $\partial \mathbf{B}/\partial t$ may be taken over the original surface S, since the contribution arising from an integration over the change ΔS in S is a second-order term.

The term $\oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$ is the motional emf contribution. A further insight into the connection between this term and the changing-flux concept may be obtained as follows. With reference to Fig. 8.4*b*, it is clear that an element $d\mathbf{l}$ of C sweeps out an area $d\mathbf{S} = \mathbf{v} \times d\mathbf{l}\, dt$ in a time

† The field structure is similar to that described for an open-circuited battery. In the latter case chemical action sets up a nonconservative field within the battery (analogous to the Lorentz force field) and also an electrostatic field which pervades all space but cancels the nonconservative field within the battery (within the generator in the present case). Accordingly, $V = \text{emf} = vBL$ may be similarly viewed as an open-circuit voltage.

interval dt. The change in flux caused by the displacement of C is equal to the integral of $\mathbf{B}$ through the swept-out area, i.e., equal to

$$d\psi_1 = \oint_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l})\, dt \tag{8.7}$$

Hence, $-d\psi_1/dt = -\oint_C \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l} = \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$, which is the usual form for the motional emf term. Consequently, we have shown that $V_{ind} = -d\psi/dt$, that is, equals the negative total time rate of change of flux linkage. Thus, a generalization of Faraday's law may be written

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{8.8}$$

In the above case it was quite clear how the total change in flux linkage could be evaluated since a definite closed contour C was involved. In the case of a single conductor, as in Fig. 8.3, it is not clear how to evaluate a change in flux linkage since a definite closed contour is not involved. In a situation like this the use of the Lorentz force equation is the most straightforward.

Example 8.2. Motional EMF. Figure 8.5 illustrates a single-turn rectangular coil, with sides b and a, which is rotating with an angular

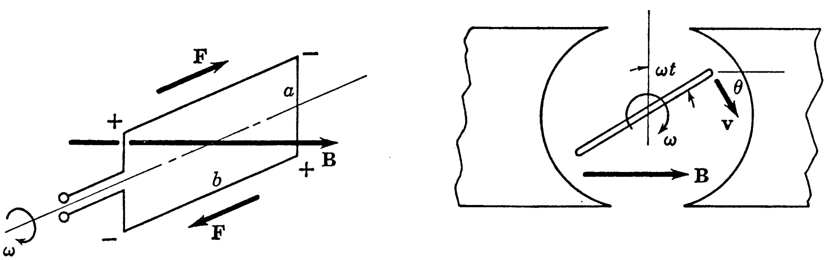

FIG. 8.5. Rotating coil in a magnetic field.

velocity ω about its axis. The coil is located between the pole pieces of a magnet which sets up a uniform magnetic field $\mathbf{B}$. Since the magnetic field $\mathbf{B}$ does not vary with time, that is, $\partial\mathbf{B}/\partial t = 0$, the induced voltage is entirely of the motional type. We may calculate the induced voltage from the negative time rate of change of the total magnetic flux linking the coil. At any instant of time t the flux through the coil is

$$abB \cos \omega t$$

and hence the induced voltage is

$$V = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{d\psi}{dt} = \omega abB \sin \omega t$$

The above result may also be obtained by an application of the Lorentz force equation. The velocity of an electron along the sides of the coil is $v = a\omega/2$, and the sine of the angle between $\mathbf{v}$ and $\mathbf{B}$ is given by $\sin\theta = \sin\omega t$, as in Fig. 8.5. The force on an electron is then

$$F = e|\mathbf{v} \times \mathbf{B}| = evB\sin\theta = e\frac{a}{2}\omega B\sin\omega t$$

This is equivalent to the presence of an electric field E, where

$$E = \frac{F}{e} = \frac{a}{2}\omega B\sin\omega t$$

In each side arm of the coil the induced voltage is Eb. Consequently, the total voltage is just twice this amount; that is,

$$V = 2Eb = ab\omega B\sin\omega t$$

which is the same as that given earlier. The above result neglects the effect of the ends of the coil; however, there is no induced voltage in the ends since $\mathbf{F}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$. This analysis is seen to be equivalent to a formal evaluation of the motional emf term $\oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$.

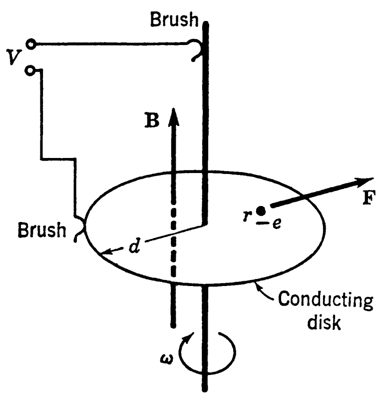

FIG. 8.6. The Faraday disk dynamo.

Example 8.3. Faraday Disk Dynamo. The Faraday disk dynamo is illustrated in Fig. 8.6. It consists of a circular conducting disk rotating in a uniform magnetic field B. Brushes make contact with the disk at the center and along the periphery. The problem is to determine if an induced voltage will be measured between the brushes. The answer is yes, and the magnitude of the voltage is readily found from the Lorentz force equation. An electron at a radial distance r from the center has a velocity ωr and hence experiences a force $e\omega rB$ directed radially outward. The electric field acting on the electron at equilibrium is also ωrB but is directed radially inward. The potential from the center to the outer rim of the disk is thus

$$V = \int_0^d E(r)\,dr = -\omega B\int_0^d r\,dr = -\frac{\omega B d^2}{2} \qquad (8.9)$$

The value computed by (8.9) is the open-circuit voltage of the Faraday disk dynamo and therefore also represents the emf of the generator.

8.3. Inductance

Consider a single current-carrying loop in which a constant current has been established. A magnetic field is set up which could be calculated from the given geometry of the loop and which is proportional to the current magnitude. If the current is caused to change, so will the magnetic field. But this means that the total flux linking the loop also changes and, by Faraday's law, a voltage is induced in the loop. If the problem is analyzed quantitatively, it will be discovered that the self-induced voltage always has such a polarity that tends to oppose the original change in current. For example, if the current begins to decrease, the induced voltage acts in a direction to offset this decrease.

If the problem involves two current loops, a somewhat more involved sequence of events takes place, but with the same qualitative outcome.

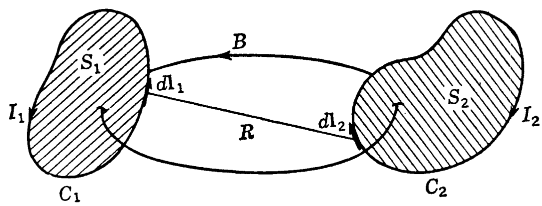

FIG. 8.7. Two circuits with magnetic coupling.

Thus Fig. 8.7 illustrates two circuits C_1 and C_2, with currents I_1 and I_2. The current I_1 produces a partial field $\mathbf{B}_1$, which causes a magnetic flux $\psi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S}$ to link C_2 and $\psi_{11} = \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{S}$ to link C_1 (itself). Similarly, the partial field $\mathbf{B}_2$ due to I_2 is responsible for the flux

$$\psi_{21} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S}$$

linking C_1 and $\psi_{22} = \int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}$, which links itself. If now the current I_1 is allowed to change, this causes a corresponding variation in ψ_{11} and ψ_{12}. The latter effect results in an induced voltage in C_2, hence a change in I_2. This in turn causes ψ_{21} to be disturbed from its previous value, so that the net flux linking C_1 (that is, $\psi_{11} + \psi_{21}$) is altered. Again, if all possible cases are considered analytically, it turns out that the change in both ψ_{11} and ψ_{21} is always such that the induced voltage in C_1 is opposed to the original perturbation of I_1. The fact that the induced voltage always acts to oppose the change in current that produces the induced voltage is known as Lenz's law.

The property of a single circuit, such as C_1, that results in an induced voltage which opposes a change in the current flowing in the circuit is known as self-inductance. The similar effect of a changing current in one circuit producing an induced voltage in another circuit is known as mutual inductance. Inductance is analogous to inertia in a mechanical system. The symbol L is used for self-inductance, and M for mutual inductance. The symbol L, with appropriate subscripts, is also used for mutual inductance and is the notation we shall adopt. The unit for inductance is the henry, in honor of Joseph Henry, who contributed much to the early knowledge of magnetic fields and inductance.

There are several equivalent mathematical definitions of inductance. One definition is in terms of flux linkages. If ψ_{12} is the magnetic flux linking circuit C_2, due to a current I_1 flowing in circuit C_1, the mutual inductance L_{12} between circuits C_1 and C_2 is defined by

$$L_{12} = \frac{\text{flux linking } C_2 \text{ due to current in } C_1}{\text{current in } C_1} = \frac{\psi_{12}}{I_1} \tag{8.10}$$

The mutual inductance is considered to be positive if the flux ψ_{12} links C_2 in the same direction as the self-flux linkage ψ_{22} due to the field from the current I_2. If ψ_{12} and ψ_{22} are in opposite directions, the mutual inductance is negative. Reversal of either I_1 or I_2 will change the sign of the mutual inductance L_{12}. The self-inductance L_{11} of circuit C_1 is defined in a similar way; that is,

$$L_{11} = \frac{\text{flux linking } C_1 \text{ due to current in } C_1}{\text{current in } C_1} = \frac{\psi_{11}}{I_1} \tag{8.11}$$

The mutual inductance between C_1 and C_2 may be defined by

$$L_{21} = \frac{\psi_{21}}{I_2} \tag{8.12}$$

as well, where ψ_{21} is the flux linking C_1 due to a current I_2 in C_2. We shall show that $L_{12} = L_{21}$, so that (8.10) and (8.12) are equivalent.

Since C_1 and C_2 are two very thin current-carrying loops, it is a simple matter to formulate the expressions for the flux linkage. However, in the limit of zero cross section, (8.11) leads to an infinite value for L_{11}, although the magnetic energy associated with the field remains finite. The extension of the definition (8.11) to current loops or filaments of finite (large) cross section can be made and is done in a later section. The proper interpretation of ψ_{11} follows from a consideration of the magnetic energy associated with the circuit and is fully discussed later.

The above definition of inductance is satisfactory only for quasi-stationary magnetic fields where the current and the magnetic field have the same phase angle over the whole region of the circuit. At high frequencies the magnetic field does not have the same phase angle over the whole region of the circuit because of the finite time required to propagate the effects of a changing current and field through space. A more general definition in terms of the magnetic energy associated with a circuit will be given in the next section.

Neumann Formulas

Consider two very thin wires bent into two closed loops C_1 and C_2, as in Fig. 8.7. Let a current I_1 flow in C_1. Since the wire is assumed to be very thin, the value computed for $\mathbf{B}_1$ will not be much in error if the current is assumed concentrated in an infinitely thin filament along the center of the conductor, provided only field points external to the wire are considered. With this limitation in mind, the field $\mathbf{B}_1$ produced by I_1 is given by

$$\mathbf{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\mathbf{l}_1 \times \mathbf{a}_R}{R^2} = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \left[\nabla\left(\frac{1}{R}\right)\right] \times d\mathbf{l}_1 \tag{8.13}$$

since $\nabla(1/R) = -\mathbf{a}_R/R^2$. The integration is over the source coordinates, while ∇ affects only the field coordinates; so we have

$$\nabla \times \frac{d\mathbf{l}_1}{R} = \left[\nabla\left(\frac{1}{R}\right)\right] \times d\mathbf{l}_1$$

since $d\mathbf{l}_1$ is a constant vector as far as ∇ is concerned. Then, in place of (8.13), we can write

$$\mathbf{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \nabla \times \frac{d\mathbf{l}_1}{R}$$

and hence the flux ψ_{12} linking circuit C_2 is

$$\begin{aligned}\psi_{12} &= \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S} = \frac{\mu_0 I_1}{4\pi} \int_{S_2} \oint_{C_1} \nabla \times \frac{d\mathbf{l}_1}{R} \cdot d\mathbf{S} \\ &= \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \int_{S_2} \nabla \times \frac{d\mathbf{l}_1}{R} \cdot d\mathbf{S}\end{aligned}$$

upon changing the order of integration. By using Stokes' law the surface integral may be converted to a contour integral around C_2; so we get

$$\psi_{12} = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R} \tag{8.14}$$

From the definition of mutual inductance stated in (8.10) we obtain Neumann's formula:

$$L_{12} = \frac{\psi_{12}}{I_1} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R} \tag{8.15}$$

Since in (8.15) R is the distance between a point on C_1 to a point on C_2, the integral as a whole is symmetrical; that is, the subscripts 1 and 2 may be interchanged without changing the end result. This proves the reciprocity relation stated earlier:

$$L_{12} = L_{21} = \frac{\psi_{12}}{I_1} = \frac{\psi_{21}}{I_2} \tag{8.16}$$

Equation (8.15) may be derived in an alternative way by noting that $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$, where the vector potential $\mathbf{A}_1$ is given by

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\mathbf{l}_1}{R}$$

Thus

$$\psi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S} = \int_{S_2} \nabla \times \mathbf{A}_1 \cdot d\mathbf{S} = \oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l}_2$$

Using the expression for $\mathbf{A}_1$ and dividing by I_1 leads to the desired end result.

A formula similar to (8.15) may be written for the self-inductance also. However, it is not permissible to assume that the current is concentrated in a thin filament at the center since it is necessary to include values of **B** at the wire itself where the approximation breaks down. For an idealized infinitely thin wire, the analogous formula is

$$L_{11} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_1} \frac{d\mathbf{l}_1' \cdot d\mathbf{l}_1}{R} \tag{8.17}$$

where $d\mathbf{l}_1$ and $d\mathbf{l}_1'$ are differential elements of length along C_1 and separated by a distance R. Since R can become zero, the integral is an improper one and leads to an infinite value of self-inductance, which is actually consistent with the assumption of infinitesimal wire diameter. To evaluate the self-inductance of a practical loop, the finite thickness of the conductor must be taken into account. A suitable procedure to be followed will be presented later, but first we shall consider some typical applications of (8.15) and also introduce the concept of internal inductance.

Example 8.4. Inductance of a Coaxial Line. Figure 8.8 illustrates a coaxial transmission line made of two thin-walled conducting cylinders

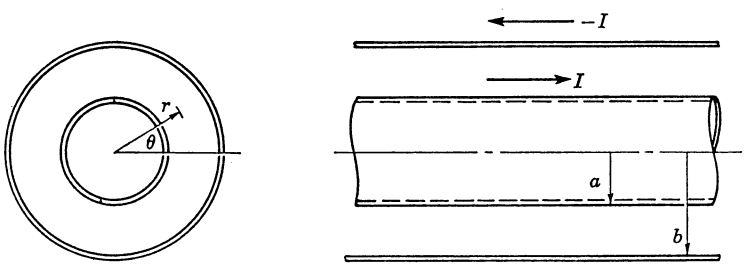

FIG. 8.8. A coaxial line made of two thin-walled cylinders.

with radii a and b. A current I flows along the inner cylinder, and a return current $-I$ along the outer cylinder. The inductance per unit length is to be evaluated.

It will be noted that this geometry does not correspond precisely to that of the thin-wire loops for which the definition of inductance has been formulated. At a later time a more fundamental definition of inductance will be given which allows generalization in terms of distributed current-carrying bodies. For the present we shall try to extend the definitions of (8.11) in a plausible way and with the understanding that future work will confirm its usefulness.

The field $\mathbf{B}$ is in the θ direction only and is given by $\mathbf{B} = (\mu_0 I/2\pi r)\mathbf{a}_\theta$. The total magnetic flux linking the inner conductor per unit length of line is

$$\psi = \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}$$

and hence the inductance per unit length of line is given by

$$L = \frac{\psi}{I} = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \tag{8.18}$$

If the center conductor is solid, the above result is not valid, since the current I is distributed uniformly over the cross section of area πa^2. To treat this case the concept of partial flux linkages is required. The current flowing in the portion of the inner conductor between 0 and r is $I\pi r^2/\pi a^2 = Ir^2/a^2$. The field in the coaxial line is given by (see Example 6.7)

$$B = \frac{\mu_0 I r}{2\pi a^2} \qquad 0 < r < a$$

$$B = \frac{\mu_0 I}{2\pi r} \qquad a < r < b$$

Since the field is circularly symmetric, each element of current in the annular ring between r and $r + dr$ is linked by the same flux. The value of the magnetic flux linking this current is

$$\begin{aligned} d\psi' &= \int_r^b B\,dr = \frac{\mu_0 I}{2\pi a^2} \int_r^a r\,dr + \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r} \\ &= \frac{\mu_0 I}{4\pi a^2} (a^2 - r^2) + \frac{\mu_0 I}{2\pi} \ln \frac{b}{a} \end{aligned}$$

In the earlier calculation of inductance for the thin-walled inner conductor the flux linked the total current flow I. Within the solid conductor, however, we have flux which links only part of the current. Now since the flux $d\psi'$ does not link the entire current I, it seems plausible that we should reduce its contribution to the total flux linkage, for purposes of inductance calculation, by the ratio that the actual current linked bears to the total current. Since the current which is linked by $d\psi'$ is the current in an annular ring of area $2\pi r\,dr$, the reduction factor is $2\pi r\,dr/\pi a^2$, and the equivalent flux linkage $d\psi$ is given by

$$d\psi = \frac{2\pi r\,dr}{\pi a^2}\,d\psi'$$

At a later point a firmer basis for this procedure will be given. For the present example we have

$$d\psi = \frac{I2\pi r\,dr}{\pi a^2 I}\left[\frac{\mu_0 I}{4\pi a^2}(a^2 - r^2) + \frac{\mu_0 I}{2\pi}\ln\frac{b}{a}\right]$$

The total flux linkage is

$$\psi = \int_0^a d\psi = \frac{\mu_0 I}{\pi a^2}\left(\int_0^a \frac{a^2 r - r^3}{2a^2}\,dr + \ln\frac{b}{a}\int_0^a r\,dr\right)$$

$$= \frac{\mu_0 I}{8\pi} + \frac{\mu_0 I}{2\pi}\ln\frac{b}{a} \qquad (8.19)$$

Hence the inductance per unit length is

$$L = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a} \qquad (8.20)$$

The first term $\mu_0/8\pi$ is known as the internal inductance of the center conductor since this term arises from the flux linkages internal to the conductor. The second term is known as the external inductance since this corresponds to the external flux linkages.

Evaluation of Self-inductance

For an infinitely long single wire of circular cross section the internal inductance per unit length is obviously $\mu_0/8\pi$, since a single wire has a field internal to itself of the same form as the center conductor of the coaxial line we just considered. The external inductance per unit length is infinite, a result which may be obtained by letting b tend to infinity in (8.20). In practice, we do not have infinitely long wires; so this latter result is of no consequence. However, for a thin wire of total length l bent into an arbitrary loop, the magnetic field near the surface is very nearly the same as that for an infinitely long wire provided the radius of curvature of the loop is much greater than the conductor radius at all points. In other words, we may treat the wire locally as though it were part of an infinitely long wire. It follows that the internal inductance of any loop of mean length l is $\mu_0 l/8\pi$. This result is of great importance since it leads to a simple method of formulating an expression for the self-inductance of a circuit.

Consider a conductor of radius r_0 bent into a closed loop C_1, as in Fig. 8.9. Let the contour C_1 coincide with the interior edge of the conductor. The self-inductance of the circuit consists of the sum of the internal inductance and the external inductance. The external inductance arises from the flux linking the contour C_1. To evaluate this flux linkage we may assume that the current I is concentrated in an infinitely thin filament along the center C_0 of the conductor with negligible error. The problem is equivalent to that of evaluating the mutual inductance between the contours C_0 and C_1. Thus

$$L_e = \frac{\mu_0}{4\pi} \oint_{C_0} \oint_{C_1} \frac{d\mathbf{l}_0 \cdot d\mathbf{l}_1}{R}$$

The self-inductance L is thus given by

$$L = L_i + L_e = \frac{\mu_0 l}{8\pi} + \frac{\mu_0}{4\pi} \oint_{C_0} \oint_{C_1} \frac{d\mathbf{l}_0 \cdot d\mathbf{l}_1}{R} \tag{8.21}$$

Example 8.5. Self-inductance of a Circular Loop. Consider a conductor of radius r_0 bent into a circular loop of mean radius a, as in Fig.

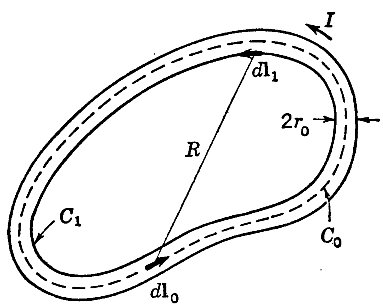

FIG. 8.9. A conductor of finite radius r_0 bent into a closed loop.

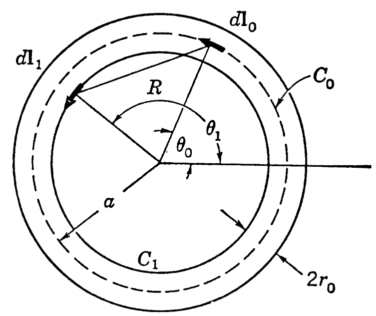

FIG. 8.10. A circular conducting loop.

8.10. The internal inductance of the loop is $\mu_0 2\pi a/8\pi = \mu_0 a/4$. The magnitudes of $d\mathbf{l}_0$ and $d\mathbf{l}_1$ are given by

$$|d\mathbf{l}_0| = a\, d\theta_0 \qquad |d\mathbf{l}_1| = (a - r_0)\, d\theta_1 \approx a\, d\theta_1$$

The angle between $d\mathbf{l}_0$ and $d\mathbf{l}_1$ is $\theta_1 - \theta_0$, and hence

$$d\mathbf{l}_0 \cdot d\mathbf{l}_1 = a^2 \cos(\theta_1 - \theta_0)\, d\theta_0\, d\theta_1$$

The distance R between the two elements of arc length is given by

$$R^2 = a^2 + (a - r_0)^2 - 2a(a - r_0)\cos(\theta_1 - \theta_0)$$

Hence the external inductance is given by

$$L_e = \frac{\mu_0 a^2}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(\theta_1 - \theta_0)\, d\theta_1\, d\theta_0}{[2a(a - r_0) + r_0^2 - 2a(a - r_0)\cos(\theta_1 - \theta_0)]^{1/2}}$$

If we integrate over θ_1 first, we may change variables and replace $\theta_1 - \theta_0$ by θ and $d\theta_1$ by $d\theta$; thus

$$L_e = \frac{\mu_0 a^2}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos\theta\, d\theta\, d\theta_0}{[r_0^2 + 2a(a - r_0)(1 - \cos\theta)]^{1/2}}$$

It is not necessary to alter the limits of the integral because the origin for θ_1 is arbitrary in view of the circular symmetry. Since the result of the integration in θ is independent of θ_0, we may perform the integration over θ_0 at once, thereby obtaining a factor 2π. We now have

$$L_e = \frac{\mu_0 a^2}{2} \int_0^{2\pi} \frac{\cos\theta\, d\theta}{[r_0^2 + 2a(a - r_0)(1 - \cos\theta)]^{1/2}}$$

This expression can be evaluated in terms of elliptic integrals. Only the final result will be given here. It is found that

$$L_e = \mu_0 a \left[\left(\frac{2}{k} - k \right) K - \frac{2}{k} E \right] \tag{8.22}$$

where $k^2 = 4a(a - r_0)/(2a - r_0)^2$ and K and E are elliptic integrals given by

$$K = \int_0^{\pi/2} \frac{d\alpha}{(1 - k^2 \sin^2 \alpha)^{1/2}} \qquad E = \int_0^{\pi/2} (1 - k^2 \sin^2 \alpha)^{1/2}\, d\alpha$$

The above integrals are tabulated.† For $r_0 \ll a$, the result (8.22) reduces to

$$L_e = \mu_0 a \left(\ln \frac{8a}{r_0} - 2 \right) \tag{8.23}$$

Thus the self-inductance of a circular loop of mean radius a is

$$L = L_i + L_e = \mu_0 a \left(\ln \frac{8a}{r_0} - 1.75 \right) \qquad r_0 \ll a \tag{8.24}$$

8.4. Energy of a System of Current Loops

Consider two closed conducting loops C_1 and C_2, as in Fig. 8.7, with currents i_1 and i_2, which are initially zero. In the process of increasing i_1 and i_2 from zero to final values I_1 and I_2, work is done on the system. According to the field theory, this work results in stored energy in the magnetic field surrounding the conductors. To evaluate this quantity, let us maintain i_2 at zero while increasing i_1 from zero to its final value I_1

† E. Jahnke and F. Emde, "Tables of Functions," 4th ed., Dover Publications, New York, 1945.

first. When we change i_1 by an amount di_1 in a time interval dt, the magnetic field $\mathbf{B}_1$ due to i_1 changes at an average rate $d\mathbf{B}_1/dt$. Consequently, an induced voltage $\mathcal{E}_1 = -d\psi_{11}/dt$ is produced in C_1, and similarly, an induced voltage $\mathcal{E}_2 = -d\psi_{12}/dt$ is produced in C_2. Thus, in order to change i_1 an amount di_1 in a time interval dt, we must apply a voltage $-\mathcal{E}_1$ in the circuit C_1. At the same time we must apply a voltage $-\mathcal{E}_2$ in C_2 to maintain i_2 at zero. In the time interval dt the applied voltage $-\mathcal{E}_1$ does work of amount

$$dW_1 = -\mathcal{E}_1 i_1\,dt = i_1\,d\psi_{11} = L_{11} i_1\,di_1$$

since by definition $L_{11}i_1 = \psi_{11}$ and because L_{11} is constant $L_{11}\,di_1 = d\psi_{11}$. The applied voltage $-\mathcal{E}_2$ does zero work since i_2 is kept equal to zero. The total work done in increasing i_1 from zero to I_1 is thus

$$W_1 = \int_0^{I_1} L_{11} i_1\,di_1 = \tfrac{1}{2} L_{11} I_1^2 \tag{8.25}$$

This is the energy stored in the magnetic field surrounding a single circuit.

Next we keep I_1 constant and increase i_2 by an amount di_2 in a time interval dt. This results in an induced voltage

$$\mathcal{E}_2 = \frac{-d\psi_{22}}{dt} = -L_{22}\frac{di_2}{dt} \qquad \text{in } C_2$$

and

$$\mathcal{E}_1 = \frac{-d\psi_{12}}{dt} = -L_{12}\frac{di_2}{dt} \qquad \text{in } C_1$$

To maintain i_1 constant at its value I_1, we must apply a voltage $-\mathcal{E}_1$. In time dt this voltage does work (or has work done upon it, depending on whether $\mathcal{E}_1$ tends to increase or decrease i_1):

$$dW_{12} = -\mathcal{E}_1 I_1\,dt = I_1 L_{12}\,di_2$$

Similarly, the voltage $-\mathcal{E}_2$ that must be applied to change i_2 by an amount di_2 does work of amount

$$dW_2 = -\mathcal{E}_2 i_2\,dt = L_{22} i_2\,di_2$$

The total work done in changing i_2 from zero to a final value I_2 is

$$\begin{aligned} W_{12} + W_{22} &= I_1 L_{12} \int_0^{I_2} di_2 + L_{22}\int_0^{I_2} i_2\,di_2 \\ &= I_1 I_2 L_{12} + \tfrac{1}{2} I_2^2 L_{22} \end{aligned} \tag{8.26}$$

The work done on the system is the sum of (8.25) and (8.26) and represents the energy W_m stored in the magnetic field. This energy is given by

$$\begin{aligned} W_m &= \tfrac{1}{2} L_{11} I_1^2 + L_{12} I_1 I_2 + \tfrac{1}{2} L_{22} I_2^2 \\ &= \tfrac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} L_{ij} I_i I_j \end{aligned} \tag{8.27}$$

This result is easily generalized to a system of N loops; the result is given by (8.27) by increasing the summations in (8.27) for both i and j, up to N.

The magnetic energy in the field around a single current loop of finite cross section may be written in a form analogous to (8.27). We may divide the current-loop cross section into a large number of current filaments of cross-sectional area ΔS_i for the ith filament (see Fig. 8.11). If J_i is the current density in the ith filament, then $I_i = J_i \, \Delta S_i$ is the total current flow in this flow tube or filament. Let ψ_i be the flux linking the ith filament due to all the other current filaments in the current loop. This in turn is given by

$$\psi_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} L_{ij} I_j = \sum_{\substack{j=1 \\ j \neq i}}^{N} L_{ij} J_j \, \Delta S_j$$

where N is the total number of current filaments or flow tubes making up the total current loop, L_{ij} is the mutual inductance between filaments i and j, and I_j is the current flowing in the jth filament. Since we have divided our original current loop into N current filaments, we have reduced the problem to one of a collection of N filamentary current loops and (8.27) may be applied to give

$$\begin{aligned} W_m &= \tfrac{1}{2} \sum_{i=1}^{N} L_{ii} I_i^2 + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} L_{ij} I_i I_j \\ &= \tfrac{1}{2} \sum_{i=1}^{N} L_{ii} I_i^2 + \tfrac{1}{2} \sum_{i=1}^{N} \psi_i I_i \end{aligned} \tag{8.28a}$$

Now, as demonstrated in Example 8.5, the self-inductance L_{ii} of a thin current filament of cross-sectional radius r_0 becomes infinite as $\ln r_0$. However, the total current in the filament decreases as r_0^2 as the cross-sectional area is made smaller, so that in the limit as r_0 goes to zero for each current filament, $L_{ii}I_i^2$ vanishes as $r_0^4 \ln r_0$. The number of current flux tubes N is inversely proportional to the cross-sectional area of the flux tube, that is, $N \propto r_0^{-2}$. Thus, for infinitely thin current filaments, the sum of the "self-energy" terms in (8.28a), i.e., the terms $L_{ii}I_i^2$, vanishes as $r_0^2 \ln r_0$. Each term in the double summation of (8.28a) is also of order r_0^4; however, the total number of such terms is $N^2 \propto r_0^{-4}$. Consequently, this summation may be expected to remain finite in the limit $r_0 \to 0$. Thus we are left with

$$W_m = \tfrac{1}{2} \sum_{i=1}^{N} \psi_i I_i \tag{8.28b}$$

This result expresses the energy of a single "thick" current loop in terms of the mutual energy between the current filaments that comprise the current loop. Equation (8.28*b*) will be used in the next section to establish a suitable definition for partial and total flux linkages.

Equation (8.27) gives an interpretation of the coefficients of inductance L_{ij} as the coefficients in the quadratic expression for the energy stored in the magnetic field. The terms L_{ij} $(i \neq j)$ may be either positive or negative, depending on the direction in which the mutual magnetic flux links the respective circuits. For two circuits with currents I_1 and I_2, we have

$$W_m = \tfrac{1}{2}I_1^2L_{11} + \tfrac{1}{2}I_2^2L_{22} \pm I_1I_2L_{12}$$

which may be written as

$$W_m = \tfrac{1}{2}[(I_1\sqrt{L_{11}} - I_2\sqrt{L_{22}})^2 + I_1I_2(\sqrt{L_{11}L_{22}} \pm L_{12})]$$

The first term is always positive or zero. If we choose

$$I_1\sqrt{L_{11}} = I_2\sqrt{L_{22}}$$

so that the first term is zero, then since the energy stored in the field is always positive, we see that the mutual inductance L_{12} must satisfy the relation

$$L_{12} \leq \sqrt{L_{11}L_{22}}$$

in order that the second term may also always be positive. The coefficient of coupling k is defined by

$$L_{12} = k\sqrt{L_{11}L_{22}} \tag{8.29}$$

and has a maximum value of unity when all the magnetic flux set up by the magnetic field of circuit 1 links circuit 2.

8.5. Energy as a Field Integral

In the preceding section the work done in setting up a system of current-carrying loops was evaluated in order to determine the energy stored in the magnetic field. As in electrostatics, it should be possible to express this energy in terms of the field alone. The analogy with the electrostatic field turns out to be a very close one, for we shall show that the energy in the magnetic field is given by the following integral:

$$W_m = \tfrac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}\,dV = \tfrac{1}{2}\mu\int_V \mathbf{H}\cdot\mathbf{H}\,dV \tag{8.30}$$

where the integration is to be taken over the whole volume occupied by the field. The second expression in (8.30) is valid only if μ is a constant.

We shall prove the above result for the special case of a single conducting loop with finite thickness and carrying a current I, as in Fig. 8.11. Replacing $\mathbf{B}$ by $\nabla \times \mathbf{A}$ in (8.30), we obtain

$$W_m = \frac{1}{2} \int_V (\nabla \times \mathbf{A}) \cdot \mathbf{H} \, dV$$

Next we use the expansion $\nabla \cdot (\mathbf{A} \times \mathbf{H}) = (\nabla \times \mathbf{A}) \cdot \mathbf{H} - (\nabla \times \mathbf{H}) \cdot \mathbf{A}$ and replace $\nabla \times \mathbf{H}$ by $\mathbf{J}$, thereby obtaining

$$\begin{aligned} W_m &= \frac{1}{2} \int_V \mathbf{J} \cdot \mathbf{A} \, dV + \frac{1}{2} \int_V \nabla \cdot (\mathbf{A} \times \mathbf{H}) \, dV \\ &= \frac{1}{2} \int_V \mathbf{J} \cdot \mathbf{A} \, dV + \frac{1}{2} \oint_S \mathbf{A} \times \mathbf{H} \cdot d\mathbf{S} \end{aligned}$$

where the divergence theorem has been used to convert the second volume integral to a surface integral over the closed surface S. If we choose S to

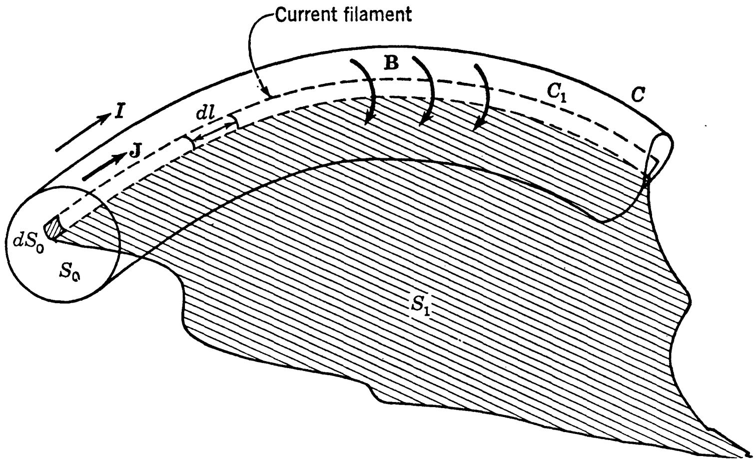

FIG. 8.11. Cross section of conductor C.

be a spherical surface at infinity, then, assuming that the sources are in a finite region, $A \propto 1/R$ and $H \propto 1/R^2$ on S, as we may confirm from (6.9) and (6.11). Thus, while $S \propto R^2$, the integral behaves as $1/R$, and since S is at infinity, this integral vanishes.† Hence

$$W_m = \frac{1}{2} \int_V \mathbf{J} \cdot \mathbf{A} \, dV \qquad (8.31)$$

Now $\mathbf{J} = 0$ everywhere except along the circuit C, where $\mathbf{J} \, dV = J \, dS_0 \, d\mathbf{l}$ and dS_0 is an element of area in the cross section of C, as in Fig. 8.11. We

† When we come to examine general time-varying fields, we shall discover that a radiation field can exist for which $H \propto 1/R$. Under these conditions the integral in question does not vanish but represents radiated energy.

may write (8.31) as follows:

$$W_m = \tfrac{1}{2} \int_{S_0} J \, dS_0 \oint_{C_1} \mathbf{A} \cdot d\mathbf{l}$$

where S_0 is the cross-sectional area of the conductor and C_1 is the contour of an elementary filament of current. If J is constant over the cross section S_0, we have $I = JS_0$, and we get

$$W_m = \tfrac{1}{2} I \int_{S_0} \left(\frac{dS_0}{S_0} \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} \right) \tag{8.32}$$

Equation (8.32) is readily seen to be the integral form of (8.28*b*) since $\oint_{C_1} \mathbf{A} \cdot d\mathbf{l}$ is the flux that links the current filament $I \, dS_0/S_0$ and the integral over S_0 is merely the limit of the sum in (8.28*b*) as the number N of current filaments is made infinite; i.e., the cross section of each filament is made infinitesimally small. This result thus verifies the equivalence between the field integral (8.30) and the expression (8.28*b*) for the energy in the magnetic field surrounding a current loop.

The term $(dS_0/S_0) \oint_{C_1} \mathbf{A} \cdot d\mathbf{l}$ is called the partial flux linkage $d\psi$ because

$$\oint_{C_1} \mathbf{A} \cdot d\mathbf{l} = \int_{S_1} \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{S_1} \mathbf{B} \cdot d\mathbf{S}$$

and is the flux linking the contour C_1, where S_1 is the surface bounded by C_1, as illustrated. It should be noted that the flux linking the contour C_1 is multiplied by the fraction of the total current that flows in the thin filament of cross-sectional area dS_0 to obtain the partial flux linkage. Completing the integration we have

$$W_m = \tfrac{1}{2} I \int_{S_0} d\psi = \tfrac{1}{2} I\psi = \tfrac{1}{2} LI^2 \tag{8.33}$$

This equation shows how the total flux linkage ψ of a single circuit must be defined in order that $\tfrac{1}{2} I\psi$ will give a correct result for the energy stored in the field. The alternative expression $W_m = \tfrac{1}{2} LI^2$ follows simply by defining L as equal to ψ/I, with ψ understood as the sum of all the partial flux linkages.

We now see that by a consideration of the energy stored in the magnetic field we are able to give a consistent and useful definition for the total flux linkage ψ. The resulting definition for the self-inductance L is thus based indirectly on energy considerations. We may, however, omit the intermediate step which introduced the flux linkage ψ and define L directly in terms of the magnetic energy stored in the field. Thus, consider a device with two terminals through which a current I enters and leaves. Let W_m be the energy stored in the magnetic field surrounding the device. Its inductance may now be defined as [Eq. (8.33)]

$$L = \frac{2W_m}{I^2} \tag{8.34}$$

This definition is often easier to apply in practice in order to evaluate L than the original definition in terms of flux linkages. A device of the type above is called an inductor, and its circuit applications are discussed at the end of Chap. 9.

It has thus been proved that (8.33) and (8.30) are equivalent expressions for the energy stored in the magnetic field. In the proof of this equivalence (8.30) was reduced to the form given by (8.32). The integrand in (8.32) was next identified as the partial flux linkage of the total current. This corresponds to the definition that was used in Example 8.4, where we chose

$$d\psi = \frac{dS_0}{S_0} \oint_{C_1} \mathbf{A} \cdot d\mathbf{l} \tag{8.35}$$

as the definition of the partial flux linkage. If the current density J is not constant over the cross section, the partial flux linkage must be taken as

$$d\psi = \frac{J\, dS_0}{I} \oint_{C_1} \mathbf{A} \cdot d\mathbf{l}$$

instead.

The above proof may be generalized to a system of N current loops as well, and hence (8.30) is a valid expression under all circumstances. At times it is convenient to think of the integrand $\mathbf{B} \cdot \mathbf{H}/2$ as the density of magnetic energy at a given point in space. However, it must be kept in mind that it is not possible to state where energy is located. Only the total energy associated with a given field has a physical meaning.

8.6. Forces as Derivatives of Coefficients of Inductance

The force between two separate current-carrying loops or circuits may be evaluated by means of Ampère's law of force. However, an alternative method that is much easier to apply in many cases may also be used. This alternative method consists essentially in evaluating the derivatives of mutual-inductance coefficients with respect to arbitrary virtual displacements of the circuits with respect to each other. When two circuits are displaced relative to each other, the mutual inductance, and hence the energy stored in the magnetic field, changes. The change in the magnetic energy is in turn related to the work done against the forces of the field in displacing the circuits.

Consider two circuits C_1 and C_2 with currents I_1 and I_2, as in Fig. 8.12. The force $\mathbf{F}$ exerted on C_2 by C_1 will be evaluated by finding the work done and the change in field energy when C_2 is displaced by an amount $d\mathbf{r}$. During the displacement the currents I_1 and I_2 will be kept constant. Initially, the flux ψ_{12} linking C_2 due to the current I_1 in C_1 is given by $\psi_{12} = L_{12}I_1$ by definition of L_{12}. The energy stored in the magnetic field is†

$$W_m = \tfrac{1}{2}I_1^2L_{11} + I_1I_2L_{12} + \tfrac{1}{2}I_2^2L_{22} \tag{8.36}$$

Consider that the displacement of C_2 by the amount $d\mathbf{r}$ occurs in a time

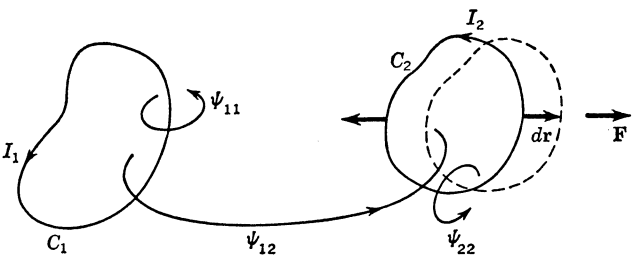

FIG. 8.12. Illustration of two circuits and their relative displacement (L_{12} negative).

interval dt. In this displacement the flux linking C_2 changes by an amount

$$d\psi_{12} = I_1\,dL_{12}$$

As a result of this change in flux linkage an induced voltage

$$\mathcal{E}_2 = -\frac{d\psi_{12}}{dt}$$

is produced in C_2. In order to keep I_2 constant we must apply a voltage $-\mathcal{E}_2$ in C_2. This voltage does work of amount

$$dW_{12} = -\mathcal{E}_2I_2\,dt = I_1I_2\,dL_{12}$$

† The interpretation of

$$W_m = \tfrac{1}{2}L_1I_1^2 + \tfrac{1}{2}L_2I_2^2 + L_{12}I_1I_2$$

may be made either in terms of infinitely thin circuit elements or in terms of finite cross-sectional current-carrying conductors. In the latter case the modified definition of self-inductance in terms of partial flux linkages must be used. For nonfilamentary conductors we define the mutual inductance in terms of the mutual energy and, by a derivation similar to that for self-inductance, are led to a generalized expression $L_{ij} = \psi_{ij}/I_i$, where ψ_{ij} is now the total partial flux linking the jth circuit due to I_i. Specifically, $\psi_{ij} = \int_{\text{cross section}} \psi_{0i}(dS_0/S_0)$, where ψ_{0i} is the flux due to i linking the current tube dS_0 of the jth circuit and S_0 is the total cross-sectional area of the jth circuit. From a practical standpoint the internal flux is often negligible, in which case $L_{ij} = \psi_{ij}/I_i$ and ψ_{ij} is the flux linking any mean current tube in the jth circuit.

in the time interval dt. Similarly, the flux linking C_1 changes by $d\psi_{12}$, and in order to keep I_1 constant, we must apply a voltage $-\mathcal{E}_1 = d\psi_{12}/dt$ in C_1. This voltage does work of amount $dW_{21} = -\mathcal{E}_1 I_1\,dt = I_1 I_2\,dL_{12}$ in the time dt. At the same time the energy stored in the magnetic field changes. This can be evaluated from (8.36) to be

$$dW_m = I_1 I_2\,dL_{12}$$

If we now assume that the force $\mathbf{F}$ due to C_1 on C_2 is in the direction of $d\mathbf{r}$, then we must apply a force $-\mathbf{F}$ along $d\mathbf{r}$ in order to displace C_2 relative to C_1. The work we shall do during the displacement is

$$dW = -\mathbf{F} \cdot d\mathbf{r}$$

In order to satisfy the law of energy conservation, the mechanical work done plus the work performed by the voltage sources in keeping I_1 and I_2 constant must be equal to the change in the field energy. Thus we get

$$-\mathbf{F} \cdot d\mathbf{r} + 2I_1 I_2\,dL_{12} = I_1 I_2\,dL_{12}$$

and hence the force exerted on C_2 in the direction $d\mathbf{r}$ is given by

$$F = I_1 I_2 \frac{dL_{12}}{dr} \tag{8.37}$$

The force between two circuits acts in the direction of increasing mutual inductance.

If we have N circuits and displace the jth circuit by an amount dr_j, we shall find in a similar way that the force F_j exerted on C_j by all the other circuits is given bv

$$F_j = \sum_{\substack{n=1 \\ n \neq j}}^{N} I_j I_n \frac{dL_{jn}}{dr_j} \tag{8.38}$$

where F_j is the component of force along dr_j acting on the jth circuit. The result expressed by (8.38) is equivalent to

$$F_j = \frac{dW_m}{dr_j} \tag{8.39}$$

since
$$W_m = \tfrac{1}{2} \sum_{n=1}^{N} \sum_{s=1}^{N} I_n I_s L_{ns}$$

and we are assuming as a constraint that the currents be kept constant. When circuit j is displaced by an amount dr_j, it must be recalled that in differentiating W_m, the right-hand side is differentiated with respect to L_{nj} and L_{js} $(s = n)$ and the factor $\tfrac{1}{2}$ is thus canceled.

Usually one associates forces with the negative change in energy of a system; i.e., the system moves in such a manner that the energy stored in the system is decreased. The reason why the sign in (8.39) is positive is that, because of the changing flux linkages, the batteries in each circuit do work of twice the amount given by (8.39). Thus the batteries supply not only the increase in the field energy, but also an amount of energy equal to the work done by the field on the circuit during the displacement. The situation here is similar to the electrostatic one when a constant potential constraint is involved. Again it is important to note that the force exerted by the field is unique for a given system of loops with specified currents. The use of a constant-current constraint under a hypothetical displacement is only a matter of convenience; any other assumed constraint would lead to the same value for the force.

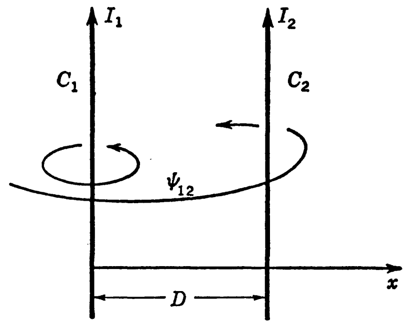

FIG. 8.13. Two infinite linear current-carrying conductors.

Example 8.6. Force on Two Parallel Wires. Consider two thin infinitely long and parallel conductors, as in Fig. 8.13. The conductors are separated by a distance D. The currents in the two conductors are I_1 and I_2. The flux linking C_2 due to the current I_1 in C_1 is

$$\psi_{12} = \frac{\mu_0 I_1}{2\pi} \int_D^\infty \frac{dx}{x} \qquad \text{per unit length}$$

The integral cannot be evaluated since it is not bounded at infinity. However, since we are going to differentiate it with respect to D, we do not need to evaluate it. From (8.37) the force per unit length exerted on C_2 by C_1 is

$$F = I_1 I_2 \frac{dL_{12}}{dD}\bigg|_{I=\text{constant}} = I_2 \frac{d\psi_{12}}{dD} = -\frac{\mu_0 I_1 I_2}{2\pi D}$$

a result in accord with Ampère's law of force. The negative sign signifies that the force is an attractive one for currents I_1 and I_2, having the directions assumed in Fig. 8.13.

Example 8.7. Force between a Long Wire and a Rectangular Loop. Figure 8.14 illustrates a rectangular loop C_2 carrying a current I_2 and placed with its nearest side a distance D from an infinitely long conductor C_1 carrying a current I_1. With the assumed directions of current flow, the flux linking C_2 due to I_1 is oppositely directed to that due to I_2. Hence the mutual inductance is negative. The flux linking C_2 due to

I_1 is

$$\psi_{12} = \frac{\mu_0 I_1}{2\pi} a \int_D^{b+D} \frac{dx}{x} = \frac{\mu_0 I_1 a}{2\pi} \ln \frac{b+D}{D}$$

The mutual inductance L_{12} is thus

$$L_{12} = -\frac{\psi_{12}}{I_1} = -\frac{\mu_0 a}{2\pi} \ln \frac{b+D}{D}$$

The force exerted by the field on C_2 in the direction of increasing D is

$$F = I_1 I_2 \frac{dL_{12}}{dD}\bigg|_{I=\text{constant}} = \frac{\mu_0 a b I_1 I_2}{2\pi D(b+D)}$$

This result is readily verified by Ampère's law of force.

Example 8.8. Torque on a Rectangular Loop. The rectangular loop C_2 in Fig. 8.14 is rotated about its axis by an amount θ so that the resulting

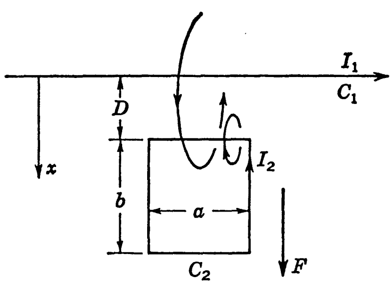

FIG. 8.14. Evaluation of force on a rectangular loop.

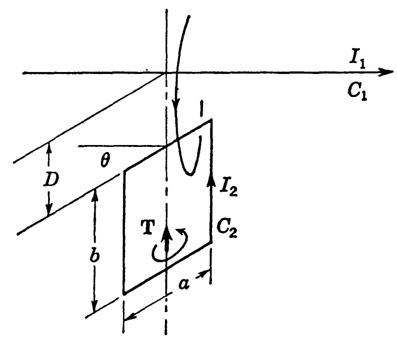

FIG. 8.15. Evaluation of torque on a rectangular loop.

configuration is given by Fig. 8.15. The torque exerted on the loop by the field is required. The flux linking the loop due to the field set up by I_1 differs from that of the previous example approximately by a factor $\cos\theta$ when $a \sin\theta \ll D$. Hence we have in the present case

$$L_{12} = -\frac{\mu_0 a \cos\theta}{2\pi} \ln \frac{b+D}{D}$$

The torque exerted on the loop by the field, by a slight modification of (8.38), is

$$T = I_1 I_2 \frac{dL_{12}}{d\theta}\bigg|_{I=\text{constant}} = \frac{\mu_0 a \sin\theta}{2\pi} I_1 I_2 \ln \frac{b+D}{D}$$

8.7. Lifting Force of Magnets

An equation for the lifting force of a magnet may be obtained by means of an analysis similar to that used to obtain the force equation (8.39) in the previous section. Consider two U-shaped pieces of iron with uniform cross sections of area S and with mean lengths l. The two sections are separated by a small air gap of thickness x, as in Fig. 8.16. The upper section is wound with N turns of wire to produce an electromagnet. A current I flows in the coil. The permeability of the iron is assumed constant at the value μ. The case when μ varies is considered later. A solution of the magnetic circuit shows that the total flux in the cross section of the circuit is given by

$$\psi = \frac{NI}{2l/\mu S + 2x/\mu_0 S} \tag{8.40}$$

The inductance of the coil under the given conditions is

$$L = \frac{N\psi}{I} = \frac{SN^2}{2l/\mu + 2x/\mu_0} \tag{8.41}$$

since the flux ψ links N turns. If the lower U section is displaced by an amount dx in a time interval dt, then, assuming constant current, the flux ψ changes by an amount

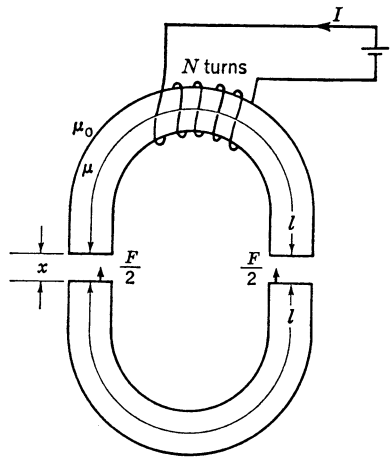

FIG. 8.16. An electromagnet.

$$d\psi = \frac{d\psi}{dx}\bigg|_{I=\text{constant}} dx = -\frac{2NI\,dx}{\mu_0 S\left(\dfrac{2l}{\mu S} + \dfrac{2x}{\mu_0 S}\right)^2} = -\frac{\psi\,dx}{\mu_0\left(\dfrac{l}{\mu} + \dfrac{x}{\mu_0}\right)} \tag{8.42}$$

This results in an induced voltage $\mathcal{E}$ in the coil, where

$$\mathcal{E} = -N\frac{d\psi}{dt}\bigg|_{I=\text{constant}} = -I\frac{dL}{dt}$$

In order to keep the current I constant, we must apply a voltage $-\mathcal{E}$ in series with the battery in the coil. This applied voltage does work of amount

$$dW_1 = -\mathcal{E}I\,dt = I^2\,dL$$

in the time interval dt. During the displacement, the energy in the magnetic field changes by an amount $dW_m = I^2\,dL/2$. If the field exerts a force F on the lower U section, we must apply a force $-F$ in order to increase the air gap by an amount dx. During the displacement we do work of amount $dW = -F\,dx$. Equating the work done on the system to the change in field energy, we get

$$dW + dW_1 \equiv -F\,dx + I^2\,dL = \tfrac{1}{2}I^2\,dL$$

and hence

$$F = \tfrac{1}{2}I^2\frac{dL}{dx} \tag{8.43}$$

Replacing IL by $N\psi$ and using (8.42), we obtain

$$\frac{F}{2} = -\frac{NI\psi}{4\mu_0 S(l/\mu S + x/\mu_0 S)} \tag{8.44}$$

for the force exerted on one pole or arm of the U section. In the air gap we have

$$B = \frac{\psi}{S}$$

$$SH \equiv \frac{\psi}{\mu_0} = \frac{NI}{2\mu_0(l/\mu S + x/\mu_0 S)}$$

and hence (8.44) may also be written as

$$\frac{F}{2} = -\frac{BH}{2}S = -\frac{\mu_0 H^2}{2}S \tag{8.45}$$

and the negative sign indicates that the force is attractive under the given conditions. The interpretation of this result is that the magnetic field exerts an attractive force along the direction of the field with a magnitude or density given by

$$f = \frac{BH}{2} = \frac{\mu_0 H^2}{2} \qquad \text{newtons/sq m} \tag{8.46}$$

In (8.46) f is a force per unit area, the factor 2 having been absorbed by the two pole faces. With this interpretation it is clear that it does not matter if the magnet illustrated in Fig. 8.16 is an electromagnet or a permanent magnet. As long as the flux density in the air gap is the same, the lifting force produced by the field will be the same. This conclusion can be demonstrated to have very general application, and a discussion of this is reserved for the following section. First, however, we shall consider the effect of variable μ on the expression for the force exerted by the field.

We have seen that under a constant-current constraint the force exerted by the magnetic field could be expressed as [see Eq. (8.39)]

$$F_j = \frac{dW_m}{dr_j}$$

The derivation implicitly assumed that the permeability of all material bodies involved was constant. When the permeability is a function of the field **H**, then it will be found that the above equation is not correct, although a very similar equation will be found to apply. For a ferromagnetic material the flux density **B** is a function of the field **H**, and hence of the currents I that exist in the circuit. Thus the total flux in the magnetic circuit of Fig. 8.16 is a function of the current I as determined by the B-H curve for the iron involved. In Fig. 8.17 the total magnetic flux ψ is plotted as a function of the current I. If the current is changed by an amount dI in a time interval dt, an induced voltage $\mathcal{E} = -d\psi/dt$ is produced in the circuit. The battery consequently does work of amount $dW_1 = I\,d\psi$ in changing the current by the amount dI. The total work done in increasing ψ from zero to the final value ψ, that is, bringing the material from the point P_1 to P_2 in Fig. 8.17, is given by

$$W_1 = W_m = \int_0^\psi I\,d\psi \tag{8.47}$$

This is also equal to the energy stored in the magnetic field and is seen to be given by the area that is singly hatched in Fig. 8.17.

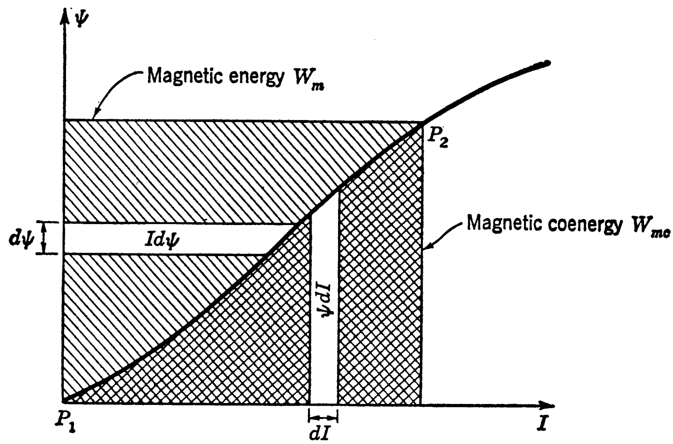

FIG. 8.17. Illustration of magnetic coenergy.

With the aid of (8.47) we shall now derive the expression for the force exerted by the magnetic field on the pole piece of the electromagnet illustrated in Fig. 8.16. If the field exerts a force F on the lower U-shaped section, then the external mechanical work done in a virtual displacement dx is $dW = -F\,dx$. If we assume a constant-current constraint as before, then, because of a change $d\psi$ in the flux through the circuit, the battery does work of amount $dW_1 = I\,d\psi$ in keeping the current constant during the displacement. The change in magnetic energy stored in the system may be found from (8.47) and is given by

$$dW_m = d\int_0^\psi I\,d\psi$$

Equating the external work done to the change in field energy we obtain

$$dW_1 + dW = dW_m$$

or

$$-F\,dx + I\,d\psi = d\int_0^\psi I\,d\psi$$

This expression may be simplified if we integrate the right-hand side by parts once to obtain

$$\int_0^{\psi} I\,d\psi = I\psi\Big|_0^{\psi} - \int_0^{I(\psi)} \psi\,dI$$

The latter integral, that is, $\int_0^{I(\psi)} \psi\,dI$, defines a quantity called the magnetic coenergy. With reference to Fig. 8.17 it is seen to be given by the double-crosshatched area. If we denote the magnetic coenergy by W_{mc}, then we have

$$d\int_0^{\psi} I\,d\psi = I\,d\psi - dW_{mc}$$

since $d(I\psi) = I\,d\psi$, because I is kept constant.

Our energy-balance equation now becomes

$$-F\,dx + I\,d\psi = I\,d\psi - dW_{mc}$$

or

$$F = \frac{dW_{mc}}{dx} = \frac{d(\text{coenergy})}{dx} \tag{8.48}$$

Thus, when μ is variable, the force due to the field is given by the change in the magnetic coenergy instead of by the change in the magnetic energy, as is the case for constant μ. When μ is constant, then the relation between I and ψ is a linear one and $W_m = W_{mc}$ and also $dW_m = dW_{mc}$. Since many practical problems for which we desire the forces acting involve iron, the concept of coenergy is a useful one in practice.†

8.8. Magnetic Stress Tensor

Ampère's law of force between two current elements,

$$\mathbf{F} = \mu_0 \frac{I_2\,d\mathbf{l}_2 \times (I_1\,d\mathbf{l}_1 \times \mathbf{a}_R)}{R^2}$$

is an action-at-a-distance law. It gives the force of one current element on another as though the force were acting at a distance from the current producing the force. When the field $\mathbf{B}$ is introduced, we have

$$\mathbf{F} = I_2\,d\mathbf{l}_2 \times \mathbf{B}$$

This alternative form explains the force on $I_2\,d\mathbf{l}_2$ by means of the interaction of the field $\mathbf{B}$, set up by I_1, and the current I_2. In the preceding section we found that the force between an electromagnet and an iron bar could be expressed in terms of the field $\mathbf{B}$ alone. One of the aims of field theory is to express all observables such as forces and energy in terms of the field alone. Thus if we know the total field surrounding a body, the force exerted on the body is desired in terms of the field alone. This possibility was found to be true for the electrostatic field. In this section similar results for the magnetostatic field will be presented, but without proof.

The results of the previous section showed that along the lines of magnetic flux the

† An interesting discussion on the concept of coenergy from a thermodynamic viewpoint is given by O. K. Mawardi, On the Concept of Coenergy, *J. Franklin Inst.*, vol. 264, pp. 313–332, October, 1957.

field produced a force per unit area given by

$$f_t = \frac{\mu_0 H^2}{2} \tag{8.49}$$

This force per unit area is equivalent to a tension. We may therefore picture the magnetic lines of flux as elastic bands which are stretched, and hence are under tension. In addition to producing a tension along the lines of force, the magnetic field may produce a compressional force perpendicular to the lines of flux. The density of this compressional force turns out to be given by (8.49) also; i.e.,

$$f_c = \frac{\mu_0 H^2}{2} \tag{8.50}$$

We may consider (8.49) and (8.50) as the pressure that the field **H** exerts. In vector form the force acting on a body can be expressed by

$$\mathbf{F} = \oint_S \left[\mu_0 \mathbf{H}(\mathbf{n} \cdot \mathbf{H}) - \frac{\mu_0}{2} (\mathbf{H} \cdot \mathbf{H})\mathbf{n} \right] dS \tag{8.51}$$

where **n** is the unit outward normal to the surface S, and S is any closed surface surrounding the body. Since the force or pressure produced by the field has the dimensions of stress, (8.49) and (8.50) are called the components of the stress tensor. The components of the stress tensor acting on a surface element whose normal is **n** are given by the integrand in (8.51). This integrand is a vector force per unit area and was designated **T** in the analogous electrostatic case. The component of **T** along **n** is

$$\mathbf{n} \cdot \mathbf{T} = \mu_0 (\mathbf{n} \cdot \mathbf{H})^2 - \frac{\mu_0}{2} (\mathbf{H} \cdot \mathbf{H})$$

since the component of **H** along **n** is $\mathbf{n} \cdot \mathbf{H}$. This force is a pressure force. The remaining term, $\mu_0(\mathbf{n} \cdot \mathbf{H})\mathbf{H}_t$, where $\mathbf{H}_t$ is the component of **H** tangential to the surface, represents a shearing force per unit area along the surface. As in the electrostatic case, the magnitude of the surface force density $|\mathbf{T}|$ is $\mu_0 H^2/2$, and analogously, **H** bisects the angle between **T** and **n**.

Example 8.9. Force on a Current Element. The stress tensor will be used to obtain an expression for the force exerted on a linear current element placed in a uniform field $\mathbf{B}_0$. Figure 8.18 illustrates a linear conductor carrying a current I and located in a field $\mathbf{B}_0$ directed perpendicular to it. To find the force acting on the conductor per unit length we shall evaluate (8.51) over the surface of a cylinder of unit length, radius r, and concentric with the conductor. The magnetic field that is required in (8.51) is in this case the sum of $\mathbf{B}_0 = \mu_0 \mathbf{H}_0$ and the field $\mathbf{B}_\phi = \mu_0 I/2\pi r$ due to the current I.

The field $\mathbf{B}_\phi$ will be decomposed into x and y components first in order to facilitate summation. We get

$$\mathbf{B}_\phi = \frac{\mu_0 I}{2\pi r} \mathbf{a}_\phi = \frac{\mu_0 I}{2\pi r} (-\mathbf{a}_x \sin \phi + \mathbf{a}_y \cos \phi)$$

and hence

$$\mathbf{B} = \mathbf{B}_\phi + \mathbf{B}_0 = -\frac{\mu_0 I}{2\pi r} \mathbf{a}_x \sin \phi + \mathbf{a}_y \left(\mathbf{B}_0 + \frac{\mu_0 I}{2\pi r} \cos \phi \right)$$

The unit normal **n** is equal to

$$\mathbf{n} = \mathbf{a}_r = \mathbf{a}_x \cos\phi + \mathbf{a}_y \sin\phi$$

and consequently the integrand in (8.51) becomes

$$\begin{aligned}\mu_0\mathbf{H}(\mathbf{n}\cdot\mathbf{H}) - \frac{\mu_0}{2}(\mathbf{H}\cdot\mathbf{H})\mathbf{n} &= \left[-\frac{\mu_0 I}{2\pi r}\mathbf{a}_x \sin\phi + \mathbf{a}_y\left(B_0 + \frac{\mu_0 I\cos\phi}{2\pi r}\right)\right] \\ &\quad\times\left(-\frac{I\sin\phi\cos\phi}{2\pi r} + H_0\sin\phi + \frac{I\sin\phi\cos\phi}{2\pi r}\right) \\ &\quad-\frac{1}{2}\left[\frac{\mu_0 I^2}{4(\pi r)^2}\sin^2\phi + \left(B_0 + \frac{\mu_0 I\cos\phi}{2\pi r}\right)\left(H_0 + \frac{I\cos\phi}{2\pi r}\right)\right] \\ &\qquad\times(\mathbf{a}_x\cos\phi + \mathbf{a}_y\sin\phi) \qquad (8.52)\end{aligned}$$

The element of area dS is $r\,d\phi$ for a unit length of cylinder. The expression (8.52) appears rather formidable to integrate but is in actual fact rather easy to handle. As a matter of fact, because of the orthogonality

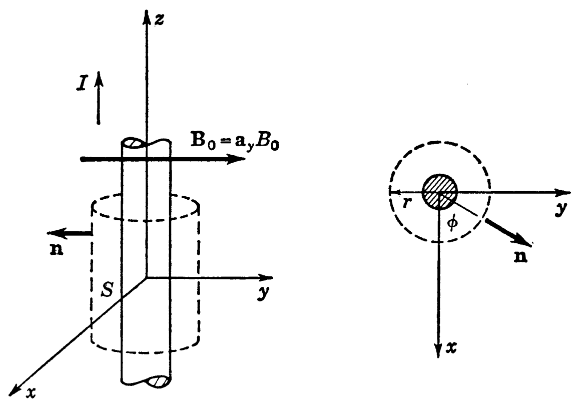

FIG. 8.18. A conductor in a field $\mathbf{B}_0$.

property of the trigonometric functions, all terms in (8.52), when integrated from zero to 2π, go to zero, except the terms in $\sin^2\phi$ or $\cos^2\phi$. Thus (8.52) simplifies to give

$$\mathbf{F} = -\frac{\mu_0 I H_0}{2\pi r}\mathbf{a}_x\int_0^{2\pi}\sin^2\phi\,d\phi - \frac{B_0 I}{2\pi r}\mathbf{a}_x\int_0^{2\pi}\cos^2\phi\,d\phi = -\mathbf{a}_x I B_0 \qquad (8.53)$$

Over the ends of the cylinder $\mathbf{H}\cdot\mathbf{n} = 0$ and the integral of the term $-(\mu_0/2)(\mathbf{H}\cdot\mathbf{H})\mathbf{n}$ vanishes, since **n** is directed in the opposite sense on time interval dt, the voltage induced in the toroid winding is

$$\mathcal{E} = -N\frac{d\psi}{dt} = -NS\frac{dB}{dt}$$

In order to increase I, the battery must do work against the induced voltage of amount

$$dW = -\mathcal{E}I\,dt = NSI\,dB \tag{8.54}$$

According to Ampère's circuital law, the field H is given by $H = NI/2\pi a$. Substituting into (8.54) gives

$$dW = 2\pi aSH\,dB = VH\,dB \tag{8.55}$$

where $V = 2\pi aS$ is the volume of the torus. In changing the field B up to B_1 along the path bc of the hysteresis loop, the work done by the battery is

$$W_1 = V\int_0^{B_1} H\,dB = V(S_1 + S_2) \tag{8.56}$$

where $S_1 + S_2$ is the area between the B axis and the portion bc of the B-H curve. When we decrease B from B_1 to B_2 along the path cd, the decreasing flux induces a voltage in the coil that tends to maintain the current I. Thus work is done against the battery. The amount of work done is

$$W_2 = V\int_{B_1}^{B_2} H\,dB = -VS_2 \tag{8.57}$$

where S_2 is the double-crosshatched area in Fig. 8.20. In reducing the field to zero along the path de, the battery again does work, since the direction of current flow is reversed but the induced voltage acts in the same direction as for (8.57). The work done is

$$W_3 = VS_3 \tag{8.58}$$

where S_3 is the area indicated in Fig. 8.20. To cycle the material from e to f and back up to b, the same amount of work is done as in bringing the material from b to e along the upper half of the loop. The total net work done by the battery in one cycle is therefore

$$W = 2V(W_1 + W_2 + W_3) = 2V(S_1 + S_2 - S_2 + S_3) = VS_t \tag{8.59}$$

where $S_t = 2(S_1 + S_3)$ is the area of the complete hysteresis loop.

Equation (8.59) shows that the area of the hysteresis loop represents the work done per unit volume in cycling a ferromagnetic material around the hysteresis loop once. This amount of energy is lost each cycle and is dissipated as heat in the material. The energy loss is caused by the work required to magnetize the material.

Chapter 8

8.1. (*a*) A rectangular loop is located near an infinite line current, as illustrated. If the current in the linear conductor is $I_0 \cos \omega t$, find the induced voltage in the rectangular loop.

(*b*) If the current in the linear conductor is a constant I_0, find the induced voltage in the rectangular loop as a function of x when the loop moves with a velocity v away from the line current.

(*c*) If the loop moves with velocity v away from the linear conductor and the current in the linear conductor is $I_0 \cos \omega t$, what is the induced voltage in the loop?

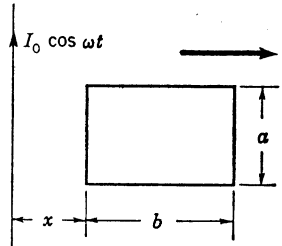

FIG. P 8.1

8.2. A conducting sphere of radius a moves with a constant velocity $v\mathbf{a}_x$ through a uniform magnetic field $\mathbf{B}$ directed along the y axis. Show that an electric dipole field given by

$$\mathbf{E} = \frac{vBa^3}{r^3}(2 \cos \theta \, \mathbf{a}_r + \sin \theta \, \mathbf{a}_\theta)$$

exists around the sphere.

8.3. A large conducting sheet of copper ($\sigma = 5.8 \times 10^7$ mhos per meter) of thickness t falls with a velocity v through a uniform magnetic field B, as illustrated. Show that a force $F = \sigma v t B^2$ per unit area resisting the motion of the conductor exists.

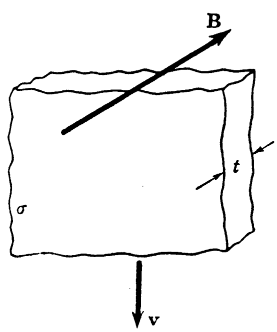

FIG. P 8.3.

8.4. A thin conducting spherical shell (radius a, thickness $t_0 \ll a$, conductivity σ) rotates about a diameter (z axis) at the rate ω in the presence of a constant magnetic field $\mathbf{B}$ directed normal to the axis of rotation (along y axis). Find the resultant current flow in the spherical shell. Assume that the self-inductance of the sphere is negligible so that the current is determined only by the induced electric field and the conductivity σ. Since $t_0 \ll a$, each portion of the spherical surface may be considered to be a plane surface locally.

HINT: Express the velocity of an arbitrary point on the sphere's surface and the field $\mathbf{B}$ in spherical components, and find the induced electric field along the surface.

Answer. For a stationary observer,

$$I_\theta = 1/2\omega a \sigma t_0 B \sin \phi$$
$$I_\phi = 1/2\omega a \sigma t_0 B \cos \theta \cos \phi$$

Note that for steady-state conditions within the sphere

$$\nabla \cdot \mathbf{J} = 0 \qquad J_r = 0 \text{ at } r = a,\ a + t_0$$

A secondary electric field will be induced and can be expressed as $-\nabla\Phi$. Since $\nabla \cdot \mathbf{v} \times \mathbf{B} = 0$, $\nabla^2\Phi = 0$. The boundary conditions on Φ are determined by those imposed on J_r.

8.5. For Prob. 8.4, show that $\cos\phi \sin^3\theta = C$ is the equation for the current flow lines on the sphere. C is the constant which determines a particular member of the family of lines.

8.6. For Prob. 8.4 let the magnetic field **B** be applied parallel to the axis of rotation, and find the resultant current flowing in the spherical shell.

8.7. A circular conducting loop rotates about a diameter at an angular rate ω in the presence of a constant magnetic field **B** normal to the axis of rotation, as illustrated.

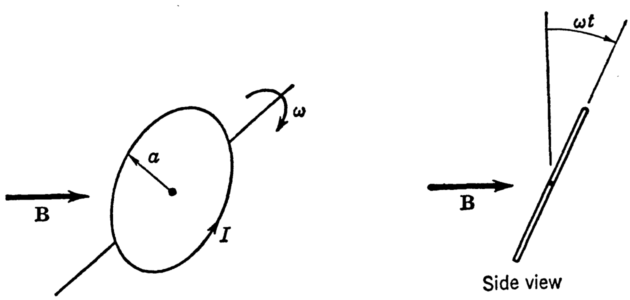

FIG. P 8.7

By making use of Faraday's law and the definition of self-inductance, show that the current flowing in the loop is given by

$$I = \frac{\pi a^2 B\omega \sin(\omega t - \phi)}{[R^2 + (\omega L)^2]^{1/2}}$$

where a is the radius of the loop, R is its resistance, L is its self-inductance, and $\tan\phi = \omega L/R$.

8.8. Show that in Prob. 8.7 the average power dissipated in the resistance R is

$$P = \frac{(\pi a^2 B\omega)^2}{R^2 + (\omega L)^2}\frac{R}{2} \qquad \text{joules/sec}$$

Show that a torque T resisting the rotation of the loop exists, where

$$T = \frac{(\pi a^2)^2 B^2 \omega}{[R^2 + (\omega L)^2]^{1/2}} \sin(\omega t - \phi) \sin \omega t$$

HINT: Consider the magnetic dipole moment of the loop.

Show that the average rate of doing work on the loop in keeping it rotating is equal to the average rate at which energy is dissipated in the resistance R. As R goes to zero, show that the peak value of the current I approaches a constant independent of ω and that for $R = 0$ the average resisting torque vanishes.

8.9. A dielectric slab of thickness t moves with a velocity **v** normal to an applied uniform magnetic field **B**, as illustrated. Find the induced polarization charge within and on the surface of the dielectric slab.

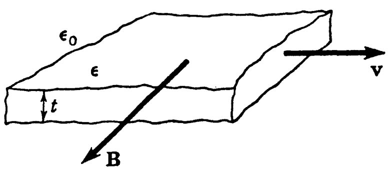

FIG. P 8.9

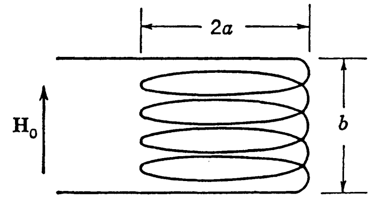

FIG. P 8.10

8.10. A small loop antenna for use in a portable radio receiver consists of N turns wound into a circular coil of radius a and height $b \ll a$. The input circuit to the radio receiver requires that the inductance of the loop be 250 microhenrys. Using the formula

$$L = \frac{0.008a^2n^2}{b} \qquad \text{microhenrys}$$

(dimensions in centimeters, n = number of turns), find the number of turns N required when $a = 6$ centimeters and $b = 1$ centimeter.

What is the voltage induced in the loop when a field $H_0 = 0.1$ microampere per meter is present and the frequency is 1 megacycle per second? Assume H_0 normal to the plane of the loop.

8.11. In place of an air-core loop as in Prob. 8.10, a ferrite rod antenna may be used as illustrated. The rod is in the shape of a prolate spheroid with a length $2d$ and a cross-section radius $a = 0.5$ centimeter. The permeability of the ferrite is $\mu = 200\mu_0$. The demagnetization factor D (see Probs. 7.15 and 7.16) is given by

$$D = \frac{a^2}{2d^2}\left(\ln\frac{4d^2}{a^2} - 2\right) \qquad d \gg a$$

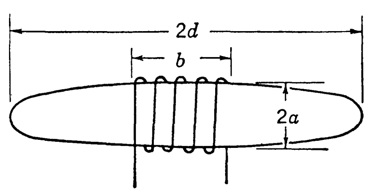

FIG. P 8.11

Find the number of turns N to give an inductance of 250 microhenrys when $d = 6$ centimeters and $b = 4$ centimeters. Use the formula

$$L = \frac{\mu_e}{\mu_0} 0.039 \frac{n^2a^2}{b} \qquad \text{microhenrys} \qquad b \gg a$$

where μ_e is given in Prob. 7.16 and a, b are in centimeters. What is the voltage induced in the coil for the same applied field as in Prob. 8.10? Note that the flux density in the core will be $\mu_e H_0$. What do you conclude regarding the merits of the ferrite rod antenna vs. the air-core loop? The ferrite rod antenna is much smaller, a factor of considerable importance for a portable radio receiver.

8.12. For the infinitely long thin conductor and the rectangular loop arranged as illustrated, show that the mutual inductance is given by

$$M = -\frac{\mu_0 a}{2\pi} \ln \frac{R}{[2b(R^2 - c^2)^{1/2} + b^2 + R^2]^{1/2}}$$

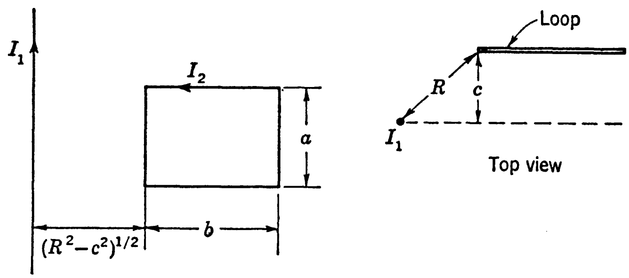

FIG. P 8.12

8.13. For Prob. 8.12 show that the component of force acting on the loop in the direction of R increasing is given by

$$F = \frac{\mu_0 a I_1 I_2}{2\pi R(R^2 - c^2)^{1/2}} \frac{bR^2 - 2bc^2 + b^2(R^2 - c^2)^{1/2}}{2b(R^2 - c^2)^{1/2} + b^2 + R^2}$$

when c is held constant. What is the component of force acting when R is held constant and c is allowed to vary?

8.14. A toroidal coil of mean radius b and cross-sectional radius a consists of N closely wound turns. Show that its self-inductance is given by

$$L = \frac{\mu_0 N^2 a^2}{2b}$$

when $b \gg a$. If the variation in B over the cross section is taken into account, show that

$$L = \mu_0 N^2[b - (b^2 - a^2)^{1/2}]$$

8.15. Two circular loops of radii r_1 and r_2 carry currents I_1 and I_2. The loops are coplanar and separated by a large distance R. By using a dipole approximation for the magnetic field set up by one loop at the position of the second loop, obtain an expression for the mutual inductance between the two loops. What is the force existing between the two loops?

8.16. Design a winding for a U-shaped electromagnet capable of lifting a 1,000-kilogram mass. The cross-section area of each leg is to be 20 square centimeters. The air gap between the electromagnet and the lower bar is 0.1 millimeter, as illustrated. What is the required number of ampere-turns when the magnet mean length is 30 centimeters and its effective relative permeability is 4,000? Assume that the reluctance of the iron bar is negligible.

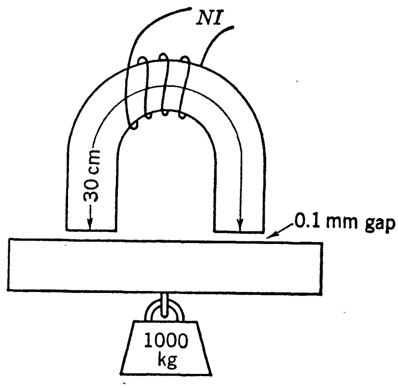

FIG. P 8.16

8.17. A round conductor of radius r_0 is bent into a circular loop of mean radius a. A current I flows in the circuit. Determine if a compressional or tension force acts on the conductor, and also find the magnitude of the force.