CHAPTER 7

MAGNETIC FIELD IN MATERIAL BODIES

According to the atomic theory, all matter is composed of a number of basic elementary particles of which the electron, proton, and neutron are perhaps the best known. These particles combine to form atoms, which we may, for our present purpose, think of as a number of neutrons and protons collected together into a central heavy core around which a number of electrons rotate along closed orbital paths. The electrons rotating in orbital paths are equivalent to circulating currents on an atomic scale. We may associate with each electron orbit a magnetic dipole moment in order to account for the external effects of the material body. The total dipole moment of an atom due to the orbital motion of electrons is called the orbital magnetic moment and is the vector sum of the moments of each orbit with its circulating electron. For a single electron with an electronic charge e rotating around a circular orbit of radius r and with a velocity v, the average circulating current is equal to the rate of movement of charge ev divided by the length $2\pi r$ of the orbit. Thus the magnetic dipole moment of the circulating electron is

$$\begin{aligned} m &= \text{circulating current} \times \text{area of loop} \\ &= \frac{ev}{2\pi r}\pi r^2 = \frac{evr}{2} \end{aligned} \tag{7.1}$$

In addition to the orbital magnetic moment, the electron itself is endowed with an intrinsic magnetic moment which plays an important role in the magnetic properties of material bodies. The physical structure of the electron that gives rise to its characteristic properties mass, charge, and mechanical and magnetic moment is not known, but this does not prevent us from discussing the external effects produced by these intrinsic properties. The presence of a mechanical and a magnetic moment has led physicists to think of the electron as having a spin, and its magnetic moment is therefore referred to as the spin magnetic moment.

The above discussion was introduced in order to provide some insight into the essential properties of matter which give rise to an interaction with externally applied magnetic fields. A more complete account of the physics of the magnetic properties of materials is given in a later section.

7.1. Magnetic Dipole Polarization per Unit Volume

Consider a material body of volume V and bounded by a surface S, as in Fig. 7.1a. In view of our earlier discussion we may think of this material body as consisting of a large number of small circulating current loops having a magnetic dipole moment $\mathbf{m}$. In the absence of any external applied magnetic field $\mathbf{B}_0$, the dipoles of individual atoms are randomly orientated and the body as a whole will have a zero net magnetic moment (this is not true for permanent magnets, which have a nonzero moment even in the absence of applied external fields). When we apply an external magnetic field $\mathbf{B}_0$, two things happen:

1. All atoms which have a nonzero magnetic moment tend to have their magnetic dipoles aligned with the applied field in accordance with the torque relations for magnetic dipoles, as discussed in Sec. 6.4.

2. Even if each atom has a zero moment, the presence of an external field distorts the electron orbits and thus creates or induces magnetic dipole moments.

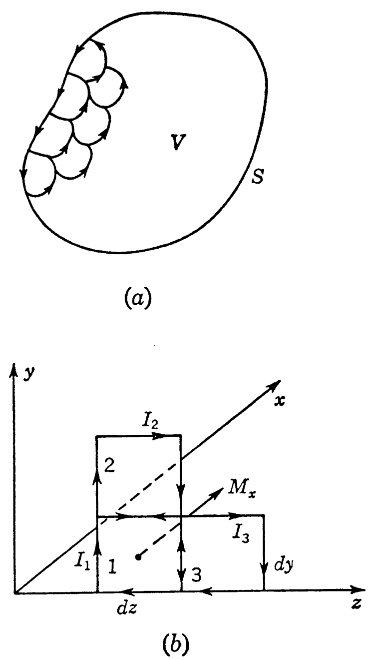

FIG. 7.1. A material body and the equivalent circulating atomic currents.

In many materials the individual atoms do not have an intrinsic magnetic dipole moment, so that only the second effect above takes place when an external field is applied. When induced dipoles are produced, they are always aligned in opposition to the inducing external field. This results in a net magnetic flux density in the interior of the material body, which is less than that of the external field itself. Such materials are referred to as diamagnetic, and the effects as diamagnetism. Some of the materials in this class are copper, bismuth, zinc, silver, lead, and mercury.

For materials with constituent atoms having nonzero dipole moments, both of the above effects are present when an external field is applied. However, the resultant dipole moment is always in the general direction of the applied field because the first effect overshadows that of the induced dipoles. Materials in this class are known as paramagnetic materials, and the effect as paramagnetism. In these materials the aligned dipoles contribute a net flux in the same direction as that of the applied field, and the resultant flux density in the interior of the material body is increased. Both diamagnetic and paramagnetic effects are usually very small. There does, however, exist a class of materials which exhibit very large paramagnetic effects, and these are referred to as ferromagnetic materials. Examples of ferromagnetic materials are iron, nickel, and cobalt. The properties of such materials will be considered in greater detail later.

When an external field is applied to a material body we may describe the external effects produced by the body by assigning to the material a volume distribution of magnetic dipoles. If $\mathbf{m}$ is the magnetic dipole moment of an individual atom or molecule and there are N such effective dipoles per unit volume, then the magnetic dipole polarization $\mathbf{M}$ per unit volume is given by

$$\mathbf{M} = N\mathbf{m} \qquad \text{amp/m}^\dagger \tag{7.2}$$

Having assigned this dipole polarization to the material, we may evaluate the external effects of the material body without any further consideration as to the exact physical process responsible for the existence of such magnetic dipoles.

7.2. Equivalent Volume and Surface Polarization Currents

Using (6.25), the vector potential $\mathbf{A}$ at a point (x,y,z) due to an infinitesimal magnetic dipole $\mathbf{m}$ at (x',y',z') a distance R away is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \nabla \times \frac{\mathbf{m}}{R}$$

The contribution to the vector potential from the magnetic polarization of a material body will be given by an integral over the body of the contribution from each elementary dipole element $\mathbf{M}\, dV'$. Thus we have

$$\mathbf{A}(x,y,z) = \frac{\mu_0}{4\pi} \int_V \nabla \times \frac{\mathbf{M}}{R}\, dV' \tag{7.3}$$

as illustrated in Fig. 7.2.

Since the physical basis of the magnetic dipole moment is the circulating atomic currents, it would appear that we should be able to describe the field produced by a magnetized body in terms of equivalent volume and surface currents. This is indeed the case, and by a suitable juggling of the expression (7.3), it can be put into a form that shows this explicitly.

† Because of the very limited size of the alphabet (both English and Greek) it is sometimes necessary that the same letter mean several things. In Chap. 6, the letter **M** designated the total dipole moment of an arbitrary current loop. For the remainder of the book, beginning in this chapter, **M** represents the dipole moment per unit volume. With a little care it should be possible to avoid confusing the two.

The desired result, which we shall prove, is that

$$\mathbf{A}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \mathbf{M}(x',y',z')}{R}\,dV' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{M}\times\mathbf{n}}{R}\,dS' \tag{7.4}$$

where $\mathbf{n}$ is the unit outward normal to the surface S of the body, and ∇' indicates differentiation with respect to the source variables x', y', z'. If we compare (7.4) with the expression for the vector potential from a distribution of true currents, we are led to interpret the term $\nabla' \times \mathbf{M}$ in the volume integral as an equivalent volume density of polarization current $\mathbf{J}_m$ and similarly to interpret the term $\mathbf{M} \times \mathbf{n}$ in the surface integral as the equivalent surface polarization current $\mathbf{J}_{ms}$. Hence we may write

$$\nabla' \times \mathbf{M} = \mathbf{J}_m \tag{7.5a}$$

$$\mathbf{M} \times \mathbf{n} = \mathbf{J}_{ms} \tag{7.5b}$$

and (7.4) becomes

$$\mathbf{A} = \frac{\mu_0}{4\pi}\left(\int_V \frac{\mathbf{J}_m}{R}\,dV' + \oint_S \frac{\mathbf{J}_{ms}}{R}\,dS'\right) \tag{7.5c}$$

Equations (7.5*a*) and (7.5*b*) are more than mathematical equivalents in that they represent net effective, though inaccessible, currents. These currents arise when adjacent molecular current loops do not quite cancel each other. This will take place if the magnetization is inhomogeneous, including, particularly, the discontinuity at the boundary of a magnetic material. Figure 7.1*b* illustrates three adjacent molecular current loops, which, for simplicity, are chosen to be square and to lie in the yz plane. They represent the x component of the net magnetic moment per unit volume for the cube with sides dx, dy, and dz. If $\mathbf{M}$ is the dipole moment per unit volume, then

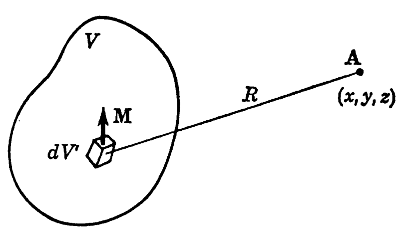

FIG. 7.2. Illustration of evaluation of vector potential from a volume distribution of magnetic dipoles.

$$\mathbf{m} = \mathbf{M}\,dx\,dy\,dz = I\,d\mathbf{S}$$

is the moment of a molecular loop of area $d\mathbf{S}$ with a net circulating current I. Applying this to loops 1, 2, and 3 in Fig. 7.1*b* leads to

$$I_1 = \frac{M_x\,dx\,dy\,dz}{dy\,dz} = M_x\,dx$$

$$I_2 = \left(M_x + \frac{\partial M_x}{\partial y}\,dy\right)dx = M_x\,dx + \frac{\partial M_x}{\partial y}\,dx\,dy$$

$$I_3 = \left(M_x + \frac{\partial M_x}{\partial z}\,dz\right)dx = M_x\,dx + \frac{\partial M_x}{\partial z}\,dx\,dz$$

A net current flows in the z direction if $I_2 \neq I_1$ and in the y direction if $I_1 \neq I_3$. The net current is given by

$$I_z = -\frac{\partial M_x}{\partial y}\,dx\,dy \qquad I_y = \frac{\partial M_x}{\partial z}\,dx\,dz$$

or in terms of a current density,

$$J_z = -\frac{\partial M_x}{\partial y} \qquad J_y = \frac{\partial M_x}{\partial z}$$

The remaining components follow from cyclic permutation; that is, we replace x by y, y by z, and z by x, and the final result is simply (7.5*a*). Consequently, that expression represents the physical picture presented here, of a net effective molecular current due to incomplete cancellation of adjacent current loops in the interior of a magnetizable body. A similar demonstration would show the physical basis for (7.5*b*).

If the magnetization of the body is uniform, **M** is constant throughout, the curl of **M** is zero, and the equivalent volume polarization current is zero. The surface polarization current vanishes only when **M** is perpendicular to the surface so that $\mathbf{M} \times \mathbf{n}$ is zero.

Derivation of Eq. (7.4)

From (7.3) we have

$$\mathbf{A}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \nabla \times \frac{\mathbf{M}(x',y',z')}{R}\,dV' = \frac{-\mu_0}{4\pi}\int_V \mathbf{M}(x',y',z') \times \nabla\left(\frac{1}{R}\right)dV'$$

since $\nabla \equiv \mathbf{a}_x\,\partial/\partial x + \mathbf{a}_y\,\partial/\partial y + \mathbf{a}_z\,\partial/\partial z$ does not operate on $\mathbf{M}(x',y',z')$. Now

$$-\mathbf{M}(x',y',z') \times \nabla(1/R) = \mathbf{M}(x',y',z') \times \nabla'(1/R)$$

and

$$\nabla' \times \frac{\mathbf{M}(x',y',z')}{R} = \frac{1}{R}\nabla' \times \mathbf{M}(x',y',z') - \mathbf{M} \times \nabla'\left(\frac{1}{R}\right)$$

so that we obtain

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \mathbf{M}}{R}\,dV' - \frac{\mu_0}{4\pi}\int_V \nabla' \times \frac{\mathbf{M}}{R}\,dV' \tag{7.6}$$

If we make use of the following vector identity,

$$-\frac{\mu_0}{4\pi}\int_V \nabla' \times \frac{\mathbf{M}}{R}\,dV' = \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{M} \times \mathbf{n}}{R}\,dS' \tag{7.7}$$

then (7.6) is converted into (7.4), which is the desired relation. Equation (7.7) can be thought of as a form of Stokes' theorem; a proof is given below.

Let **C** be a constant vector, and **F** a variable vector function. Then we have $\nabla' \cdot [\mathbf{F}(x',y',z') \times \mathbf{C}] = (\nabla' \times \mathbf{F}) \cdot \mathbf{C} - (\nabla' \times \mathbf{C}) \cdot \mathbf{F} = \mathbf{C} \cdot \nabla' \times \mathbf{F}$, since **C** is a constant vector and $\nabla' \times \mathbf{C} = 0$. Using this result and applying the divergence theorem permits us to establish the following vector relations, namely,

$$\int_V \nabla' \cdot (\mathbf{F} \times \mathbf{C})\,dV' = \oint_S \mathbf{F} \times \mathbf{C} \cdot \mathbf{n}\,dS' = \mathbf{C} \cdot \int_V \nabla' \times \mathbf{F}\,dV'$$
$$= -\mathbf{C} \cdot \oint_S \mathbf{F} \times \mathbf{n}\,dS' \tag{7.8}$$

where $\mathbf{F} \times \mathbf{C} \cdot \mathbf{n} = -\mathbf{C} \cdot \mathbf{F} \times \mathbf{n}$ has also been used. The constant vector $\mathbf{C}$ is arbitrary, so that the result above can hold only if $\int_V \nabla' \times \mathbf{F}\, dV' = -\oint_S \mathbf{F} \times \mathbf{n}\, dS'$. If we now let $\mathbf{F}$ be equal to $\mathbf{M}/R$, the relation (7.7) follows at once and the proof of the equivalence of (7.3) and (7.4) is completed.

7.3. The Magnetic Field Intensity H

When dealing with dielectric bodies in the presence of electrostatic fields, it was convenient to introduce the displacement vector $\mathbf{D}$ in order to eliminate the necessity of taking the electric dipole polarization $\mathbf{P}$ of the material into account explicitly. A similar procedure is used for the magnetic case.

Let us consider a material body of infinite extent in which a true current distribution $\mathbf{J}$ exists. This current gives rise to a partial magnetic field $\mathbf{B}_0$, which in turn gives rise to a magnetic polarization of the material. The secondary field $\mathbf{B}_i$ from the magnetic dipole polarization may be evaluated from the equivalent volume polarization current $\mathbf{J}_m$. The total magnetic field is thus given by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_i = \nabla \times \mathbf{A}_0 + \nabla \times \mathbf{A}_i = \nabla \times \mathbf{A}$$

where $\mathbf{A}_0$ is the vector potential from the true current $\mathbf{J}$, and $\mathbf{A}_i$ arises from the polarization current $\mathbf{J}_m$. Introducing the current sources into the expression for $\mathbf{A}$ results in

$$\mathbf{B} = \nabla \times \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J} + \mathbf{J}_m}{R}\, dV' \tag{7.9}$$

In Sec. 6.6 it was shown that the curl of $\mathbf{B}$ was given by $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ for currents located in vacuum. The derivation was performed by taking the curl of both sides of an expression similar to (7.9), replacing $\nabla \times \nabla \times$ by $\nabla\nabla \cdot - \nabla^2$ and then using the singularity property of the function $\nabla^2(1/R)$ to obtain the final result. If we take the curl of (7.9), we have a completely analogous development, the only difference being that the term $\mathbf{J} + \mathbf{J}_m$ appears instead of $\mathbf{J}$. Therefore (7.9) leads to

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_m) \tag{7.10}$$

This result should not surprise us since the separation of current into the components $\mathbf{J}$ and $\mathbf{J}_m$ is, in a sense, an artifice. We have seen that $\mathbf{J}_m$ has all the physical characteristics of current flow, and if measurements were made on an atomic scale, there would be no basis for distinguishing $\mathbf{J}_m$ from $\mathbf{J}$. Consequently, (7.10) may also be thought of as evolving from (6.41) by generalizing the total current to include the sum of the true current $\mathbf{J}$ and magnetization current $\mathbf{J}_m = \nabla \times \mathbf{M}$. The viewpoint noted here is quite similar to that presented in electrostatics. In the absence of material bodies, the electric field $\mathbf{E}$ is related to true charge, just as magnetic field is related to true current. The presence of material bodies can be considered in terms of equivalent sources that are set up, sources which act in every way similarly to true sources. In electrostatics these are equivalent polarization charges ($\rho_p = -\nabla \cdot \mathbf{P}$, $\rho_{sp} = \mathbf{n} \cdot \mathbf{P}$), while in magnetostatics they are equivalent magnetization currents as given in (7.5*a*) and (7.5*b*).

Since the equivalent volume polarization current $\mathbf{J}_m$ is equal to the curl of the magnetic dipole polarization $\mathbf{M}$, in place of (7.10) we also have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{M}$, or

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J} \tag{7.11}$$

The vector $\mathbf{B}/\mu_0 - \mathbf{M}$ on the left-hand side of (7.11) has as its source only the true current $\mathbf{J}$. Therefore, to eliminate the necessity of dealing directly with the polarization $\mathbf{M}$, a new vector $\mathbf{H}$ is introduced by the relation

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{7.12}$$

The vector function $\mathbf{H}$ is called the magnetic-field-intensity vector, and its (vortex) source is the true current distribution $\mathbf{J}$; that is,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{7.13}$$

Note that $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$, so that an irrotational component of $\mathbf{H}$ exists for which the sources correspond, mathematically, to an equivalent magnetic charge. The unit for $\mathbf{H}$ is amperes per meter.

For most materials (ferromagnetic materials excluded) it is found that the polarization $\mathbf{M}$ is directly proportional to the applied external field $\mathbf{B}_0$, and hence, because of the linear relation (7.12), $\mathbf{M}$ is then directly proportional to $\mathbf{H}$ also. Following the conventional procedure, we may therefore write

$$\mathbf{M} = \chi_m \mathbf{H} \tag{7.14}$$

where the dimensionless constant of proportionality χ_m is called the magnetic susceptibility of the material. It is a measure of how susceptible the material is to polarization by an applied field. For diamagnetic materials χ_m is negative, while for paramagnetic materials it is positive. A relation like (7.14) may be written for ferromagnetic materials also, but for these materials χ_m is a function of the field intensity $\mathbf{H}$ as well as the past magnetization history of the material.

If (7.14) is substituted into (7.12), we get

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \tag{7.15}$$

where $\mu = \mu_0(1 + \chi_m)$ and is called the magnetic permeability of the material. In practice, the permeability μ is determined experimentally. When it is known, the effects of material bodies may be accounted for by using (7.13) to evaluate **H** and (7.15) to find the total net magnetic flux in the body. This process eliminates the necessity of taking the dipole polarization **M** into account explicitly.

In the interior of a material body both **B** and **M** vary rapidly in the region between individual atoms or molecules. However, in any practical procedure that might be used to measure **B** or **M**, the field is sampled in a space containing many atoms. Consequently, what is important are the space-average fields, where the average is taken over the domain of many atoms but which at the same time is small from the macroscopic viewpoint. Hence, in future work, when we speak of the polarization **M** and the total field **B**, such a space average is assumed. This concept is equivalent to the one introduced in discussing polarization in electrostatics.

For paramagnetic and diamagnetic materials μ differs from μ_0 by an amount which is entirely negligible for many practical situations; e.g., for bismuth, which is the strongest diamagnetic material known, $\mu = 0.99983\mu_0$, while for aluminum $\mu = 1.00002\mu_0$. For ferromagnetic materials μ is much greater than μ_0; for example, for silicon iron $\mu = 7{,}000\mu_0$; for permalloy 78 μ is about 10^5 greater than μ_0. Alloys with even larger values of μ exist. It must be kept in mind, however, that a unique value for μ does not exist for ferromagnetic materials because of strong nonlinearities. In ferromagnetic materials the relation between **B** and **H** is usually presented graphically in the form of a curve known as the hysteresis curve, or *B-H* curve. The nature of this relationship is discussed in the next section from an engineering viewpoint. The physical processes which give rise to the hysteresis curve are considered in Sec. 7.8.

7.4. *B-H* Curve

Consider a specimen of ferromagnetic material in the shape of a toroid, as in Fig. 7.3. The cross-section radius is r_0, and the mean toroid radius

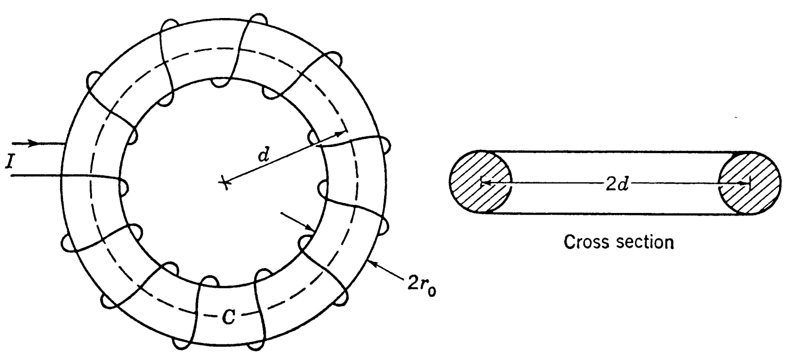

FIG. 7.3. A toroidal specimen of ferromagnetic material.

is d. The toroid is wound uniformly with N turns of wire through which an adjustable current I flows. The cross-sectional radius r_0 is assumed to be much smaller than the toroid radius d, so that the length of any closed circumferential path within the toroid may be taken as $2\pi d$.

The magnetic field intensity H in the interior is essentially uniform over the cross section and circumferentially directed when r_0 is small compared with d. The magnitude of H may be found from Ampère's circuital law. Starting with $\nabla \times \mathbf{H} = \mathbf{J}$, an application of Stokes' law gives

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

If the line integral is taken along the path C in the interior of the toroid, as shown in Fig. 7.3, we get

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = 2\pi dH = NI$$

since the total current linked by C, that is, cutting through a plane surface with boundary C, is NI. Hence the field intensity inside the toroid is $H = NI/2\pi d$ and may be varied by changing the current I.

If the ferromagnetic material is in an unmagnetized state (this condition can be arrived at by reversing the current I several times while at the same time gradually reducing the current magnitude to zero), it is found that as I, and hence H, is increased, the flux density B in the specimen increases from zero along a curve such as P_1P_2 in Fig. 7.4. If the current is now reduced from I_1 to $-I_1$ so as to change H from H_1 to $-H_1$, the flux density B will change according to the pattern illustrated by the curve $P_2P_3P_4P_5$. Increasing H from $-H_1$ to H_1 again causes the flux density to return to its value at the point P_2 along the curve $P_5P_6P_2$. The resultant closed curve $P_2P_3P_5P_6P_2$ is called a hysteresis curve. Further reversal of the field H between H_1 and $-H_1$ causes the same hysteresis loop to be retraced. The initial part of the curve from P_1 to P_2 is called the initial magnetization curve. When H is reduced from H_1, a value of $H = 0$ is reached for which the corresponding flux density B is nonzero (i.e., at P_3). The particular value $B = B_r$ at the point $H = 0$ is called the remanent flux density, i.e., the remaining flux density in the material. The value of H required to reduce B to zero is $-H_C$ and is called the coercive force and is the value of H at the point P_4 on the curve.

If H is increased from H_1 to H_2 and then reversed between the limits H_2 and $-H_2$, a new hysteresis loop with extremities P_7 and P_8 is traced out. When the magnitude of H increases beyond a certain range, the magnetization M does not increase any further because the material becomes magnetically saturated. In this condition all the magnetic dipoles are aligned with the field. In the saturated condition incremental changes in H cause changes in B according to their vacuum relations. For the hysteresis loop that brings the material into saturation, the value of remanent flux density is called the retentivity and the value of the field intensity H required to reduce B to zero is called the coercivity of the material. In the next chapter it will be shown that the area of the hysteresis loop is proportional to the energy dissipated in cycling the material around the hysteresis loop. This energy is dissipated as heat in the material.

The ratio of B to H is the slope of a line joining the origin to a point on the hysteresis curve and gives the permeability μ. Clearly, there is no

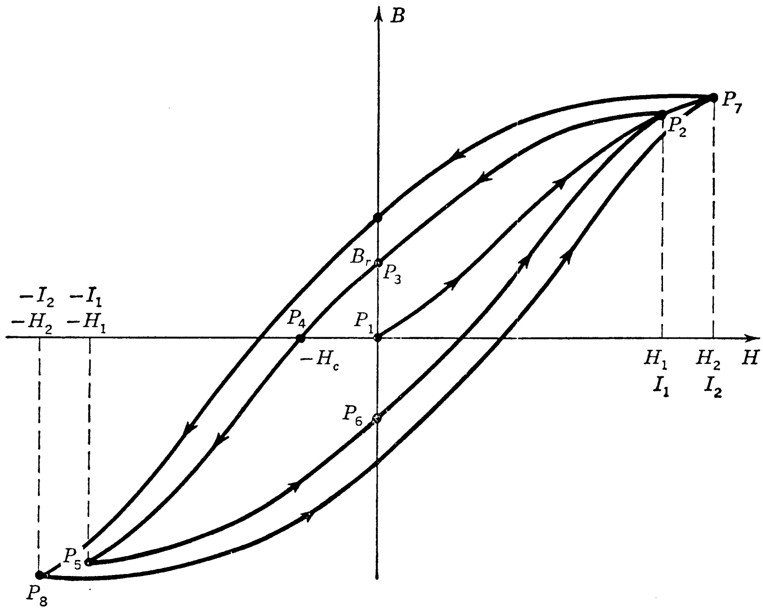

FIG. 7.4. A typical hysteresis (B-H) curve for a ferromagnetic material.

unique value of permeability. The tangent to the hysteresis curve at any particular point gives the differential permeability at that point. However, if H is varied between the limits $H_0 + \Delta H$ and $H_0 - \Delta H$, a new minor hysteresis loop centered about H_0 is traced out so that the differential permeability has little significance. The slope of the line joining the tips of the minor hysteresis loop is called the incremental permeability. It is the effective permeability for a small a-c current superposed on a d-c current.

7.5. Boundary Conditions for B and H

In the discussion leading to the introduction of the field intensity vector **H**, we considered an infinite medium and thereby avoided consideration of the equivalent surface polarization current $\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{n}$. Consequently, the relationships that have been developed are applicable everywhere within or external to a finite body, but for points at the surface, either they must be modified to take account of the surface magnetization current, or a limiting procedure adopted that includes the discontinuity in **M**. Rather than deal with the surface currents explicitly, we shall demonstrate that they may be accounted for by specifying certain boundary conditions for the field vectors **B** and **H** at the boundary between two different media. This procedure duplicates that adopted in electrostatics where the effect of surface polarization charge on the **E** and **D** fields was considered in terms of certain boundary conditions.

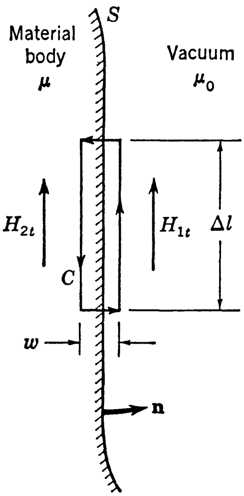

FIG. 7.5. Determination of boundary condition on the tangential component of **H**.

Consider a material body with a permeability μ, different from μ_0 and located in vacuum, as in Fig. 7.5. Let H_{1t} be the component of **H** which is tangent to the surface S on the vacuum side of the boundary, and let H_{2t} be the tangential component of **H** at the surface in the interior of the body. From (7.13) we have $\nabla \times \mathbf{H} = \mathbf{J}$. The integral form of this law is obtained by applying Stokes' theorem and leads to Ampère's circuital law for **H**; that is,

$$\int_{\Delta S} \nabla \times \mathbf{H} \cdot dS = \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{\Delta S} \mathbf{J} \cdot d\mathbf{S} \qquad (7.16)$$

where C is the contour bounding the open surface ΔS. Let C be a small loop with sides parallel to the boundary surface S in Fig. 7.5 and an infinitesimal distance on either side. Applying (7.16) we have $\oint_C \mathbf{H} \cdot d\mathbf{l} = 0$, if, as we suppose, no true current is enclosed by the contour. For the small rectangular path illustrated, whose length is Δl, the line integral gives essentially $H_{1t}\,\Delta l - H_{2t}\,\Delta l = 0$, or

$$H_{1t} = H_{2t} \qquad (7.17)$$

since the contribution from the ends is negligible; that is, the width w of the rectangular path can be made as small as we wish without affecting Δl. This relation tells us that the tangential magnetic field is continuous across a surface discontinuity separating a material body and vacuum. The same result is obviously also true at the boundary between two different material bodies.

Since $\mathbf{B} = \mu\mathbf{H}$, the corresponding boundary conditions for the tangential components of $\mathbf{B}$ are

$$\frac{B_{1t}}{\mu_0} = \frac{B_{2t}}{\mu} \tag{7.18}$$

Thus the tangential components of $\mathbf{B}$ are not continuous since μ and μ_0 are not equal. At first we might be somewhat puzzled over this result, but it can be explained quite readily. In fact, the tangential component of $\mathbf{B}$ is discontinuous because of the presence of the surface polarization current $\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{n}$. This result may be demonstrated as follows. We have

$$\mathbf{B}_1 = \mu_0\mathbf{H}_1$$
$$\mathbf{B}_2 = \mu\mathbf{H}_2 = \mu_0(\mathbf{H}_2 + \mathbf{M})$$

and subtracting the tangential components of $\mathbf{B}$ gives

$$B_{2t} - B_{1t} = \mu_0(H_{2t} - H_{1t}) + \mu_0 M_t = \mu_0 M_t \tag{7.19}$$

since $H_{2t} = H_{1t}$ from (7.17). The polarization vector $\mathbf{M}$ may be written as the sum of a tangential component $\mathbf{M}_t$ and a normal component $\mathbf{M}_n$. The cross product of $\mathbf{M}$ with the surface normal $\mathbf{n}$ thus gives

$$\mathbf{M} \times \mathbf{n} = \mathbf{M}_t \times \mathbf{n} + \mathbf{M}_n \times \mathbf{n} = \mathbf{M}_t \times \mathbf{n} = \mathbf{J}_{ms}$$

The magnitude of $\mathbf{J}_{ms}$ is M_t since $\mathbf{M}_t$ and $\mathbf{n}$ are perpendicular. Thus the tangential components of $\mathbf{B}$ are discontinuous by an amount equal to μ_0 times the magnitude of the polarization current $\mathbf{J}_{ms}$. With reference to Fig. 7.6, let the surface polarization current flow into the paper. The field produced by this current sheet is denoted by $\mathbf{B}_s$, and the field produced by all other current sources as $\mathbf{B}_0$. The field $\mathbf{B}_0$ is continuous across the surface S, but the field $\mathbf{B}_s$ is oppositely directed on the two sides of the current sheet. It is for this reason that the tangential component of the total field $\mathbf{B}_0 + \mathbf{B}_s$ is discontinuous across S.

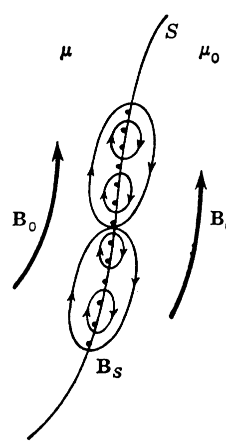

FIG. 7.6. Discontinuity in the tangential component of B as produced by the polarization surface current.

A further demonstration of the discontinuity in $\mathbf{B}$ may be had by applying (7.10) to the contour shown in Fig. 7.5. In this case the current term in the right-hand side of (7.10) is the total current, which includes polarization currents. Consequently, in place of the expression leading to (7.17), we get $B_{2t}\,\Delta l - B_{1t}\,\Delta l = \mu_0 J_{ms}\,\Delta l$, and this reduces to the result of (7.19). Note that since J_{ms} is a surface current, it does not disappear no matter how small the loop width w is allowed to get. The boundary conditions (7.17) and (7.18) take into account the effects of the surface polarization current, so that we do not need to include these explicitly when we are dealing with the interaction of material bodies with magnetic fields.

To determine the boundary conditions on the normal components of **B** and **H**, we note that since **B** is always equal to the curl of a vector potential **A**, the divergence of **B** is always zero; that is, $\nabla \cdot \mathbf{B} = 0$. An application of the divergence theorem gives

$$\int_V \nabla \cdot \mathbf{B}\, dV = \int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

We now apply this result to the closed surface of a small coin-shaped box with end faces on adjacent sides of the surface S of the material body, as in Fig. 7.7. The height w of the box is assumed to be so small that the magnetic flux through the side is negligible. Since the total flux through the closed surface is zero, just as much flux leaves face 1 as enters in through face 2. Thus $B_{1n}\, \Delta S_0 = B_{2n}\, \Delta S_0$, or

$$B_{1n} = B_{2n} \tag{7.20}$$

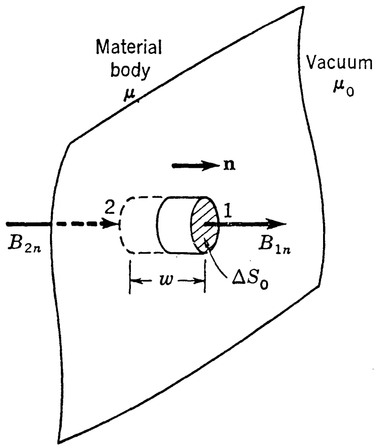

FIG. 7.7. Derivation of boundary conditions for the normal component of **B**.

The corresponding boundary condition on the normal component of **H** is obtained by using the relationship between **B** and **H** to get

$$\mu_0 H_{1n} = \mu H_{2n} \tag{7.21}$$

It is seen that the normal component of **B** is continuous, but not so for the normal component of **H**. This result for the normal component of **H** is the consequence of the magnetic polarization of the material. We shall show later that we can think of the magnetic polarization as equivalent to a magnetic charge distribution in analogy with the similar situation in electrostatics. With this alternative viewpoint, the discontinuity in **H** arises because of an equivalent surface layer of fictitious magnetic charge.

Refraction of Magnetic Lines of Flux

At a boundary surface between two different materials with permeabilities μ_1 and μ_2, the boundary conditions on **B** and **H** are

$$B_{1n} = B_{2n} \tag{7.22a}$$
$$\mu_2 B_{1t} = \mu_1 B_{2t} \tag{7.22b}$$
$$H_{1t} = H_{2t} \tag{7.22c}$$
$$\mu_1 H_{1n} = \mu_2 H_{2n} \tag{7.22d}$$

These boundary equations are of a similar nature to those occurring in electrostatics as well as those occurring in the study of stationary current flow fields. For this reason one expects to have a refraction or bending of the flux lines associated with **B** at a surface separating two different materials.

Thus let us consider the geometry shown in Fig. 7.8, where the field **B** makes the angles θ_1 and θ_2 with the interface normal in mediums 1 and 2, respectively. In accord with (7.22) we have

$$B_1 \cos \theta_1 = B_2 \cos \theta_2$$
$$\mu_2 B_1 \sin \theta_1 = \mu_1 B_2 \sin \theta_2$$

It is possible to eliminate B_1 and B_2 by taking the quotient of the two equations, with the result that $\mu_2 \tan \theta_1 = \mu_1 \tan \theta_2$, or

$$\tan \theta_2 = \frac{\mu_2}{\mu_1} \tan \theta_1 \tag{7.23}$$

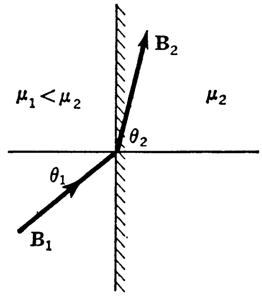

FIG. 7.8. Refraction of magnetic flux lines.

Equation (7.23) shows that the magnetic flux lines are bent away from the normal in the medium with the highest value of permeability. If medium 2 is a ferromagnetic material and medium 1 is vacuum or air, μ_2 is much greater than $\mu_1 = \mu_0$. In this case $\tan \theta_1$ is very small and the flux lines are for all practical purposes normal to the surface on the air side, provided θ_2 does not equal exactly 90°.

Discontinuity in **H** *at a Current Sheet*

In this section we shall examine the behavior of the magnetic field **H** in the vicinity of a thin sheet of conduction current. Such current sheets are of particular importance in the general boundary-value problem for time-varying magnetic fields. The surface of a closely wound solenoid also essentially constitutes a surface current sheet. Applying the right-hand rule, i.e., with the thumb of the right hand turned in the direction of current flow the fingers of the right hand are along the direction of the magnetic lines of flux, we see that the magnetic field is perpendicular to the current flow lines, as illustrated in Fig. 7.9. The line integral of **H** around the small contour C is equal to the total current passing through the contour C, that is, passing through the surface ΔS_0. If the enclosed current has a volume density **J**, we obtain

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{\Delta S_0} \mathbf{J} \cdot d\mathbf{S}$$

Because the contour is of infinitesimal size and we assume $h \ll \Delta l$,

$$(H_{1t} - H_{2t})\, \Delta l = J\, \Delta S_0 \tag{7.24}$$

since in the contour integral the contribution from the ends is negligible. If we let the thickness h of the current sheet tend to zero and increase the

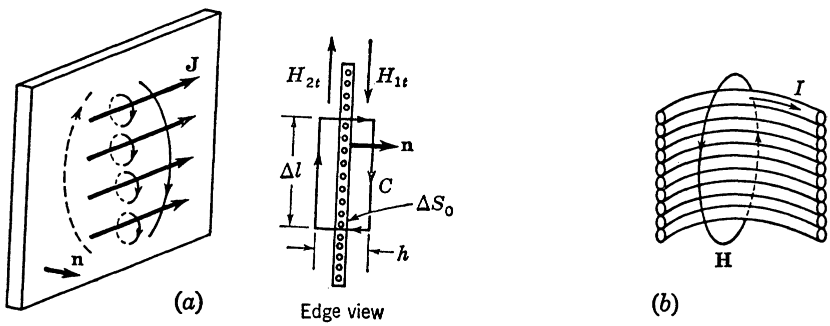

FIG. 7.9. Illustration of current sheets. (a) A uniform current sheet, (b) cross section of a solenoid.

volume density of current **J** so that the total enclosed current per unit length remains constant, i.e.,

$$\lim_{h \to 0} hJ = J_s$$

we obtain a true surface current sheet with a current J_s amperes per meter. In the limit, (7.24) gives

$$\begin{aligned}(H_{1t} - H_{2t})\, \Delta l &= \lim_{h \to 0} hJ \frac{\Delta S_0}{h} \\ &= J_s\, \Delta l\end{aligned}$$

or

$$H_{1t} - H_{2t} = J_s \tag{7.25}$$

since $\Delta S_0 = h\, \Delta l$. Thus the tangential component of **H** is discontinuous across a current sheet by an amount equal to the surface current density. This behavior is similar to that noted for the tangential component of **B** at a polarization current sheet. In vector form (7.25) may be written as

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \tag{7.26}$$

since $\mathbf{n} \times \mathbf{H}$ selects the tangential component of **H**. In (7.26) **n** is the unit outward normal to the current sheet.

The normal component of **H** is continuous across the current sheet since the normal component of **B** is, and $\mathbf{H} = \mathbf{B}/\mu_0$ on both sides.

Example 7.1. Far Field from a Long Solenoid. Consider a long uniform solenoid having N turns per meter and carrying a current I, as in Fig. 7.10. The solenoid is of length L and has a radius a. We choose the coordinate system with the z axis coinciding with the axis of the solenoid.

We wish to determine the field at the point (x,y,z), which is at a much greater distance from the solenoid than the solenoid radius a. We may treat the solenoid as a stack of current loops of total height L. For a

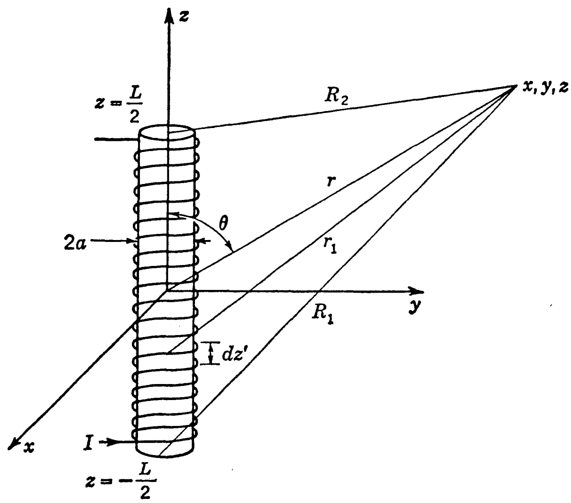

FIG. 7.10. A long uniform solenoid.

section of length dz' there are $N\,dz'$ turns, each with a current I, so that the dipole moment of such a section is

$$\mathbf{m} = NI\,dz'\,\pi a^2 \mathbf{a}_z \tag{7.27}$$

We could now proceed to compute the total vector potential set up by integrating over the length of the solenoid the contribution from each infinitesimal section. However, since we are looking for the field $\mathbf{B}$, we shall use the relation (6.26) of Sec. 6.4 and find $\mathbf{B}$ directly. The field $d\mathbf{B}$ at a long distance from a dipole $\mathbf{m}$ is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\nabla\left[\mathbf{m}\cdot\nabla\left(\frac{1}{r_1}\right)\right]$$

where r_1 is the distance from the center of the dipole at z' to the field point (x,y,z). The total field B is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\nabla\int_{-L/2}^{L/2}\mathbf{m}\cdot\nabla\left(\frac{1}{r_1}\right)dz' = \frac{\mu_0}{4\pi}\nabla\int_{-L/2}^{L/2}\nabla\left(\frac{1}{r_1}\right)\cdot\mathbf{a}_z NI\pi a^2\,dz' \tag{7.28}$$

Now $\nabla(1/r_1) = -\mathbf{r}_1/r_1^3$, where $\mathbf{r}_1 = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z(z - z')$. Hence (7.28) gives

$$\mathbf{B} = \frac{-NIa^2\mu_0}{4}\nabla\int_{-L/2}^{L/2}\frac{(z-z')\,dz'}{[x^2+y^2+(z-z')^2]^{3/2}}$$

$$= \frac{-NIa^2\mu_0}{4}\nabla[x^2+y^2+(z-z')^2]^{-1/2}\Big|_{-L/2}^{L/2}$$

$$= \frac{-NIa^2\mu_0}{4}\left[\nabla\left(\frac{1}{r_1}\right)\right]_{-L/2}^{L/2} = \frac{-NIa^2\mu_0}{4}\nabla\left(\frac{1}{R_2}-\frac{1}{R_1}\right)$$

where R_1 and R_2 are the distances from the ends of the solenoid to the field point, as in Fig. 7.10. We may approximate R_1 and R_2 as follows:

$$R_2^{-1} = \left[x^2+y^2+\left(z-\frac{L}{2}\right)^2\right]^{-1/2}$$

$$= \left[x^2+y^2+z^2+\left(\frac{L}{2}\right)^2 - Lr\cos\theta\right]^{-1/2}$$

$$\approx (r^2 - Lr\cos\theta)^{-1/2}$$

$$\approx r^{-1}\left(1+\frac{L}{2r}\cos\theta\right)$$

where z is replaced by $r\cos\theta$, $(L/2)^2$ is dropped in comparison with r^2, and the binomial expansion used. We similarly find that

$$R_1^{-1} \approx r^{-1}\left(1-\frac{L}{2r}\cos\theta\right)$$

and hence

$$\mathbf{B} = \frac{-\mu_0 NIa^2L}{4}\nabla\left(\frac{\cos\theta}{r^2}\right) = \frac{\mu_0}{4\pi}M\left(\mathbf{a}_r\frac{2\cos\theta}{r^3}+\mathbf{a}_\theta\frac{\sin\theta}{r^3}\right) \quad (7.29)$$

where $NI\pi a^2 L$, the total dipole moment of the solenoid, is designated by M. The relation (7.29) is of the same form as that derived in Sec. 6.4 for a small elementary magnetic dipole. It is, of course, valid only at a distance r that is large compared with the length L.

Example 7.2. Far Field from a Long Cylindrical Bar Magnet. Figure 7.11 illustrates a cylindrical bar magnet of length L and radius a. The magnet is assumed to be permanently magnetized with a uniform magnetic dipole moment $\mathbf{M}_0$ per unit volume, where $\mathbf{M}_0$ is directed in the positive z direction.

The external field produced by the bar magnet may be computed in terms of the equivalent polarization currents. A cross section of the magnet is shown in Fig. 7.11b and illustrates the nature of the equivalent atomic circulating currents. In the interior the volume density of polarization current is given by $\mathbf{J}_m = \nabla \times \mathbf{M}_0$. If $\mathbf{M}_0$ is constant, $\mathbf{J}_m$ vanishes. This result is intuitively obvious from Fig. 7.11b, since with $\mathbf{M}_0$ constant, the circulating current loops all carry the same current, and hence all the interior currents effectively cancel. If $\mathbf{M}_0$ were not uniform throughout, there would be incomplete cancellation from one loop to the next and a residual volume current density would be left, as we have already discussed. Along the outer boundary there is no cancellation of the currents of each adjacent small loop. Thus all the individual current loops combine to produce a net surface polarization current flowing along the surface of the bar magnet. The value of this current has already been

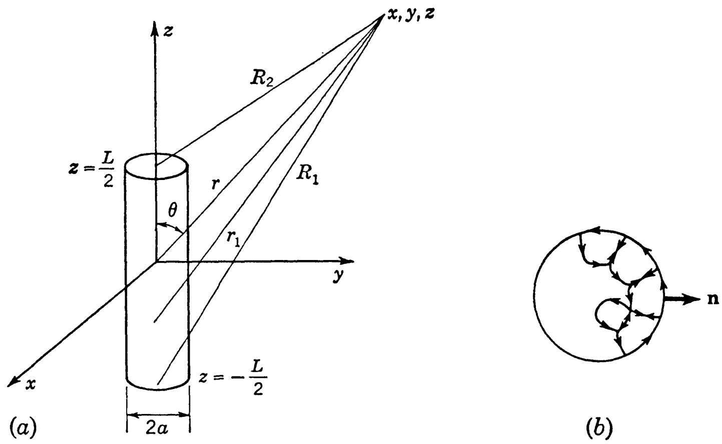

FIG. 7.11. (a) A cylinder bar magnet; (b) enlarged cross section illustrating equivalent circulating polarization currents.

determined to equal $\mathbf{M}_0 \times \mathbf{n}$. Since $\mathbf{M}_0 = \mathbf{a}_z M_0$ and $\mathbf{n} = \mathbf{a}_r$, the surface current is given by

$$\mathbf{J}_{ms} = \mathbf{M}_0 \times \mathbf{n} = M_0\mathbf{a}_z \times \mathbf{a}_r = M_0\mathbf{a}_\phi$$

and flows circumferentially around the cylindrical bar magnet. It is now apparent that the field from the bar magnet will be the same as that from an equivalent solenoid having an effective surface current of M_0 amperes per meter. A closely wound solenoid of N turns per meter carrying a current I would be equivalent to the bar magnet if $NI = M_0$. The total moment of the magnet is $\pi a^2 L M_0 = M$, and substitution of this into (7.29) gives the resultant field.

7.6. Scalar Potential for H

In the introduction to Chap. 6 it was pointed out that the early theory of magnetism developed along lines parallel to that of electrostatics. This alternative theory is useful for computing the fields produced by magnetized bodies, and it will therefore be worthwhile to examine it in some detail.

Let us consider a permanently magnetized body of volume V bounded by a surface S and located in a region of space where the true conduction current density $\mathbf{J}$ is zero. Since $\mathbf{J}$ is zero, we have $\nabla \times \mathbf{H} = \mathbf{J} = 0$ also. But this is just the condition that the vector field may be derived from the gradient of a scalar potential. Thus let

$$\mathbf{H} = -\nabla\Phi_m \tag{7.30}$$

where Φ_m is called the magnetic scalar potential. The negative sign in (7.30) is chosen only to make the analogy with electrostatics a closer one. A field $\mathbf{H}$ determined by (7.30) will have zero curl since $\nabla \times \nabla\Phi_m$ is identically zero.

In electrostatics the source for the scalar potential is the charge density ρ. In magnetostatics we do not have a physical magnetic charge (or unit magnetic pole), but we are nevertheless able to recast our expressions for the magnetic field produced by magnetic dipole polarization of a body into a form which gives the potential Φ_m in terms of equivalent magnetic charges.

According to (6.26) of Sec. 6.4, the field $\mathbf{B}$ from a single isolated magnetic dipole $\mathbf{m}$ is given by

$$\mathbf{B}(x,y,z) = \frac{\mu_0}{4\pi}\left\{\nabla\left[\nabla \cdot \frac{\mathbf{m}(x',y',z')}{R}\right] - \nabla^2\left(\frac{\mathbf{m}}{R}\right)\right\} \tag{7.31}$$

At all points except $R = 0$ we have $\nabla^2(1/R) = 0$, so that if we restrict attention to the fields external to a magnetized body, the second term is absent. If the body has a polarization $\mathbf{M}$ per unit volume, the field from the dipoles in a volume element dV' at (x',y',z') is given by

$$d\mathbf{H} = \frac{d\mathbf{B}}{\mu_0} = \frac{1}{4\pi}\nabla\left[\nabla \cdot \frac{\mathbf{M}(x',y',z')}{R}\right] dV'$$

The total field $\mathbf{H}$ is then

$$\begin{aligned}\mathbf{H} &= \frac{1}{4\pi}\nabla\left(\int_V \nabla \cdot \frac{\mathbf{M}}{R}\, dV'\right)\\ &= \frac{1}{4\pi}\nabla\left[\int_V \mathbf{M} \cdot \nabla\left(\frac{1}{R}\right) dV'\right]\end{aligned}$$

If we compare this with (7.30), we see that the scalar potential Φ_m is given by

$$\Phi_m = -\frac{1}{4\pi}\int_V \mathbf{M} \cdot \nabla\left(\frac{1}{R}\right) dV' \tag{7.32}$$

Now

$$\nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right) \qquad \text{and} \qquad \nabla' \cdot \left(\frac{\mathbf{M}}{R}\right) = \mathbf{M} \cdot \nabla'\left(\frac{1}{R}\right) + \frac{1}{R}\nabla' \cdot \mathbf{M}$$

and hence $$-\mathbf{M} \cdot \nabla\left(\frac{1}{R}\right) = \mathbf{M} \cdot \nabla'\left(\frac{1}{R}\right) = \nabla' \cdot \frac{\mathbf{M}}{R} - \frac{1}{R}\nabla' \cdot \mathbf{M}$$

Substituting into (7.32) we get

$$\Phi_m = \frac{1}{4\pi}\left(\int_V \nabla' \cdot \frac{\mathbf{M}}{R}\, dV' - \int_V \frac{\nabla' \cdot \mathbf{M}}{R}\, dV'\right)$$
$$= \frac{1}{4\pi}\left(\oint_S \frac{\mathbf{M} \cdot \mathbf{n}}{R}\, dS' - \int_V \frac{\nabla' \cdot \mathbf{M}}{R}\, dV'\right) \qquad (7.33)$$

after converting the first volume integral to a surface integral by means of the divergence theorem. We now compare (7.33) with the expression for the scalar potential Φ due to a volume and a surface distribution of charge in electrostatics and are thus led to interpret $\mathbf{M} \cdot \mathbf{n}$ as an equivalent surface density of magnetic charge ρ_{ms} and to interpret $-\nabla' \cdot \mathbf{M}$ as equivalent to a volume density of magnetic charge ρ_m. It must be stressed that this equivalence is a purely mathematical one and does not prove a physical existence of magnetic charge.

The result obtained in (7.33) can also be derived by analogy with the relationships found for the electrostatic field. We recall that the $\mathbf{E}$ field due only to a polarization source satisfies the following equations:

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{E} = -\frac{\nabla \cdot \mathbf{P}}{\epsilon_0} \qquad (7.34a)$$

where $\mathbf{P}$ is the dipole moment per unit volume. Under these conditions it is also true, according to (2.97), that $\mathbf{E}$ is related to the sources that produce it by

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\nabla\left(\int_V \frac{\nabla' \cdot \mathbf{P}}{R}\, dV' - \oint_S \frac{\mathbf{P} \cdot \mathbf{n}}{R}\, dS'\right)$$

Now if the only source of magnetic field is a permanent magnetization $\mathbf{M}$, the $\mathbf{H}$ field satisfies

$$\nabla \times \mathbf{H} = 0 \qquad \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \qquad (7.34b)$$

If we compare (7.34*a*) with (7.34*b*), it is noted that the equations are intrinsically the same, it being necessary only to exchange $\mathbf{E}$ for $\mathbf{H}$ and $\mathbf{P}/\epsilon_0$ for $\mathbf{M}$. Now the Helmholtz theorem states that a vector field is completely determined by its divergence and curl. Thus (7.34*a*) completely specifies $\mathbf{E}$, as does (7.34*b*) specify $\mathbf{H}$. Then since $\mathbf{E}$ also satisfies (2.97), it must be that the latter inevitably follows from (7.34*a*). Consequently, $\mathbf{H}$ must also satisfy such an equation, it being necessary only to replace $\mathbf{E}$ by $\mathbf{H}$ and $\mathbf{P}/\epsilon_0$ by $\mathbf{M}$. If we carry out this substitution, the result obtained is precisely that given by (7.33).

The above is an excellent illustration of the power of the Helmholtz theorem. This theorem enables us to recognize basic common properties of vector fields independent of their individual physical properties. The results (7.33) and (2.97) are essentially embodied in the general vector formulation of (1.103).

The magnetic field intensity **H** is obtained by taking the negative gradient of (7.33) with respect to the coordinates x, y, z of the field point. The field **B** is given by $\mathbf{B} = \mu_0\mathbf{H}$. It is important to note that these results hold only for points exterior to the magnetized body since the term $-(\mu_0/4\pi)\nabla^2(\mathbf{m}/R)$ was dropped in (7.31). In the interior of the body Φ_m is still given by (7.32) and **H** by (7.30), as we shall show presently. However, the relation between **B** and **H** is

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{7.35}$$

and this equation must be used to find **B** in the interior.

The above results for **B** and **H** in the interior of a magnetized body are readily proved. In the interior of the body we see from (7.31) that the field **B** is given by

$$\begin{aligned}\mathbf{B}(x,y,z) &= \frac{\mu_0}{4\pi}\nabla\int_V \nabla\cdot\frac{\mathbf{M}}{R}\,dV' - \frac{\mu_0}{4\pi}\int_V \nabla^2\left(\frac{\mathbf{M}}{R}\right)dV' \\ &= -\mu_0\,\nabla\Phi_m - \frac{\mu_0}{4\pi}\int_V \nabla^2\left(\frac{\mathbf{M}}{R}\right)dV'\end{aligned}$$

since the first term defines the magnetic scalar potential Φ_m. We have already noted that the remaining volume integral is zero for external field points since in this case $R \neq 0$ and $\nabla^2(1/R) = 0$. However, if the point (x,y,z) lies interior to V, then $R = 0$ is included in the integral. In this case it is necessary to consider the singularity property of $\nabla^2(1/R)$, as was done in the derivation of (6.41). With this result we have

$$-\frac{\mu_0}{4\pi}\int_V \nabla^2\left(\frac{\mathbf{M}}{R}\right)dV' = \begin{cases}\mu_0\mathbf{M}(x,y,z) & x,\ y,\ z \text{ in } V \\ 0 & x,\ y,\ z \text{ outside } V\end{cases}$$

Hence in the interior of the magnetized body the field **B** is given by $\mathbf{B} = -\mu_0\,\nabla\Phi_m + \mu_0\mathbf{M}$. Since **B** is also given by (7.35), it follows that (7.33) is a valid expression for the scalar magnetic potential Φ_m for **H** both interior and exterior to the magnetized body.

At this point it will be worthwhile to summarize briefly the major results that have been obtained thus far in this chapter. This is desirable in order to correlate the different approaches that may be used to treat the magnetic effects of material bodies. In the absence of any material bodies the magnetic field **B** is computed from the true current distribution **J**, either directly or by means of the vector potential **A**, according to the methods of Chap. 6. When we are dealing with the field produced by a permanently magnetized body in a region where **J** is zero, there are two equivalent approaches available.

For one, we represent the state of polarization of the body by an equivalent volume polarization current $\mathbf{J}_m = \nabla \times \mathbf{M}$ and an equivalent surface polarization current $\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{n}$. From these current sources the field

B may be evaluated either directly or through the intermediate step of finding the vector potential **A** first. In the region surrounding the body the field **H** is given by $\mathbf{H} = \mathbf{B}/\mu_0$, while at all points interior to the body **H** is given by $\mu_0\mathbf{H} = \mathbf{B} - \mu_0\mathbf{M}$.

In the second approach the field intensity **H** is computed first. The magnetic polarization of the material is represented by an equivalent volume distribution of fictitious magnetic charge $\rho_m = -\nabla \cdot \mathbf{M}$, together with an equivalent surface charge $\rho_{ms} = \mathbf{M} \cdot \mathbf{n}$. The field **H** is given by $-\nabla\Phi_m$, and the magnetic scalar potential Φ_m is evaluated by methods analogous to those used in electrostatics. From the known value of **H**, the field **B** may be found from the relations

$$\mathbf{B} = \mu_0\mathbf{H} \qquad \text{outside body}$$
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \qquad \text{interior to body}$$

The parameters μ and χ_m are unnecessary with a theory that takes explicit account of the magnetic polarization of the material.

When the problem involves both magnetizable material bodies and true conduction currents **J**, the use of the magnetic scalar potential is usually avoided since **H** is no longer a conservative field and Φ_m becomes multivalued. An exception is the class of problems that are dual of the d-c electric circuit, where this approach is particularly useful. The best procedure is to obtain appropriate solutions for **B** in the regions internal and external to all material bodies and then match these solutions at the boundaries according to the boundary conditions on **B** and **H** which were presented in Sec. 7.5. For problems of this sort, if $\nabla \times \mathbf{M} = 0$, then it is preferable not to introduce the magnetic polarization of the material explicitly but to use the relation $\mathbf{B} = \mu\mathbf{H}$, together with the boundary conditions on **B** and **H** instead. The examples to be discussed now will clarify some of the above concepts.

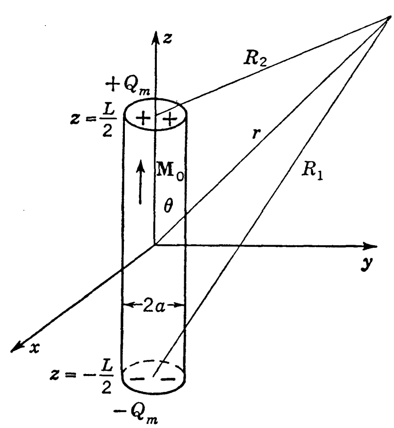

FIG. 7.12. A cylindrical bar magnet.

Example 7.3. Use of Scalar Potential to Find Field from a Bar Magnet. The cylindrical magnet is of length L and radius a, as in Fig. 7.12. It is assumed to be uniformly magnetized with a magnetic dipole polarization $\mathbf{M}_0$ per unit volume. We shall compute **H** at all exterior points, remote from the magnet, by means of the scalar potential Φ_m. Since $\mathbf{M}_0$ is constant, $\nabla \cdot \mathbf{M}_0 = 0$ and there is no equivalent volume charge. On the two end faces at $z = \pm L/2$, we have an equivalent surface charge given by $\rho_{ms} = \mathbf{M}_0 \cdot \mathbf{n} = \mathbf{M}_0 \cdot \mathbf{a}_z$ or $\rho_{ms} = M_0$ at $z = L/2$ and $-M_0$ at $z = -L/2$. There are no surface charges on the sides since $\mathbf{M}_0$ is parallel to the sides. The total charge on each face is of magnitude $Q_m = \pi a^2 \rho_{ms} = \pi a^2 M_0$. For $r \gg a$ we may consider this charge to be lumped together at the center of each face. The scalar potential from two such charges is given by an expression similar to what one has in electrostatics and is

$$\Phi_m = \frac{Q_m}{4\pi}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = \frac{\pi a^2 M_0}{4\pi}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \tag{7.36}$$

The field $\mathbf{H}$ is thus given by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = -\nabla\Phi_m = \frac{-\pi a^2 M_0}{4\pi}\nabla\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \tag{7.37}$$

which is the same as that obtained in Example 7.2 by treating the magnet as an equivalent solenoid.

For $r \gg L$ the dipole relations obtained in Sec. 2.11 apply, and we obtain

$$\Phi_m = \frac{Q_m L \cos\theta}{4\pi r^2} = \frac{M_T \cos\theta}{4\pi r^2}$$

where $M_T = \pi a^2 L M_0$ is the total effective dipole moment of the magnetized rod.

Example 7.4. Field around a Cut Toroid. Figure 7.13a illustrates a highly permeable ($\mu \gg \mu_0$) toroid wound with N turns of wire carrying a current I. The mean radius of the toroid is d. The cross section of the toroid is circular, with a radius a much smaller than d.

The tangential field $\mathbf{H}$ is continuous across the boundary separating the toroid and the air region just outside but inside the helix winding. Therefore the flux density $\mathbf{B}$ in the interior is much greater than that outside the toroid since $\mathbf{B} = \mu\mathbf{H}$ inside and $\mu_0\mathbf{H}$ outside and we are assuming that μ is much greater than μ_0. Most of the flux lines are concentrated in the interior except for a small amount of leakage flux on the outside, as illustrated in Fig. 7.13a. If we apply Ampère's circuital law for $\mathbf{H}$ around a closed circular path in the interior of the toroid (path C in Fig. 7.13a), we obtain

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = NI \tag{7.38}$$

since the total current cutting through the surface of a disk with boundary C, that is, total current linked by C, is NI. From symmetry considerations we conclude that $\mathbf{H}$ is not a function of the angle θ around the toroid and also that **H** is tangent to the curve C; so (7.38) gives

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = H_\theta \int_0^{2\pi} r\, d\theta = 2\pi r H_\theta = NI$$

and

$$H_\theta = \frac{NI}{2\pi r} \approx \frac{NI}{2\pi d} \tag{7.39}$$

where r is the radius of the path C. Since $a \ll d$ we have $r \approx d$ for all closed paths within the toroid; so for a first approximation we may assume that H_θ is given by $NI/2\pi d$ at all points interior to the toroid. The flux density B_θ is given by μH_θ.

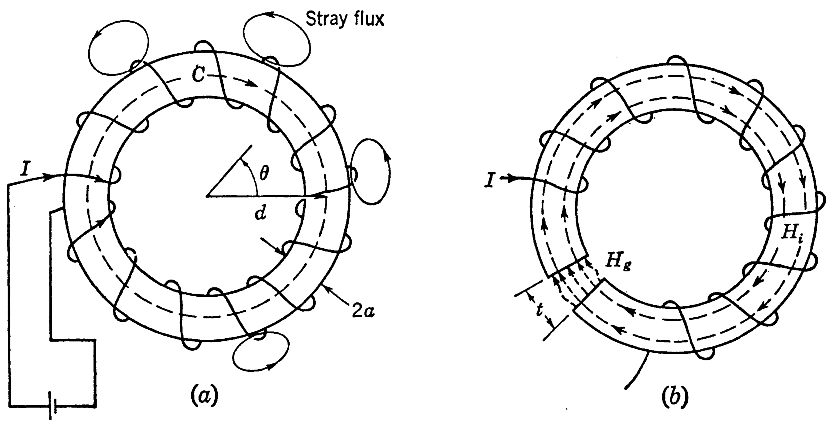

FIG. 7.13. (a) A toroid wound with N turns; (b) the same toroid with a small section removed (gap shown enlarged).

In the second situation illustrated in Fig. 7.13b, a small section of thickness t has been removed from the toroid. If $t \ll a$, the flux in the air gap is essentially uniform apart from a small amount of fringing or bulging of the field near the edge. The field B_θ, which is normal to the faces, must be continuous across the face, and hence B_θ has the same value in the air gap as in the interior of the toroid. The field H_θ is equal to B_θ/μ_0 in the gap and B_θ/μ in the toroid. If we let H_i be the interior field and H_g be the gap field, Ampère's circuital law gives

$$H_i(2\pi d - t) + H_g t = NI$$

and hence, substituting for H_i and H_g in terms of B_θ, we get

$$\frac{B_\theta}{\mu}(2\pi d - t) + \frac{B_\theta}{\mu_0} t = NI$$

Solving for B_θ gives

$$B_\theta = \frac{NI\mu\mu_0}{2\pi d\mu_0 + (\mu - \mu_0)t} \tag{7.40}$$

From (7.40) the fields H_g and H_i are readily found. When t is zero, the field H_i is given by $NI/2\pi d$, but when t is not equal to zero, H_i is reduced in value to

$$H_i = \frac{NI}{2\pi d + (\mu - \mu_0)t/\mu_0}$$

When the toroid is made of ferromagnetic material, the magnetic polarization will generally not be zero when the current I is reduced to zero. A determination of the exact values of B and H when $I = 0$ can be made only if the relationship of B to H is specified, as would be the case with an appropriate B-H curve. Let us consider how the desired result may be obtained. When I is zero, let the flux density in the toroid and air gap be B. In the air gap the magnetic field intensity is

$$H_g = \frac{B}{\mu_0}$$

while in the interior to the toroid

$$H_i = \frac{B}{\mu}$$

Applying Ampère's circuital law we get

$$H_i(2\pi d - t) + H_g t = 0$$

since $I = 0$, and hence

$$H_i = \frac{-H_g t}{2\pi d - t} \tag{7.41}$$

Equation (7.41) shows that inside the toroid H_i must be oppositely directed to the field H_g in the gap. However, in the gap H_g and B are in the same direction and B is continuous into the toroid. Therefore, in the toroid, B and H_i are oppositely directed, as illustrated in Fig. 7.14a.

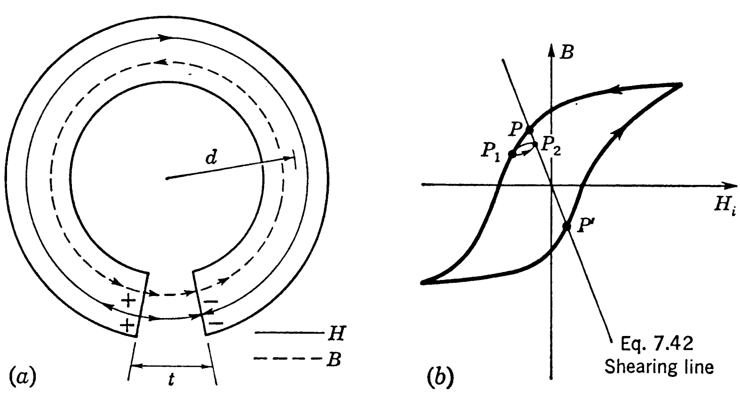

FIG. 7.14. The field in a magnetized cut toroid (gap shown enlarged).

Hence μ must be negative. a situation which is possible if the state of magnetization of the material corresponds to a point such as P on the B-H curve illustrated in Fig. 7.14b. If we substitute B/μ_0 for H_g in (7.41), we get $H_i = -Bt/\mu_0(2\pi d - t)$, or

$$B = \frac{-\mu_0 H_i(2\pi d - t)}{t} \tag{7.42}$$

This equation determines a relation between B and H_i. It is a straight line, called the shearing line, with negative slope, as shown plotted on the B-H curve in Fig. 7.14b. It intersects the B-H curve at the points P and P', which are the only two points that can simultaneously satisfy (7.42) and the relation between B and H_i given by the B-H curve. The flux density in the cut toroid is thus that given by the point P (or P').

The above behavior may be understood by recalling that the residual polarization $\mathbf{M}_0$ of the material when I is reduced to zero is equivalent to a magnetic surface charge density $\mathbf{M}_0 \cdot \mathbf{n}$ and $-\mathbf{M}_0 \cdot \mathbf{n}$ on the faces of the toroid at the gap. These charges result in a field $\mathbf{H}$, which is oppositely directed on the two sides of each face. From the point P on the B-H curve, B and H are determined. Thus, using the relation

$$B = \mu H = \mu_0(H + M_0)$$

we may calculate M_0 and μ. The value of M_0 turns out to be

$$M_0 = \frac{\mu - \mu_0}{\mu_0} H = \frac{B}{\mu_0} - H$$

This last result may be used to find the equivalent magnetic charge density on the end faces of the toroid.

An alternative interpretation of the results for the cut toroid involves consideration of the equivalent currents due to the residual magnetic polarization. Because of the assumed uniformity of $\mathbf{M}_0$, we have $\nabla \times \mathbf{M}_0 = 0$, so that $\mathbf{J}_m = 0$. However, $\mathbf{M}_0 \times \mathbf{n} \neq 0$ along the toroidal surface and thus leads to equivalent circulation currents along the meridians of the torus. These currents behave as a source for $\mathbf{B}$ just as do the true currents in the helical winding when $I \neq 0$. The residual $\mathbf{J}_{ms}$ accounts for the solenoidal nature of $\mathbf{B}$ and the persistence of flux when I is reduced to zero.

The above example illustrates several points which are of interest in the design of permanent magnets. For permanent magnets it is desirable to use materials that have a high retentivity and also a high coercivity. A magnet in the form of a "horseshoe" is magnetized by completing the magnetic circuit with an iron bar and winding the magnet with a coil through which a large steady current is passed. Upon removal of the coil and iron bar, the magnetization decreases in the manner indicated for the cut toroid because of the demagnetizing effect of the free poles, i.e., because of the equivalent magnetic charge on the end faces of the magnet.

A point such as P on the B-H curve in Fig. 7.14*b* is not a stable point since the application of a small magnetic field causes the magnetization to move along a new small hysteresis loop away from P. Since a magnet may at times be subjected to small stray fields, it is stabilized by applying a small negative field to bring the magnetization to the point P_1 and then removing this field to permit the magnetization to move along a new hysteresis loop up to the final point P_2, as in Fig. 7.14*b*. Application of a stray field now causes the magnetization to move along this new hysteresis loop. However, now upon removal of the stray field, the state of magnetization returns very nearly to the point P_2, whereas without the above stabilization technique the magnetization would not return to the initial value at P upon removal of the stray fields. In the absence of stray fields the magnetization will always lie on the shearing line; that is, P_2 lies on the shearing line.

7.7. The Magnetic Circuit

The solution to the general magnetostatic boundary-value problem involving conduction currents in the presence of ferromagnetic material bodies is extremely difficult to obtain. Fortunately, for many engineering applications involving ferromagnetic materials, good approximate solutions can be obtained by means of an analysis that parallels that used to analyze d-c circuits composed of series and parallel combinations of resistors. The ideas and limitations involved in this equivalent-circuit approach will be discussed below.

If we return to the cut-toroid problem of Example 7.4 and reexamine the method of solution presented, it will be seen to be similar to that which we would use for a simple d-c circuit of two resistors in series together with an applied voltage source. We consider the line integral of **H** around the circuit as being the magnetomotive force $\mathcal{H}$ which causes a total flux ψ to flow through the circuit. The magnitude of ψ is determined by $\mathcal{H}$ and a property of the circuit called the reluctance $\mathcal{R}$, where $\mathcal{R}$ is analogous to resistance.

According to (7.40), the flux density in the cut toroid is

$$B = \frac{NI\mu\mu_0}{2\pi d\mu_0 + (\mu - \mu_0)t}$$

The total flux ψ through the circuit is $\psi = AB = \pi a^2 B$, where A is the cross-sectional area of the toroid. The line integral of **H** around the toroid is equal to the ampere-turns NI and gives the magnetomotive force $\mathcal{H}$. The solution for ψ may be written as

$$\psi = BA = \frac{\mathcal{H}}{(2\pi d - t)/\mu A + t/\mu_0 A} \tag{7.43}$$

The first term in the denominator is similar to that for the d-c resistance of a conductor of length $2\pi d - t$, of cross-sectional area A, and having a specific conductivity μ. Thus this term is interpreted as the reluctance $\mathcal{R}_1$ of the section of toroid. For the same reasons the second term is interpreted as the reluctance $\mathcal{R}_2$ of the gap. We may now rewrite (7.43) as

$$\psi = \frac{\mathcal{H}}{\mathcal{R}_1 + \mathcal{R}_2} \tag{7.44}$$

where

$$\mathcal{R}_1 = \frac{2\pi d - t}{\mu A}$$

$$\mathcal{R}_2 = \frac{t}{\mu_0 A}$$

The close analogy between the d-c current circuit and the magnetic circuit may be seen from a comparison of the two following examples.

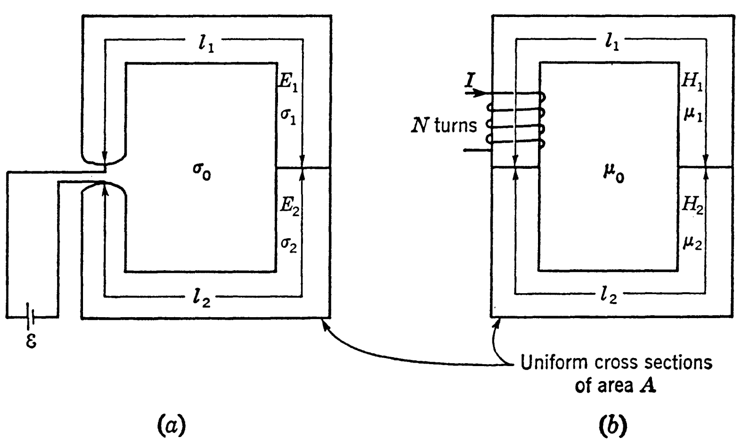

FIG. 7.15. (*a*) A d-c current circuit; (*b*) a similar magnetic circuit.

Figure 7.15*a* illustrates a d-c current circuit consisting of two sections of mean lengths l_1 and l_2 and having a uniform cross-sectional area A. The two sections have specific conductivities σ_1 and σ_2 and are submerged in a medium having a much lower conductivity σ_0. The two sections are connected in series with an applied electromotive force $\mathcal{E}$.

Since σ_0 is much smaller than σ_1 or σ_2, most of the current flow is confined to the highly conducting sections. The line integral of **E** around the circuit gives the applied electromotive force:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} = E_1 l_1 + E_2 l_2 \tag{7.45a}$$

The current density **J** is given by $\sigma\mathbf{E}$, and the total current is obtained by multiplying this quantity by the cross-sectional area A. Since the current is continuous, we get

$$I_1 = I_2 = I = \sigma_1 E_1 A = \sigma_2 E_2 A \tag{7.45b}$$

where the leakage current through σ_0 is neglected. Hence $E_2 = (\sigma_1/\sigma_2)E_1$, and from (7.45*a*)

$$E_1 = \frac{\mathcal{E}}{l_1 + (\sigma_1/\sigma_2)l_2} \tag{7.45c}$$

Thus

$$I = I_1 = \frac{\sigma_1 A \mathcal{E}}{l_1 + \sigma_1 l_2/\sigma_2} = \frac{\mathcal{E}}{l_1/A\sigma_1 + l_2/A\sigma_2} = \frac{\mathcal{E}}{R} \tag{7.45d}$$

where $R = l_1/A\sigma_1 + l_2/A\sigma_2$ and is the total resistance of the circuit.

Figure 7.15*b* illustrates a magnetic circuit which is similar to the above d-c circuit. Provided μ_1 and μ_2 are much larger than μ_0, most of the magnetic flux is confined to the highly permeable sections. The line integral of **H** around the circuit gives the applied magnetomotive force

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \mathcal{H} = NI = H_1 l_1 + H_2 l_2 \tag{7.46a}$$

The flux density **B** is given by $\mu\mathbf{H}$, while the total flux, analogous to current, is obtained by multiplying by the area A. Since the flux ψ is continuous,

$$\psi = \psi_1 = \psi_2 = \mu_1 H_1 A = \mu_2 H_2 A \tag{7.46b}$$

Hence $H_2 = (\mu_1/\mu_2)H_1$, and from (7.46*a*)

$$H_1 = \frac{\mathcal{H}}{l_1 + (\mu_1/\mu_2)l_2} \tag{7.46c}$$

Thus

$$\psi = \frac{\mathcal{H}}{l_1/A\mu_1 + l_2/A\mu_2} = \frac{\mathcal{H}}{\mathcal{R}} \tag{7.46d}$$

where $\mathcal{R} = l_1/\mu_1 A + l_2/\mu_2 A$ and is the reluctance of the circuit.

The validity of the solutions to the above two circuit problems depends on the accuracy of the assumption that the fields are uniform over the cross-sectional areas and that the current flow or flux is also uniform and confined to the sections of the circuit only, i.e., confined to the cross section of the two segments of length l_1 and l_2. In the case of the current circuit, this assumption is usually more nearly correct since the ratio

σ/σ_e is generally very much larger than μ/μ_0 for the magnetic materials. In spite of these limitations, the concept of a magnetic circuit is of great utility in the engineering application of ferromagnetic materials. The ease with which it provides a solution for the magnetic flux ψ in a circuit makes it superior to other approximate methods of solution when the requirements on the accuracy of the solution are not too stringent.

Example 7.5. Iron-core Transformer. Figure 7.16*a* illustrates an iron-core transformer of uniform cross section and having a small air gap of thickness t in the center leg. Such air gaps are frequently used in practice in order to increase the reluctance of the magnetic circuit so as to prevent magnetic saturation of the material. The left leg is wound with N_1 turns of wire, and the right leg with N_2 turns. The problem is to compute the flux linkage for the coil with N_2 turns when a current I flows in the other coil.

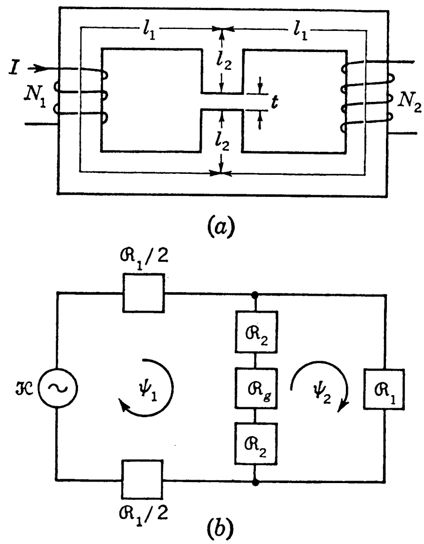

FIG. 7.16. An iron-core transformer and its equivalent magnetic circuit.

The equivalent magnetic circuit is illustrated in Fig. 7.16*b*. The magnetomotive force applied to the circuit is $N_1 I$ and acts in series with the reluctance $\mathcal{R}_1$ of the left side leg. This reluctance is shown arbitrarily split into two parts of magnitude $\mathcal{R}_1/2$ each. In series with $\mathcal{R}_1$ is the reluctance $\mathcal{R}_1$ of the right side leg in parallel with the reluctance of the center leg. The reluctances are given by the following expressions:

$$\mathcal{R}_1 = \frac{l_1}{\mu A}$$
$$\mathcal{R}_2 = \frac{l_2}{\mu A}$$
$$\mathcal{R}_g = \frac{t}{\mu_0 A}$$

Following the usual circuit-analysis approach, the following two equations may be written:

$$\mathcal{H} = \psi_1(\mathcal{R}_1 + 2\mathcal{R}_2 + \mathcal{R}_g) - \psi_2(2\mathcal{R}_2 + \mathcal{R}_g)$$
$$0 = \psi_1(2\mathcal{R}_2 + \mathcal{R}_g) - \psi_2(\mathcal{R}_1 + 2\mathcal{R}_2 + \mathcal{R}_g)$$

Solving for ψ_2 gives $\psi_2 = N_1I(2\mathcal{R}_2 + \mathcal{R}_g)/\mathcal{R}_1(\mathcal{R}_1 + 4\mathcal{R}_2 + 2\mathcal{R}_g)$. The flux linkage for the secondary coil is thus given by $N_2\psi_2$.

We should like to summarize in a more formal way the duality between electric and magnetic circuits that have been illustrated in the previous examples. As a starting point we note the similarity in the relations

$$\oint \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} \qquad \oint \mathbf{H} \cdot d\mathbf{l} = NI = \mathcal{H}$$

In the study of stationary currents in Sec. 5.2 it was pointed out that external to the sources (i.e., battery, etc.) $\oint \mathbf{E} \cdot d\mathbf{l} = 0$, so that for the external region the electric field could be related to a scalar potential function. The potential concept must be handled carefully, however, since if a closed path is taken which includes the battery, the net change in potential is not zero but equals the emf of the source; that is, the potential is actually multivalued because the $\mathbf{E}$ field is not conservative. In electric-circuit theory this difficulty is avoided by assigning a potential rise of $\mathcal{E}$ to the integral of $\mathbf{E}$ through a source of emf. In this case, if $\Delta\Phi$ is the potential drop across a portion of a loop, then the requirement that $\oint \mathbf{E} \cdot d\mathbf{l} = \Sigma\mathcal{E}$ (the total emf) can be written

$$\sum_{\text{loop}} \Delta\Phi = \sum_{\text{loop}} \mathcal{E}$$

or

$$\sum_{\text{loop}} \Delta\phi + \sum_{\text{loop}} (-\mathcal{E}) = 0$$

The above relation is one of Kirchhoff's laws and symbolizes the statement that the sum of the potential drops taken around any closed loop equals zero.

In an analogous fashion the magnetic field $\mathbf{H}$ can be derived from a scalar magnetic potential provided the multivalued nature of the field is taken into account.† If we adopt a similar convention to that in electric circuits we can write

$$\sum_{\text{loop}} \mathcal{H} = \sum_{\text{loop}} \Delta\Phi_m$$

The magnitude of the mmf taken around a closed path equals the total number of ampere-turns linking that path. Often this quantity is con-

† We are assuming that the true current is not zero; otherwise $\nabla \times \mathbf{H} = 0$, in which case $\mathbf{H}$ can be derived from a scalar potential unambiguously. The multivalued nature of Φ_m in the presence of a current loop can be demonstrated directly, for it can be shown that if we were to require $\mathbf{H} = -\nabla\Phi_m$, then Φ_m must satisfy $\Delta\Phi_m = (I/4\pi)\,\Delta\Omega$; that is, in the presence of a current loop, the change in magnetic scalar potential as a consequence of a change in position of the field point equals the change in subtended solid angle times the current in the loop divided by 4π. For a path encircling the loop, $\Delta\Omega$ will change discontinuously by 4π at some point on this contour, hence producing the multivalued nature of Φ_m. The total change in potential for one complete loop equals I; that is, $\oint \mathbf{H} \cdot d\mathbf{l} = I$, as we already know.

centrated over a small region of the magnetic circuit, just as in the electric circuit the source of emf is usually highly localized. If this is not the case, as in a uniformly wound toroid, it is not possible to separate the sources from the circuit. It is as if in an electric circuit the battery were split up and continuously distributed through the circuit.

In addition to the above equations we also have the following similar relations:

$$\oint \mathbf{J} \cdot d\mathbf{S} = 0 \qquad \oint \mathbf{B} \cdot d\mathbf{S} = 0$$
$$\mathbf{J} = \sigma \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

In view of the fact that the three electric-circuit equations given above form the basis for all d-c circuit theory, magnetic-circuit theory follows immediately by duality. It is only necessary to replace **J** by **B**, **E** by **H**, $\mathcal{E}$ by $\mathcal{H}$, and σ by μ. For example, since the total current is obtained from

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

then by duality, total flux ψ is

$$\psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

As a further illustration we may obtain the reluctance of an arbitrary shaped magnetic material from (5.36). By duality this is clearly

$$\mathcal{R} = \int_0^L \frac{du_1}{\int_S (\mu h_2 h_3 / h_1)\, du_2\, du_3}$$

It should be noted that the derivation in Sec. 5.5 tacitly assumed no current flow in the external medium, a relatively easy thing to achieve. In magnetic circuits the medium will invariably carry some leakage flux, so that true dual conditions cannot be provided, as we noted in an earlier example. However, with highly permeable materials, good results can be expected even though leakage is neglected.

7.8. Physical Properties of Magnetic Material†

As demonstrated earlier in this chapter, the magnetic properties of materials may be described in terms of a magnetic polarization or distribution of magnetic dipoles per unit volume. In this section we shall take a closer look at the properties of materials that give rise to a volume

† For a much more complete discussion, see, for example, A. J. Dekker, "Solid State Physics," 1957, or "Electrical Engineering Materials," 1959, Prentice-Hall, Inc., Englewood Cliffs, N.J.

density of magnetic dipoles. There are essentially three mechanisms, or properties, of atoms that give rise to a magnetic dipole moment:

1. The orbital motion of electrons around the nucleus is equivalent to a circulating current loop.

2. The spinning electron has an intrinsic magnetic dipole moment of magnitude $eh/4\pi w$, where e is the electron charge, w is the mass of the electron, and h is Planck's constant (6.62×10^{-34} joule-second). The magnitude of the spin magnetic moment of the electron is called a Bohr magneton.

3. The nucleus of an atom contains charged particles, and since the nucleus also has a spin, there is a magnetic dipole moment associated with the nucleus.

In most materials the dipole moment of the nucleus is negligible in comparison with the orbital and spin magnetic moments of the electron. The reason for this is the large mass of the nucleus in comparison with that of the electron (at least 10^3 larger). The angular momentum of the nuclear spin is about the same as that for the electron spin. Since the mass is so much greater, it follows that the angular velocity is much smaller, and hence the equivalent circulating current and dipole moment are also much smaller. It is the orbital magnetic dipole moment and, even more important, the electron-spin magnetic dipole moment that are largely responsible for the magnetic properties of materials.

Magnetic materials are classified according to the following scheme:

1. Diamagnetic Materials. These are materials which do not have a permanent magnetic dipole moment in the absence of an external applied magnetic field.

2. Paramagnetic Materials. These materials have a permanent magnetic dipole moment, but the interaction between neighboring dipoles is negligible, with the result that in the presence of an external field the flux density in the material is increased by only a small amount.

3. Ferromagnetic Materials. In these materials the dipoles interact strongly and all tend to line up parallel with the applied field so as to produce a large increase in the flux density in the material.

4. Antiferromagnetic Materials. In these materials the permanent dipoles tend to align themselves so that alternate dipoles are antiparallel to the applied field. The result is a cancellation of the effects of each dipole and a zero net increase in flux density in the material.

5. Ferrimagnetic Materials. In these materials magnetically polarized domains of unequal magnitude align with alternate domains parallel and antiparallel to the applied field. Usually a relatively large increase in flux density is produced since the strong dipoles align themselves with the field and the weak dipoles are aligned antiparallel to the applied field.

Diamagnetism

The nature of diamagnetism may be understood from the following example. Consider an atom with several electrons which rotate in orbits orientated so that the net magnetic dipole moment is zero. Instead of considering each orbit in detail, we consider a particular orbit for which the magnetic field has maximum effects. We shall then assume that the change in magnetic moment produced by the magnetic field is of the order of magnitude of the net change per orbit in the atom. The simplicity achieved outweighs the loss in quantitative precision. Accordingly, we specify our model to consist of an electron orbit whose plane is perpendicular to the externally applied field B_0, as in Fig. 7.17, and which is electrically balanced by a $+e$ nuclear charge. Let the radius of the orbit be r_0 and the angular velocity be ω_0 in the absence of the external field.

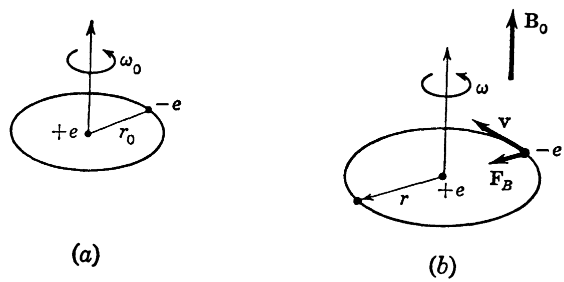

FIG. 7.17. Illustration of perturbation of electron orbit by an external field $\mathbf{B}_0$.

When B_0 is not present, the orbital angular velocity is determined by the condition that the outward centrifugal force $w\omega_0^2 r_0$ equals the inward coulomb attraction force $e^2/4\pi\epsilon_0 r_0^2$. Solving for ω_0 gives

$$\omega_0^2 = \frac{e^2}{4\pi\epsilon_0 w r_0^3} \tag{7.47}$$

In the presence of the external field B_0 there is an additional inward (Lorentz) force F_B whose magnitude is $evB_0 = er\omega B_0$. In place of the relation that led to (7.47) we must now have

$$\frac{e^2}{4\pi\epsilon_0 r^2} + er\omega B_0 = w\omega^2 r \tag{7.48}$$

To a first approximation the radius may be assumed to remain constant at the value r_0, and hence, using (7.47) in (7.48), we get

$$\omega^2 = \omega_0^2 + \frac{eB_0}{w}\omega \tag{7.49}$$

The field B_0 produces only a small perturbation in ω, so that

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega_0(\omega - \omega_0)$$

With this result, (7.49) may be written as

$$\omega - \omega_0 = \frac{eB_0}{2w} \tag{7.50}$$

In the absence of the field, the dipole moment of the rotating electron is $-ev_0\pi r_0^2/2\pi r_0 = -e\omega_0 r_0^2/2$, while in the presence of the field B_0 the total dipole moment is $-e\omega r_0^2/2$. Prior to the application of B_0 the contribution from all electronic orbits leads to a zero net dipole moment. The change due to B_0 is in a common direction for all orbits, hence resulting in a total net moment. The order of magnitude per orbit is the difference in dipole moments just evaluated and is

$$\begin{aligned} m_i &= \frac{-er_0^2(\omega - \omega_0)}{2} \\ &= \frac{-e^2r_0^2B_0}{4w} \\ &= \frac{-e^2r_0^2\mu_0H_0}{4w} \end{aligned} \tag{7.51}$$

where $\mu_0 H_0 = B_0$. It should be noted that this induced moment is directed antiparallel to the applied field. If N is the effective number of such orbits per unit volume, the induced dipole polarization M per unit volume will be $M = -Ne^2r_0^2\mu_0H_0/4w$, and hence the susceptibility χ_m is given by

$$\chi_m = \frac{-Ne^2r_0^2\mu_0}{4w} \tag{7.52}$$

Substitution of known typical values of N and r_0 shows that χ_m is of the order of -10^{-5}, a result in essential agreement with measured values. The small resultant value of χ_m justifies the assumption that the orbital radius remains essentially constant and that only a small change in the angular velocity ω_0 results upon application of an external field.

Although the above result was based on a classical physics approach, it still explains the basic process responsible for diamagnetism and may be verified by the methods of quantum mechanics.

Paramagnetism

Paramagnetism is due almost entirely to the spin magnetic dipole moment of the electron. Consider a material with N spinning electrons per unit volume. For paramagnetic materials the interaction between neighboring dipoles is negligible; so we may assume that the field acting on each dipole is just the applied field B_0. In the presence of the field B_0 some of the magnetic dipoles (each with a moment equal to one Bohr magneton) are aligned with the field while the remainder are aligned antiparallel to the field. The relative number in each position depends on the energy difference of the two positions and is essentially governed by Boltzmann's statistics. If N_p represents the number of dipoles per unit volume that are aligned parallel to the field and N_a is the number aligned antiparallel to the field, it is found that†

$$N_p = \frac{N}{1 + \exp\,(-ehB_0/2\pi wkT)}$$

$$N_a = \frac{N}{1 + \exp\,(ehB_0/2\pi wkT)}$$

where k is Boltzmann's constant and T is the absolute temperature. The net magnetic polarization is given by

$$M = \frac{N_p - N_a}{4\pi w}\,eh = \frac{Neh \tanh\,(ehB_0/4\pi wkT)}{4\pi w} \tag{7.53}$$

For normal temperatures and fields the argument of the hyperbolic tangent is small, so the approximation $\tanh x \approx x$ may be used and (7.53) gives

$$M = \frac{Ne^2h^2\mu_0 H_0}{(4\pi w)^2 kT} \tag{7.54}$$

Hence the susceptibility is given by

$$\chi_m = \frac{M}{H_0} = \frac{Ne^2h^2\mu_0}{(4\pi w)^2 kT} = \frac{C}{T} \tag{7.55}$$

where the constant C, defined by this equation, is called the Curie constant. Typical values of χ_m for paramagnetic materials are of the order of 10^{-3} in magnitude. As (7.55) shows, χ_m is inversely proportional to the absolute temperature T (Curie law). This is very reasonable, since at higher temperatures molecular activity opposes the effect of the applied field to orient the dipole moments. At the low temperatures the Curie law breaks down since the approximation used to obtain (7.55) is no longer valid.

Diamagnetic effects are also present in paramagnetic materials, but since the contribution to the susceptibility by the induced dipoles is of the order of -10^{-5}, it is completely masked by the paramagnetic effect.

Ferromagnetism

In ferromagnetic materials the effective field acting on each spin magnetic dipole is the vector sum of the applied field plus a strong inter-

† See Sec. 3.1, where an analogous derivation for electric dipoles is given.

action field arising from all the neighboring dipoles. This interaction field has been found experimentally to be much greater than the classically calculated magnetic field from the neighboring dipoles. These large "exchange forces" can be explained only by quantum mechanics and exist because of the wave nature of the electrons.

A theory for ferromagnetism was proposed by Weiss in 1907 and has been verified by quantum mechanics. The two basic postulates made by Weiss are: (1) There exists a strong interaction field from neighboring dipoles that tends to aid the alignment of the dipoles with the applied field. Thus the internal field acting on a dipole may be expressed as $B_i = \mu_0(H_0 + \alpha M)$, where α is called the internal field constant. (2) A ferromagnetic material consists of a number of domains with linear dimensions as large as or larger than 10^{-4} centimeter. In each domain all the spins are aligned in parallel but the direction of magnetization differs from one domain to the next. Also, each domain is spontaneously magnetized even in the absence of applied fields. However, because of the random orientation of the domains, the net flux density in the material is small.

By means of the domain theory a satisfactory explanation of the characteristic hysteresis curve for ferromagnetic materials can be given. Each domain can have its spins aligned along several possible directions or axes. The field strength required to produce magnetization along the various permissible axes differs, so that there are "easy" and "hard" directions of magnetization. In a ferromagnetic specimen which is originally unmagnetized, the application of an external field causes the following sequence of events to take place:

1. For weak applied fields those domains whose easy direction of magnetization is in the direction of the applied field, i.e., which are spontaneously magnetized in this direction, grow at the expense of the other domains. For small applied fields the domain wall movement is reversible. For large applied fields the wall movements are irreversible and a negative field must be applied to return the domain walls to their original positions. This irreversible wall movement gives rise to the hysteresis effect.

2. As the applied field is increased in strength, this domain growth continues until the whole specimen is essentially a single domain. At high field strengths some domain rotation takes place also. This process continues until the specimen becomes magnetically saturated. Figure 7.18 illustrates schematically this magnetization process.

Upon reducing the field the domain walls begin to move so as to produce more nearly equal sized domains again. When the applied field has been reduced to zero, a net magnetization remains, since many of the domains are still magnetized in the direction of the applied field. A microscopic examination of the magnetization curve shows that the process of magnetization is not a smooth one. This effect, known as the Barkhausen effect, is produced by random motion of the domain walls.

Curie-Weiss Law

If the expression $\mu_0(H_0 + \alpha M)$ for the internal field in a ferromagnetic material is substituted into (7.53), which gives the magnetization for

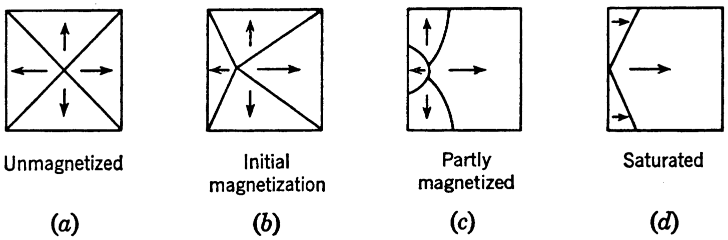

FIG. 7.18. Illustration of the growth of domains in the magnetization process.

paramagnetic materials, the equation for the magnetization in a ferromagnetic material is obtained; i.e.,

$$M = \frac{Neh}{4\pi w} \tanh \frac{eh\mu_0(H_0 + \alpha M)}{4\pi wkT} \tag{7.56}$$

This result is based on the paramagnetic model that electron spins are aligned either parallel or antiparallel to the applied field. For weak fields and high temperatures, (7.56) reduces to

$$M = N\left(\frac{eh}{4\pi w}\right)^2 \frac{\mu_0(H_0 + \alpha M)}{kT}$$

which may be solved for M to give

$$M = \frac{C}{T - \theta} \tag{7.57}$$

where $$C = \frac{N}{k}\left(\frac{eh}{4\pi w}\right)^2 \quad \text{and} \quad \theta = \alpha C$$

This law is known as the Curie-Weiss law, where C is the Curie constant and θ is the Curie temperature. For $T > \theta$, the behavior of a ferromagnetic material is similar to that of a paramagnetic material. Below the Curie temperature ($T < \theta$) the Curie-Weiss law is no longer applicable, since the hyperbolic tangent cannot be replaced by its argument in this range. For $T < \theta$ a finite value of the magnetization can exist even though the applied field H_0 is zero. This magnetization is called the spontaneous magnetization. With $H_0 = 0$ the magnetization is given by (7.56) as

$$M = \frac{Neh}{4\pi w} \tanh \frac{\mu_0 eh\alpha M}{4\pi wkT} \tag{7.58}$$

The value of M may be solved for as a function of the temperature T, and a curve of the form illustrated in Fig. 7.19 is obtained. At the Curie temperature $T = \theta$, the spontaneous magnetization vanishes. Above

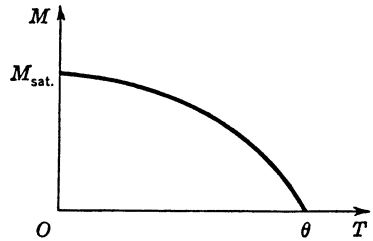

FIG. 7.19. Spontaneous magnetization as a function of temperature.

this temperature, (7.58) does not have a solution. At zero temperature the magnetization is equal to the saturation value $Neh/4\pi w$ corresponding to the condition where all the spin magnetic dipoles are aligned in the same direction. In a ferromagnetic medium it is actually each domain that is spontaneously magnetized in accord with (7.58). This spontaneous magnetization accounts for the residual magnetization in the absence of an external applied field.

Chapter 7

7.1. A permeable sphere of radius a is magnetized so that **M** within the sphere is uniform.

(a) What is the distribution of magnetization current in and on the sphere?

(b) In Prob. 7.15 we show that for this case **B** is also uniform within the sphere. From this information design a current winding that will set up a uniform **B** field over a spherical region of space.

7.2. A permeable sphere of radius a, whose center is at the origin of a system of coordinates, is magnetized so that

$$\mathbf{M} = (Az^2 + B)\mathbf{a}_z$$

Determine the equivalent magnetization currents and charges.

7.3. The magnetic moment of a magnetized body is given by the integral $\int_V \mathbf{M}\,dV$ taken over the body. If the body is placed in a uniform **B** field, determine the total torque in terms of the total moment.

7.4. A spherical shell of magnetic material is uniformly magnetized so that $\mathbf{M} = M_0\mathbf{a}_z$ in the shell. Find the scalar potential produced along the polar axis inside and outside of the shell. The inside radius of the shell is R_i, and the outside radius is R_0.

7.5. A very thin cylindrical iron rod and a thin (compared with radius) iron disk are placed in a magnetic field $\mathbf{B}_0$ with their axes parallel to the field. Find **B** and **H** internal to the iron specimens. Calculate the values of **M** in each case, given that $\mathbf{B}_0$ is 1.0 weber per square meter and $\mu = 5{,}000\mu_0$.

7.6. An infinitely long straight copper wire of circular cross section has a radius of 1 centimeter. It is surrounded coaxially by a permeable hollow cylinder which extends from a radius of 2 to 3 centimeters and whose relative permeability is 2,000. A current of 25 amperes flows in the wire.

(a) Calculate the total flux in the magnetic material per unit meter.

(b) Calculate the magnetization **M** in the permeable material.

(c) Find the induced magnetization currents in the magnetic medium.

(d) Show that the field for $r > 3$ centimeters is the same as it would be in the absence of the magnetic material by considering the net effect of the magnetization currents.

7.7. Find the field produced by a line current I located parallel to and above the plane interface of a magnetic material occupying the half space below the line current. The permeability of the material is μ.

HINT: The problem may be solved by an image method. Show that all boundary conditions can be satisfied if the field in the air is calculated from I and a current I' at a mirror-image position (assuming that both I and I' lie in free space), while the field in the magnetic material is due to I and I'', where I'' is located at I, and I and I'' are assumed to lie in an infinite material with permeability μ. The values of I' and I'' can be determined from the boundary conditions at the interface. Show in general that, for $\mu \to \infty$, lines of **B** on the air side must be perpendicular to the interface, while **H** in the iron goes to zero, and confirm that the above solution reduces to these results. Sketch the field lines.

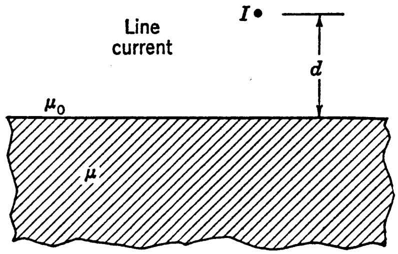

FIG. P 7.7

Answer

$$I' = \frac{\mu - \mu_0}{\mu + \mu_0} I$$

$$I'' = -I'$$

7.8. (*a*) Consider an arbitrary current loop as illustrated. Show that the magnetic scalar potential Φ_m at an arbitrary field point P can be expressed as

$$\Phi_m = -\frac{I\Omega}{4\pi}$$

where Ω is the solid angle subtended at P by an arbitrary surface whose periphery is C; that is,

$$\Omega = \int_S \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3}$$

HINT: Divide the surface into a large number of small circulating current loops as was done in Fig. 6.13.

(*b*) Consider the surface (with periphery C) chosen in (*a*) and assume that a uniform electric dipole layer lies on this surface. ($\mathbf{P}_s$ is the dipole moment per unit area such that $\mathbf{P}_s \cdot d\mathbf{S}$ is the electric dipole moment of a differential surface $d\mathbf{S}$. We assume $\mathbf{P}_s$ to be uniform and normal to the surface.) Show that an electric scalar potential at P can be expressed as

$$\Phi = -\frac{|\mathbf{P}_s|\Omega}{4\pi\epsilon_0}$$

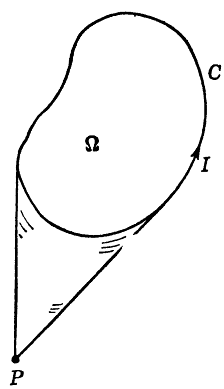

FIG. P 7.8

Note that the potential is discontinuous as the dipole layer is crossed. In the case of the current loop, the discontinuity is not associated with a physical surface, but may be any surface whose periphery is C. If one integrates the field (either $\mathbf{E} = -\nabla\Phi$ or $\mathbf{H} = -\nabla\Phi_m$) over a closed path that intersects this surface, a discontinuity in Ω of 4π results such that for the magnetic field

$$\oint \mathbf{H} \cdot d\mathbf{l} = I$$

as we expect. The multivalued electric field that results, while mathematically correct, violates the known conservative nature of the **E** field and cautions us that a true electric double layer cannot be achieved physically.

7.9. A toroid has the dimensions of 15 centimeters mean radius and 2 centimeters radius of circular cross section and is wound with 1,000 turns of wire. The toroid material is iron, with an effective relative permeability of 1,400 when the current is 0.7 ampere.

(*a*) Calculate the total flux.

(*b*) If a narrow air gap of length x is introduced, determine how the total flux depends on x (assuming $x \ll 2$ centimeters and μ remains the same in the iron). Compute total flux for $x = 0.1$, 1.0, 5.0 millimeters.

7.10. A ferromagnetic material is formed into the illustrated shape, where the cross section is everywhere square and 2 centimeters on a side, and the remaining mean dimensions are illustrated. If the winding carries 500 turns and a current of 0.3 ampere and if $\mu = 2{,}500\mu_0$, calculate the total flux in the central and right-hand leg. (Neglect leakage.)

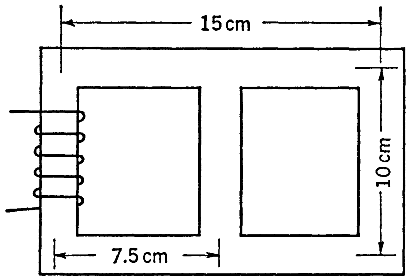

FIG. P 7.10

7.11. Repeat Prob. 7.10, but assume that a 2-millimeter air gap is cut in the central leg. Neglect leakage at air gap, and assume that μ remains constant.

7.12. Repeat Prob. 7.9, except that instead of assuming μ to be constant, the actual *B-H* relationship will be utilized. For this purpose the following data are available:

H, amp-turns/m	50	100	150	200	250	300	400	500	600	800
B, webers/sq m	0.07	0.23	0.60	0.85	1.00	1.07	1.18	1.25	1.30	1.33

7.13. A region V contains a permanent magnet of arbitrary shape. No additional sources of magnetic field exist. Show that

$$\tfrac{1}{2}\int \mathbf{B} \cdot \mathbf{H}\, dV = 0$$

where the integral is over all space. (In Chap. 8 we show that this integral evaluates the stored magnetic energy.)

7.14. The magnetic circuit shown consists of a permanent magnet of length 8 centimeters, two lengths of soft iron of 10 centimeters each, and an air gap of 1 centimeter.

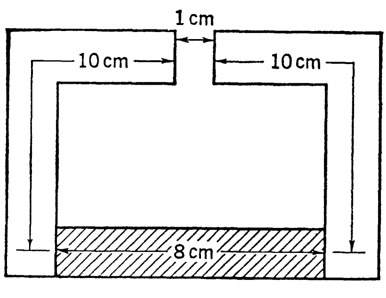

FIG. P 7.14

The cross sections of each are the same, and fringing is to be neglected. The *B-H* data for the magnet are given below, and $\mu = 5{,}000\mu_0$ for the soft iron.

(*a*) What is the flux density in the air gap?

(*b*) Sketch the **B** and **H** fields in the magnetic circuit.

H, amp-turns/m.....	0	−10,000	−20,000	−30,000	−35,000	−40,000	−45,000
B, webers/sq m......	1.25	1.22	1.18	1.08	1.00	0.80	0.00

7.15. A permeable (magnetizable) sphere of radius a is placed in a uniform external magnetic field $\mathbf{a}_z B_0 = \mathbf{a}_z \mu_0 H_0$. Introduce a scalar magnetic potential Φ_m, and find the induced magnetic field intensity $\mathbf{H}_i$ inside and outside of the sphere, as in Prob. 2.15. Show that for $r > a$, $\mathbf{H}_i$ is a dipole field, while for $r < a$, $\mathbf{H}_i$ is uniform. In the interior of the sphere, show that the relation between the magnetic polarization **M** and the applied field $\mathbf{H}_0$ is

$$\mathbf{M} = 3\,\frac{\mu - \mu_0}{\mu + 2\mu_0}\,\mathbf{H}_0$$

by using the relation $\mathbf{B} = \mu\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M})$, where **B** and **H** are the total fields in the sphere. Next show that the field **H** is also given by

$$\mathbf{H} = \mathbf{H}_0 - \frac{\mathbf{M}}{3} \qquad r < a$$

Eliminate **M** from the relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ to get $\mathbf{B} = -2\mu_0\mathbf{H} + 3\mu_0\mathbf{H}_0$. This is the equation of the shearing line. Plot this equation on a *B-H* plane, and indicate by the point of intersection with the *B-H* curve (sketch a suitable *B-H* curve for the purpose) what the remanent flux density B_r in the sphere will be when H_0 is reduced to zero.

7.16. The only shapes for which a rigorous solution such as that in Prob. 7.15 can be found are the ellipsoids and their degenerate forms such as spheroids and the sphere. For these bodies the interior field is uniform and the distant induced field is a dipole field when the external field is applied along an axis. The solution is of the form $\mathbf{H} = \mathbf{H}_0 - D\mathbf{M}$, and D is called the demagnetization factor. Show that in order to obtain a high remanent flux when the applied field $\mathbf{H}_0$ is reduced to zero, D should be as small as possible. In the interior of the body show that

$$\mathbf{B} = \frac{\mu\mathbf{H}_0}{1 + D(\mu/\mu_0 - 1)}$$

by using the relations $\mathbf{B} = \mu\mathbf{H} = \mu_0(\mathbf{H} + \mathbf{M})$ and $\mathbf{H} = \mathbf{H}_0 - D\mathbf{M}$. Thus the effective permeability of the body may be considered to be

$$\mu_e = \frac{\mu}{1 + D(\mu/\mu_0 - 1)}$$

since the flux density **B** is increased by μ_e/μ_0 in the body relative to its value $\mu_0\mathbf{H}_0$ for the applied field.