CHAPTER 6

STATIC MAGNETIC FIELD IN VACUUM

Our knowledge of magnetism and magnetic phenomena is as old as science itself. According to the writings of the great Greek philosopher Aristotle, the attractive power of magnets was known by Thales of Miletus, whose life spanned the period 640?–546 B.C. It was not until the sixteenth century, however, that any significant experimental work on magnets was performed. During this century the English physician Gilbert studied the properties of magnets, realized that a magnetic field existed around the earth, and even magnetized an iron sphere and showed that the magnetic field around this sphere was similar to that around the earth. Several other workers also contributed to the knowledge of magnetism during this same period.

The eighteenth century was a period of considerable growth for the theory and understanding of electrostatics. It is therefore not surprising to find that in the eighteenth century the theory of magnetism developed along lines parallel to that of electrostatics. The basic law that evolved was the inverse-square law of attraction and repulsion between unlike and like magnetic poles. Indeed, it would have been difficult for the theory to develop along any other path since batteries for producing a steady current were nonexistent. With the development of the voltaic cell by Volta, it was not long before the magnetic effects of currents were discovered by Oersted in 1820. This was followed by the formulation, by Biot and Savart, of the law for the magnetic field from a long straight current-carrying wire. Further studies by Ampère led to the law of force between conductors carrying currents. In addition, Ampère's studies on the magnetic field from current-carrying loops led him to postulate that magnetism itself was due to circulating currents on an atomic scale. Thus the gap between the magnetic fields produced by currents and those produced by magnets was effectively closed.

Today it is expedient to base our entire theory of magnetism and static magnetic fields on the work of Biot, Savart, and Ampère. A formulation in terms of fields produced by currents or charges in motion is perfectly general and can account for all the known static magnetic effects. The magnetic effects of material bodies is accounted for by equivalent volume and surface currents. This is not to say that the early theory, based on concepts similar to those used in electrostatics, is of no value. On the contrary, it is often much simpler to use this alternative formulation when dealing with problems involving magnetized bodies (magnets) and the perturbing fields set up by permeable bodies placed in external magnetic fields. Throughout the next two chapters we shall have an opportunity to examine both theories. Our main efforts will be devoted to the study of the magnetic effects of currents, since this provides us with a general foundation for the understanding of all static magnetic phenomena. Ampère's law of force between two closed current-carrying conducting loops will be elevated to the position of the fundamental law or postulate from which we shall proceed.

6.1. Ampère's Law of Force

With reference to Fig. 6.1, let C_1 and C_2 be two very thin closed conducting loops (wires) in which steady currents I_1 and I_2 flow. The

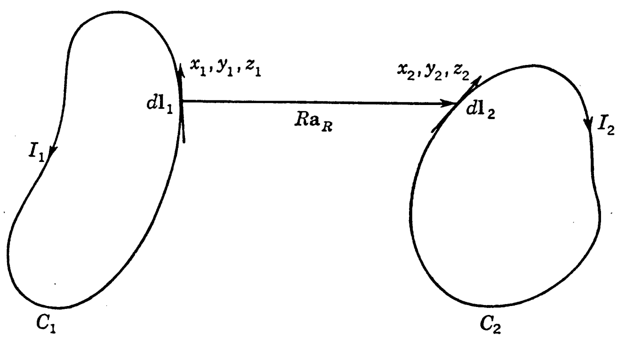

FIG. 6.1. Illustration of Ampère's law of force.

coordinates along the loop C_1 will be designated by x_1, y_1, z_1 and the vector arc length by $d\mathbf{l}_1$. Points along C_2 are designated by the variables x_2, y_2, z_2 and the vector arc length by $d\mathbf{l}_2$. The vector distance from $d\mathbf{l}_1$ to $d\mathbf{l}_2$ is $\mathbf{r}_2 - \mathbf{r}_1 = R\mathbf{a}_R$, where $\mathbf{a}_R$ is a unit vector directed from x_1, y_1, z_1 to x_2, y_2, z_2 and $R = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$. From the work of Ampère it is found that the vector force $\mathbf{F}_{21}$ exerted on C_2 by C_1, as caused by the mutual interaction of the currents I_1 and I_2, may be expressed as

$$\mathbf{F}_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2\, d\mathbf{l}_2 \times [I_1\, d\mathbf{l}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)]}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2\, d\mathbf{l}_2 \times (I_1\, d\mathbf{l}_1 \times \mathbf{a}_R)}{R^2} \quad (6.1)$$

The force $\mathbf{F}_{21}$ is measured in newtons, the current in amperes, and all lengths in meters. The currents are assumed to be located in vacuum. The constant μ_0 arises because of the system of units (mks units) which we are using and is equal to $4\pi \times 10^{-7}$ henry per meter. Thus μ_0 has the dimensions of inductance per unit length. This constant is called the permeability of vacuum, and for practical purposes one may take the permeability of air equal to μ_0 also, with negligible error. An appreciation of the term permeability will have to be postponed until we take up the properties of magnetizable material bodies. It will suffice to note that permeability has much the same significance for magnetostatics that permittivity has for electrostatics.

Equation (6.1) reveals the inverse-square-law relationship. The differential element of force $d\mathbf{F}_{21}$ between $I_1\,d\mathbf{l}_1$ and $I_2\,d\mathbf{l}_2$ may be regarded as given by the integrand in (6.1) and is

$$d\mathbf{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\, d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{a}_R) \tag{6.2a}$$

The triple-vector product may be expanded to give

$$d\mathbf{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi R^2}\left[(d\mathbf{l}_2 \cdot \mathbf{a}_R)\, d\mathbf{l}_1 - (d\mathbf{l}_2 \cdot d\mathbf{l}_1)\mathbf{a}_R\right] \tag{6.2b}$$

One should note that (6.2) does not correspond to a physically realizable condition since a steady-current element cannot be isolated. All steady currents must flow around continuous loops or paths since they have zero divergence.

A further difficulty with the relation (6.2*a*) or (6.2*b*) is that it is not symmetrical in $I_1\,d\mathbf{l}_1$ and $I_2\,d\mathbf{l}_2$. This superficially appears to contradict Newton's third law, which states that every action must have an equal and opposite reaction; i.e., the force exerted on $I_2\,d\mathbf{l}_2$ by $I_1\,d\mathbf{l}_1$ is not necessarily equal and opposite to the force exerted on $I_1\,d\mathbf{l}_1$ by $I_2\,d\mathbf{l}_2$. However, if the entire closed conductor, such as C_1 and C_2, is considered, no such difficulty arises and Newton's law is satisfied. Recalling that $\mathbf{a}_R/R^2 = -\nabla(1/R)$, we can replace the first term in (6.2*b*) with

$$\frac{-\mu_0 I_1 I_2\, d\mathbf{l}_1}{4\pi}\nabla\left(\frac{1}{R}\right) \cdot d\mathbf{l}_2$$

In an integration around C_2, this term vanishes, since $\nabla(1/R) \cdot d\mathbf{l}_2$ is a complete differential; that is, $\nabla(1/R) \cdot d\mathbf{l}_2$ is the directional derivative of $1/R$ along C_2 and is equal to

$$\frac{d(1/R)}{dl_2}\, dl_2 = d\left(\frac{1}{R}\right)$$

The integral of $d(1/R)$ is $1/R$, and since this is a single-valued function, it is equal to zero when evaluated around the closed contour C_2. For closed current loops, an equivalent form of Ampère's law of force may now be obtained by integrating (6.2*b*) and using the result that the first term vanishes; thus

$$\mathbf{F}_{21} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \oint_{C_1} \frac{(d\mathbf{l}_2 \cdot d\mathbf{l}_1)\mathbf{a}_R}{R^2} \qquad (6.3)$$

This alternative relation is symmetrical with respect to loops 1 and 2 (that is, $\mathbf{F}_{21} = \mathbf{F}_{12}$) and therefore obeys Newton's third law.

Using the expansion of the integrand as given in (6.2*b*) shows that $d\mathbf{F}_{21}$ is a vector in the plane containing the vectors $d\mathbf{l}_1$ and $\mathbf{a}_R$ and in addition is perpendicular to $d\mathbf{l}_2$, as in Fig. 6.2*a*. When $d\mathbf{l}_2$ is perpendicular to $\mathbf{a}_R$, the force is entirely radial, as in Fig. 6.2*b*. When $d\mathbf{l}_2$ and $d\mathbf{l}_1$ are perpendicular, the force is directed parallel to $d\mathbf{l}_1$, since the component proportional to $d\mathbf{l}_2 \cdot d\mathbf{l}_1$ along $\mathbf{a}_R$ is zero. Finally, when $d\mathbf{l}_2$ is perpendicular to $\mathbf{a}_R$ and $d\mathbf{l}_2$ and $d\mathbf{l}_1$ are also mutually perpendicular, the force vanishes, as illustrated in Fig. 6.2*c*.

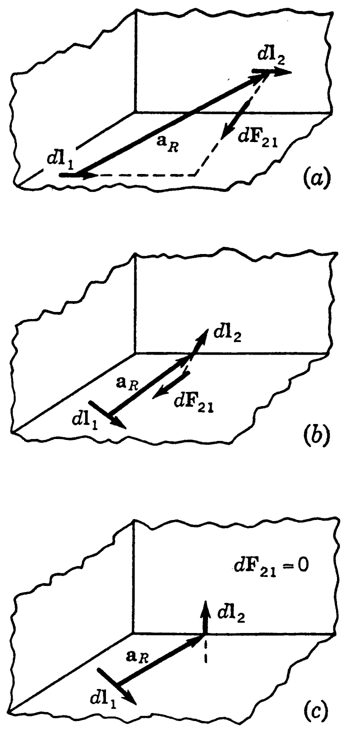

FIG. 6.2. Space relation between $d\mathbf{l}_1$, $d\mathbf{l}_2$, $\mathbf{a}_R$, and $d\mathbf{F}_{21}$.

6.2. The Magnetic Field B

In electrostatics the concept of an electric field was developed and found to be of great importance. This work stemmed from the definition of the electric field as the force acting on a unit charge. The field concept proves equally important in the magnetic case, and we find it possible to set up an analogous definition of the magnetic field **B**. In place of (6.1) we can write

$$\mathbf{F}_{21} = \oint_{C_2} I_2\, d\mathbf{l}_2 \times \mathbf{B}_{21} \qquad (6.4)$$

where

$$\mathbf{B}_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1\, d\mathbf{l}_1 \times \mathbf{a}_R}{R^2} \qquad (6.5)$$

Equation (6.1) may be thought of as evaluating the force between current-carrying conductors through an action-at-a-distance formulation. In contrast, (6.4) evaluates the force on a current loop in terms of the interaction of this current with the magnetic field **B**, which in turn is set up by the remaining current in the system. The current-field interaction that produces $\mathbf{F}_{21}$ in (6.4) takes place over the extent of the current loop C_2, while the magnetic field $\mathbf{B}_{21}$ depends only on the current and geometry of C_1 which sets up the field.

Except that the relations are vector ones, this work reiterates the field concept as developed in electrostatics. If we always assume orientation for maximum force, then **B** is the force per unit current element. In particular, (6.5) can be considered as a formulation for the magnetic field at any point in space which is independent of the existence of a test loop C_2 to detect the field. Furthermore, each element of current may be considered to contribute an amount

$$d\mathbf{B}(x,y,z) = \frac{\mu_0 I(x',y',z')\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|^3}$$

to the total field $\mathbf{B}(x,y,z)$. (The generalized notation here follows the definitions introduced in Sec. 1.17.) From this formula the field of an arbitrary current distribution can be found by superposition.

One of the advantages of the field formulation of (6.5) is that when **B** is known, this relation permits one to evaluate the force exerted on a current-carrying conductor placed in the field **B** without consideration of the system of currents which give rise to **B**. Equation (6.5) is the law based on the experimental and theoretical work of Biot and Savart and is therefore usually called the Biot-Savart law. Since this law may also be extracted from Ampère's law of force, it is sometimes referred to as Ampère's law as well.

A charge q moving with a velocity v is equivalent to an element of current $I\, d\mathbf{l} = q\mathbf{v}$ and hence in the presence of a magnetic field experiences a force **F** given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{6.6}$$

This force is called the Lorentz force, and (6.6) is often taken as the defining equation for **B**.

In practice, one does not always deal with currents flowing in thin conductors, and hence it is necessary to generalize the defining equation (6.5) for **B** so that it will apply for any arbitrary volume distribution of current. The steady-current flow field is divergenceless, and all flow lines form closed loops. Let us single out a short length dl of one current flow tube of cross-sectional area dS and compute its contribution to the total field **B**. Let the current density in the current tube under consideration be **J**, as in Fig. 6.3. We may associate the direction with the current density vector **J** rather than with the arc length dl, and hence the current-flow-tube element of length dl at (x',y',z') produces a partial field $d\mathbf{B}$ at (x,y,z) given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi R^2}\mathbf{J}(x',y',z') \times \mathbf{a}_R\, dl\, dS$$

since the total current is $\mathbf{J}\,dS$. The total current contained in a volume V will therefore produce a field $\mathbf{B}$ given by

$$\mathbf{B}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(x',y',z') \times \mathbf{a}_R}{R^2}\,dV' \tag{6.7}$$

where the integration is over the source coordinates x', y', z' and dV' is an element of volume $dS\,dl$. For a surface current $\mathbf{J}_s$ amperes per

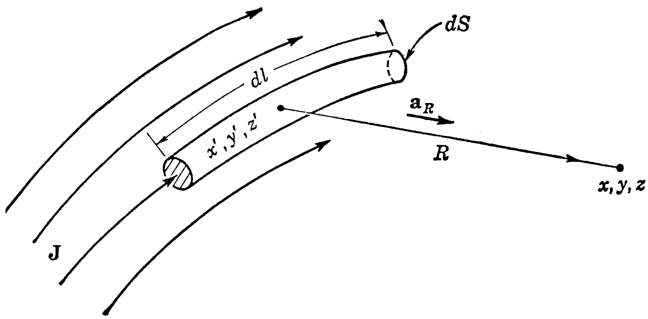

FIG. 6.3. A current flow tube.

meter flowing on a surface S, a similar derivation shows that the field produced is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_s \times \mathbf{a}_R}{R^2}\,dS' \tag{6.8}$$

The unit for $\mathbf{B}$ is the weber per square meter, which is also equal to volt-seconds per square meter.

The equation for $\mathbf{B}$ is a vector equation, and its evaluation in practice is usually carried out by decomposing the integrand into components along three mutually perpendicular directions. If the current $\mathbf{J}$ is referred to a rectangular coordinate frame and has components J_x, J_y, and J_z, then

$$\mathbf{J} \times \mathbf{a}_R = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ J_x & J_y & J_z \\ \dfrac{x - x'}{R} & \dfrac{y - y'}{R} & \dfrac{z - z'}{R} \end{vmatrix} = \mathbf{a}_x[J_y(z - z') - J_z(y - y')]\frac{1}{R}$$
$$+ \mathbf{a}_y[J_z(x - x') - J_x(z - z')]\frac{1}{R} + \mathbf{a}_z[J_x(y - y') - J_y(x - x')]\frac{1}{R}$$

The x component of magnetic field is thus given by

$$B_x(x,y,z) = \frac{\mu_0}{4\pi}\int_V \frac{(z - z')J_y(x',y',z') - (y - y')J_z(x',y',z')}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}\,dx'\,dy'\,dz'$$

with similar expressions for B_y and B_z. Integrals of the above form are not always very easy to handle, and in practice it is convenient to compute an auxiliary potential function first from which **B** may subsequently be found by suitable differentiation. Such a procedure was used in electrostatics and found to be of considerable value as an intermediate step in finding the electric field. The next section will consider the potential function from which **B** may be obtained.

6.3. The Vector Potential

For convenience, the general equation defining the static magnetic field **B** is repeated here:

$$\mathbf{B}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(x',y',z') \times \mathbf{a}_R}{R^2}\,dV' \tag{6.9}$$

If we replace $\mathbf{a}_R/R^2$ by $-\nabla(1/R)$, the integrand becomes $-\mathbf{J} \times \nabla(1/R)$. The vector differential operator ∇ affects only the variables x, y, z, and since $\mathbf{J}$ is a function of the source coordinates x', y', z' only, this latter relation may also be written as follows: $-\mathbf{J} \times \nabla(1/R) = \nabla \times (\mathbf{J}/R)$; that is, $\nabla \times (\mathbf{J}/R) = (1/R)\nabla \times \mathbf{J} - \mathbf{J} \times \nabla(1/R) = -\mathbf{J} \times \nabla(1/R)$, since $\nabla \times \mathbf{J} = 0$. Thus in place of (6.9) we may write

$$\mathbf{B}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \nabla \times \frac{\mathbf{J}}{R}\,dV' = \nabla \times \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(x',y',z')}{R}\,dV' \tag{6.10}$$

The curl operation could be brought outside the integral sign since the integration is over the x', y', z' coordinates and the differentiation is with respect to the x, y, z coordinates. Equation (6.10) expresses the field **B** at the point (x,y,z) as the curl or circulation of a vector potential function given by the integral. From (6.10), the definition of the vector potential function, denoted by **A**, is

$$\mathbf{A}(x,y,z) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(x',y',z')}{R}\,dV' \tag{6.11}$$

The integral for **A** is a vector integral and must be evaluated by decomposing the integrand into components along the coordinate axis; e.g., the x component of **A** is given by $A_x = (\mu_0/4\pi)\int_V (J_x/R)\,dV'$. Note that the integral for each component of the vector potential **A** is of the same type as the integral for the scalar potential from a volume distribution of charge in electrostatics. Having computed **A**, the field **B** is obtained by taking the curl of **A**:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.12}$$

The integral for **A** is easier to evaluate than the original expression (6.9) for **B**, and since the curl operation is readily performed, the use of (6.11) as an intermediate step provides us with a simpler procedure for finding **B**.

Example 6.1. Field from an Infinite Wire Carrying a Current *I*. Consider an infinitely long straight wire in which a steady current I flows, as in Fig. 6.4. The magnetic field **B** will be determined at points which are much farther away from the wire than the diameter of the wire, so that we may assume that the wire is infinitely thin with negligible error. To form a closed loop we may imagine that the wire is closed by an infinitely large semicircular loop which does not contribute to the field in any finite region (as we could verify). According to the Biot-Savart law,

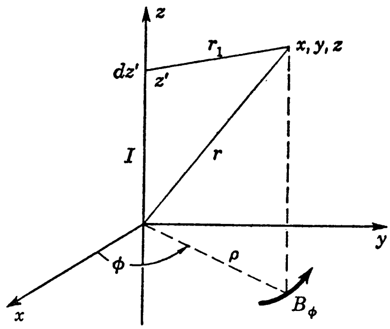

FIG. 6.4. An infinitely long wire with a current I.

$$\mathbf{B}(x,y,z) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{d\mathbf{l}' \times \mathbf{r}_1}{r_1^3}$$

The vector $\mathbf{r}_1$ is given by $\mathbf{r}_1 = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z(z - z')$, and $d\mathbf{l}' = \mathbf{a}_z\, dz'$, and hence

$$d\mathbf{l}' \times \mathbf{r}_1 = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & dz' \\ x & y & z - z' \end{vmatrix} = -\mathbf{a}_x y\, dz' + \mathbf{a}_y x\, dz'$$

We may evaluate the x and y components of **B** separately. For later work it will be desirable to have an expression for the field contributed by a finite length L of wire so that the integral will be evaluated between $\pm L/2$ first. The x component is given by

$$B_x = \frac{-\mu_0 I y}{4\pi} \int_{-L/2}^{L/2} \frac{dz'}{[x^2 + y^2 + (z - z')^2]^{3/2}} \tag{6.13}$$

The integral may be evaluated by making the substitution

$$\tan \alpha = \frac{z - z'}{\rho}$$

where $\rho = (x^2 + y^2)^{1/2}$ and is the radial coordinate in a cylindrical coordinate system ρ, ϕ, z. The differential dz' becomes

$$-\rho d \tan \alpha = -\rho \sec^2 \alpha\, d\alpha$$

The term $x^2 + y^2 + (z - z')^2$ becomes $\rho^2(1 + \tan^2 \alpha) = \rho^2 \sec^2 \alpha$. When $z' = \pm L/2$, the corresponding values of the angle $\alpha = \alpha_{1,2}$ are given by $\tan \alpha_{1,2} = (z \mp L/2)/\rho$, or since $\tan^2 \alpha = \sec^2 \alpha - 1$, we get

$\cos \alpha_{1,2} = \rho/[\rho^2 + (z \mp L/2)^2]^{1/2}$ and

$$\sin \alpha_{1,2} = \frac{z \mp L/2}{[\rho^2 + (z \mp L/2)^2]^{1/2}}$$

Hence the component B_x is given by

$$B_x = \frac{\mu_0 I y}{4\pi\rho^2} \int_{\alpha_2}^{\alpha_1} \cos \alpha \, d\alpha = \frac{\mu_0 I y}{4\pi\rho^2} (\sin \alpha_1 - \sin \alpha_2)$$

Substituting $\sin^{-1} (z \mp L/2)/[\rho^2 + (z \mp L/2)^2]^{1/2}$ for α_1 and α_2 yields the final result

$$B_x = \frac{\mu_0 I y}{4\pi\rho^2} \left\{ \frac{z - L/2}{[\rho^2 + (z - L/2)^2]^{1/2}} - \frac{z + L/2}{[\rho^2 + (z + L/2)^2]^{1/2}} \right\} \tag{6.14}$$

The evaluation of B_y is similar and can be found from the expression for B_x by replacing y by $-x$. If we now note that the unit vector $\mathbf{a}_\phi$ is given by $\mathbf{a}_\phi = -\mathbf{a}_x \sin \phi + \mathbf{a}_y \cos \phi = (-y/\rho)\mathbf{a}_x + (x/\rho)\mathbf{a}_y$, we see that the two components of $\mathbf{B}$ combine to form a vector along the direction of the unit vector $\mathbf{a}_\phi$. The total field $\mathbf{B}$ is thus given, in this case, by

$$\begin{aligned} \mathbf{B} &= B_x\mathbf{a}_x + B_y\mathbf{a}_y = B_\phi\mathbf{a}_\phi \\ &= \frac{\mu_0 I}{4\pi\rho} \left\{ \frac{z + L/2}{[\rho^2 + (z + L/2)^2]^{1/2}} - \frac{z - L/2}{[\rho^2 + (z - L/2)^2]^{1/2}} \right\} \mathbf{a}_\phi \end{aligned} \tag{6.15}$$

For an infinitely long wire, L tends to infinity, and the limiting form of the expression for B_ϕ becomes

$$B_\phi = \frac{\mu_0 I}{2\pi\rho} \tag{6.16}$$

since for $L \gg \rho$ and $L \gg z$ the terms $(z + L/2)/[\rho^2 + (z + L/2)^2]^{1/2}$ and $(z - L/2)/[\rho^2 + (z - L/2)^2]^{1/2}$ approach 1 and -1, respectively. That $\mathbf{B}$ should have only a component B_ϕ could have been anticipated from the cylindrical symmetry of the problem.

Example 6.2. Field from a Conducting Ribbon with a Current I_s per Unit Width. Consider a thin conducting strip of width d, infinitely long and carrying a uniform current I_s amperes per meter, as in Fig. 6.5. The field from a strip of width dx' carrying a current $I_s\, dx'$ and located at $y = 0$, $x = x'$ is equivalent to that from a thin wire similarly located. From (6.16) this field is seen to have components dB_x and dB_y given by

$$dB_x = \frac{-\mu_0 I_s}{2\pi} \frac{y \, dx'}{y^2 + (x - x')^2}$$

$$dB_y = \frac{\mu_0 I_s}{2\pi} \frac{(x - x') \, dx'}{y^2 + (x - x')^2}$$

when dB_ϕ is decomposed into components along the x and y axis and x is replaced by $x - x'$ (new origin) throughout. The total field is found

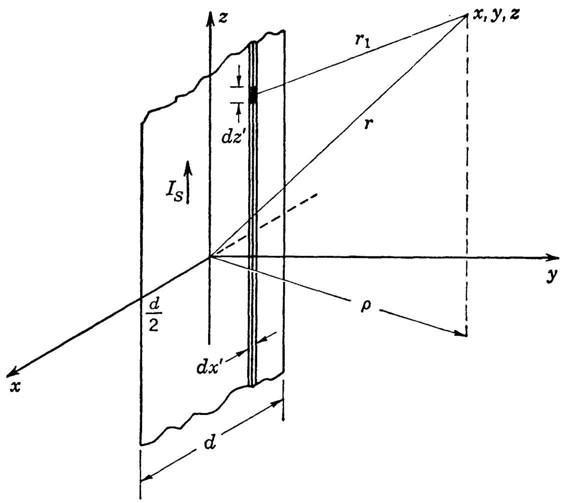

FIG. 6.5. Current I_s in an infinitely long strip.

by integrating over x' from $-d/2$ to $d/2$. The essential integrals to be evaluated are

$$\int_{-d/2}^{d/2} \frac{dx'}{y^2 + (x - x')^2} = -\frac{1}{y} \tan^{-1} \frac{x - x'}{y} \bigg|_{-d/2}^{d/2}$$

$$\int_{-d/2}^{d/2} \frac{(x - x')\,dx'}{y^2 + (x - x')^2} = -\tfrac{1}{2} \ln\,[y^2 + (x - x')^2] \bigg|_{-d/2}^{d/2}$$

Utilizing the above results, the components of the magnetic field become

$$B_x(x,y,z) = \frac{-\mu_0 I_s}{2\pi}\left(\tan^{-1}\frac{x + d/2}{y} - \tan^{-1}\frac{x - d/2}{y}\right) \tag{6.17a}$$

$$B_y(x,y,z) = \frac{\mu_0 I_s}{4\pi} \ln \frac{y^2 + (x + d/2)^2}{y^2 + (x - d/2)^2} \tag{6.17b}$$

Example 6.3. Force between Two Infinite Wires. Consider two thin infinite wires which are parallel and spaced at a distance d. The currents flowing in the wires are I_1 and I_2, as in Fig. 6.6. From (6.16) the magnetic field at C_2 due to C_1 has a ϕ component only and is given by

$$B_\phi = \frac{\mu_0 I_1}{2\pi d}$$

The force exerted on C_2 per unit length is given by Ampère's law of force (6.4) and is

$$\begin{aligned}\mathbf{F}_{21} &= I_2\mathbf{a}_z \times \mathbf{a}_\phi B_\phi \\ &= \frac{-\mu_0 I_1 I_2}{2\pi d}\mathbf{a}_\rho \qquad \text{newtons/m}\end{aligned} \tag{6.18}$$

When I_1 and I_2 are in the same direction, the two conductors experience

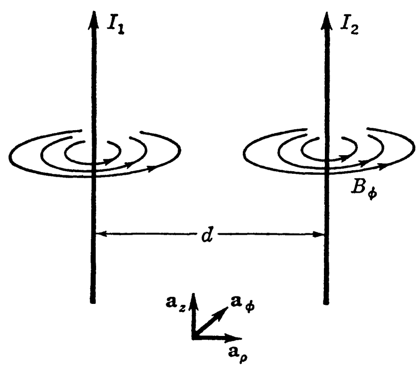

FIG. 6.6. Two parallel current-carrying wires.

an attractive force. When I_1 and I_2 are oppositely directed, the conductors repel each other.

Example 6.4. Field from a Circular Loop and Use of the Vector Potential. Consider a thin wire bent into a circular loop and carrying a current I. The radius of the loop is a, and it is located in the xy plane at the

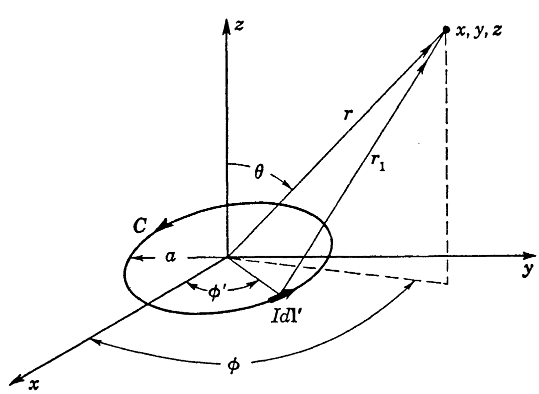

FIG. 6.7. A circular current loop.

origin, as in Fig. 6.7. We shall compute the field **B** at all points whose distance from the origin is much greater than the loop radius a by the direct method and by using the vector potential **A**. By the direct method **B** is given by the integral (6.5) as

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{l}' \times \mathbf{r}_1}{r_1^3}$$

where in this case

$$\begin{aligned} d\mathbf{l}' &= \mathbf{a}_\phi a\, d\phi' = (-\mathbf{a}_x \sin \phi' + \mathbf{a}_y \cos \phi')a\, d\phi' \\ \mathbf{r}_1 &= \mathbf{a}_x(x - a \cos \phi') + \mathbf{a}_y(y - a \sin \phi') + \mathbf{a}_z z \end{aligned}$$

Consequently,

$$d\mathbf{l}' \times \mathbf{r}_1 = [\mathbf{a}_x z \cos \phi' + \mathbf{a}_y z \sin \phi' - \mathbf{a}_z(y \sin \phi' + x \cos \phi' - a)]a\, d\phi'$$

The expression for r_1^3 is

$$\begin{aligned} r_1^3 &= [(x - a \cos \phi')^2 + (y - a \sin \phi')^2 + z^2]^{3/2} \\ &= (x^2 + y^2 + z^2 + a^2 - 2ax \cos \phi' - 2ay \sin \phi')^{3/2} \\ &\approx r^3 \left(1 - \frac{2ax}{r^2} \cos \phi' - \frac{2ay}{r^2} \sin \phi'\right)^{3/2} \end{aligned}$$

since $a^2 \ll r^2$. For r_1^{-3} we have approximately

$$r_1^{-3} \approx r^{-3} \left(1 + \frac{3ax}{r^2} \cos \phi' + \frac{3ay}{r^2} \sin \phi'\right)$$

upon using the binomial expansion and retaining only the leading terms.

We now obtain for the field **B** the expression

$$\begin{aligned} \mathbf{B} = \frac{\mu_0 I a}{4\pi r^3} \int_0^{2\pi} [\mathbf{a}_x z \cos \phi' + \mathbf{a}_y z \sin \phi' - \mathbf{a}_z(y \sin \phi' \\ + x \cos \phi' - a)] \left(1 + \frac{3ax}{r^2} \cos \phi' + \frac{3ay}{r^2} \sin \phi'\right) d\phi' \end{aligned}$$

The integration is straightforward, with most terms going to zero, and we are left with

$$\mathbf{B} = \frac{\mu_0 I \pi a^2}{4\pi r^3} \left[\mathbf{a}_x \frac{3xz}{r^2} + \mathbf{a}_y \frac{3yz}{r^2} - \mathbf{a}_z \left(\frac{3y^2}{r^2} + \frac{3x^2}{r^2} - 2\right)\right] \tag{6.19}$$

It will be convenient to refer this field to a spherical coordinate system r, θ, ϕ. For the purpose the following substitutions are required, namely, $z = r \cos \theta$, $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$. The component of **B** along the direction of the unit vector $\mathbf{a}_r$ is given by the projection of **B** on $\mathbf{a}_r$ and is $(\mathbf{B} \cdot \mathbf{a}_r)\mathbf{a}_r$; along the unit vector $\mathbf{a}_\theta$ it is $(\mathbf{B} \cdot \mathbf{a}_\theta)\mathbf{a}_\theta$, and similarly for the ϕ component. To evaluate these components note that

$$\begin{aligned} &\mathbf{a}_x \cdot \mathbf{a}_r = \sin \theta \cos \phi \qquad \mathbf{a}_y \cdot \mathbf{a}_r = \sin \theta \sin \phi \\ &\mathbf{a}_x \cdot \mathbf{a}_\theta = \cos \theta \cos \phi \qquad \mathbf{a}_y \cdot \mathbf{a}_\theta = \cos \theta \sin \phi \\ &\mathbf{a}_z \cdot \mathbf{a}_r = \cos \theta \qquad \mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin \theta \\ &\mathbf{a}_x \cdot \mathbf{a}_\phi = -\sin \phi \qquad \mathbf{a}_y \cdot \mathbf{a}_\phi = \cos \phi \qquad \mathbf{a}_z \cdot \mathbf{a}_\phi = 0 \end{aligned}$$

Using the above relations in (6.19), the following expression for **B** is obtained:

$$\mathbf{B} = \frac{\mu_0 I \pi a^2}{4\pi r^3} (\mathbf{a}_r 2 \cos\theta + \mathbf{a}_\theta \sin\theta) \tag{6.20}$$

As may be anticipated from the symmetry about the z axis, **B** has no component along the direction of the unit vector $\mathbf{a}_\phi$.

For the second method we must evaluate the following integral for **A**:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{l}'}{r_1}$$

For $r^2 \gg a^2$ we have

$$r_1^{-1} \approx r^{-1}\left(1 + \frac{ax}{r^2}\cos\phi' + \frac{ay}{r^2}\sin\phi'\right)$$

by using the binomial expansion, as was done to obtain an approximate expression for r_1^{-3}. The integral for **A** becomes

$$\mathbf{A} = \frac{\mu_0 I a}{4\pi r}\int_0^{2\pi} (-\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi')\left(1 + \frac{ax}{r^2}\cos\phi' + \frac{ay}{r^2}\sin\phi'\right) d\phi'$$

and integrates to give

$$\mathbf{A} = \frac{\mu_0 I \pi a^2}{4\pi r^3}(-\mathbf{a}_x y + \mathbf{a}_y x) \tag{6.21}$$

Referred to a spherical coordinate system, **A** is given by

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{a}_r)\mathbf{a}_r + (\mathbf{A}\cdot\mathbf{a}_\phi)\mathbf{a}_\phi + (\mathbf{A}\cdot\mathbf{a}_\theta)\mathbf{a}_\theta = \frac{\mu_0 I \pi a^2}{4\pi r^2}\mathbf{a}_\phi \sin\theta = A_\phi \mathbf{a}_\phi \tag{6.22}$$

and has only a component A_ϕ.

The magnetic field **B** is given by the curl of **A**. In spherical coordinates we have

$$\nabla \times \mathbf{A} = \mathbf{B} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix} \tag{6.23}$$

where A_ϕ is given by (6.22). Expansion of this determinant gives

$$\mathbf{B} = \mathbf{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

Substitution for A_ϕ from (6.22) will now yield the same expression as given by (6.20). The use of the vector potential in the present example leads to the end result in a simpler manner than the use of the defining relation (6.5) for **B**.

The previous results are restricted by the condition that $r \gg a$. If the field point is chosen to be along the axis of the loop (z axis), then a rigorous solution is easy to obtain. In the first method we note that since $x = y = 0$, we have

$$r_1^3 = (a^2 + z^2)^{3/2}$$
$$d\mathbf{l}' \times \mathbf{r}_1 = (\mathbf{a}_x z \cos \phi' + \mathbf{a}_y z \sin \phi' + a\mathbf{a}_z)a\, d\phi'$$

The expression for magnetic field

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}' \times \mathbf{r}_1}{r_1^3} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{(\mathbf{a}_x z \cos \phi' + \mathbf{a}_y z \sin \phi' + a\mathbf{a}_z)a\, d\phi'}{(a^2 + z^2)^{3/2}}$$

then evaluates to

$$\mathbf{B} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \mathbf{a}_z$$

6.4. The Magnetic Dipole

The distant field **B** produced by a small current loop will be shown to be similar to the electric field from a small electric dipole. For this reason a small current loop is called a magnetic dipole. Its dipole moment **M** is defined as equal to the product of the area of the plane loop and the magnitude of the circulating current, and the vector direction of the moment is perpendicular to the plane of the loop and along the direction a right-hand screw would advance when rotated in the same sense as the current circulates around the loop. For a circular loop of radius a the magnitude of the dipole moment is $\pi a^2 I$, as in Fig. 6.8.

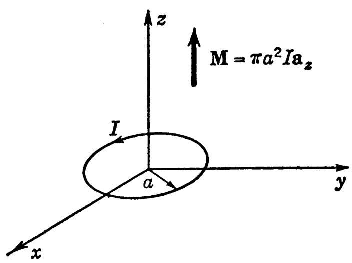

FIG. 6.8. The magnetic dipole.

From (6.21) in Example 6.4, we have $\mathbf{A} = (\mu_0 M/4\pi r^3)(-\mathbf{a}_x y + \mathbf{a}_y x)$ for the vector potential from a small circular current loop described as in Fig. 6.8. We now note that

$$\mathbf{a}_z \times \mathbf{r} = \mathbf{a}_z \times (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) = -\mathbf{a}_x y + \mathbf{a}_y x$$

and hence we may write

$$\mathbf{A} = \frac{\mu_0 M}{4\pi} \mathbf{a}_z \times \frac{\mathbf{r}}{r^3} = \frac{-\mu_0 M}{4\pi} \mathbf{a}_z \times \nabla \left(\frac{1}{r}\right) \tag{6.24}$$

since $\nabla(1/r) = -\mathbf{r}/r^3$. Now $\mathbf{M} = \mathbf{a}_z M$ is a constant; so we may also write in place of (6.24)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \nabla \times \frac{\mathbf{M}}{r} \tag{6.25}$$

The steps involved in arriving at (6.25) are the same as those used to derive the integrand of (6.10).

The magnetic field **B** is given by

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \nabla \times \left(\nabla \times \frac{\mathbf{M}}{r} \right)$$

This expression may be simplified by using the definition

$$\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2$$

and the fact that $\nabla^2(1/r) = 0$ for $r \neq 0$. Remembering that **M** is a constant, we now get

$$\mathbf{B} = \frac{\mu_0}{4\pi} \left[\nabla \left(\nabla \cdot \frac{\mathbf{M}}{r} \right) - \nabla^2 \frac{\mathbf{M}}{r} \right] = \frac{\mu_0}{4\pi} \nabla \left(\nabla \cdot \frac{\mathbf{M}}{r} \right) = \frac{\mu_0}{4\pi} \nabla \left[\mathbf{M} \cdot \nabla \left(\frac{1}{r} \right) \right] \tag{6.26}$$

This last result is of the same form as the expression

$$\frac{1}{4\pi\epsilon_0} \nabla \left[\mathbf{p} \cdot \nabla \left(\frac{1}{r} \right) \right]$$

for the electric field from an electric dipole **p**. Thus the electric and magnetic dipoles give rise to similar fields, as illustrated in Fig. 6.9. There is a fundamental difference, however, in that the electric lines of flux leave and terminate on charges while the magnetic lines of flux are continuous closed loops. In fact, we shall show in a later section that this property is always true for the magnetic field **B**. This difference is not revealed by (6.26) and the equivalent electric dipole expression because these expressions are valid only for field points whose distance from the dipole is much greater than the extent of the dipole. Even for infinitesimal dipoles we may not use those expressions to reveal the behavior of the fields in the immediate vicinity of the sources.

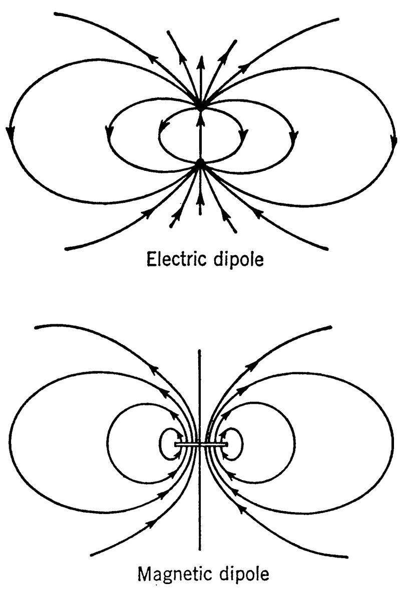

FIG. 6.9. Comparison of electric and magnetic dipoles.

For an arbitrary current loop the magnetic dipole moment is defined as equal to the product of the loop current I and the vector area **S** of the surface bounded by the loop; thus

$$\mathbf{M} = I\mathbf{S} \tag{6.27}$$

The positive direction of current flow is related to the positive direction of the surface by the usual right-hand-screw rule. This definition is illustrated in Fig. 6.10. It is not hard to show that the magnetic field from an arbitrary current loop of moment **M**, as defined in (6.27), is also given by (6.26), provided the distance to the field point is large

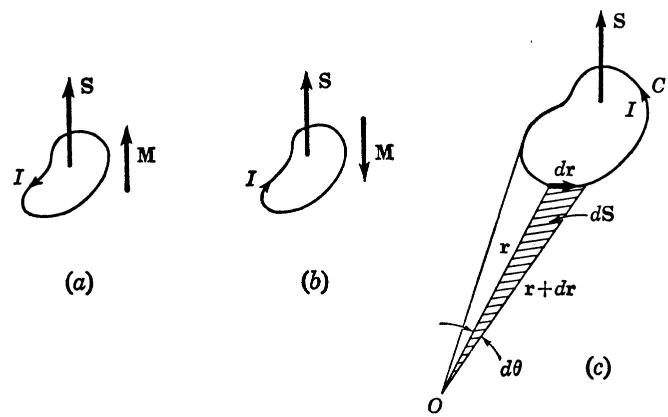

FIG. 6.10. General magnetic dipole.

compared with a characteristic linear dimension of the loop (see Prob. 6.16).

To facilitate the generalization of the magnetic dipole moment to a volume distribution of stationary currents, we first express the vector area S as an integral. Let **r** be a vector from some convenient origin to a point on the loop C. The arc length along the loop is $d\mathbf{r}$. The vector area of the infinitesimal triangle shown shaded in Fig. 6.10c is

$$\tfrac{1}{2}\mathbf{r} \times d\mathbf{r} = d\mathbf{S}$$

Note that this element of surface is in the positive direction as defined by the sense of the contour indicated in Fig. 6.10. The vector area of the cone with apex at O and subtended by the contour C is

$$\mathbf{S} = \oint_C d\mathbf{S} = \tfrac{1}{2} \oint_C \mathbf{r} \times d\mathbf{r} \tag{6.28}$$

This result is the same for any surface whose periphery is C, as we know from vector-analysis considerations. For example, if C is a plane curve, $|\mathbf{S}|$ is the plane area circumscribed. The magnetic dipole moment of an arbitrary shaped loop is therefore given by

$$\mathbf{M} = \frac{I}{2} \oint_C \mathbf{r} \times d\mathbf{r} \tag{6.29}$$

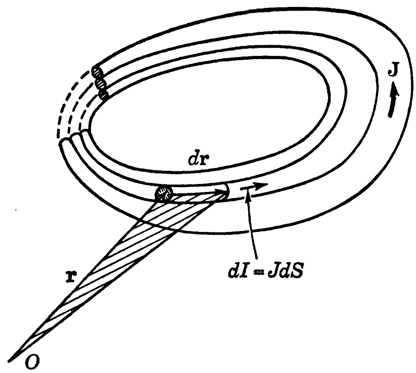

FIG. 6.11. A volume distribution of current separated into infinitesimal flow tubes.

The extension to a volume distribution of current is now a trivial one. Since the current is divergenceless, we may separate it into a large number of infinitesimal closed current flow tubes, as in Fig. 6.11. For any one flow tube the total current is $dI = J\,dS$, where dS is the cross-sectional area and J is the current density. Although dS may vary along the tube, the product $J\,dS$ is constant. A single flow tube contributes an amount

$$d\mathbf{M} = \frac{dI}{2}\oint \mathbf{r} \times d\mathbf{r} = \oint \frac{dS}{2}\,\mathbf{r} \times \mathbf{J}\,dr$$

to the total dipole moment, since we may associate the vector direction with $\mathbf{J}$ instead of with $d\mathbf{r}$. Now $dS\,dr$ is an element of volume dV, and summing over all current flow tubes (integration over dS), we get the general result

$$\mathbf{M} = \tfrac{1}{2}\int_V \mathbf{r} \times \mathbf{J}\,dV \qquad (6.30)$$

Torque on a Magnetic Dipole

An electric dipole $\mathbf{p}$ placed in a uniform electrostatic field $\mathbf{E}$ experiences a torque $\mathbf{T} = \mathbf{p} \times \mathbf{E}$ but no translational force. A similar result holds for a magnetic dipole in a field $\mathbf{B}$ with the torque $\mathbf{T}$ given by

$$\mathbf{T} = \mathbf{M} \times \mathbf{B} \qquad (6.31)$$

The torque is such that it tends to align the dipole with the field. The relation (6.31) is readily proved for a rectangular-loop dipole, as in Fig. 6.12.

Let the sides of the loop be L. The current in the loop is I, while $\mathbf{B}$ is chosen so that the plane defined by $\mathbf{B}$ and the surface normal is orthogonal to two sides of the loop, that is, C_2 and C_4 in Fig. 6.12*a*. $\mathbf{B}$ makes an angle θ with the surface normal. By Ampère's law of force (6.4), the force on the segments C_2 and C_4 is IBL and has the directions indicated in Fig. 6.12*b*, i.e., perpendicular to $\mathbf{B}$ and I. The forces on the segments C_1 and C_3 are equal, opposite, and directed along the axis of rotation and hence produce neither a torque nor a translational force. The product of the force F and the moment arm $L \sin\theta$ gives the torque as

$$T = IBL^2 \sin\theta = MB \sin\theta$$

or in vector form, $\mathbf{T} = \mathbf{M} \times \mathbf{B}$, since the magnitude of the dipole moment M is IL^2. If the direction of $\mathbf{B}$ is arbitrary, then it may be resolved into components of the above type and the resultant torque found by superposition. The analysis shows that the relation $\mathbf{T} = \mathbf{M} \times \mathbf{B}$ holds for arbitrary orientations of $\mathbf{B}$ relative to $\mathbf{M}$.

The generalization to an arbitrary current loop is obtained by considering the loop as made up of a large number of infinitesimal square loops, each with a dipole moment $d\mathbf{M}$, as in Fig. 6.13. The torque on each square-loop dipole is given by $d\mathbf{M} \times \mathbf{B} = I(d\mathbf{S} \times \mathbf{B})$, where I is the current magnitude and is constant. Integrating over all the infinitesimal dipoles gives the relation (6.31) if $\mathbf{B}$ is constant over the whole region. If $\mathbf{B}$ varies across the region occupied by the dipole, then (6.31) must be replaced by

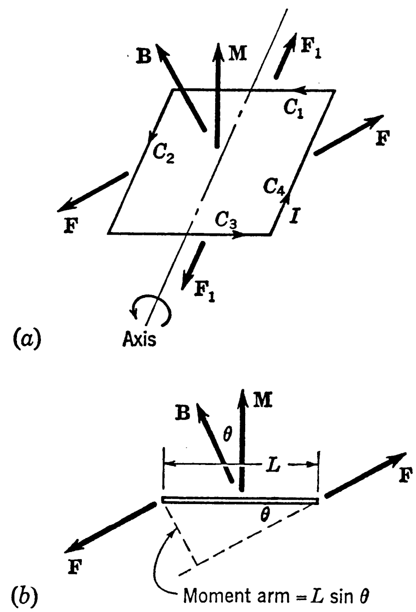

FIG. 6.12. Illustration of torque on a square-loop dipole.

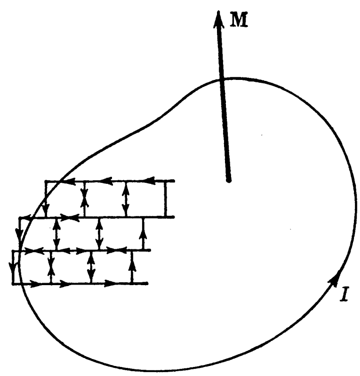

FIG. 6.13. Decomposition of dipole $\mathbf{M}$ into elementary square-loop dipoles $d\mathbf{M}$. (Currents along all interior boundaries cancel.)

$$\mathbf{T} = \int d\mathbf{M} \times \mathbf{B} \tag{6.32}$$

Example 6.5. Torque on a D'Arsonval Movement. The D'Arsonval moving-coil instrument for measuring current consists of a rectangular coil, of n turns, which is free to rotate against the restoring torque of a hair spring. The coil is placed between the poles of a permanent magnet, which produces a field $\mathbf{B}$, as in Fig. 6.14, which we may assume to be uniform. The current I to be measured passes through the coil of cross-sectional area S. The magnetic dipole moment of the movement is nSI. The torque produced on the movement is then $BnSI \sin \theta = BnSI \cos \alpha$, where α is the angle of rotation from the zero current equilibrium position. Rotation stops when the restoring torque $k\alpha$ of the hair spring is equal to the torque produced by the field, i.e., when $k\alpha = BnSI \cos \alpha$. For small angles $\cos \alpha$ is approximately unity, and hence α is directly proportional to the current I. In practical instruments special shaped pole pieces

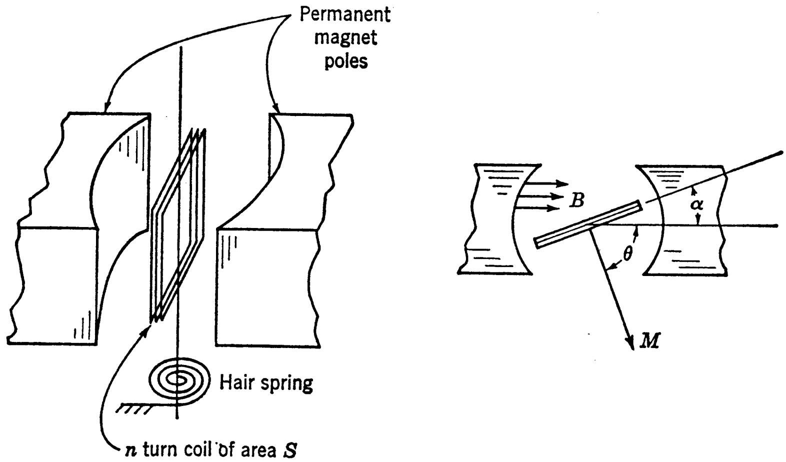

FIG. 6.14. The D'Arsonval moving-coil instrument.

are often used so as to produce a field **B** that will result in a linear scale over a larger range than that available with a uniform field.

6.5. Magnetic Flux and Divergence of B

We have seen that the magnetic field **B** can be derived from the curl of an auxiliary vector potential function **A**. This result leads at once to an important physical property for the field **B**. The divergence of the curl of any vector is identically equal to zero, and hence $\nabla \cdot \nabla \times \mathbf{A} = 0$, from which it follows that the divergence of **B** is also identically zero; i.e.,

$$\nabla \cdot \mathbf{B} = 0 \tag{6.33}$$

In the next chapter we shall show that the effects of material bodies can be accounted for by equivalent volume and surface magnetization currents. Thus even in the presence of material media it is possible to derive **B** from the curl of a vector potential, and hence the relation (6.33) is true in general. It now follows that the flux lines of **B** are always continuous and form closed loops. This property of **B** is the mathematical consequence of the formulation of Ampère's law, which in turn is based on experiments that do not reveal the existence of free magnetic poles, or "magnetic charge." All magnets have both a north and a south pole, and the field **B** is continuous through the magnet. For this reason the magnetostatic field **B** is fundamentally a different kind of field from the electrostatic field **E**. As discussed at several points earlier in this book, it is frequently advantageous to consider a vector field as representing the flow of something. The magnetic field **B** is often thought of as representing a magnetic flux density. Then the flux through an element of area

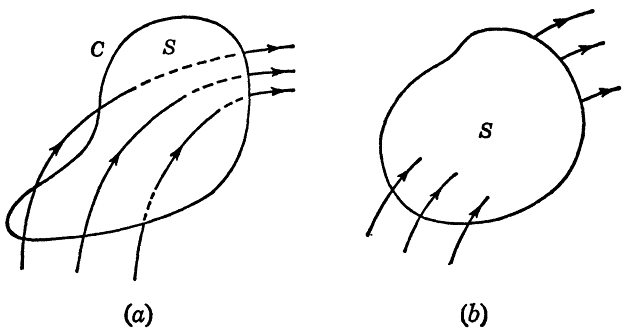

FIG. 6.15. Magnetic flux through a surface S. (a) Open surface; (b) closed surface.

$d\mathbf{S}$ is given by the dot product of **B** with $d\mathbf{S}$, $\mathbf{B} \cdot d\mathbf{S}$. The dot product selects the normal component of **B** through the surface $d\mathbf{S}$. For an arbitrary surface S bounded by a closed contour C, as in Fig. 6.15a, the total magnetic flux Ψ passing through the surface is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{6.34}$$

The flux passing through the surface S bounded by the contour C is said to link the contour C and is commonly referred to as the "flux linkage."

For a closed surface S, as in Fig. 6.15b, just as much flux leaves the surface as enters because of the continuous nature of the flux lines. Hence the integral of $\mathbf{B} \cdot d\mathbf{S}$ over a closed surface is equal to zero. Mathematically, this result follows from (6.33) by an application of the divergence theorem. In the present case $\nabla \cdot \mathbf{B} = 0$; so we have

$$\int_V \nabla \cdot \mathbf{B}\, dV = \int_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{6.35}$$

The flux which links a contour C may be expressed in terms of the vector potential **A** also. Since $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

The latter integral may be transformed to a contour integral by using

Stokes' law; thus

$$\Psi = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l} \tag{6.36}$$

This latter integral is sometimes more convenient to evaluate than (6.34) is.

6.6. Ampère's Circuital Law

So far we have dealt only with integrals that give the field **B** or the vector potential **A**. What we need to do next is to obtain an equation for **B** that relates **B** to the current which exists at the point in space where **B** is being evaluated. A general vector field is a field which has both a divergence and a curl, neither of which is identically zero throughout all space. When the divergence and curl are both identically zero, the field vanishes everywhere. A field with a zero divergence but a nonzero curl is known as a pure rotational or solenoidal field. A field with a zero curl and a nonzero divergence is called an irrotational or lamellar field, of which the static electric field is a well-known example. We have seen that **B** is a solenoidal field with a zero divergence everywhere. Therefore the source **J** for the field **B** must be related to the curl of **B** in some manner.

To obtain the relation we are seeking we begin with (6.10) and take the curl of both sides to get

$$\nabla \times \mathbf{B}(x,y,z) = \nabla \times \nabla \times \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(x',y',z')}{R}\, dV' \tag{6.37}$$

The curl-curl operator may be expanded into the $\nabla\nabla \cdot - \nabla^2$ operator. We may also carry out the differentiation first and then the integration, because of the independence of the variables x, y, z and x', y', z'. Hence we have

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int_V \left[\nabla\nabla \cdot \frac{\mathbf{J}(x',y',z')}{R} - \mathbf{J}(x',y',z')\nabla^2 \left(\frac{1}{R}\right) \right] dV' \tag{6.38}$$

We shall show later that the integral of the first term vanishes, so that

$$\nabla \times \mathbf{B} = -\frac{\mu_0}{4\pi} \int_V \mathbf{J}(x',y',z')\nabla^2 \left(\frac{1}{R}\right) dV' \tag{6.39}$$

By direction differentiation one readily finds that $\nabla^2(1/R) = 0$ for all finite values of R. Thus if the field point (x,y,z) is outside a finite source region, then R will never vanish, and it is clear that $\nabla \times \mathbf{B} = 0$. But if the field point is within the source region, then in the process of integration it is possible for R to be zero. This condition requires more careful attention since the integrand of (6.39) has a singularity at $R = 0$. Actually, as we shall now verify, the singularity of $\nabla^2(1/R)$ in (6.39) is integrable and yields a finite result. We proceed by noting that in the immediate neighborhood of the point $R = 0$ the current density function does not vary much from its value at the point $R = 0$, that is, at the point $x' = x$, $y' = y$, $z' = z$. Since the integrand is zero everywhere except at $R = 0$, we need only integrate (6.39) over a small sphere with center at (x,y,z), as in Fig. 6.16. We may take $\mathbf{J}(x',y',z')$ equal to $\mathbf{J}(x,y,z)$ throughout the volume of the sphere for the reason just given, and hence (6.39) becomes

$$\nabla \times \mathbf{B} = -\frac{\mu_0}{4\pi}\mathbf{J}(x,y,z)\int_{V_s}\nabla^2\left(\frac{1}{R}\right)dV' \tag{6.40}$$

Now since $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$, we have the result $\nabla(1/R) = -\nabla'(1/R)$ and $\nabla^2(1/R) = \nabla'^2(1/R) = \nabla' \cdot \nabla'(1/R)$, where ∇' signifies differentiation with respect to x', y', and z'. In place of (6.40) we have

$$-\frac{\mu_0}{4\pi}\mathbf{J}(x,y,z)\int_{V_s}\nabla'\cdot\nabla'\left(\frac{1}{R}\right)dV' = -\frac{\mu_0}{4\pi}\mathbf{J}(x,y,z)\oint_S\nabla'\left(\frac{1}{R}\right)\cdot d\mathbf{S}'$$

where we have also made use of the divergence theorem. The element of area in spherical coordinates is $\mathbf{n}R^2\,d\Omega$, where $\mathbf{n}$ is the unit outward normal and $d\Omega$ is an element of solid angle, that is,

$$d\Omega = \sin\theta\,d\phi\,d\theta$$

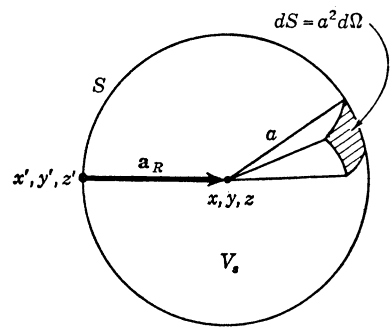

FIG. 6.16. Small sphere of radius a surrounding singularity point $x' = x$, $y' = y$, $z' = z$.

Also we have $\nabla'(1/R) = \mathbf{a}_R/R^2$, and hence

$$\nabla'\left(\frac{1}{R}\right)\cdot d\mathbf{S}' = \mathbf{n}\cdot\mathbf{a}_R\,d\Omega = -d\Omega$$

since $\mathbf{a}_R$ points toward the center of the sphere. Therefore (6.40) gives

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\mathbf{J}(x,y,z)\int_S d\Omega = \mu_0\mathbf{J}(x,y,z) \tag{6.41}$$

since there are 4π steradians in the solid angle of a closed surface. Hence we see that the curl or rotation of $\mathbf{B}$ is equal to $\mu_0\mathbf{J}$.

Proof of $\int_V \nabla\nabla\cdot\frac{\mathbf{J}}{R}\,dV' = 0$

Having obtained the desired relation (6.41) between $\mathbf{B}$ and $\mathbf{J}$, we must now return to (6.38) and show that the first integral vanishes as stated.

The integral in question may be rewritten as

$$\int_V \nabla\nabla \cdot \frac{\mathbf{J}(x',y',z')}{R}\, dV' = \nabla \int_V \nabla \cdot \frac{\mathbf{J}}{R}\, dV'$$

where one of the differentiation operators has been brought outside the integral sign. To prove the desired result, rewrite the integrand as follows:

$$\nabla \cdot \frac{\mathbf{J}(x',y',z')}{R} = \mathbf{J} \cdot \nabla\left(\frac{1}{R}\right) = -\mathbf{J} \cdot \nabla'\left(\frac{1}{R}\right) = -\nabla' \cdot \frac{\mathbf{J}}{R}$$

since $\nabla' \cdot (\mathbf{J}/R) = (1/R)\nabla' \cdot \mathbf{J} + \mathbf{J} \cdot \nabla'(1/R)$ and $\nabla' \cdot \mathbf{J}$ is zero for stationary currents. The integral becomes

$$\nabla \int_V \nabla \cdot \frac{\mathbf{J}}{R}\, dV' = -\nabla \int_V \nabla' \cdot \frac{\mathbf{J}}{R}\, dV' = -\nabla \oint_S \frac{\mathbf{J}}{R} \cdot d\mathbf{S}' \qquad (6.42)$$

by using the divergence theorem, where S is a closed surface surrounding V.† Since $\mathbf{J}$ is a stationary current and confined to a finite region of space, we may choose S so large that all currents lie within and in particular so that $\mathbf{J} \cdot d\mathbf{S}'$ equals zero on the surface S. Hence the integral vanishes as stated.

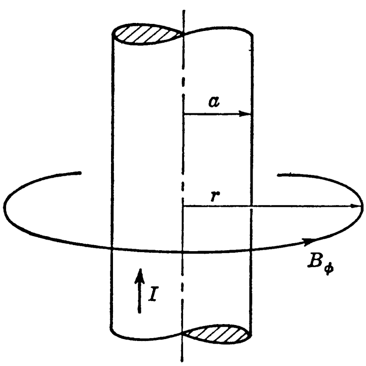

FIG. 6.17. An infinitely long coaxial line.

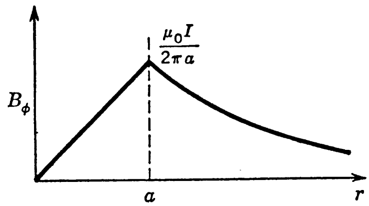

FIG. 6.18. Field B_ϕ as a function of radial distance from center of wire.

Equation (6.41) is Ampère's circuital law in differential form. By applying Stokes' law, an integral form of this law may be obtained. If we integrate $\nabla \times \mathbf{B}$ over a surface S bounded by a closed contour C and use Stokes' law, we get

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_S \mu_0 \mathbf{J} \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{l} \qquad (6.43)$$

This equation states that the line integral of $\mathbf{B} \cdot d\mathbf{l}$ around any closed contour C is equal to μ_0 times the total net current passing through the contour C. The law is particularly useful in solving magnetostatic prob-

† Note that the divergence theorem can be applied only to the second integral in (6.42), where the variables of the differential operator and of integration are the same.

lems having cylindrical symmetry, as the following examples will demonstrate.

Example 6.6. Field from an Infinite Wire of Finite Radius. Consider an infinite wire of radius a with total current I (Fig. 6.17). The current density J is equal to $I/\pi a^2$ and uniform over the cross section of the wire. From symmetry considerations the field $\mathbf{B}$ has only a component B_ϕ, which is a function of r only. Using Ampère's circuital law (6.43) and integrating around a circular contour of radius r gives

$$\oint B_\phi\, dl = \int_0^{2\pi} B_\phi r\, d\phi = \mu_0 \int_0^r \int_0^{2\pi} Jr\, d\phi\, dr$$

$$= \frac{\mu_0 I}{\pi a^2} \int_0^{2\pi} \int_0^r r\, d\phi\, dr \qquad r \le a$$

or

$$B_\phi = \frac{\mu_0 I r}{2\pi a^2} \qquad r \le a$$

For $r \geq a$, the total current enclosed is I; so

$$\int_0^{2\pi} B_\phi r\, d\phi = \mu_0 I$$

or

$$B_\phi = \frac{\mu_0 I}{2\pi r} \qquad a \le r$$

A plot of the intensity of B_ϕ as a function of r is given in Fig. 6.18.

Example 6.7. Magnetic Field in a Coaxial Line. Consider an infinitely long coaxial line consisting of an inner conductor of radius a, an outer conductor of inner radius b, and thickness t. A current I flows along the inner conductor, and a return current $-I$ along the outer conductor, as in Fig. 6.19.

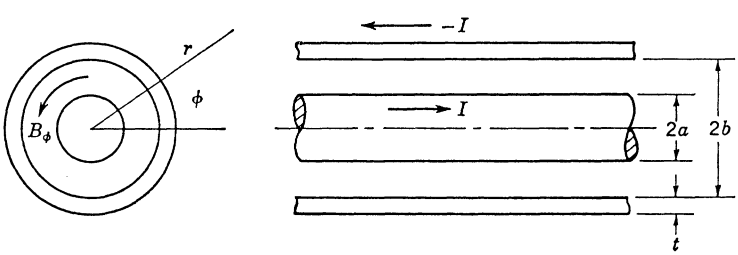

FIG. 6.19. An infinitely long coaxial line.

In the region $r \leq b$ the solution for B_ϕ is the same as in Example 6.6:

$$B_\phi = \frac{\mu_0 I r}{2\pi a^2} \qquad r \le a$$

$$B_\phi = \frac{\mu_0 I}{2\pi r} \qquad a \le r \le b$$

In the region $b \leq r \leq b + t$ we have

$$\int_0^{2\pi} B_\phi r \, d\phi = \mu_0 I - \frac{\mu_0 I}{\pi[(b+t)^2 - b^2]} \int_b^r \int_0^{2\pi} r \, d\phi \, dr$$

or

$$2\pi r B_\phi = \mu_0 I - \frac{\mu_0 I (r^2 - b^2)}{(b+t)^2 - b^2}$$

since the current density in the outer conductor is $I/\pi[(b+t)^2 - b^2]$. Hence

$$B_\phi = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I (r^2 - b^2)}{2\pi r[(b+t)^2 - b^2]} \qquad b \leq r \leq b + t$$

For $r \geq b + t$ the field B_ϕ is zero, since no net current is enclosed by the contour of integration. The above expression for B_ϕ is seen to vanish when r is placed equal to $b + t$. A plot of B_ϕ as a function of r is given in Fig. 6.20.

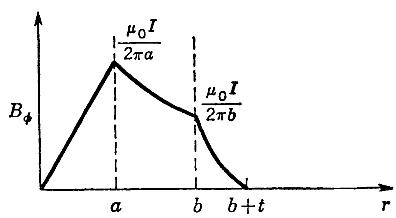

FIG. 6.20. The field B_ϕ as a function of r in a coaxial line.

A word in retrospect on the subject of vector fields. By definition, the vector field of a physical quantity is the totality of all points in a given region for which the direction and magnitude of the quantity are specified. Physically realizable fields are given by vector functions of position which are mathematically well behaved. To the list of vector fields given in Chap. 1, we may now add the fundamental field of electrostatics **E** and magnetostatics **B**.

The formal mathematical description of a vector field gives no insight into its physical properties. This means that an **E** or **B** field could also be thought of as representing a hydrodynamic velocity field, and vice versa. The hydrodynamic analogy was introduced in Chap. 1 in several places in order to develop some "feel" for the abstract vector calculus. We should like to follow this a bit further, but would like it clearly understood that only an analogy is being depicted. Other analogies could be formulated; indeed the reader may be satisfied with no analogy at all.

We plan to represent an arbitrary vector field by assigning a proportional value of velocity to each corresponding point in an infinite volume of incompressible fluid. Thus the condition that the fluid is at rest corresponds to a null field. There are two ways whereby we may bring about motion in the fluid and hence correspondingly set up a field. The simplest way is to "stir things up"; we could do this, for example, with a paddle wheel. But to allow for greatest generality we can think of the paddle wheel as infinitesimal since this permits us to synthesize a more arbitrary "stirrer" by means of an aggregate of paddle wheels (vortex source) whose direction and magnitude vary with position. Thus if a large paddle wheel is used to set the fluid in motion, i.e., create a vector field, this macroscopic source can be synthesized by an appropriate summation of infinitesimal vortex sources. Since the curl of the vector field is a measure of how effectively the source has stirred things up, it is an appropriate measure of the source strength.

In the case of the magnetostatic field we determined that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

which establishes $\mu_0\mathbf{J}$ as a vortex source. The existence of a current density in space may be thought of as causing a **B** field to exist by "stirring up" of the media.

A second way of causing fluid motion is to inject fluid or to remove fluid. If the total source plus sink is zero, then the net amount of fluid remains constant. Point sources refer to an idealization of sources where the fluid enters in a spherically symmetric uniform flow from a mathematical point. Similarly, a negative point source (or sink) removes fluid in an analogous pattern. While the over-all quantity of fluid is constant, there can be a net positive or negative flow in a limited region, and this net amount is proportional to the sum of sources within the region. The divergence of the vector function is a measure of the source strength at each point, where the source is considered as being distributed throughout a volume. In electrostatics the **E** field arises from divergence-producing types of sources and is given by the formula

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

In the most general case the fluid may be set in motion, hence represent an arbitrary field, by a combination of sources such as described above. In this case the vector function has a nonzero value of both divergence and curl in at least some region of space. An example of such a field is the **D** field in electrostatics in the presence of dielectric materials, with permittivity that is a function of the coordinates. For in this case

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \times \mathbf{D} &= \nabla \times (\epsilon_0 \mathbf{E}) + \nabla \times \mathbf{P} \\ &= \nabla \times \mathbf{P} \end{aligned}$$

The true charge causes an outflow of **D** from their positions, while the source $\nabla \times \mathbf{P}$ acts to stir up the **D** field.

6.7. Differential Equation for Vector Potential

In electrostatics we showed that the scalar potential Φ was related to the charge sources that produce it by means of the following equation:

$$\Phi = \frac{1}{4\pi\epsilon_0} \int \rho \, dV'$$

We determined further that the scalar potential Φ satisfied the following partial differential equation (Poisson's):

$$\nabla^2\Phi = \frac{-\rho}{\epsilon_0}$$

so that the integral formulation could also be considered as a solution to Poisson's equation.

In magnetostatics we have found, analogously, that the vector potential $\mathbf{A}$ is related to the current sources through the following expression:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \, dV'}{R}$$

If we consider the rectangular components of this equation, that is,

$$A_x = \frac{\mu_0}{4\pi} \int \frac{J_x \, dV'}{R}$$
$$A_y = \frac{\mu_0}{4\pi} \int \frac{J_y \, dV'}{R}$$
$$A_z = \frac{\mu_0}{4\pi} \int \frac{J_z \, dV'}{R}$$

then each is a scalar equation of precisely the type dealt with in electrostatics. Then by analogy it is clear that each component must satisfy a Poisson equation with the corresponding current component as a source; that is,

$$\nabla^2 A_x = -\mu_0 J_x \qquad \nabla^2 A_y = -\mu_0 J_y \qquad \nabla^2 A_z = -\mu_0 J_z$$

If, now, each equation is multiplied by the corresponding unit vector and all three summed, we obtain the following vector Poisson equation:

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

This result can be obtained in a more formal mathematical way. If we take the double curl of $\mathbf{A}$, which is equal to the curl of $\mathbf{B}$, and use (6.41) to replace $\nabla \times \mathbf{B}$ by $\mu_0\mathbf{J}$, we get $\nabla \times \nabla \times \mathbf{A} = \nabla \times \mathbf{B} = \mu_0\mathbf{J}$. Expanding the curl-curl operator gives

$$\nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \tag{6.44}$$

Since $\mathbf{A}$ is given by $\mathbf{A} = (\mu_0/4\pi)\int(\mathbf{J}/R)\,dV'$, the divergence of $\mathbf{A}$ is given by $\nabla\cdot\mathbf{A} = (\mu_0/4\pi)\int\nabla\cdot(\mathbf{J}/R)\,dV'$. This latter integral occurred, and was shown to vanish, in the derivation of (6.41). It follows then that $\nabla\cdot\mathbf{A} = 0$ and (6.44) reduces to

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{6.45}$$

which is an alternative proof that $\mathbf{A}$ is a solution of the vector Poisson equation.

We can also confirm that

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{R}\,dV' \tag{6.46}$$

is a particular integral of (6.44). This can be established by using the singularity property of $\nabla^2(1/R)$, as was done to obtain (6.41). The details are left as a problem. To the particular solution (6.46) may be added any solution to the homogeneous equation $\nabla^2\mathbf{A} = 0$ as dictated by the boundary conditions which $\mathbf{A}$ must satisfy.

Gauge Transformation

In the earlier work $\mathbf{A}$ was defined by (6.11), as a consequence of which $\nabla\cdot\mathbf{A} = 0$ and $\nabla\times\mathbf{A} = \mathbf{B}$. If, alternatively, $\mathbf{A}$ was simply specified by the requirement that $\nabla\times\mathbf{A} = \mathbf{B}$, then there is a certain arbitrariness in the choice of $\mathbf{A}$. We could equally well use a new vector potential $\mathbf{A}'$ which differs from $\mathbf{A}$ by the addition of the gradient of a scalar function Φ, since $\nabla\times\nabla\Phi = 0$. Thus let

$$\mathbf{A}' = \mathbf{A} + \nabla\Phi \tag{6.47}$$

The field $\mathbf{B}$ is given by $\mathbf{B} = \nabla\times\mathbf{A}' = \nabla\times\mathbf{A} + \nabla\times\nabla\Phi = \nabla\times\mathbf{A}$ and is invariant to such a transformation to a new potential $\mathbf{A}'$. The transformation (6.47) is called a gauge transformation, and the invariant property of $\mathbf{B}$ is known as gauge invariance.

In view of the gauge invariance of $\mathbf{B}$ it is always possible to introduce a gauge transformation, as in (6.47), to make the new potential $\mathbf{A}'$ have zero divergence in the case when $\mathbf{A}$ does not have zero divergence. It is only necessary to choose Φ so that

$$\nabla\cdot\mathbf{A}' = 0 = \nabla\cdot\mathbf{A} + \nabla^2\Phi \tag{6.48}$$

Consequently, we can always work with a vector potential which has zero divergence if we wish, or on the other hand, if more convenient, we can choose a vector potential with nonzero divergence. In the general case $\mathbf{A}$ is then a solution of (6.44) rather than (6.45).

From another point of view we note that in defining the vector potential $\mathbf{A}$, only its curl has been specified; that is, $\nabla\times\mathbf{A} = \mathbf{B}$. From the Helmholtz theorem we understand that this does not completely specify the vector $\mathbf{A}$; in fact, the divergence of $\mathbf{A}$ is completely at our disposal. If we choose as a fundamental relation $\mathbf{A} = (\mu_0/4\pi)\int(\mathbf{J}/R)\,dV'$, then, as we have seen, we are inherently specifying $\nabla\cdot\mathbf{A} = 0$. This condition is usually a satisfactory one, but as pointed out above, it is not necessary. We could think of (6.45) as arising from a choice of integration constant equal to zero. This is equivalent to establishing an arbitrary reference potential, just as was discussed in the electrostatic case.

Chapter 6

6.1. Use the Biot-Savart law [Eq. (6.5)] to find the field **B** set up by two infinitely long line currents located at $x = \pm 1$, $y = 0$ and parallel to the z axis. The currents flowing in the line sources are I and $-I$.

6.2. For the line sources in Prob. 6.1, find an equation for the lines of flux and show that these are the same as the constant-potential contours around two line charges of opposite sign.

6.3. Use Eq. (6.14) to find the field **B** at the center of a square current loop with sides d and current I.

6.4. Consider the rectangular U-shaped conductor illustrated. The circuit is completed by means of a sliding bar. When a current I flows in the circuit, what is the force acting on the sliding bar? When $a = 4$ centimeters, $b = 10$ centimeters, and $I = 5$ amperes, what is the value of the force?

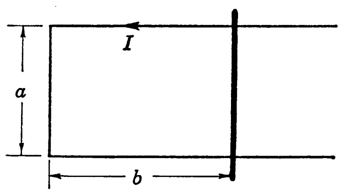

FIG. P 6.4

6.5. Consider two square loops with sides d and equal currents I. One loop is located a distance h above the other loop, as illustrated. Find the force acting on one loop due to the other loop.

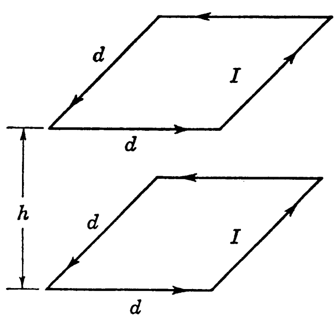

FIG. P 6.5

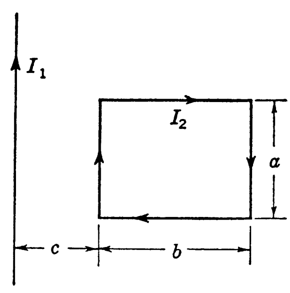

FIG. P 6.6

6.6. A rectangular loop is located near a current line source as illustrated. Find the force acting on the loop.

6.7. Find B at any point along the axis of a circular current loop of radius a and current I.

6.8. A solenoid of length $L \gg a$, where a is the radius, has n turns per meter. A current I flows in the winding. Find the field **B** along the axis.

6.9. A regular polygon of N sides has a current I flowing in it. Show that at the center

$$B = \frac{\mu_0 N I}{2\pi d} \tan \frac{\pi}{N}$$

where d is the radius of the circle circumscribing the polygon. Show that as N becomes large, the result reduces to that called for in Prob. 6.7.

6.10. Use Ampère's circuital law to find the field due to the two line sources specified in Prob. 6.1.

6.11. A z-directed current distribution is given by

$$J_z = r^2 + 4r \qquad r \leq a$$

Find **B** by means of Ampère's circuital law.

6.12. The vector potential due to a certain current distribution is given by

$$\mathbf{A} = x^2y\mathbf{a}_x + y^2x\mathbf{a}_y - 4xyz\mathbf{a}_z$$

Find the field **B**.

6.13. A current distribution is given by

$$\mathbf{J} = \mathbf{a}_z J_0 r \qquad r \leq a$$

where r is the radial coordinate in a cylindrical coordinate system. Find the vector potential **A** and the field **B**. Also find **B** directly by using Ampère's circuital law.

HINT: Solve the differential equation for **A** in cylindrical coordinates in the two regions $r < a$ and $r > a$. The arbitrary constants of integration may be found from the condition that **A** is continuous at $r = a$, equals zero at $r = 0$, and for $r \to \infty$ must be asymptotic to $C \ln r$, where C is a suitable constant.

6.14. A square loop with sides d and current I_1 is free to rotate about the axis illustrated. If the plane of the loop makes an angle θ with respect to an infinite line current I_2, find the torque acting to rotate the loop.

HINT: Consider the loop to be made up of infinitesimal dipoles of moment $d\mathbf{M} = I_1\, d\mathbf{S}$.

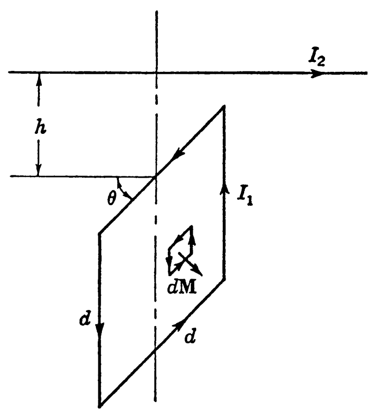

FIG. P 6.14

6.15. Show that Eq. (6.46) for **A** is a solution of (6.45) by substituting (6.46) into (6.45) and using the singularity property of $\nabla^2(1/R)$.

6.16. Given a current loop of arbitrary shape with current magnitude I. Show that (6.26) correctly gives the magnetic field if the magnetic moment is defined as in (6.27).

HINT: First find the vector potential **A** starting with

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|}$$

where **r** is the position vector of the field point and **r**′ that of the source point. Make use of the approximation

$$|\mathbf{r} - \mathbf{r}'|^{-1} = (r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}')^{-1/2} \approx \frac{1}{r}\left(1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{2}\right)$$

and show that

$$d\mathbf{r}' \, (\mathbf{r}' \cdot \mathbf{r}) = \tfrac{1}{2}(\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} + \tfrac{1}{2}d[\mathbf{r}'(\mathbf{r} \cdot \mathbf{r}')]$$

to eliminate all terms except

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\left(\tfrac{1}{2}\oint \mathbf{r}' \times d\mathbf{r}'\right) \times \frac{\mathbf{r}}{r^3}$$

Note that $\frac{1}{2}\oint \mathbf{r}' \times d\mathbf{r}'$ equals the vector area of the loop and that the integral of $d[\mathbf{r}'(\mathbf{r} \cdot \mathbf{r}')]$ around a closed loop is zero since the latter is a complete differential.