CHAPTER 5

STATIONARY CURRENTS

In the previous chapters the characteristics of an electrical conductor were noted, namely, that it was a repository of free electronic charge which would readily move under the influence of an applied field. A particular consequence of this, that the electric field within a conductor must be zero in the presence of an electrostatic field, has already been described.

In this chapter we look into the conditions required for the production of steady current flow in conductors and for a description of the properties of current flow fields. We could take advantage of the existence of a large body of knowledge dealing with the flow of current in electric circuits that is available in electric circuit theory. We prefer, however, to describe current flow in terms of an appropriate electric field. The field-theory approach may then be related to the circuit approach.

In the region external to batteries, the field producing a current flow is the electrostatic field. Since the current density is linearly related to the electric field, an interesting duality between the current flow field and the displacement flux exists. This duality may often be made use of in the solution of current flow problems. In particular, we shall show that there exists a simple relationship between the capacitance and resistance between two electrodes.

Many current flow problems cannot be solved analytically. Therefore a discussion of flux-plotting techniques and experimental techniques, such as the electrolytic tank, is included in the latter part of this chapter.

5.1. Ohm's Law

In the conducting medium it is found experimentally that the current is related to the electric field by the following expression:

$$\mathbf{J} = \sigma \mathbf{E} \tag{5.1}$$

In this equation σ is the conductivity of the medium in mhos per meter and $\mathbf{J}$ is the current density in amperes per square meter. The past chapters have dealt with vector fields where, for conceptual reasons, we interpreted them in terms of flow (flux) fields. The current flow field, however, truly involves flow, and $\mathbf{J}$ represents the quantity of coulombs flowing across a unit cross-sectional area per second. As usual, to calculate total current flow across a surface, the following surface integral must be evaluated:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{5.2}$$

The phenomenon of conduction in a metal can be considered from an atomic viewpoint, in which case a fundamental understanding of the dependence of σ on atomic structure, impurities, and temperature can be developed. It will be sufficient for our purposes, however, to have in mind a very simple physical model. We may think of the conductor as composed of a lattice of fixed positive ions containing an electron gas free to move about. Ordinarily, these free electrons are in a state of random motion because of their thermal energy. The space-time-average charge density, however, is zero. Conduction arises from the drift of electrons because of the action of an applied electric field. Except for an initial transient, the electron velocity reaches a steady-state value when the accelerating force of the applied field is exactly balanced by the scattering effect of electron collisions with the lattice. These collisions may also be viewed as the mechanism whereby some of the energy of the electrons, hence of the field, is dissipated as heat. At equilibrium the current density at any point is simply the electron charge density at the point times its drift velocity. Thus it can be shown that the time-average drift velocity is†

$$\mathbf{v} = -\frac{e\mathbf{E}\lambda}{2mv_0} \tag{5.3}$$

where λ is the mean free path of the electrons, and v_0 the mean thermal velocity. Then, if the density of charge is N electrons per cubic meter,

$$\mathbf{J} = -Ne\frac{e\mathbf{E}\lambda}{2mv_0} = -\frac{Ne^2\lambda\mathbf{E}}{2mv_0} \tag{5.4}$$

This expression reveals the linear relation between current density and field and also relates the conductivity to the atomic quantities. Note that, by convention, positive current is associated with the flow of positive charge.

Equation (5.1) implies that conduction is both linear and isotropic.

† In obtaining this expression it is assumed that the scattering of electrons by the heavy-metal atoms occurs in a completely random manner so that the average velocity after collision is zero. Consequently, the average drift velocity is that which is acquired between collisions under the action of the electric field force, that is, $v = \frac{1}{2}at$, where $a = eE/m$ is the acceleration and $t = \lambda/v_0$ is the time between collisions. Ordinarily, the thermal velocity $v_0 \gg v$; hence the dependence of t on v_0 only.

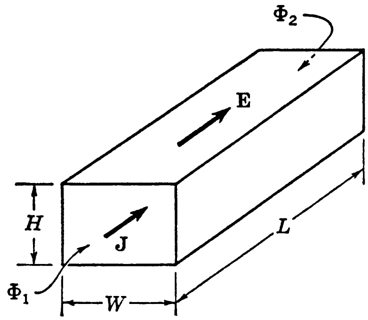

FIG. 5.1. Uniform current flow in a rectangular bar.

This is not always true. However, for metals, under a wide range of current densities it does apply. We shall assume that (5.1) correctly relates the current density and electric field in conductors.

Equation (5.1) is a point relationship and is true even if σ is a function of the coordinates. For a homogeneous body with uniform current, it is relatively easy to find the total current and to obtain a relationship between it and the applied field. Such a formulation would be desirable in a circuit analysis. A simple case is illustrated in Fig. 5.1, where a uniform axial applied field exists in a conductor of rectangular cross section. The total current that flows is

$$I = JWH = \sigma EWH \tag{5.5}$$

If we assume, for the moment, that over the extent of the conductor $\mathbf{E}$ is conservative, then $\mathbf{E} = -\nabla\Phi$, and the difference of potential between the ends of the conductor, V, is

$$V = \Phi_1 - \Phi_2 = EL \tag{5.6}$$

Combining (5.5) and (5.6) gives

$$I = \frac{\sigma WH}{L} V = GV \tag{5.7}$$

where

$$G = \frac{\sigma WH}{L}$$

is the total conductance of the parallelepiped.

It is more common to specify the resistance of the conductor. This is the reciprocal of the conductance and so may be written

$$R = \frac{L}{\sigma WH} = \frac{\rho L}{WH} \tag{5.8}$$

where ρ (not to be confused with charge density), the resistivity in ohm-meters, is the reciprocal of the conductivity; i.e.,

$$\rho = \frac{1}{\sigma} \tag{5.9}$$

From (5.6) and (5.7) we get the well-known statement of Ohm's law as applied to the macroscopic circuit:

$$V = IR \tag{5.10}$$

5.2. Nonconservative Fields—EMF

We should like to produce a steady current, and we inquire now into methods whereby this may be accomplished. As we know from (5.1), it will be necessary to start with an electric field. So far, however, we have considered only the production of an electrostatic field by stationary charges. Will this suffice?

Suppose we consider the electrostatic field set up within the parallel-plate capacitor of Fig. 5.2, into which we now insert the conductor of Fig. 5.1. At this instant the conductor finds itself in a uniform axial **E**

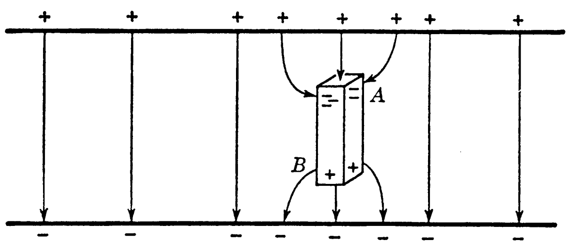

FIG. 5.2. A rectangular conducting bar placed in the electrostatic field between two charged plates.

field, and consequently a uniform current $I = \sigma WHE$ flows. But this current is nothing more than the movement of charge, and as time increases, it must be that negative charge accumulates at A while B becomes positively charged. These charges represent secondary sources of field, and it is not hard to see that they set up a field within the conductor that opposes the primary (capacitor) field. Actually, we have already considered this kind of problem and were led to the conclusion that the total field within the conductor would soon reach the equilibrium value of zero. In that case, though, the current would stop. It seems that an electrostatic field is not capable of setting up steady currents.

Further consideration explains why an electrostatic field alone cannot be the cause of steady currents. Consider an electron which is an element of a steady conduction stream. Since steady-state conditions exist, it must make a complete circuit and return to an arbitrary starting point, thence to repeat the circuit ad infinitum. In any such circuit the electron gives up energy to the conductors in the form of heat, as a consequence of the finite resistance of the conductors. The energy, however, comes ultimately from the field, since this is the basis for the current flow. But an electrostatic field is conservative; it is not capable of giving up energy indefinitely. As a matter of fact, if we assume the field to remain unchanged, as must be true where steady current exists, then an electron making a closed circuit in an electrostatic field gains no net energy from the field. Clearly, another source of field is required for the maintenance of steady currents, and this field must be nonconservative.

The action of a chemical battery may be interpreted from a field standpoint as producing such a nonconservative field $\mathbf{E}'$. In general, an electrostatic field will also be created by a battery as a result of the accumulation, at the battery terminals and elsewhere in the circuit, of stationary (capacitor) charge. Designating the latter field by $\mathbf{E}$, the total field is then

$$\mathbf{E}_t = \mathbf{E} + \mathbf{E}' \tag{5.11}$$

Equation (5.1) applies to the total field, so that

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}') \tag{5.12}$$

This equation holds at all points, but it is important to note that $\mathbf{E}'$ may be zero at some points in the circuit; e.g., outside the battery $\mathbf{E}'$ is zero but $\mathbf{E}$ is not. If we integrate (5.11) over a closed circuit in which steady current flows and make use of (5.12) and the conservative nature of $\mathbf{E}$ (that is, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$), then

$$\oint_C \mathbf{E}_t \cdot d\mathbf{l} = \oint_C \mathbf{E}' \cdot d\mathbf{l} = \mathcal{E} = \oint_C \frac{\mathbf{J} \cdot d\mathbf{l}}{\sigma} \tag{5.13}$$

where $\mathcal{E}$ is a measure of the strength of the nonconservative source. It is called the emf, an abbreviation of electromotive force. The current that flows depends on the conductivity, geometry, and the value of $\mathcal{E}$. If the current density is uniform over a constant cross section A, then we have

$$\mathcal{E} = I \oint \frac{dl}{\sigma A} = IR \tag{5.14}$$

In (5.14)

$$R = \oint \frac{dl}{\sigma A} \tag{5.15}$$

is the total circuit resistance and is a simple extension of the result given by (5.8).

Under open-circuit conditions (consider a battery with no circuit connections) an electrostatic field exists everywhere because of the accumulation of charge on the battery terminals. From a field standpoint the chemical action of the battery may be described by postulating a nonconservative field within the battery which just neutralizes the electrostatic field there. As viewed by a test charge, it is possible to acquire energy in moving from the positive to the negative terminal external to the battery, but by completing the circuit through the battery, where no field exists, this energy is not returned to the field (as would be the case in a purely electrostatic field). The test charge thus makes a complete circuit with a net accumulation of energy. With an actual circuit and real batteries, the accumulated energy is simultaneously dissipated as heat. The electron is capable of making repeated circuits, hence constituting a steady current.

With special arrangements, a nonconservative field can be set up so that the energy accumulated in a complete circuit by a unit of charge is available as kinetic energy. Repeated circuits continue to add energy to the charge, yielding the high-energy particles produced by devices such as the cyclotron, betatron, etc.

Making use of (5.11) under open-circuit conditions and integrating through the battery from terminals 1 to 2, we have

$$-\int_1^2 \mathbf{E} \cdot d\mathbf{l} = +\int_1^2 \mathbf{E}' \cdot d\mathbf{l} = \mathcal{E} \tag{5.16}$$

Thus the total emf just equals the open-circuit electrostatic voltage between the battery terminals. In general, points 1 and 2 may be arbitrarily chosen provided that the line integral just traverses the entire nonconservative field. Then the open-circuit electrostatic voltage between those points also equals the total emf of the source.

In the region external to that containing the nonconservative field, e.g., external to the battery, only an electrostatic field exists. Since the external region contains only a conservative field, it is possible to derive this field from the gradient of a scalar potential. This accounts for our ability to discuss d-c circuits in terms of potentials and potential difference, in spite of the nonconservative nature of the field as a whole. By describing the line integral of $\mathbf{E}_t$ through a battery from negative to positive terminal as a voltage rise (an increase in potential) equal to the emf of the battery, the multivalued nature of potential energy in a nonconservative field is avoided,† and one may now state the Kirchhoff loop equation

$$\text{Potential change over a closed loop} = 0 \tag{5.17a}$$

or

$$\text{emf} = \Sigma IR \text{ (over a closed loop)} \tag{5.17b}$$

5.3. Conservation of Charge

Since current consists of the flow of charge, a relationship between the two should be available. This is indeed the case. If we consider a volume V, then the net flow of current into this volume must be accom-

† As an analogy, consider the polar angle of a vector rotating in a counterclockwise direction. The angle increases from zero to 2π, thence from 2π to 4π, 4π to 6π, etc. But if we make a cut along $\theta = 0°$ and agree every time we cross it in a counterclockwise sense to subtract 2π, then we avoid the multivalued nature of the angle. The battery is analogous to the cut, and its strength is not -2π but whatever its emf might be.

panied by an increase of charge within V. We can express this as

$$-\int_S \mathbf{J} \cdot d\mathbf{S} = \frac{\partial}{\partial t} \int_V \rho \, dV \tag{5.18}$$

where the left-hand side of (5.18) gives the net inflow of current (coulombs per second) and the right-hand side represents the net rate of increase of total charge (coulombs per second). In (5.18) the surface S bounds the volume V, of course.

Using Gauss' theorem, (5.18) may be transformed to

$$\int_V \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dV = 0 \tag{5.19}$$

Since this must be true regardless of the choice of V, the integrand must itself be zero and we are led to the differential form of the law for conservation of charge.

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5.20}$$

This equation is commonly referred to as the continuity equation also. Where steady currents are involved, then of course $\partial \rho / \partial t = 0$. In this case

$$\nabla \cdot \mathbf{J} = 0 \tag{5.21}$$

That is, for "stationary currents," the current density is solenoidal. We are restricting our attention in this chapter to steady currents and have already noted that such currents must form closed loops.

If (5.21) is applied to a volume that contains a junction of conductors in a network, then the second of the Kirchhoff equations results, namely,

$$\sum_i I_i = 0 \tag{5.22}$$

This equation states that the algebraic sum of the currents flowing into (or out of) a terminal is zero.

5.4. Relaxation Time

If a charge distribution is placed within a conducting body, the charge will move to the surface and distribute itself in such a way that zero field exists within and tangent to the conductor surface. The length of time required for this process to essentially take place is called the relaxation time. Whether this time is measured in millimicroseconds or in days is, of course, extremely important. A quantitative evaluation of this characteristic time is presented below.

Consider a homogeneous conducting region with a permittivity ϵ and a conductivity σ. From the divergence equation for $\mathbf{D}$ we have $\nabla \cdot \mathbf{E} = \rho/\epsilon$

when ϵ is constant. Within the conductor $\mathbf{J} = \sigma\mathbf{E}$, so that (5.20) can be written as

$$\nabla \cdot \mathbf{J} = \nabla \cdot \sigma\mathbf{E} = -\frac{\partial \rho}{\partial t} \tag{5.23}$$

From (5.23) and replacing $\nabla \cdot \mathbf{E}$ by ρ/ϵ, we obtain

$$\frac{\rho}{\epsilon} = -\frac{1}{\sigma}\frac{\partial \rho}{\partial t} \tag{5.24}$$

Thus

$$\frac{\sigma}{\epsilon}\int_0^t dt = -\int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$$

and hence

$$\rho(x,y,z,t) = \rho_0(x,y,z)e^{-\sigma t/\epsilon} \tag{5.25}$$

where ρ_0 is the initial charge density when $t = 0$. The relaxation time τ is defined as

$$\tau = \frac{\epsilon}{\sigma} \tag{5.26}$$

and is the value of elapsed time required for the initial charge distribution to decay to $1/\epsilon$ of its initial value.

Table 5.1 gives the value of τ for several common materials. We note the extremely short duration for good conductors and the relatively large value for insulators. As a matter of fact, it is the relaxation time itself which truly measures what we choose to call a conductor or an insulator. When τ is extremely short compared with a measurement time, the material is considered as a "conductor"; however, if τ is very long compared with the duration of a measurement, we consider the material to behave like an "insulator." Note that our prior assumption of zero charge and field within a metallic conductor is amply justified by the numerical results in Table 5.1.

TABLE 5.1. RELAXATION TIMES FOR SOME COMMON MATERIALS

Material	*Relaxation time* τ
Copper	1.5×10^{-19} sec
Silver	1.3×10^{-19} sec
Sea water	2×10^{-10} sec
Distilled water	10^{-6} sec
Fused quartz	10 days

5.5. Resistance of Arbitrary Shaped Conductors

Equation (5.15) gives the total resistance of a uniform cylindrical conductor. For a homogeneous body of conductivity σ, but of an arbitrary shape, a more general formula is required. In order to derive it, we start with the conductor illustrated in Fig. 5.3, which is representative of a generalized shape.

If we visualize a battery connected to the ends of the body, then a current will flow and its density will be nonuniform. For simplicity we take the end surfaces A_1 and A_2 to be equipotentials; this could be assured, for example, by coating these surfaces with a perfect conductor.† Since,

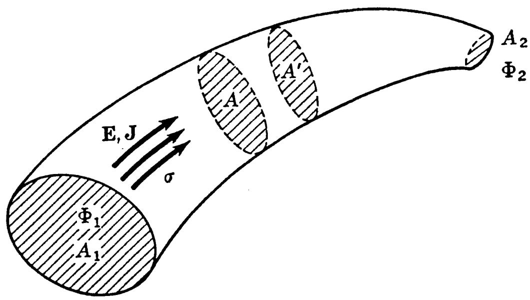

FIG. 5.3. An arbitrary conductor.

as we have already noted, the field external to the battery is conservative and can be derived from a scalar potential,

$$\int_C \mathbf{E} \cdot d\mathbf{l} = \Phi_1 - \Phi_2 \tag{5.27}$$

where C is any path starting at A_1 and terminating at A_2 and $\Phi_1 - \Phi_2$ is the difference of potential between the surfaces A_1 and A_2. If the battery and lead resistances are negligible, then it is also true that

$$\int_C \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} \tag{5.28}$$

where $\mathcal{E}$ is the emf of the battery.

Consider any cross-sectional surface in the conductor such as A or A' in Fig. 5.3. Since the current is solenoidal, the same total current crosses surface A_1, A, A', and A_2. We can evaluate this current over any surface A as given by

$$I = \int_A \mathbf{J} \cdot d\mathbf{S} \tag{5.29}$$

Now $\mathbf{J} = \sigma\mathbf{E}$; so

$$I = \sigma \int_A \mathbf{E} \cdot d\mathbf{S} \tag{5.30}$$

By definition the resistance between the two faces A_1 and A_2 is

$$R = \frac{\Phi_1 - \Phi_2}{I} = \frac{\int_C \mathbf{E} \cdot d\mathbf{l}}{\sigma \int_A \mathbf{E} \cdot d\mathbf{S}} \tag{5.31}$$

† Coating the ends with a material whose conductivity was very much greater than that of the body would suffice.

Although the above formula is rather simple in concept and form, the integrals cannot be evaluated before a detailed solution for the field **E** (or current flow density **J**) has been obtained. For a general shaped conductor this is usually not feasible and one is forced to resort to approximate methods of analysis or experimental methods in order to determine the resistance R.

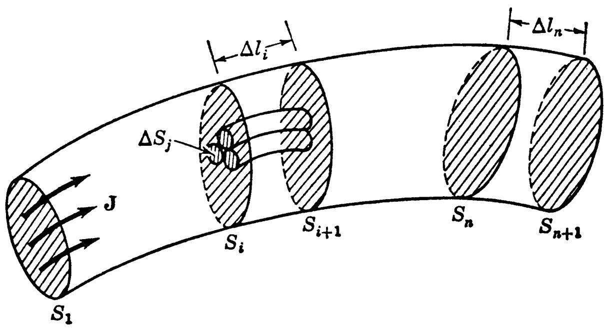

FIG. 5.4. Two equipotential surfaces within an arbitrary current-carrying conductor.

Another expression for R can be formulated that demonstrates the geometrical properties of the resistance more clearly. Again, it is necessary to know the field and current distribution everywhere within the conductor. In Fig. 5.4, a general conductor is illustrated and two equipotential cross-sectional surfaces S_i and S_{i+1} are indicated. Let the potential difference between these surfaces be designated by $\Delta\Phi_i$. The volume between S_i and S_{i+1} may be decomposed into a number of elementary flow tubes of length Δl_i and cross-sectional area ΔS_j. For each small tube the resistance r_j is given by

$$r_j = \frac{\Delta\Phi_i}{\Delta I_j} = \frac{E\,\Delta l_i}{\sigma E\,\Delta S_j} = \frac{\Delta l_i}{\sigma\,\Delta S_j} \tag{5.32}$$

The conductance of this flow tube is

$$g_j = r_j^{-1} = \frac{\sigma\,\Delta S_j}{\Delta l_i} \tag{5.33}$$

Since conductances in parallel add directly, the total conductance between the surfaces S_i and S_{i+1} is

$$\Delta G_i = \sum_j g_j = \sum_j \frac{\sigma\,\Delta S_j}{\Delta l_i} \tag{5.34a}$$

and the corresponding resistance is

$$\Delta R_i = \frac{1}{\displaystyle\sum_j \frac{\sigma\,\Delta S_j}{\Delta l_i}} \tag{5.34b}$$

In general, Δl_i will vary over the cross section since the spacing between the equipotential surfaces is not necessarily uniform. If we break up the whole conductor into n such sections, then (5.34b) is the resistance of the ith section. The total resistance is the series combination of all the ΔR_i and is given by

$$R = \sum_i \frac{1}{\sum_j \frac{\sigma \, \Delta S_j}{\Delta l_i}} \tag{5.35}$$

From (5.35) we can see how the resistance formula may be obtained by passing to the limit ΔS_j and Δl_i approaching zero. In order to obtain

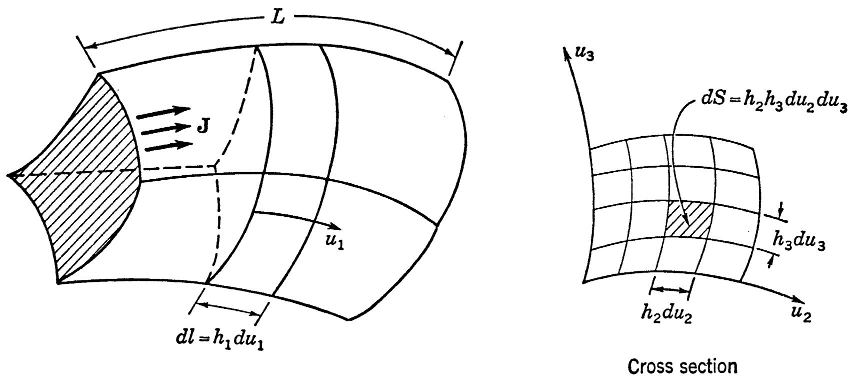

FIG. 5.5. Orthogonal curvilinear coordinates used to describe current flow in a conductor.

a meaningful formula, we have to digress for a moment and introduce a suitable set of curvilinear coordinates to express the variables in. Since $\sigma\mathbf{E} = \mathbf{J} = -\sigma \, \nabla\Phi$ and $\nabla\Phi$ is normal to the constant potential surfaces, we may introduce a curvilinear coordinate u_1 which increases in the direction parallel to the current flow lines and is normal to the constant potential surfaces. Distance dl along the flow lines is then given by $h_1 \, du_1$, where h_1 is a scale factor and will, in general, vary over the cross section of the conductor. Over the constant potential surface we shall assume that two additional orthogonal curvilinear coordinates u_2, u_3 can be introduced in order to measure the cross-sectional area of an elementary flow tube.† The cross-sectional area ΔS_j is now given by $\Delta S_j = h_2 h_3 \, \Delta u_2 \, \Delta u_3$, as in Fig. 5.5. The factors h_2 and h_3 are scale factors introduced so that $h_2 \, du_2$ and $h_3 \, du_3$ are differential lengths in the direction of increasing

† If u_2 and u_3 do not form an orthogonal system, the problem cannot, in general, be solved analytically anyway, so that the restriction is not a serious one from a practical standpoint.

u_2 and u_3, respectively. For (5.35) we may now write

$$R = \sum_i \frac{1}{\sum_j \frac{\sigma h_2 h_3 (\Delta u_2 \, \Delta u_3)_j}{h_1 (\Delta u_1)_i}} = \sum_i \frac{(\Delta u_1)_i}{\sum_j \frac{\sigma h_2 h_3}{h_1} (\Delta u_2 \, \Delta u_3)_j}$$

since u_1, u_2, u_3 are independent variables because of their mutual orthogonality. In the limit we obtain

$$R = \int_0^L \frac{du_1}{\int_S \frac{\sigma h_2 h_3}{h_1} du_2 \, du_3} \tag{5.36}$$

This equation is clearly a function of the geometry of the conductor only. In a later section we present a flux-mapping technique which is essentially a graphical procedure for evaluating the above expression for resistance. The following example will also help to clarify some of the concepts involved.

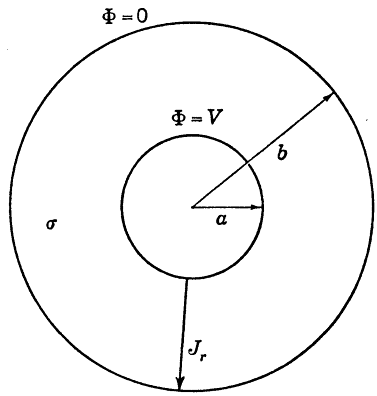

FIG. 5.6. A spherical resistor.

Example 5.1. Resistance of a Spherical Section. Consider two concentric spheres of radii a and b, as in Fig. 5.6. Let the inner sphere be kept at a potential V relative to the outer sphere, and let the medium between the spheres have a conductivity σ. From the work of previous chapters it is clear that the potential Φ is given by

$$\Phi = \frac{ab}{b-a} V \left(\frac{1}{r} - \frac{1}{b} \right) \tag{5.37}$$

since this makes $\Phi = V$ at $r = a$, $\Phi = 0$ at $r = b$, and $\nabla^2 \Phi = 0$. The radial electric field E_r is given by $-\partial\Phi/\partial r$, and hence the current density is

$$J_r = \sigma E_r = \frac{ab}{b-a} V\sigma \frac{1}{r^2} \tag{5.38}$$

The total current is equal to $4\pi a^2$ times the current density at $r = a$ and is given by

$$I = \frac{4\pi ab}{b-a} \sigma V \tag{5.39}$$

The total resistance between the two shells is

$$R = \frac{V}{I} = \frac{b - a}{4\pi ab\sigma} \tag{5.40}$$

The above solution is a direct application of (5.31).

Let us now consider just a portion of this spherical resistor as obtained by lifting out a section contained within a cone of semiangle θ_0, as in Fig. 5.7a. The end surfaces $r = a, b$ are kept at a potential V and 0 as

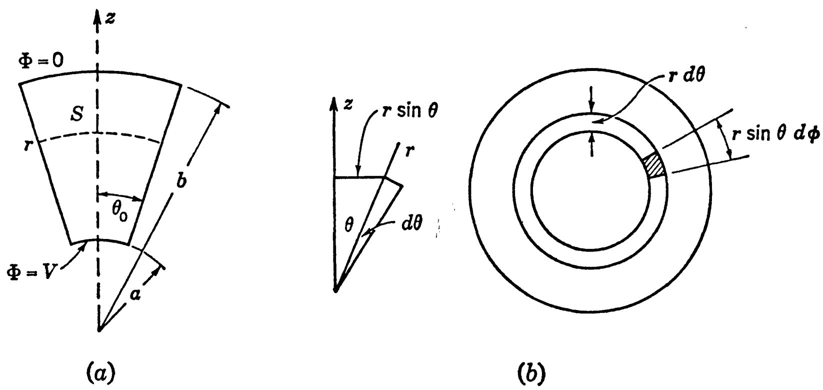

FIG. 5.7. (a) A conical resistor; (b) orthogonal curvilinear coordinates θ, ϕ on an equipotential surface.

before. Consequently, all surfaces r = constant are equipotential surfaces. On an equipotential surface the element of area dS may be described in terms of the spherical coordinates θ and ϕ, as in Fig. 5.7b. The separation between equipotential surfaces is simply dr. Our curvilinear coordinates u_1, u_2, u_3 and scale factors h_1, h_2, h_3 in the present case are

$$u_1 = r,\ h_1 = 1 \qquad u_2 = \theta,\ h_2 = r \qquad u_3 = \phi,\ h_3 = r\sin\theta$$

From (5.36) we obtain the following expression for the resistance:

$$R = \int_a^b \frac{dr}{\int_0^{2\pi}\int_0^{\theta_0} \sigma r^2 \sin\theta\, d\theta\, d\phi} = \int_a^b \frac{dr}{\sigma r^2 2\pi(1 - \cos\theta_0)}$$
$$= \frac{1}{2\pi\sigma(1 - \cos\theta_0)}\,\frac{b - a}{ab} \tag{5.41}$$

If $\theta_0 = \pi$, this result reduces to (5.40), as it should.

The result expressed by (5.41) could have been arrived at in another way also. Let the solid angle subtended by the end surface of the conical resistor be Ω. In the volume between the two spheres $4\pi/\Omega$, such resistors may be placed in parallel. The resistance of any one individual resistor is equal to $4\pi/\Omega$ times the combined total resistance, i.e., equal to $4\pi/\Omega$ times (5.40). For a surface such as that in Fig. 5.7*a*, the solid angle Ω is given by $2\pi(1 - \cos\theta_0)$, and hence (5.41) follows at once. For example, if $\theta_0 = \pi/2$, the resistor of Fig. 5.6 consists of two of the resistors of Fig. 5.7*a* in parallel. Therefore the resistance of a half-spherical section is twice the value given by (5.40). This result is verified at once from (5.41) by placing θ_0 equal to $\pi/2$.

In addition to the two formulas (5.31) and (5.36), the resistance of a circuit may be defined on an energy basis. Only the results are presented here; the derivation is given in Sec. 5.8. For the conductor as a whole, the power dissipated is given by I^2R. The power dissipation is given by one of the following volume integrals also, so that

$$I^2R = \int_V \mathbf{J}\cdot\mathbf{E}\,dV = \int_V \sigma\mathbf{E}\cdot\mathbf{E}\,dV = \frac{1}{\sigma}\int_V \mathbf{J}\cdot\mathbf{J}\,dV \tag{5.42}$$

Once the field has been found, the volume integral may be evaluated and R is then determined by the above equation. This latter method is often the easiest one to formulate.

5.6. Boundary Conditions and Refraction of Current Flow Lines

An examination of the flow of current across an interface between two media of different conductivity, σ_1 and σ_2, reveals that the flow lines are refracted. What happens is analogous to what has already been noted in electrostatics with respect to electric flux lines at a dielectric interface.

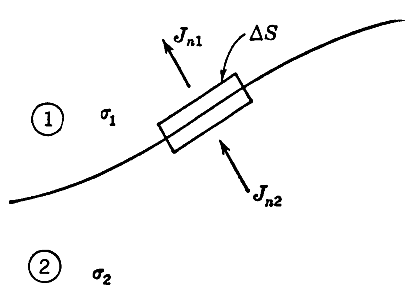

FIG. 5.8. Boundary between two conducting media.

Figure 5.8 shows the interface of medium 1 (conductivity σ_1) and medium 2 (conductivity σ_2). If a coin-shaped surface is considered which has a broad face in medium 1 and a broad face in medium 2, then over the volume V occupied by this surface

$$\int_V (\nabla\cdot\mathbf{J})\,dV = \oint_S \mathbf{J}\cdot d\mathbf{S} = 0 \tag{5.43}$$

Let the surface area of the coin faces be ΔS, and let the thickness of the coin approach zero. Then no contribution to the surface integral in (5.43) comes from the edges. The remainder of the integral can be written

$$(J_{n1} - J_{n2})\,\Delta S = 0$$

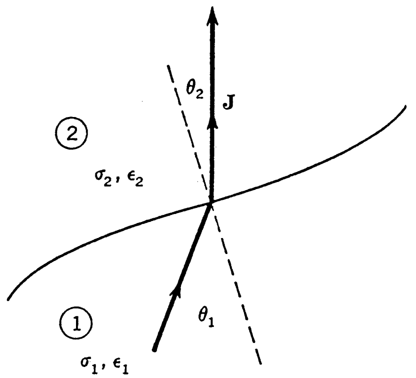

FIG. 5.9. Refraction of current flow lines.

The notation J_{n1} refers to the normal component of **J** at the interface in region 1, and similarly for J_{n2} in region 2. The element of surface ΔS is arbitrary; so we have

$$J_{n1} = J_{n2} \tag{5.44}$$

Since the electric field is conservative throughout the region in question (we assume this region to be outside the nonconservative source), it follows that $\nabla \times \mathbf{E} = 0$. As a consequence, the development in Sec. 3.3, which leads to continuity of tangential **E**, applies in this case also. This result can be stated in terms of the tangential current density as

$$\frac{J_{t1}}{\sigma_1} = \frac{J_{t2}}{\sigma_2} \tag{5.45}$$

By combining (5.44) and (5.45) and noting the definition of θ_1 and θ_2, as illustrated in Fig. 5.9, we may write

$$\tan\theta_1 = \frac{J_{t1}}{J_{n1}} \qquad \tan\theta_2 = \frac{J_{t2}}{J_{n2}} = \frac{\sigma_2 J_{t1}}{\sigma_1 J_{t2}}$$

Therefore

$$\tan\theta_2 = \frac{\sigma_2}{\sigma_1}\tan\theta_1 \tag{5.46}$$

If region 1 is a good conductor and region 2 an insulator, then $\sigma_1 \gg \sigma_2$, and the current leaves the surface in medium 2 at right angles. This corresponds to the requirement that the electric field be normal to the surface of a good conductor.

At the interface of lossy dielectrics, the above boundary condition which holds for the normal component of the current density is, in general, incompatible with the boundary conditions on the normal component of the displacement flux density **D**, for a dielectric material with finite conductivity, unless a layer of surface charge is assumed to exist on the boundary separating the two media. With reference to Fig. 5.9, let the permittivity of the two media be ϵ_1 and ϵ_2. From (5.44) we have $J_{n1} = \sigma_1 E_{n1} = J_{n2} = \sigma_2 E_{n2}$, or

$$\sigma_1 E_{n1} = \sigma_2 E_{n2} \tag{5.47}$$

If a surface layer of charge of density ρ_s exists on the boundary, the boundary condition on D_n gives

$$D_{n2} - D_{n1} = \rho_s$$

or

$$\epsilon_2 E_{n2} - \epsilon_1 E_{n1} = \rho_s \tag{5.48}$$

Only if $\epsilon_2/\epsilon_1 = \sigma_2/\sigma_1$ will ρ_s vanish. Combining (5.47) and (5.48) gives

$$\rho_s = \left(\epsilon_2 - \epsilon_1 \frac{\sigma_2}{\sigma_1}\right) E_{n2} = \left(\epsilon_2 \frac{\sigma_1}{\sigma_2} - \epsilon_1\right) E_{n1} \tag{5.49}$$

During the transient state while the current is building up to its final steady-state value, charge accumulates on the boundary. Once steady-state conditions have been reached, no further accumulation of charge takes place. If we introduce the relaxation time constants $\tau_1 = \epsilon_1/\sigma_1$, $\tau_2 = \epsilon_2/\sigma_2$ and replace σE_n by J_n, we have in place of (5.49)

$$\rho_s = (\tau_2 - \tau_1) J_n \tag{5.50}$$

Some of the practical implications of the above results are presented in the following example.

Example 5.2. Capacitor Filled with Lossy Dielectric Material. For simplicity consider a parallel-plate capacitor with spacing $2d$ and plate area A, as in Fig. 5.10. The region between the plates is filled with two

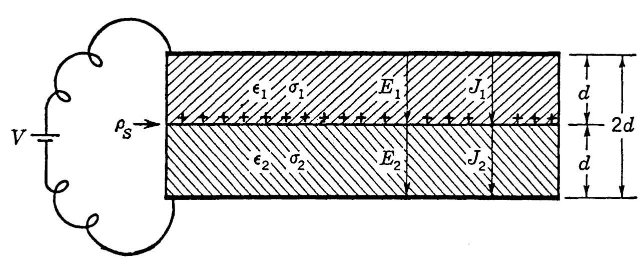

FIG. 5.10. Capacitor filled with lossy dielectric slabs.

lossy dielectric slabs of thickness d and with parameters ϵ_1, σ_1 and ϵ_2, σ_2. A potential V is applied across the plates. When steady-state conditions have been reached, the electric field between the plates must satisfy the following conditions:

$$E_1 d + E_2 d = V \tag{5.51}$$

$$J_1 = \sigma_1 E_1 = J_2 = \sigma_2 E_2 \tag{5.52}$$

$$\rho_s = \epsilon_2 E_2 - \epsilon_1 E_1 \tag{5.53}$$

and consequently

$$E_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \frac{V}{d} \tag{5.54a}$$

$$E_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2} \frac{V}{d} \tag{5.54b}$$

The surface charge density on the boundary separating the two dielectric media can now be found from (5.53) with the aid of (5.54) and is

$$\rho_s = \frac{\epsilon_2\sigma_1 - \epsilon_1\sigma_2}{\sigma_1 + \sigma_2}\frac{V}{d} \tag{5.55}$$

If now we turn our attention to the transient interval during which ρ_s increases to its steady-state value, then (5.52) no longer applies. This is because (5.52) is derived from the continuity equation under stationary conditions. For the time-varying case it is necessary to use (5.20). If this equation is applied to a coin-shaped surface centered at the interface using the, by now, familiar arguments, it is possible to establish the following general boundary conditions:

$$\frac{\partial \rho_s}{\partial t} = J_1 - J_2 = \sigma_1 E_1 - \sigma_2 E_2 \tag{5.56}$$

Note that if $\partial\rho_s/\partial t = 0$, then (5.56) reduces to (5.52). An expression for E_1 and E_2 in terms of ρ_s and V can be obtained from the simultaneous solution of (5.51) and (5.53). Substituting these values into (5.56) yields the following differential equation:

$$\frac{\partial \rho_s}{\partial t} = -\rho_s \frac{\sigma_1 + \sigma_2}{\epsilon_1 + \epsilon_2} + \frac{\epsilon_2\sigma_1 - \sigma_2\epsilon_1}{\epsilon_1 + \epsilon_2}\frac{V}{d} \tag{5.57}$$

The general solution to (5.57) is

$$\rho_s = A \exp\left(-\frac{\sigma_1 + \sigma_2}{\epsilon_1 + \epsilon_2}t\right) + \frac{\epsilon_2\sigma_1 - \sigma_2\epsilon_1}{\sigma_1 + \sigma_2}\frac{V}{d} \tag{5.58}$$

where A is an arbitrary constant. When $t \rightarrow \infty$, (5.58) correctly reduces to the steady-state value already found. The constant A is determined from the initial condition that $\rho_s = 0$ when $t = 0$. Consequently, we finally have

$$\rho_s = \frac{V}{d}\frac{\epsilon_2\sigma_1 - \sigma_2\epsilon_1}{\sigma_1 + \sigma_2}\left[1 - \exp\left(-\frac{\sigma_1 + \sigma_2}{\epsilon_1 + \epsilon_2}t\right)\right] \tag{5.59}$$

If the dielectric conductivity is very small, as is usual, then the time constant in (5.59) will be very large. Suppose that measurements are to be made which involve a lossy dielectric, under d-c conditions. If the duration of the experiment is short compared with the relaxation time $(\epsilon_1 + \epsilon_2)/(\sigma_1 + \sigma_2)$, then (5.59) ensures that the dielectric may be considered to be essentially perfect.

The problem discussed here can be analyzed by setting up an equivalent lumped-parameter circuit and proceeding to the analysis of this circuit by conventional techniques. While no new information can be expected by this procedure, it is of considerable interest since it relates field theory to circuit theory. Accordingly, we present and analyze a circuit structure in Fig. 5.11 which represents the lossy dielectric problem of Fig. 5.10. The conductances G_1, G_2 and capacitances C_1, C_2 are given by

$$R^{-1} = G_1 = \frac{\sigma_1 A}{d} \qquad R_2^{-1} = G_2 = \frac{\sigma_2 A}{d}$$

$$C_1 = \frac{\epsilon_1 A}{d} \qquad C_2 = \frac{\epsilon_2 A}{d}$$

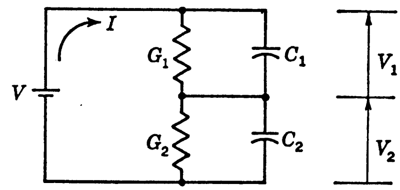

FIG. 5.11. Equivalent circuit of capacitor filled with lossy dielectric.

Under steady-state conditions it is clear that the division in voltage between the two halves of the capacitor is determined by the conductances only. From the equivalent circuit it is seen that for steady-state conditions

$$\frac{V_1}{V_2} = \frac{G_2}{G_1}$$

and

$$V_1 = \frac{G_2}{G_1 + G_2} V \qquad V_2 = \frac{G_1}{G_1 + G_2} V$$

But $E_1 d = V_1$, $E_2 d = V_2$, so that this is just the circuit equivalent of the field relations (5.54).

On the capacitor C_1 a total charge $Q_1 = C_1 V_1$ exists on the upper plate and a total charge $-Q_1$ on the lower plate. Similarly, on C_2 a total charge $Q_2 = C_2 V_2$ exists on the upper plate and a charge $-Q_2$ on the lower plate, under steady-state conditions. On the lower plate of C_1 and the upper plate of C_2, which together represent the boundary surface between the two dielectric slabs, the net total charge is $Q_2 - Q_1$ and is given by

$$Q_2 - Q_1 = C_2 V_2 - C_1 V_1 \tag{5.60}$$

This result is the same as that given by (5.53), since

$$A\rho_s = \frac{\epsilon_2 A}{d} E_2 d - \frac{\epsilon_1 A}{d} E_1 d = C_2 V_2 - C_1 V_1$$

Because of the finite conductivity, the structure of Fig. 5.10 does not behave as a pure capacitor; rather it is a parallel-series combination of resistance and capacitance, as illustrated in Fig. 5.11. The transient build-up of surface charge ρ_s may be found from a study of the transient behavior of the equivalent circuit. A transient analysis is readily carried out if it is assumed that the internal resistance of the battery is negligible (this assumption is really not valid, and its consequences will be pointed out later). Referring to the equivalent circuit, it is seen that the current flowing through the C_1, G_1 combination is

$$I = G_1(V - V_2) + C_1 \frac{d}{dt}(V - V_2)$$

The same current flows through the C_2, G_2 combination; so we have

$$G_1(V - V_2) - C_1 \frac{dV_2}{dt} = G_2 V_2 + C_2 \frac{dV_2}{dt}$$

since dV/dt is zero because V is constant. This equation may be written as

$$\tau \frac{dV_2}{dt} + V_2 = \frac{G_1}{G_1 + G_2} V \tag{5.61}$$

where
$$\tau = \frac{C_1 + C_2}{G_1 + G_2} = \frac{(C_1 + C_2)R_1 R_2}{R_1 + R_2}$$

i.e., the product of the parallel combination of C_1, C_2 and R_1, R_2. The solution to (5.61) is

$$V_2 = \frac{G_1}{G_1 + G_2} V + Be^{-t/\tau} \tag{5.62}$$

where B is a constant to be determined. If we assume that the battery is connected at time $t = 0$, then at $t = 0$,

$$V_2 = \frac{C_1}{C_1 + C_2} V \qquad V_1 = V - V_2 = \frac{C_2}{C_1 + C_2} \tag{5.63}$$

However, this initial condition is not physically possible, since if we begin with zero charge on the capacitors, it implies an infinite current flow as soon as the battery is connected. If we actually had a battery with zero internal resistance, this could be accomplished since an uncharged capacitor behaves as a short circuit. An actual battery has finite internal resistance R_b, and the initial flow of current is finite. If, however, R_b is very small compared with R_1 and R_2, the current is initially limited only by R_b, C_1, and C_2, since the current flow into C_1 and C_2 will be much greater than the small amount of current flowing through R_1 and R_2. Thus the voltage across C_1 and C_2 builds up very rapidly. These remarks apply also to the field solution where the assumption that $\Phi = V$ when $t = 0$ implied a source with no internal losses. Using the idealized condition (5.63) in (5.62) gives

$$V_2 = \frac{G_1 V}{G_1 + G_2}\left(1 - \frac{\tau - \tau_1}{\tau} e^{-t/\tau}\right) \tag{5.64a}$$

$$V_1 = V - V_2 = \frac{G_2 V}{G_1 + G_2}\left(1 + \frac{G_1}{G_2}\frac{\tau - \tau_1}{\tau} e^{-t/\tau}\right) \tag{5.64b}$$

where $\tau_1 = C_1R_1$. As $t \to \infty$, these expressions clearly give the correct steady-state values of V_1 and V_2. From (5.60), the surface charge density ρ_s is found to be

$$\rho_s = \frac{Q_2 - Q_1}{A} = \frac{C_2}{A} V_2 - \frac{C_1}{A} V_1 = \frac{C_2G_1 - C_1G_2}{G_1 + G_2} \frac{V}{A} (1 - e^{-t/\tau})$$
$$= \frac{\epsilon_2\sigma_1 - \epsilon_1\sigma_2}{\sigma_1 + \sigma_2} \frac{V}{d} (1 - e^{-t/\tau}) \tag{5.65}$$

where $\tau = (\epsilon_2 + \epsilon_1)/(\sigma_1 + \sigma_2)$. This result checks with that found from the field point of view as given in (5.59).

As we have already noted, for time intervals that are short compared with τ, we can assume that ρ_s is negligible, and hence the boundary condition $D_{n2} = D_{n1}$ is a good approximation. This boundary condition could be assumed if the applied voltage were sinusoidal and the period much shorter than τ (high frequencies); that is, if the following inequality holds,

$$\frac{1}{2\pi f} = \frac{1}{\omega} \ll \frac{C_1 + C_2}{G_1 + G_2}$$

then the capacitor behaves essentially as if there were no losses. This inequality may be rewritten as

$$\frac{1}{\omega(C_1 + C_2)} \ll \frac{R_1R_2}{R_1 + R_2}$$

which from a circuit standpoint simply states that the resistances may be neglected if their parallel combination is much greater than the parallel capacitive reactance.

If we are interested in time intervals comparable with τ, that is, low frequencies, then the boundary condition $D_{n2} - D_{n1} = \rho_s$ must be used since the surface charge density will not be negligible. Since most dielectrics have a finite conductivity, these considerations are of importance and some care must be exercised in using the assumption of zero surface charge density, as is commonly done by many authors.

As a point of further interest, if the time constants are equal, that is, if $\tau = \tau_1$, then ρ_s is always equal to zero. An examination of (5.64) now shows that $V_1/V_2 = R_1/R_2$. Thus the circuit of Fig. 5.11 provides a frequency-independent voltage divider.

5.7. Duality between J and D

The current density **J** and displacement flux density **D** are both linearly related to the electric field **E** in many materials. A consequence of this property is the existence of dual relationships between **J** and **D**. In a region where nonconservative fields are absent, i.e., external to the battery, the following equations apply for linear, isotropic materials:

Conducting media	*Dielectric media*	
$\nabla \times \mathbf{E} = 0$	$\nabla \times \mathbf{E} = 0$	
$\mathbf{J} = \sigma \mathbf{E}$	$\mathbf{D} = \epsilon \mathbf{E}$	(5.66)
$\nabla \cdot \mathbf{J} = 0$	$\nabla \cdot \mathbf{D} = 0$	

In a homogeneous material where ϵ and σ are constant, we also have

$$\nabla \times \mathbf{J} = 0 \qquad \nabla \times \mathbf{D} = 0 \tag{5.67}$$

This latter property shows that both **J** and **D** may be derived from a scalar potential; hence

$$\mathbf{J} = -\nabla\Phi \qquad \mathbf{D} = -\nabla\Phi \tag{5.68}$$

and $\nabla^2\Phi = 0$ in both cases by virtue of the divergence relations given in (5.66). It should be noted that the divergence and curl of **J** and **D** can be zero only over part of space, or else **J** and **D** would vanish everywhere. In the present case we are limiting consideration to regions that are external to all sources, so that both the divergence and curl of **J** and **D** may be zero. The solution for **J** and **D** is uniquely determined by finding a scalar potential function Φ that satisfies Laplace's equation and any imposed boundary conditions.

An examination of the above equations shows that any solution for **J** can be transformed into a solution for **D**, and vice versa, by means of the following interchange of quantities:

$$\mathbf{J} \leftrightarrow \mathbf{D} \tag{5.69a}$$

$$\sigma \leftrightarrow \epsilon \tag{5.69b}$$

This means that if a solution to a boundary-value problem in electrostatics is known, it is also the solution to a corresponding problem in steady current flow. This procedure is valid only if the boundary conditions are equivalent in both cases. Where a boundary is an interface between different media, then

Conducting media	*Dielectric media*	
$J_{n1} = J_{n2}$	$D_{n1} = D_{n2}$	
$\dfrac{J_{t1}}{\sigma_1} = \dfrac{J_{t2}}{\sigma_2}$	$\dfrac{D_{t1}}{\epsilon_1} = \dfrac{D_{t2}}{\epsilon_2}$	(5.70)

Note that (5.70) conforms to the duality relations expressed by (5.69), as we should expect.

Certain boundary-value problems involving steady current do not have a dual in electrostatics. This occurs when a region with $\sigma = 0$ is involved, i.e., a perfect insulator. Duality requires an electrostatic region with a relative dielectric constant of zero; however, it is not possible to achieve $\kappa < 1$. Thus consider, for example, the steady flow between parallel plates as illustrated in Fig. 5.12*a*. Since $\sigma_2 = 0$, the flow lines will be

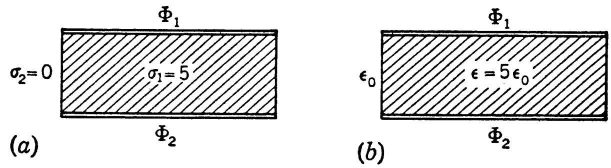

FIG. 5.12. Equivalent conductance and capacitance problems.

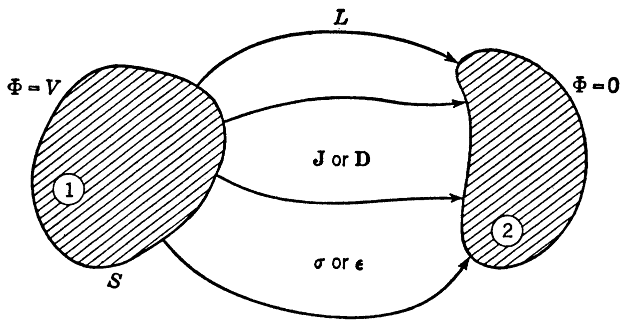

FIG. 5.13. Duality between conductance and capacitance for two arbitrary bodies.

uniform and directed normal to the parallel plates through the conducting medium of conductivity σ_1. In Fig. 5.12*b* a dielectric is placed between the parallel conducting plates. This is not quite the dual of Fig. 5.12*a* because it is impossible to provide a zero permittivity region surrounding the capacitor. As a consequence, fringing of the **D** lines occurs, such as did not happen in the case of current flow.

In Fig. 5.13 we show two arbitrarily shaped conducting bodies in a uniform, infinite medium. If the medium is a conductor, then the total resistance between the bodies is

$$R = \frac{\int_L \mathbf{E} \cdot d\mathbf{l}}{\oint_S \sigma\mathbf{E} \cdot d\mathbf{S}} \tag{5.71}$$

where L is any path from one body to the other and S is any surface enclosing either body. The numerator of (5.71) is the difference of potential between the bodies; the denominator evaluates the total current that flows between them.

If the conducting bodies are immersed in a dielectric, with a permittivity ϵ, then the capacitance C is given by

$$C = \frac{\int_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}} \tag{5.72}$$

where S and L are taken as before. That this evaluates the capacitance is clear, since the numerator determines the total charge on either conductor by application of Gauss' flux theorem, while the denominator is the difference of potential between the conductors. If we assume that the same difference of potential is maintained in both cases, then in view of the uniqueness theorem, $\mathbf{E}$ is the same in (5.71) and (5.72).

In comparing (5.71) and (5.72), one notes a dual relationship between $1/R = G$ and C. Provided that the conducting boundaries are identical and that everywhere σ is replaced by ϵ, then

$$RC = \frac{\epsilon}{\sigma} = \frac{C}{G} \tag{5.73}$$

Equation (5.73) does not hold for the geometry of Fig. 5.12 unless fringing effects are negligible. In the case of the spherical resistor in Fig. 5.6, the capacitance between the two spherical shells, when the intervening medium has a permittivity ϵ, may be found from (5.40) by using the relation (5.73). The result is

$$C = \frac{4\pi ab\epsilon}{b - a} \tag{5.74}$$

5.8. Joule's Law

The energy required to maintain a steady flow of current through an arbitrary conducting body of total resistance R (e.g., Fig. 5.3) can be found from basic principles. Let the total difference of potential across the body be V. Then the work done on a charge Q moving through this potential difference is $W = QV$. The rate at which work is expended by the field is then

$$\frac{dW}{dt} = P = V\frac{dQ}{dt} = VI \tag{5.75}$$

Or, since $V = IR$,

$$P = I^2R \tag{5.76}$$

This is known as Joule's law. As already noted, the work done by the field in moving the electrons through the conductor is in turn dissipated as heat as a consequence of electron-lattice interaction.

A formulation of Joule's law in differential form will also be useful.

Consider the differential volume element described in Fig. 5.14. The axial extent of the element is dl, and this is taken in the direction of current flow. If the electric field is $\mathbf{E}$, then the difference of potential across the ends of the element is

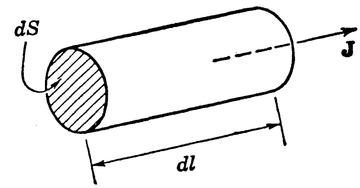

FIG. 5.14. A differential element of a resistor.

$$d\Phi = \mathbf{E} \cdot d\mathbf{l} = \frac{\mathbf{E} \cdot \mathbf{J}}{J}\, dl$$

The total current through the element, dI, equals $J\, dS$. According to (5.75), the total power dissipated in this element is

$$dP = dI\, d\Phi = \mathbf{E} \cdot \mathbf{J}\, dl\, dS$$

Consequently, the differential form of Joule's law is

$$\frac{dP}{dV} = U_J = \mathbf{E} \cdot \mathbf{J} \tag{5.77}$$

where dV is an element of volume $dS\, dl$. The electric field $\mathbf{E}$ therefore gives up $\mathbf{E} \cdot \mathbf{J}$ watts of power per unit volume to the steady electron flow $\mathbf{J}$. We have already noted that, since the current flows in a conductor, this power is converted into heat.

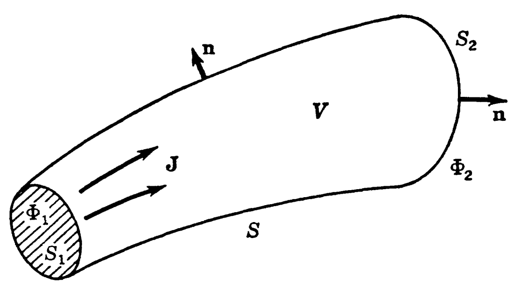

FIG. 5.15. An arbitrary resistor.

We are now in a position to derive the energy formula (5.42) for resistance. Figure 5.15 illustrates a conductor with end surfaces S_1, S_2 kept at constant potentials Φ_1 and Φ_2. The current density J flows normal to these end surfaces and parallel to the sides of the conductor. From (5.76) and (5.77) the total power (energy per second) dissipated in the resistor is

$$P = I^2R = \int_V \mathbf{E} \cdot \mathbf{J}\, dV = \int_V \frac{\mathbf{J} \cdot \mathbf{J}}{\sigma}\, dV = \int_V \sigma\mathbf{E} \cdot \mathbf{E}\, dV$$

and hence

$$R = \frac{\int_V \mathbf{E} \cdot \mathbf{J}\, dV}{I^2} \tag{5.78}$$

We may readily show that this definition of total resistance is identical with that given by (5.31). We note that $\mathbf{E} = -\nabla\Phi$ and

$$-(\nabla\Phi) \cdot \mathbf{J} = -\nabla \cdot (\Phi\mathbf{J})$$

since $\nabla \cdot \mathbf{J} = 0$. The volume integral in (5.78) becomes, upon application of the divergence theorem,

$$-\int_V \nabla \cdot (\Phi\mathbf{J})\, dV = -\oint_{S+S_1+S_2} \Phi\mathbf{J} \cdot \mathbf{n}\, dS$$

where $S + S_1 + S_2$ is the total surface of the resistor and $\mathbf{n}$ is the outward normal. Since $\mathbf{J} \cdot \mathbf{n} = 0$ except on the constant potential end surfaces, we obtain

$$I^2R = \Phi_1 \int_{S_1} \mathbf{J} \cdot (-\mathbf{n})\, dS - \Phi_2 \int_{S_2} \mathbf{J} \cdot \mathbf{n}\, dS = (\Phi_1 - \Phi_2)I$$

and hence $R = (\Phi_1 - \Phi_2)/I$, which is (5.31).

5.9. Convection Current

The flow of current in conductors, under the action of an electric field, that has been considered so far is called *conduction current*. In contrast, if an insulator carries charges and the entire body is in motion, an equivalent current also flows. An important special case is that of the movement of the charges themselves in a vacuum, under the influence, perhaps, of an electric field. In these cases the relation between current and electric field is no longer described by Ohm's law. The current developed by moving media depends on the mechanics of the moving particles; the motion of charges in an evacuated region containing an electric field can be described by the laws of mechanics with the inclusion of forces of electric origin. The term *convection current* is applied to describe this latter type of current.

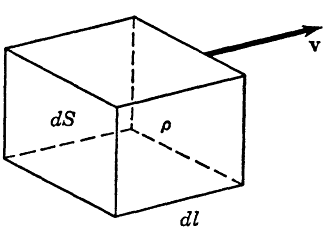

FIG. 5.16. A volume element of convection current.

Given a flow of convection current, then the electron stream is specified by a charge density ρ and a velocity $\mathbf{v}$ at any point. The current density at each point can be found by multiplying the charge density by the velocity; that is, if we consider a volume element with length dl in the direction of flow (see Fig. 5.16), then in a time interval $dt = dl/v$, all the charge ($\rho\, dS\, dl$) flows out of the region. The total current, by definition, is

$$dI = \frac{\rho\, dS\, dl}{dt} = \rho v\, dS$$

Hence the convection current density is given by

$$\mathbf{J} = \rho\mathbf{v} \tag{5.79}$$

The vector notation in (5.79) follows by inspection since **J** and **v** must obviously be in the same direction.

For convection currents in the presence of an electrostatic field, energy will be interchanged between the two. The development leading to (5.77) can be repeated for a conservative **E** field acting on a convection current, and the result is

$$U_J = \mathbf{E} \cdot \mathbf{J} = \rho\mathbf{E} \cdot \mathbf{v} \tag{5.80}$$

In this case the energy absorbed by the electron stream from the field, as given by (5.80), is not converted into heat, but into an increase in the kinetic energy of the particles. A further discussion of the interaction of charges and fields will be found in Chap. 12.

5.10. Flux Plotting

The solution of Laplace's equation forms the foundation not only for problems in electrostatics, but also for problems involving current flow fields. This is made clear by (5.68), which states that **J** can be derived from the gradient of a scalar potential Φ, where Φ is a solution to Laplace's equation. A full description of the **J** or **D** flow fields is consequently specified by the related function Φ. This may be of interest in itself or form the basis for further calculations such as for total resistance or capacitance.

The mathematical techniques for finding solutions of Laplace's equation are rather severely limited to certain geometry, e.g., boundaries that are spherical, circular, cylindrical, etc. For more arbitrary shapes other methods are required. This section is devoted to an explanation of an approximate graphical procedure known as the method of curvilinear squares, or simply flux plotting. In the next section the use of the electrolytic tank will be described. The technique of flux plotting is usually limited to two-dimensional problems.

By a two-dimensional problem, we refer to those cases where flow† is independent of one dimension; it can therefore be completely described in terms of the remaining two dimensions. Figure 5.17 shows the cross section of a solid, with surface S_1 at an equipotential Φ_1 and surface S_2 at an equipotential Φ_2. Flow takes place through the body, but in view of the axial uniformity, the direction of flow is confined to cross-sectional

† We have already noted the duality of **J** and **D** and that both may be derived from a scalar potential. When we refer to flow functions here, we do so in a general way; we refer to either **J** or **D**, or for that matter any vector function that can be derived as the gradient of a scalar potential.

planes. The uniformity further requires that the flux and potential distributions be the same in any transverse plane. For these reasons we can concentrate attention on finding a solution to the potential problem in such a typical plane.

The general technique involves making a guess as to the location of equipotential lines and flux lines. For reasons that we shall discuss in a

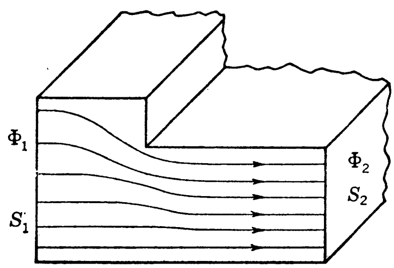

FIG. 5.17. A two-dimensional flow problem.

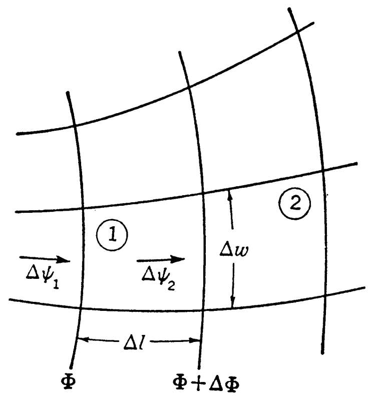

FIG. 5.18. A portion of a flux plot.

moment, equipotential lines are spaced at equal increments, with the magnitude of the spacing dependent on the coarseness or fineness of the desired plot. The flow lines will intersect these equipotentials orthogonally since the lines of flow are derived from the gradient of the potential. For example,

$$\mathbf{J} = -\sigma \nabla\Phi \tag{5.81}$$

We shall show that the flow lines should be spaced so that they form, as nearly as possible, curvilinear squares. Since certain boundary conditions are known, they serve to specify certain potential and flow lines at the outset. For example, in Fig. 5.17 the right- and left-hand edges must be equipotentials. Furthermore, if this represents a current flow problem with zero conductivity outside the body, then flow lines must be tangent to the remaining boundaries. Working from this point and by trial and error, the region can finally be covered with curvilinear squares. We proceed now to a justification of this method and an interpretation of the results.

For this purpose, consider a portion of a flux plot as just described. With respect to cell 1, in Fig. 5.18, since the width is Δw and the length in the direction of increasing potential is Δl, the field is approximately given by

$$E \approx -\frac{\Delta\Phi}{\Delta l} \tag{5.82}$$

From (5.81) the current density can be evaluated and is

$$J \approx -\sigma \frac{\Delta\Phi}{\Delta l} \tag{5.83}$$

The approximation improves as the size of the cell is made smaller. The total current in the tube of width Δw per unit length normal to the flux plot is then

$$\Delta I = \Delta\Psi_1 = -\sigma\, \Delta\Phi \frac{\Delta w}{\Delta l} \tag{5.84}$$

If the body were a dielectric and electric flux were being considered, then

$$\Delta\Psi_1 = -\epsilon\, \Delta\Phi \frac{\Delta w}{\Delta l} \tag{5.85}$$

In the above, Ψ represents total flow, either of current or of electric flux. For the latter case, since $\Delta\Psi_1 = D\,\Delta S$, the dimensions of Ψ are charge.

If the above procedure is followed at cell 2, then

$$\Delta\Psi_2 = -\sigma\, \Delta\Phi_2 \frac{\Delta w_2}{\Delta l_2} \tag{5.86}$$

Since the flux (either **J** or **D**) is solenoidal, $\Delta\Psi_2$ must equal $\Delta\Psi_1$ in order that the total flux within a tube be conserved. This can be accomplished by taking equal potential increments (itself a desirable procedure which simplifies the layout of the equipotential lines) and by making the aspect ratio $\Delta w/\Delta l$ a constant throughout the tube.

To facilitate interpretation of the entire flux plot, it is desirable that adjacent flux tubes represent equal quantities of flux. In this way the density of flow is graphically revealed by the spacing of flow lines. This will be accomplished by using the same potential increment $\Delta\Phi$ and aspect ratio $\Delta w/\Delta l$ everywhere throughout the plot. And since the eye can most readily gauge a square shape rather than a particular rectangular one, we choose $\Delta w/\Delta l$ as unity. This is the basis for the method of curvilinear squares just described. For a conducting medium, then,

$$\Delta\Psi = \Delta I = -\sigma\, \Delta\Phi \tag{5.87}$$

and in a dielectric medium

$$\Delta\Psi = -\epsilon\, \Delta\Phi \tag{5.88}$$

One objective in mapping a region may be to calculate the capacitance or resistance of a particular body. Let us see how this is done. Consider Fig. 5.19, which shows the plot of potential and flow fields between two cylindrical conductors of quite arbitrary space. The flux through any flux tube is given by (5.88). Then, if N_F is the total number of flux tubes, the total flux Ψ is

$$\Psi = N_F\, \Delta\Psi = -N_F\epsilon\, \Delta\Phi \tag{5.89}$$

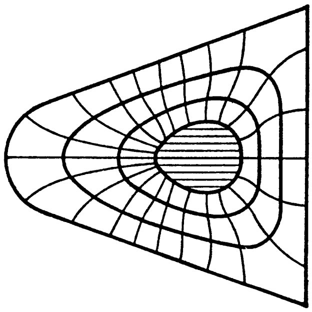

FIG. 5.19. Flux plot between two arbitrary conductors.

The total difference of potential, if N_P is the number of potential increments, is

$$V = N_P \, \Delta\Phi \tag{5.90}$$

The total charge per unit length on the conductors equals the total flux Ψ per unit length terminating thereon. Consequently, by definition, the capacitance C per unit length is

$$C = \frac{\Psi}{V} = \frac{\epsilon N_F}{N_P} \tag{5.91}$$

By duality, if the material between the conductors has a conductivity σ, the leakage conductance per unit length is

$$G = \frac{\sigma N_F}{N_P} \tag{5.92}$$

For the structure shown in Fig. 5.19, the number of potential divisions $N_P = 3$, while the number of flux divisions $N_F = 22$. Consequently, for an air dielectric,

$$C = \frac{1}{36\pi} \times 10^{-9} \times {}^{22}\!/_{3} = 65 \ \mu\mu\text{f/m}$$

A summary of the remarks concerning flux plotting, plus several additional suggestions on procedure, follows:

1. Examine the geometry to take advantage of any symmetry that may be present. For example, in Fig. 5.19, only the upper half need be plotted since mirror symmetry exists. If the cross section were elliptical, for example, only a quadrant would have to be considered.

2. Draw in the boundaries indicating those that are conducting and those that have zero conductivity.

3. Starting with known potentials and/or known flow lines, work out a rough sketch of the entire field, maintaining orthogonality between flow lines and equipotentials.

4. Refine the map to ensure that cells are curvilinear squares. If the rectangles remain very irregular, it may be desirable to cover the field with a finer net.

5. Several revisions may be necessary in order to achieve a satisfactory plot.

5.11. Electrolytic Tank

We have considered analytical solutions to Laplace's equation for conducting bodies in a homogeneous medium. We recall that solutions for potential and flow functions could be obtained only if the geometry were quite special. Under the restriction of two-dimensional variation, arbitrary shaped boundaries can be treated by means of the graphical method described in Sec. 5.10. A restricted class of such problems may also be handled analytically by means of conformal transformations, as was explained in Chap. 4.

This section is concerned with the determination of the potential field under completely arbitrary boundary conditions by means of an electrolytic analog, the electrolytic tank. Essentially, this involves setting up a model of the actual problem, using an electrolytic solution as the conducting medium and real electrodes of proper shape and positioning for the conducting boundaries. Potential and flow can now be measured using appropriate electrical instruments. Since the battery is external to the tank, the region in question contains a conservative electric field. As a consequence, a scalar potential that is a solution to Laplace's equation is, in fact, set up.

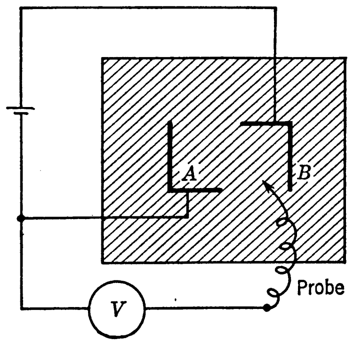

FIG. 5.20. The electrolytic tank.

Figure 5.20 illustrates a simple two-conductor problem where the shape of the field set up by electrodes A and B is desired; that is, A and B in Fig. 5.20 represent, to some scale, the actual electrodes in both shape and spacing. They are shown immersed in an electrolytic tank (shaded in the figure). A battery is connected across the electrodes, and this sets up a potential field within the electrolyte that is a solution of Laplace's equation and satisfies the appropriate boundary conditions on A and B.

The physical extent of the tank constitutes an additional boundary (the edge of the tank is characterized by requiring zero normal component of flux). Depending on the actual problem, this may represent only an approximation to actual conditions. For example, if the flow between the electrodes of Fig. 5.20 when immersed in a medium of infinite extent is desired, then the electrolytic-tank analog can be expected to be satisfactory only if the tank size is large compared with the over-all dimensions of the electrode system.

The potential field may be determined with the aid of a voltmeter. One lead is connected to an electrode (for example, A), and the free lead is used to probe the field in the electrolyte. The latter lead is insulated except for the tip. The locus of points for which the voltmeter reading is a constant establishes an equipotential surface. If the shapes of electrodes A and B do not vary in the direction normal to the page, then the potential will not vary with depth in the tank and a simpler two-dimensional problem exists. For such problems resistive coated paper may also be used in place of the electrolytic tank. In this case, electrodes can be painted on the paper with silver paint; the measurement procedure is essentially the same otherwise.

The electrolytic tank actually represents, to some scale, a current flow equivalent of the actual problem. If the original problem is, say, one in electrostatics, then both appropriate scale factors and duality conditions must be used to give the desired information. The actual problem may, itself, be a current flow problem, of course.

The electrolytic-tank technique serves to establish equipotential surfaces directly. By constructing a family of orthogonal trajectories, the flux paths may be found; i.e., these are the flux lines.

Double-sheet Electrolytic Tank

For a two-electrode problem current is confined to a finite region. As a consequence, electrodes can usually be scaled down in proportion to the size of the tank so that the medium appears infinite; that is, no serious disturbance of the current flow is caused by the limited extent of the tank.

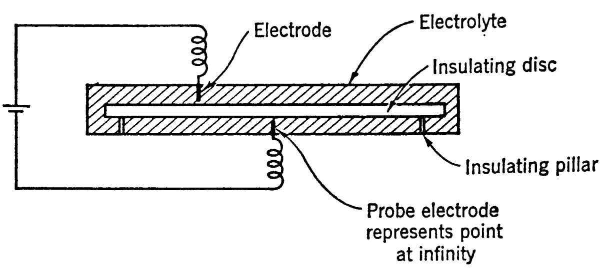

FIG. 5.21. A double-sheet electrolytic tank.

For those cases where a substantial current flow to infinity is involved, a double-sheet tank can be used. This tank, however, is applicable only to two-dimensional problems. The double-sheet tank is shown in Fig. 5.21. Let us consider the theory behind its construction, which will also serve to describe its operation.

Let us suppose the existence of an electrolytic tank of infinite extent. Electrodes could now be inserted in their proper geometric locations and a current flow set up. As before, a potential field is set up that satisfies Laplace's equation. For an arbitrary boundary at $r = R$, we could write the potential field for the region $r < R$ and for $r > R$ as follows:

$$\Phi = \Phi_1(r,\phi) \qquad r < R \tag{5.93a}$$
$$\Phi = \Phi_2(r,\phi) \qquad r > R \tag{5.93b}$$

A diagram of the coordinate system and of the location of these two regions is given in Fig. 5.22. Since Φ must be continuous, and because

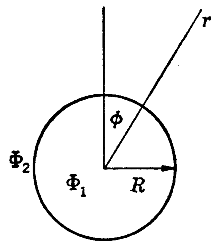

FIG. 5.22

$J_r = -\sigma\, \partial\Phi/\partial r$ is also continuous, it follows that

$$\Phi_1(R,\phi) = \Phi_2(R,\phi) \tag{5.94a}$$

$$\left.\frac{\partial\Phi_1}{\partial r}\right|_{r=R} = \left.\frac{\partial\Phi_2}{\partial r}\right|_{r=R} \tag{5.94b}$$

Assume now that all current sources lie in the region $r < R$, except possibly for a source or sink at infinity. Then

$$\nabla^2\Phi_2 = 0 \tag{5.95}$$

except possibly at infinity. Expanding (5.95) in cylindrical coordinates yields

$$\nabla^2\Phi_2 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\frac{\partial\Phi_2(r,\phi)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2\Phi_2}{\partial\phi^2} = 0$$

Let us introduce a new variable ρ such that

$$\rho = \frac{R^2}{r} \tag{5.96}$$

We desire to show that

$$\Psi(\rho,\phi) = \Phi_2\left(\frac{R^2}{\rho}, \phi\right) \tag{5.97}$$

is a solution of Laplace's equation in the variable ρ; that is, we wish to verify that

$$\nabla^2\Psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\Psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\phi^2} = 0 \tag{5.98}$$

From the relation between ρ and r we get

$$\frac{\partial}{\partial\rho} = \frac{\partial}{\partial r}\frac{\partial r}{\partial\rho} = -\frac{R^2}{\rho^2}\frac{\partial}{\partial r}$$

ticular, the point at infinity ($r \to \infty$) goes into the origin in the lower sheet ($\rho \to 0$).

The double-sheet tank not only is useful for problems of field mapping, but also can be adapted to the solution of network problems. What follows is a very brief outline of this capability.

We shall agree that any impedance function can be defined, within an arbitrary constant, by the location of its poles and zeros; that is, we can write

$$Z(\lambda) = \frac{A(\lambda - \lambda_{Z1}) \cdot \cdot \cdot (\lambda - \lambda_{Zn})}{(\lambda - \lambda_{p1}) \cdot \cdot \cdot (\lambda - \lambda_{pm})} \tag{5.102}$$

where λ_{Zi} and λ_{pi} are the coordinates of the zeros and poles, respectively, in the complex plane. Taking the natural logarithm of (5.102) and writing the equation due to the real part of both sides gives

$$\ln |Z| = \sum_{i=1}^{n} \ln |\lambda - \lambda_{Zi}| - \sum_{i=1}^{m} \ln |\lambda - \lambda_{pi}| + \ln A \tag{5.103}$$

But this is equivalent in form to the potential set up by a system of line sources located at λ_{Zi} and λ_{pi}, the source at λ_{Zi} being positive, that at λ_{pi} negative. Such a problem can be simulated in the double-sheet tank since it is capable of representing the entire complex plane. It is only necessary to locate at λ_{Zi} a current input electrode adjusted to "unit amplitude," while at λ_{pi} an output electrode extracting unit current is provided. The potential at some arbitrary point λ is then a measure of the magnitude of the impedance at the corresponding complex value. A plot of the potential variation along the imaginary λ axis, that is, ω axis, gives the frequency dependence of $\ln |Z|$. A full discussion of the application of the double-sheet tank to network analysis, including the technique for determination of the phase of $Z(\lambda)$, is given in a paper by Boothroyd, Cherry, and Makar.†

† A. R. Boothroyd, E. C. Cherry, and R. Makar, An Electrolytic Tank for the Measurement of Steady State Response, Transient Response and Other Allied Properties of Networks, *J. IEE*, vol. 96, pt. 1, pp. 163–177, 1949.

Chapter 5

5.1. A potential difference V is maintained across two very large coaxial conducting cylinders with radii r_1 and r_2 ($r_2 > r_1$).

(*a*) If the medium between the cylinders has a conductivity σ, calculate the leakage conductance G per unit length.

(*b*) If the medium were nonconducting with a dielectric constant ϵ, calculate the capacitance C per unit length from basic definitions.

(*c*) Show explicitly from (*a*) and (*b*) that $RC = \epsilon/\sigma$.

(*d*) For (*a*) compute **E** and **J** in the conducting medium.

5.2. (*a*) For the accompanying half ring, which is composed of material of conductivity σ, compute the total resistance between A and B by a rigorous treatment (the inner radius is R, and the outer radius is $R + d$).

(*b*) Find the total resistance by taking the accompanying figure to be straight with total length equal to π times the mean radius. For $R \gg d$ show that (*a*) reduces to (*b*).

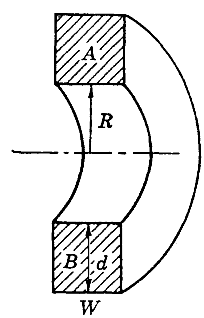

FIG. P 5.2

5.3. A long, highly conducting cylindrical wire is placed at a distance d from an infinite conducting plane and parallel to the plane. The wire diameter is a. The given conductors lie in a uniform conducting medium of conductivity σ, where $\sigma \ll$ conductivity of wire or plane. Show that the total resistance per unit length between the wire and the plane is

$$R = \frac{1}{2\pi\sigma} \cosh^{-1} \frac{d}{a}$$

5.4. Using the results of Prob. 5.3, calculate the total resistance per meter between a copper wire of radius 2 millimeters and an infinite conducting plane where the separation between the two is (*a*) 4, (*b*) 6, and (*c*) 10 millimeters. The conductivity is 10^4 mhos per meter.

5.5. Repeat Prob. 5.4, but determine the resistance per meter by means of flux plotting.

5.6. Map the flux lines and equipotentials for the adjoining figure (shown in cross section) given that (*a*) $b/d = 0.7$, (*b*) $b/d = 0.5$. Take the axial length and width to be essentially infinite, and continue the plot away from the step only to the point where the field becomes essentially uniform.

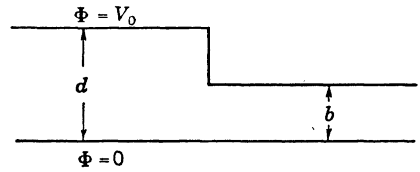

FIG. P 5.6

5.7. For Prob. 5.6 obtain a measure of the total charge per unit width and length at a sufficient distance from the step so that the field is essentially uniform. With these values determine the excess charge that is associated with the discontinuity. Evaluate the discontinuity capacitance per unit length.

5.8. The accompanying coaxial cable has a two-layered concentric cylindrical insulation with dielectric constants ϵ_1 and ϵ_2. The latter insulating materials have a leakage conductance σ_1 and σ_2, respectively. If the outer conductor is maintained at a potential V relative to the inner, what is the steady-state electric field in the dielectric and the surface charge at the dielectric interface?

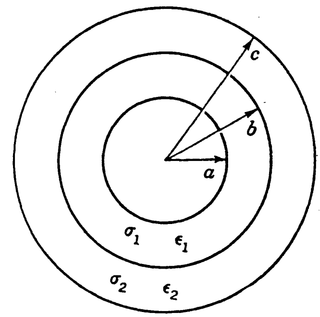

FIG. P 5.8

5.9. (*a*) Determine the transient time constant for the cable described in Prob. 5.8.
(*b*) How long would it take to reach steady state if the inner dielectric is mica ($\epsilon_1 = 6.0\epsilon_0$, $\sigma_1 = 10^{-13}$) and the outer dielectric is oil ($\epsilon_2 = 2.5\epsilon_0$, $\sigma_2 = 10^{-14}$)? The cable dimensions are $a = 1$ centimeter, $b = 1.3$ centimeters, $c = 1.8$ centimeters.

5.10. In order to obtain a good ground connection, a hemispherical conductor is embedded in the earth so that its base lies in the earth's surface. Assuming the resistivity of the ground to be 2×10^5 ohm-meters, find the total resistance to ground. (The radius of the hemisphere is 0.15 meter.)

5.11. The equation $\rho(x,y,z,t) = \rho_0(x,y,z)e^{-\sigma t/\epsilon}$ [Eq. (5.25)] required for its derivation only that σ and ϵ were uniform in the region under consideration. Suppose that at $t = 0$ a quantity of charge is localized in a small sphere in an otherwise uncharged medium. According to the above equation, at any $t > 0$, the region immediately surrounding the sphere of charge, and which was originally uncharged, remains neutral, even though the charge in the sphere is disappearing, only to reappear at the surface. A similar situation in heat flow would be quite different; the heat would flow out into the surrounding medium, instead of fading away where it stands, like the electric charge. Explain the reason for this difference.

5.12. A large sphere with uniform conductivity σ and permittivity κ has a radius R. At $t = 0$, a charge Q is placed uniformly over a small concentric spherical surface of radius a, where $a < R$. Calculate the Joule losses during the transient, and show that it is equal to the decrease in stored electric energy.

5.13. A steady current is distributed in a resistive medium which is not homogeneous (that is, σ and ϵ are functions of position). In this case a volume charge density will be set up in a similar way to that whereby a surface charge density accumulates at a lossy dielectric interface. Show that

$$\rho = \frac{-1}{\sigma} (\sigma \nabla \epsilon - \epsilon \nabla \sigma) \cdot \nabla \Phi$$

where Φ is the scalar potential for the electric field.

5.14. Show that for a problem involving N essentially perfectly conducting bodies embedded in a poorly conducting medium, where the medium is homogeneous, the current may be obtained from a scalar function Φ, and Φ is unique provided that

(a) $\Phi = \Phi_i$ $(i = 1, 2, \ldots, M)$, where Φ_i is a constant specified potential on the ith body.

(b) $I_j = -\oint_{S_j} \sigma(\partial\Phi/\partial n)\, dS$, where I_j is a specified total current from the jth body $(j = M+1, M+2, \ldots, N)$.

(c) $\Phi \propto 1/R$, $\partial\Phi/\partial n \propto 1/R^2$ for $R \to \infty$ (that is, Φ is regular at infinity).

5.15. Consider a conducting region V in which a current is caused to flow as a consequence of emf sources in an adjoining region V' (for example, V may encompass an arbitrary circuit and V' a battery). Prove that the current density in the conductor occupying V distributes itself in such a way that the generation of heat is less for the actual distribution than for any other provided the total current supplied by the sources is constant. Note that, since there are no sources in V, the current $\mathbf{J}$ is given by $-\sigma \nabla\Phi$. This problem is similar to Thomson's theorem, and the hints given in Prob. 3.23 are applicable.

5.16. A total of N conducting bodies lie in a conducting medium. If each body is at a constant potential $\Phi_1, \Phi_2, \ldots, \Phi_N$ and the total current from each is $I_1, I_2, \ldots, I_N$, show that the total Joule heat is

$$W_j = \sum_{i=1}^{N} \Phi_i I_i$$

5.17. A steady current flows into a thin conducting spherical shell at one pole and leaves at the other pole. Given the radius to be a, the conductivity σ, and the total current I, determine the potential and current density over the sphere.

5.18. A sphere of uniform conducting material of conductivity σ is placed in a potential field which is capable of maintaining a potential of $\Phi_0 \cos\theta$ over the spherical surface. (This implies a source of emf, of course.) Determine the current density $\mathbf{J}$ within the sphere.

5.19. Consider an infinite-plane conducting sheet, and let current enter at the origin and leave over a body contour at infinity. If a circular hole is cut anywhere in the sheet, not including the origin, show that the difference of potential between any two diametrically opposite points on the circumference of the hole is twice what it is prior to cutting the hole.

HINT: Consider the properties of the inversion of the original potential within the circle to the region outside the circle as discussed for the double-layer electrolytic tank. Note that the required condition at the edge of the hole can be expressed by $\partial\Phi/\partial n = 0$.

5.20. Show that, if the flux-plotting technique of Sec. 5.10 is followed, a correct and unique solution is obtained (subject, of course, to the approximation due to finite-size grids).

HINT: The graphical construction yields a family of equipotential curves $\Phi(x,y) = A_i$ and a family of flow curves $\Psi(x,y) = B_i$. The functions Φ and Ψ are related in that they intersect at right angles and form a square rather than a rectangle. From this it is possible to show that $\partial\Phi/\partial x = C\,\partial\Psi/\partial y$ and $\partial\Phi/\partial y = -C\,\partial\Psi/\partial x$, so that Φ and Ψ satisfy Laplace's equation.