PRINCIPLES AND APPLICATIONS OF

Electromagnetic Fields

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ELECTROSTATIC FIELDS IN MATERIAL BODIES, ENERGY, AND FORCES

In the previous chapter the properties of conducting materials in electrostatic fields were discussed. We turn now to a consideration of the properties of insulators in an electric field. In order to accomplish this aim, a short discussion of the microscopic (atomic) properties of insulators will be required. For this purpose simplified physical models of the atom will be used. Despite this simplicity it will be possible to satisfactorily predict the macroscopic behavior of dielectrics.

After discussing the basic properties of dielectrics, a study is made of capacitance and capacitors. This is followed by an evaluation of the work required to assemble a charge configuration. The work done is next related to the energy stored in the field. The latter part of the chapter introduces the principle of virtual work to evaluate the force acting on a body in an electrostatic field. The virtual-work principle of evaluating forces is a very powerful technique which greatly simplifies the solution of some otherwise difficult problems.

3.1. Polarizability

Electronic Polarizability

Let us begin by considering a monatomic gas in an electric field. This choice takes advantage of the fact that the spacing between the molecules of a gas is very much greater than the size of the molecule, so that, as we shall confirm, interaction between molecules can be neglected. This means that the effect of the field on any molecule is substantially the same as if it were the only particle present. A further geometric simplification arises in the choice of a single atom molecule.

Figure 3.1a illustrates a simplified model of the atom consisting of a positively charged nucleus surrounded by a spherically symmetric cloud of electrons. Since the nucleus has a diameter of the order of 10^{-15} meter while that of the electron cloud is of the order of 10^{-9} meter, the nucleus is essentially a point source. If, now, an external **E** field is applied, then a relative displacement of the nucleus from the center will occur. The

relative displacement of the nucleus z can be computed by equating the force exerted by the external field to the coulomb restoring force.

Let R be the radius of the electron cloud, as in Fig. 3.1. The electron cloud is considered to be equivalent to a uniform sphere of charge with a density $-3q/4\pi R^3$. The field due to this charge at a distance z from the
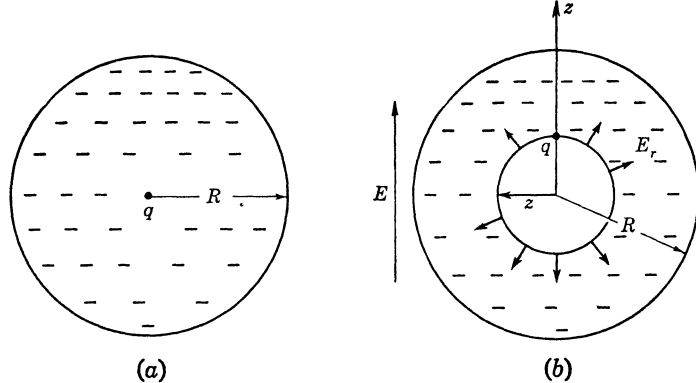

FIG. 3.1. (a) Model of atom; (b) displacement of nucleus by external field **E**.

center may be found by applying Gauss' law to the region bounded by a sphere of radius z, as in Fig. 3.1b. We have

$$\epsilon_0 4\pi z^2 E_r = -\frac{4\pi z^3}{3}\frac{3q}{4\pi R^3}$$

or

$$E_r = \frac{-qz}{4\pi R^3 \epsilon_0}$$

Since the charge on the nucleus is q, the coulomb restoring force F acting on the nucleus when displaced an amount z is

$$F = E_r q = -\frac{zq^2}{4\pi R^3 \epsilon_0} \tag{3.1}$$

Note that the restoring force, given by (3.1), is proportional to the displacement. This consequence is confirmed in atomic physics for small displacements z. Equating the external force qE to $-F$ gives

$$qz = p = 4\pi\epsilon_0 R^3 E \tag{3.2a}$$

or in vector form,

$$\mathbf{p} = 4\pi\epsilon_0 R^3 \mathbf{E} \tag{3.2b}$$

The dipole moment per atom, $\mathbf{p}$, is proportional to **E**, according to (3.2). If N is the number of atoms per cubic meter, then $\mathbf{P} = N\mathbf{p}$, where $\mathbf{P}$ is the dipole moment per unit volume. In view of the proportionality expressed in (3.2), we may relate $\mathbf{p}$ to **E** as $\mathbf{p} = \alpha\mathbf{E}$, where α is called the

polarizability. When the polarization arises in the way just described, α is designated the electronic polarizability α_e. In terms of the above parameters,

$$\alpha_e = 4\pi R^3 \epsilon_0 \tag{3.3}$$

The quantity α_e/ϵ_0 is seen, from (3.3), to be of the order of magnitude of the atomic volume. In Table 3.1 the ratio $\alpha_e/4\pi R^3\epsilon_0$ is given for several of the inert gases. Despite the crude model used, this ratio comes out in the order of unity.

TABLE 3.1. POLARIZABILITIES AND RADII FOR SOME COMMON ELEMENTS

Parameter	Gas				
	He	Ne	A	Kr	Xe
$\alpha_e \times 10^{40}$ cu m (measured)	0.222	0.43	1.8	2.7	4.4
R, (Ang)	0.95	1.15	1.4	1.6	1.75
$\alpha_e/4\pi R^3\epsilon_0$	0.28	0.29	0.66	0.67	0.83

In summary, we note that the effect of an applied electric field on the molecules of matter may be to create electrostatic dipoles. These in turn will set up a secondary (induced) field so that the net field in the presence of matter is modified from its free-space value. Before proceeding to a consideration of the fields, we mention several other polarization-producing mechanisms.

Ionic Polarizability

In a molecule characterized essentially by ionic bonds, we can think of that molecule as composed of positively and negatively charged ions. It is the coulomb forces between these ions which mainly account for the binding force. The application of an electric field to any such molecule will tend to displace the positive ions relative to the negative ones. This process will induce a dipole moment in the molecule. Note that it is quite distinct from electronic polarizability, where the displacement between the nucleus and electron cloud accounts for the polarization. In a polyatomic gas one can expect both processes to occur. We shall designate ionic polarizability by α_i.

Orientational Polarizability

In certain polyatomic molecules where the atomic bond is at least partially ionic, the individual atoms tend to be either positively or negatively charged. A two-atom molecule will thus have a permanent dipole moment, the magnitude of which depends on the time-average transfer of charge between atoms and the internuclear distance. For polyatomic

molecules several bonds may have a permanent dipole moment; the dipole moment of the entire molecule is the vector sum of the component moments. For certain kinds of symmetry the latter may come out zero.

The application of an electric field to molecules with a permanent dipole moment would ordinarily cause all molecules to align themselves with the applied field. This orienting tendency is opposed by the random thermal agitation of the molecule, and in solids and 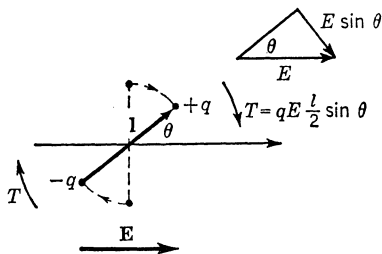some liquids by mutual interactions of the molecules. Assuming that the molecules are free to move, we can calculate their effective orientational or dipolar polarizability.

Consider a system of N molecules per unit volume, each possessing a permanent dipole moment of $\mathbf{p}$. In the absence of an electric field, their orientation is completely random, so that there is no net time-average dipole moment. Now let an electric field be applied in the z direction.

FIG. 3.2. Dipole in a field $\mathbf{E}$.

We represent in Fig. 3.2 a typical molecule, with moment $\mathbf{p} = q\mathbf{l}$, making an angle θ with the z direction.

For the potential energy of the dipole, let the zero reference be taken to be when $\mathbf{p} \cdot \mathbf{E} = 0$, that is, $\theta = 90°$. If the dipole is allowed to rotate into alignment with the field, then the decrease in potential energy of $+q$, from the position shown in Fig. 3.2, would be

$$W_+ = \int_{\pi/2}^{\theta} \frac{qlE}{2} \sin \theta \, d\theta = -\frac{qlE \cos \theta}{2} \qquad (3.4a)$$

and similarly for the negative charge,

$$W_- = -\frac{qlE \cos \theta}{2} \qquad (3.4b)$$

since the torque acting on each charge of the dipole is $T = (qlE/2) \sin \theta$. Consequently, the potential energy of the dipole in any position can be written as

$$W = -\mathbf{p} \cdot \mathbf{E} = -pE \cos \theta \qquad (3.5)$$

The contribution to the total dipole moment in the z direction from the afore-mentioned molecule is $\mathbf{p} \cdot \mathbf{a}_z = p \cos \theta$. To find the total moment it is necessary to know the relative number of molecules making different angles θ with z. With an applied field, the distribution is no longer uniform, but it can be found from the Boltzmann distribution law. The latter states that the probability that the direction of $\mathbf{p}$ lies between θ

and $\theta + d\theta$ is proportional to $e^{-W/kT} d\omega$, where $d\omega$ is an element of solid angle corresponding to $d\theta$, k is Boltzmann's constant and equals 1.38×10^{-23} joule per degree Kelvin, while T is absolute temperature. From Fig. 3.3 we note that

$$d\omega = 2\pi \sin\theta \, d\theta$$

The number of dipoles, per unit volume, whose moment lies between θ and $\theta + d\theta$, is now given by

$$dN = Ae^{(pE\cos\theta)/kT} \sin\theta \, d\theta \qquad (3.6)$$

where A is a constant of proportionality. The latter can be determined from the requirement that the total number of molecules per unit volume be given by the integral of (3.6) from $\theta = 0$ to π; that is,

Fig. 3.3. Element of solid angle on a unit sphere.

$$A = \frac{N}{\displaystyle\int_0^\pi e^{(pE\cos\theta)/kT} \sin\theta \, d\theta} \qquad (3.7)$$

The partial contribution to the polarization from dipoles lying between θ and $\theta + d\theta$ is

$$dP = dN \, p \cos\theta$$

The total polarization P is given by

$$P = Ap \int_0^\pi e^{(pE\cos\theta)/kT} \sin\theta \cos\theta \, d\theta$$

$$= pN \frac{\displaystyle\int_0^\pi e^{(pE\cos\theta)/kT} \sin\theta \cos\theta \, d\theta}{\displaystyle\int_0^\pi e^{(pE\cos\theta)/kT} \sin\theta \, d\theta} \qquad (3.8)$$

Let $x = \cos\theta$, $a = pE/kT$; then

$$P = pN \frac{\displaystyle\int_{-1}^1 e^{ax} x \, dx}{\displaystyle\int_{-1}^1 e^{ax} \, dx} = Np\left(\coth a - \frac{1}{a}\right) = NpL(a) \qquad (3.9)$$

where integration by parts was used to evaluate the numerator. The function $L(a)$ was first introduced in connection with a similar study of magnetic dipoles by Langevin (1905), and it is called the Langevin function. A plot of $L(a)$ is shown in Fig. 3.4. The curve depicts the saturation property of the orientational polarization. At room temperatures laboratory fields are too weak to approach saturation and $pE \ll kT$ can
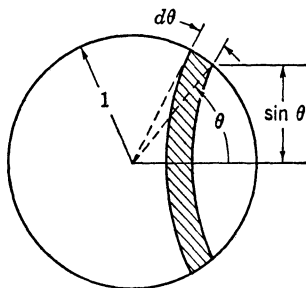
be assumed. As a consequence,

$$L(a) \approx \frac{a}{3} = \frac{pE}{3kT}$$

and the polarization per unit volume is

$$P = \frac{Np^2E}{3kT} \tag{3.10}$$

The orientational polarizability per molecule, α_0, is thus given by

$$\alpha_0 = \frac{p^2}{3kT} \tag{3.11}$$

Typical values of p are around 10^{-30} coulomb-meter, so that at room temperatures α_0 comes out around 10^{-40} cubic meter. This is the same order of magnitude as the electronic polarizability. Note, however, that while both electronic and ionic polarizability depend only on atomic configurations, and hence are essentially independent of temperature, the orientational polarizability is inversely proportional to temperature. This corresponds to the observation that at elevated temperatures it becomes more difficult to align the dipoles against the thermal motion.
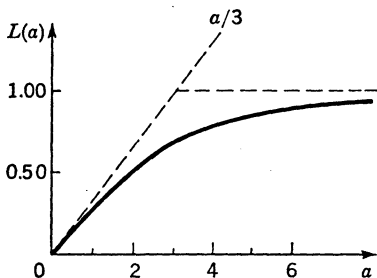

FIG. 3.4. The Langevin function.

The total polarization of a polyatomic gas may arise as a result of electronic, ionic, and orientational polarizability. Therefore, in general, we have

$$\mathbf{P} = N\left(\alpha_e + \alpha_i + \frac{p^2}{3kT}\right)\mathbf{E} \tag{3.12}$$

where the vector notation, which is implicit in the previous work, has been restored.

If the foregoing analysis is applied to solid dielectrics, then the field $\mathbf{E}$ in (3.12) cannot be taken as that which exists prior to the introduction of the dielectric. This is because adjacent dipoles become sufficiently influential in modifying the field at a point within the dielectric. The field $\mathbf{E}$ in (3.12) must be interpreted as being the molecular field in the dielectric. In a later section we shall evaluate the first-order interaction, and hence relate the molecular field to the external field.

We turn now to a consideration of the macroscopic effects of the polarizability of dielectric materials. It will be useful to lump together

the various contributions to the polarization in the following expression:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{3.13}$$

where χ_e is a dimensionless quantity called the electric susceptibility, and $\mathbf{E}$ is the total field; that is, χ_e is assumed to absorb the factor relating the molecular field to the total field. Equation (3.13) assumes that the polarization is proportional to $\mathbf{E}$, a relationship usually confirmed in practice. Note that, in part, it is based on an assumption of being far from saturation for dipolar molecules. Equation (3.13) fails to describe the polarization of certain substances which exhibit spontaneous polarization. The latter class are known as ferroelectrics. A description of their properties can be found in the references on solid-state physics.

3.2. Electric Flux Density D

In Chap. 2 we learned how to calculate the electric field from a given distribution of charge. This can be accomplished by first determining the scalar potential field that is set up. The electric field is then derived from the negative gradient of the scalar potential. The pertinent formulas follow (the reader is reminded that they apply under free-space conditions):

$$\Phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho \, dV'}{R} \tag{3.14}$$

$$\mathbf{E} = -\nabla\Phi \tag{3.15}$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

Let us consider what happens if a dielectric is introduced into the electric field set up by an arbitrary charge distribution, as described above. As a consequence of its polarizability, the volume occupied by the dielectric now contains a dipole moment distribution $\mathbf{P}$. This distribution constitutes a secondary source for the electric field. Thus, in addition to the original charges that set up the field, the dipole moment $\mathbf{P}$ must also be included as a source for the complete field, i.e., the field in the presence of the dielectric.

The scalar potential due to $\mathbf{P}$ has already been determined and is given by (2.97). The total potential, by superposition, must then be

$$\Phi = \frac{1}{4\pi\epsilon_0} \left(\int_V \frac{\rho - \nabla' \cdot \mathbf{P}}{R} \, dV' + \oint_S \frac{\mathbf{P} \cdot \mathbf{n}}{R} \, dS' \right)$$

$$= \frac{1}{4\pi\epsilon_0} \left(\int_V \frac{\rho + \rho_p}{R} \, dV' + \oint_S \frac{\rho_{sp}}{R} \, dS' \right) \tag{3.16}$$

In the last expression, $\rho_p = -\nabla' \cdot \mathbf{P}$ is the equivalent volume polarization charge, ρ_{sp} the surface polarization charge, while ρ is called the "true" charge. Note that ρ in (3.16) is, in general, not exactly the same as in

(3.14). This comes about because ρ_p may react back on the original distribution of charge, thus affecting that distribution. In many cases, however, such effects will be small.

The polarization charge does not only arise as a physical interpretation of the mathematical equivalence expressed in (2.97). If one could actually measure the excess charge in a small volume within a dielectric, one would confirm that $\rho_p = -\nabla \cdot \mathbf{P}$. [See remarks following (2.97).] Nevertheless, a distinction between true and polarization charge is made, and this difference is related to the origin of the volume charge density. True charge is essentially accessible for measurement; it is a free charge. Polarization charge would also be accessible if one could make measurements on an atomic scale within dielectric materials. However, we ordinarily consider the polarization charge as arising from constituent dipoles so that the "bound" nature of the charges (hence "inaccessibility" of individual charges) is evident. We consequently distinguish the polarization charge as a separate entity.

As a consequence of (3.16), the sources of electric field must be generalized to include the polarization charge. In the dielectric we must consider ρ_p as fully equivalent to the true charge ρ. Thus the divergence of $\mathbf{E}$ must be related to the sum of $\rho + \rho_p$ as follows:

$$\nabla \cdot \mathbf{E} = \frac{\rho + \rho_p}{\epsilon_0} \qquad (3.17)$$

This result is readily verified by taking the Laplacian of (3.16) and using the singularity property of $\nabla^2(1/R)$. Since (3.17) relates the divergence of $\mathbf{E}$ to the charge density at a point, the surface charge term does not enter as long as we are in the interior of the dielectric. Later on we shall show that the surface polarization charge is readily taken into account in practice by imposing a discontinuity condition on the normal component of $\mathbf{E}$ at the surface of the dielectric body. If we express ρ_p in terms of $\mathbf{P}$, we have

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho \qquad (3.18)$$

In practice, it is inconvenient to take explicit account of the polarization $\mathbf{P}$. We can avoid this by introducing a new field vector $\mathbf{D}$ defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (3.19)$$

Equation (3.18) now becomes

$$\nabla \cdot \mathbf{D} = \rho \qquad (3.20)$$

The vector $\mathbf{D}$ has the dimensions of coulombs per square meter and is called the electric displacement, or electric-flux-density vector. The source for the vector $\mathbf{D}$ is the true charge density ρ.

With materials for which (3.13) holds we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E} = \epsilon\mathbf{E} \qquad (3.21)$$

The parameter $\epsilon = \epsilon_0(1 + \chi_e)$ is called the permittivity of the dielectric. The permittivity relative to that of free space is $\epsilon/\epsilon_0 = 1 + \chi_e$ and is called the relative dielectric constant and will be designated by the symbol κ. If ϵ is known, we can solve for the polarization $\mathbf{P}$ since

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} \qquad (3.22)$$

Values of the relative dielectric constant of several typical materials are given in Table 3.2.

TABLE 3.2. RELATIVE DIELECTRIC CONSTANT OF SEVERAL MATERIALS

Material	Relative dielectric constant	Dielectric strength, kv/m
Air	1.00	3,000
Oil	2.3	15,000
Paper	1.5–4	15,000
Polystyrene	2.7	20,000
Glass	6.0	30,000
Paraffin	2.1	30,000
Quartz	5.0	30,000
Mica	6.0	200,000

Sometimes (3.21) is taken as a definition of $\mathbf{D}$. It is clear that such a relationship holds only for a class of dielectrics under certain conditions. It depends primarily on the linear relation between the polarization and the electric field as expressed in (3.13). In addition to this, it also requires that the material be isotropic, that is, χ_e should be independent of the direction of $\mathbf{E}$. When this is not the case, the relation between $\mathbf{P}$ and $\mathbf{E}$, and hence between $\mathbf{D}$ and $\mathbf{E}$, becomes a matrix, i.e., a tensor, relation. For example, if the molecule is not symmetrical, its dipole moment will in general not be collinear with the field $\mathbf{E}$ and each component of $\mathbf{P}$ will be related to each component of $\mathbf{E}$, so that

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

It is clear that in this case χ_e and ϵ are matrix (tensors of rank 2) quantities. Materials characterized by a tensor permittivity are called anisotropic materials. For our purposes in the remainder of this book we shall assume that ϵ is a scalar constant. Most materials fall into this class.

If (3.20) is integrated throughout a volume V and Gauss' law is used, then an integral relationship between $\mathbf{D}$ and the total true charge results; i.e.,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV = Q \qquad (3.23)$$

where Q is the total charge within V. Utilizing the flux concept, we see that the number of flux lines originating within V is proportional to the quantity of charge Q.

3.3. Boundary Conditions

In solving problems in electrostatics it is necessary to relate the inaccessible (polarization) charges to the accessible (true) charges, or to the
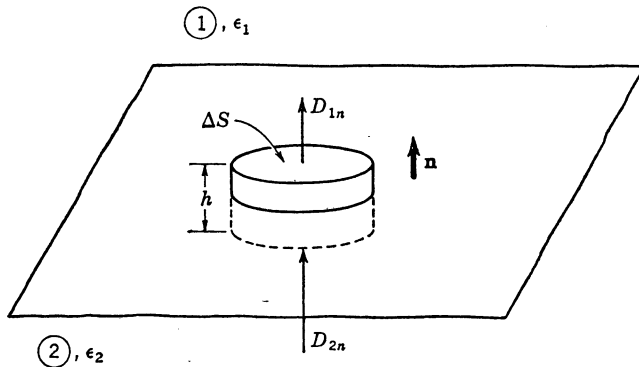

FIG. 3.5. Illustration for derivation of boundary conditions on D_n.

fields produced by the latter. Such relationships which link the inaccessible charge sources to the external fields which produce them are called the constitutive equations. An example of such an equation is (3.13), although sometimes the nomenclature is applied to (3.21) as well. Such equations depend on the properties of the material to which they apply. As was noted earlier, (3.21) is restricted to linear, isotropic materials. However, the material need not be homogeneous; that is, ϵ may be a function of position.

One very common case of nonhomogeneity occurs when the dielectric constant varies discontinuously as between two different homogeneous media. The way in which $\mathbf{D}$ and $\mathbf{E}$ behave in crossing the boundary between two dielectrics is of much interest and will be discussed now.

Figure 3.5 illustrates a very small element of the interface between dielectrics 1 and 2 whose permittivities are ϵ_1 and ϵ_2, respectively. Since the element of surface is of differential extent, it may be considered to be plane. A coin-shaped surface is placed with its broad face parallel to the

interface and so that one surface is in region 1 and the other in region 2. The area of the broad face is ΔS, and the thickness is h. Let us apply Gauss' flux theorem to the volume of the coin. If we make use of (3.23), then

$$(\mathbf{n} \cdot \mathbf{D}_1)\, \Delta S - (\mathbf{n} \cdot \mathbf{D}_2)\, \Delta S = \rho_s\, \Delta S \qquad (3.24)$$

where ρ_s is the true surface charge density on the interface. Equation (3.24) does not include the outflow of flux of $\mathbf{D}$ through the sides, because this flow can be made negligible by letting $h \to 0$, while at the same time the terms in (3.24) remain unaffected.

For the simple case of a surface charge ρ_s in a free-space medium, we can set $\mathbf{D}_1 = \epsilon_0 \mathbf{E}_1$ and $\mathbf{D}_2 = \epsilon_0 \mathbf{E}_2$ in (3.24), with the result that

$$\mathbf{n} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{\rho_s}{\epsilon_0} \qquad (3.25a)$$

In other words, the normal component of electric field is discontinuous through a charged surface, the magnitude of the discontinuity being given by (3.25a). For a dielectric interface ρ_s is ordinarily equal to zero unless a surface charge is actually placed at the interface. Taking $\rho_s = 0$, (3.24) becomes

$$\mathbf{n} \cdot \mathbf{D}_1 = \mathbf{n} \cdot \mathbf{D}_2 \qquad (3.25b)$$

That is, the normal component of $\mathbf{D}$ is continuous across a dielectric boundary. The normal component of $\mathbf{E}$, on the other hand, is discontinuous. This is clear if (3.21) is substituted into (3.25b) to give

$$\frac{\mathbf{n} \cdot \mathbf{E}_1}{\mathbf{n} \cdot \mathbf{E}_2} = \frac{\epsilon_2}{\epsilon_1} \qquad (3.26a)$$

Using a somewhat simpler notation we have

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n} \qquad (3.26b)$$

The discontinuity in $\mathbf{n} \cdot \mathbf{E}$ is readily explained physically. The field $\mathbf{E}$ arises from the total effective charge consisting of the true charge ρ, the volume polarization charge $\rho_p = -\nabla \cdot \mathbf{P}$, and the surface polarization charge ρ_{sp}. At the surface of a dielectric the normal component of $\mathbf{E}$ is discontinuous by an amount equal to ρ_{sp}/ϵ_0, just as it would be if we considered a surface layer of true charge equal to ρ_{sp}. Equation (3.26b) is readily shown to verify this result. From (3.22) the normal component of $\mathbf{P}$ at the surface is seen to be given by

$$P_{1n} = (\epsilon_1 - \epsilon_0)E_{1n} \qquad \text{medium 1}$$
$$P_{2n} = (\epsilon_2 - \epsilon_0)E_{2n} \qquad \text{medium 2}$$

The surface polarization charge is given by $P_{2n} - P_{1n}$ since this repre-

sents the amount of charge on the positive ends of the dipoles in medium 2 that is not canceled by the opposite charge on the negative ends of the dipoles in medium 1.   From the above relations we see that

$$P_{2n} - P_{1n} = \rho_{sp} = \epsilon_0(E_{1n} - E_{2n}) + \epsilon_2 E_{2n} - \epsilon_1 E_{1n} \qquad (3.27)$$

We can now see that a discontinuity of E_n by the amount ρ_{sp}/ϵ_0 corresponds to the requirement that $\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$.   In other words, satisfaction of (3.26) is consistent with the necessity that E_n be discontinuous by an amount ρ_{sp}/ϵ_0.
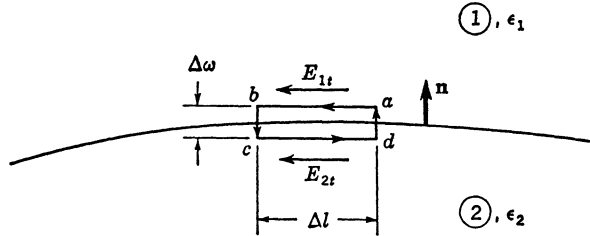
FIG. 3.6. Illustration for derivation of boundary conditions on E_t.

In practice, we usually find a suitable solution for **E** and **D** in the two dielectric regions.   We then adjust the magnitudes of these solutions so that (3.26) holds at the boundary.   By this means we avoid the necessity of taking the surface polarization charge into account explicitly.

Boundary conditions on the tangential components of the field can be found in the following manner.   Figure 3.6 is a cross section normal to the interface separating two media of different permittivity.   Considering the small rectangular path of length Δl and width $\Delta \omega$, where opposite sides of the long dimension lie in the separate media, we have

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} + \int_d^a \mathbf{E} \cdot d\mathbf{l} = 0 \qquad (3.28)$$

since the line integral of **E** around any closed path is zero.   Without affecting the remaining two integrals,

$$\int_b^c \mathbf{E} \cdot d\mathbf{l} = \int_d^a \mathbf{E} \cdot d\mathbf{l} = 0$$

by letting $\Delta \omega \to 0$, while keeping Δl fixed.   If we symbolize the tangential component of **E** in region 1 by E_{1t} and that in region 2 by E_{2t}, then, with the choice of directions given in Fig. 3.6,

$$E_{1t} \Delta l - E_{2t} \Delta l = 0$$

or
$$E_{1t} = E_{2t} \qquad (3.29a)$$

This may be written in vector form as

$$\mathbf{n} \times \mathbf{E}_1 = \mathbf{n} \times \mathbf{E}_2 \quad (3.29b)$$

For the tangential components of $\mathbf{D}$ we must now have

$$\frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2} \qquad (3.30)$$

FIG. 3.7. Refraction of $\mathbf{E}$ lines of force across a boundary.

Thus the tangential component of electric field is continuous across a boundary between two dielectrics, while tangential $\mathbf{D}$ is discontinuous. Note that this result would not be affected by the presence of a surface charge layer at the interface.

The total change in electric field in crossing an interface may be found from the above equations. The net result is analogous to the refraction of a light ray in passing from one medium to another. Thus in terms of the geometry of Fig. 3.7 and using (3.26) and (3.29), we have

$$E_1 \sin \theta_1 = E_2 \sin \theta_2 \qquad (3.31a)$$
$$\epsilon_1 E_1 \cos \theta_1 = \epsilon_2 E_2 \cos \theta_2 \qquad (3.31b)$$

Dividing (3.31a) by (3.31b) yields an electrostatic "Snell's law," relating the angle of incidence to the angle of refraction in terms of the properties of the media; i.e.,

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2} \qquad (3.32)$$

Let us consider a problem that illustrates some of the concepts just considered. Figure 3.8 depicts two very large (essentially infinite) parallel conducting planes which are maintained at a potential difference V by means of a battery. Because of the uniformity (the very large size is chosen so that fringing of the field at the edges can be neglected), the
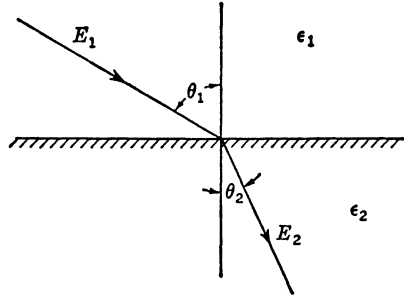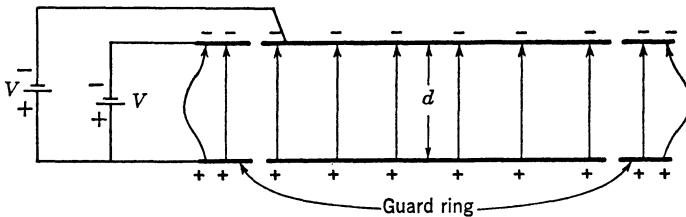

FIG. 3.8. Electric field between two parallel plates.

electric field must be given by

$$E = \frac{V}{d} \tag{3.33}$$

The uniformity of the electric field is represented in the flux plot in Fig. 3.8.†

Suppose we now insert a uniform slab of dielectric, of relative permittivity κ, between the plates, as shown in Fig. 3.9. Let the thickness of
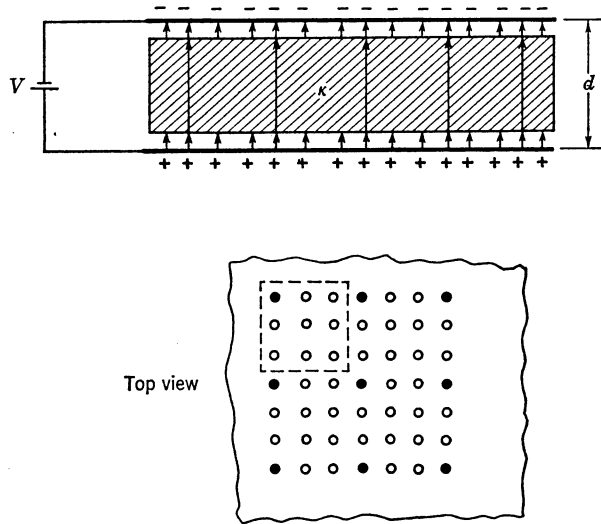

FIG. 3.9. Dielectric slab between parallel plates.

the slab be slightly less than d, leaving a small air gap. Since V is unchanged, E within the dielectric is still given by V/d; that is, $\int_0^d \mathbf{E} \cdot d\mathbf{l}$ must equal V as before. (We are neglecting the small error that the air gap introduces in this argument.)

In passing from the dielectric into the air gap, the electric field increases by virtue of (3.26). Thus, letting E_a be the field in the air gap and E_d that in the dielectric and noting that both are normal to the interface and the conducting plates, we have

$$E_a = \kappa E_d \tag{3.34}$$

where, of course, $\kappa > 1$. In Fig. 3.9, $\kappa = 9$, since the field strength is proportional to the number of lines per unit area.

† Fringing of the field at the sides of the plates may be avoided by increasing the size of the bottom plate and surrounding the upper plate by a guard ring, insulated from the upper plate and kept at the same potential $-V$ as in Fig. 3.8.

The discontinuity in **E** is explained by the polarization charge which terminates the flux lines. The polarization of the dielectric is represented in Fig. 3.9, and it should be clear that an equivalent surface charge layer is available at the dielectric surface. If the charge density on the conductor is ρ_s, then $E_a = \rho_s/\epsilon_0$. Now since $\kappa = 9$, $\chi_e = \kappa - 1 = 8$, and $P = \epsilon_0 \chi_e E_d = 8\rho_s/9$. In view of (3.27),

$$\rho_{sp} = \mathbf{n} \cdot \mathbf{P} = \begin{cases} -\tfrac{8}{9}\rho_s & \text{lower interface} \\ \tfrac{8}{9}\rho_s & \text{upper interface} \end{cases}$$

where **n** is the outward normal to the dielectric surface. From Fig. 3.9 we see that this is precisely the charge density needed to terminate the lines of flux of E_a at the surface of the dielectric.

3.4. Dielectric Strength

As we have noted, the process of induced polarization involves the displacement, from an equilibrium position, of the nucleus relative to the surrounding electron cloud. In our discussion of this phenomenon we tacitly assumed that an elastic process was involved. It might be anticipated, though, that if the field strength is increased sufficiently, new phenomena would be involved, including the possibility of a permanent change in the dielectric. This does, indeed, happen. We say that dielectric breakdown occurs, and the field strength at which this takes place is called the dielectric strength of the material. Table 3.2 lists the breakdown field strength for several common materials.

In general, breakdown begins as a result of the movement of electrons within the insulator under the influence of the applied field. The source of the electrons comes from impurities in the insulator, or crystal-lattice defects, or by field emission due to the applied field itself. If the energy gained by the electron from the accelerating field is greater than that lost in collision with the lattice structure, the electron will accumulate sufficient energy to create a hole-electron pair. The latter will then also be accelerated, so that a continuous discharge through the material can result.

If the solid dielectric contains gas bubbles or layers of gas, breakdown may first occur in that region because the breakdown field strength in a gas is lower than in the solid dielectric, while at the same time the field strength in the gas will be larger than in the dielectric [e.g., see (3.34)]. As a consequence of the gas discharge so initiated, the solid dielectric will be subjected to ionic bombardment, leading, finally, to complete breakdown.

It is not the intention to provide here a detailed physical description and analysis of dielectric breakdown. What is important is that it serves as a motivation in the solution of potential problems where exces-

sive field strengths that can cause breakdown must be avoided. In problems involving composite dielectrics one is faced with a need to reconcile the different dielectric constants and dielectric strengths. The following is a simple example of this.

Example 3.1. Dielectric Breakdown. For the parallel conducting planes as in Fig. 3.10, but with an air dielectric, the field is approximately
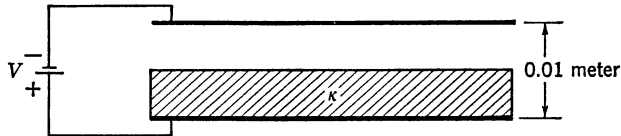

FIG. 3.10. Partially filled parallel-plate capacitor.

uniform and of magnitude V/d. Consequently, for $d = 0.01$ meter, the maximum potential is 30 kilovolts, as this results in a field of 3,000 kilovolts per meter, which is the breakdown stress for air.

If the region between the plates were filled with glass, then a voltage of 150 kilovolts would be permitted, since the dielectric strength of glass is approximately five times greater than that of air. Suppose, however, that the glass is only half the thickness of the air gap. Then since $\kappa = 6$ for glass, the field in the air is six times that in the glass, so that $\frac{6}{7}V$ exists across the air gap. In this case

$$\frac{\frac{6}{7}V_{max}}{0.005} = 3{,}000 \text{ kv/m}$$

and $$V_{max} = 17.5 \text{ kv}$$

Thus breakdown in the air will occur at a lower voltage than it would in the absence of the glass. Once breakdown occurred, the entire voltage would appear across the glass, but since the field strength would be only 17.5/0.005 = 35,000 volts per meter, the glass would not break down.
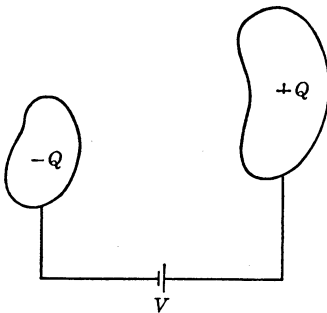
3.5. Capacitance

Consider two perfectly conducting bodies of arbitrary shape, as illustrated in Fig. 3.11, and let a quantity of charge be transferred from one to the other, such as would be accomplished by connecting a battery between the two. The charge on one is Q and on the other $-Q$. According to the uniqueness theorem, the difference in potential between the bodies

FIG. 3.11. Two charged bodies forming a capacitor.

is uniquely determined by the charge Q and the geometry of the conducting bodies. In view of the linear dependence of potential on charge, as revealed for example in (2.32), the difference of potential between the bodies, V, can be expressed as

$$V = \frac{1}{C}\, Q \tag{3.35}$$

The parameter C depends only on the geometry and for a given geometry is a constant. It is called the capacitance and is measured in units of farads or coulombs per volt.
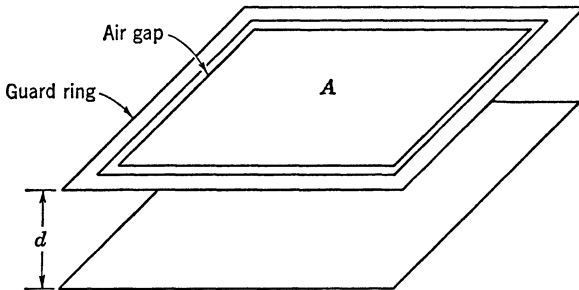

FIG. 3.12. A parallel-plate capacitor with guard ring.

The parallel-plate capacitor shown in Fig. 3.12 presents a simple geometry for illustrating the calculation of capacitance. If the smallest linear dimension of the plates is large compared with the spacing, then the field may be assumed to be uniform and the fringing at the edges neglected. Alternatively, a guard ring may be used as suggested in Fig. 3.12. In this case the relationship expressed by (3.33) applies, that is, $E = V/d$, where V is an assumed potential difference between the plates. For an air dielectric, $D = \epsilon_0 E$, and $\rho_s = D$; consequently, the total charge Q on each plate, of area A, is

$$Q = \rho_s A = \frac{\epsilon_0 V A}{d} \tag{3.36}$$

and
$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d} \tag{3.37}$$

Example 3.2. Capacitance between Concentric Spheres. It is required to find the capacitance between the two conducting concentric spherical shells illustrated in Fig. 3.13. If we assume a charge $+Q$ on the inner sphere and $-Q$ on the outer, then the electric field between the two shells is

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \tag{3.38}$$

This result can be verified by applying Gauss' flux theorem to the concentric spherical surface of radius r, where $a < r < b$. The potential difference between the spheres, V, is given by

$$V = \int_a^b E_r \, dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right)$$

(3.39)

Consequently, the capacitance between the two spheres is

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b - a}$$

(3.40)

If $b \to \infty$, the capacitance of an isolated sphere results; that is, we have a uniformly charged sphere with lines of flux extending radially outward to
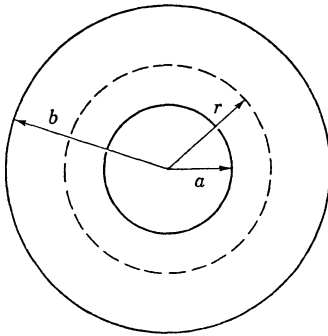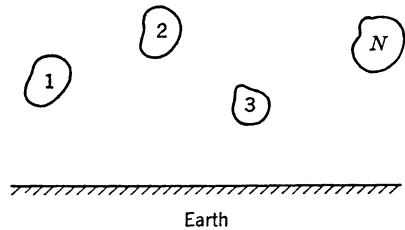

FIG. 3.13. A spherical capacitor.

FIG. 3.14. A general arrangement of conductors.

terminate at "infinity." Such a condition is essentially obtained in the case of a small charged sphere in a relatively large laboratory room, where the flux tends to terminate on the distant walls. From (3.40) the capacitance of the isolated sphere is seen to be

$$C = 4\pi\epsilon_0 a$$

(3.41)

Multicapacitor System

The concept of capacitance can be extended to a region containing more than two conducting bodies. In Fig. 3.14 we show N arbitrary conducting bodies and the earth. The latter may or may not figure in a practical problem. We can simply consider it as just another conducting body, chosen as the reference, in the following analysis.

On the basis of the uniqueness theorem in Sec. 2.9, specification of the charge on each body plus specification that the earth be a zero reference potential uniquely determines the potential everywhere. Furthermore,

because of the linear dependence of potential on charge, the following equations result:

$$\begin{aligned}
\Phi_1 &= p_{11}q_1 + p_{12}q_2 + \cdots + p_{1N}q_N \\
\Phi_2 &= p_{21}q_1 + p_{22}q_2 + \cdots + p_{2N}q_N \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Phi_N &= p_{N1}q_1 + p_{N2}q_2 + \cdots + p_{NN}q_N
\end{aligned} \tag{3.42}$$

In the above set of equations, the p_{ij}'s are constants, called the coefficients of potential, and they depend only on the geometry. We note that the potential on each body is properly a function of the total charge on each and that the relationship is linear.

The N equations can be solved to give the charges as functions of the potentials. The result will be in the form

$$\begin{aligned}
q_1 &= c_{11}\Phi_1 + c_{12}\Phi_2 + \cdots + c_{1N}\Phi_N \\
q_2 &= c_{21}\Phi_1 + c_{22}\Phi_2 + \cdots + c_{2N}\Phi_N \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
q_N &= c_{N1}\Phi_1 + c_{N2}\Phi_2 + \cdots + c_{NN}\Phi_N
\end{aligned} \tag{3.43}$$

Since the c_{ij}'s depend only on the p_{ij}'s of (3.42), they also depend only on the geometry. The terms c_{11}, c_{22}, . . . , c_{NN} are called coefficients of capacitance, while the c_{12}, c_{13}, etc., are coefficients of induction. The coefficient c_{ii} can be obtained by evaluating the ratio q_i/Φ_i of the ith body, with all others grounded. Since a positive charge on i produces a positive potential on i, the c_{ii}'s are all positive. The c_{ji} can be measured by grounding all but the ith body and evaluating the ratio q_j/Φ_i. Note that if q_i is positive, then Φ_i is positive, but the charge induced on j (q_j for $j \neq i$) will be negative. Accordingly, c_{ji} ($i \neq j$) is negative.

The coefficients of induction must satisfy a condition of reciprocity; that is, $c_{ij} = c_{ji}$. We can show this in the following way. Let all bodies be grounded but the first and second, which we take to be initially uncharged. If we begin charging body 1 to a final value of q_1 by adding very small increments of charge, then at some intermediate point the accumulated charge is q and the potential [from (3.42)] is $\Phi_1 = p_{11}q$. The energy required to add a charge dq is then, by definition of potential,

$$dW = p_{11}q\, dq$$

Accordingly, the total energy required to place a charge of q_1 on 1 is

$$W = \int_0^{q_1} p_{11}q\, dq = \frac{p_{11}q_1^2}{2} \tag{3.44}$$

If we proceed now to charge body 2 until q_2 is accumulated, then, corresponding to the presence of a charge q ($0 < q < q_2$), the potential is $\Phi_2 = p_{21}q_1 + p_{22}q$ and the work done in adding an increment of charge dq

to body 2 is

$$dW = (p_{21}q_1 + p_{22}q)\, dq$$

Altogether, the following amount of work is required to charge body 2 with q_2 coulombs:

$$W = \int_0^{q_2} (p_{21}q_1 + p_{22}q)\, dq = p_{21}q_1q_2 + \frac{p_{22}q_2^2}{2} \qquad (3.45)$$

The total energy required to put q_1 on 1 and q_2 on 2 is

$$W_T = p_{21}q_1q_2 + \frac{p_{11}q_1^2}{2} + \frac{p_{22}q_2^2}{2} \qquad (3.46)$$

If, however, we had first charged body 2 with q_2 and then body 1 with q_1, the total energy required is expressed by [we have only to interchange the subscripts 1 and 2 in (3.46)]

$$W_T = p_{12}q_2q_1 + \frac{p_{22}q_2^2}{2} + \frac{p_{11}q_1^2}{2} \qquad (3.47)$$

Since the final result is the same, the energies expressed by (3.46) and (3.47) are equal; consequently,

$$p_{12} = p_{21} \qquad (3.48)$$

The above proof can be repeated between any two bodies, say the ith and jth, so that

$$p_{ij} = p_{ji} \qquad (3.49)$$

From purely algebraic considerations it then follows that

$$c_{ij} = c_{ji} \qquad (3.50)$$

Equation (3.43) can now be rewritten in a way that is more informative. We let $C_{ij} = -c_{ij}$, noting that C_{ij} will be positive $(i \neq j)$, and let

$$C_{ii} = c_{i1} + c_{i2} + c_{i3} + \cdots + c_{iN} \qquad (3.51)$$

Then by adding and subtracting additional terms that will be clear upon examination, we can write in place of (3.43) the following:

$$q_1 = C_{11}\Phi_1 + C_{12}(\Phi_1 - \Phi_2) + C_{13}(\Phi_1 - \Phi_3) + \cdots + C_{1N}(\Phi_1 - \Phi_N)$$
$$q_2 = C_{21}(\Phi_2 - \Phi_1) + C_{22}\Phi_2 + C_{23}(\Phi_2 - \Phi_3) + \cdots + C_{2N}(\Phi_2 - \Phi_N)$$
$$\cdot\ \cdot$$
$$q_N = C_{N1}(\Phi_N - \Phi_1) + C_{N2}(\Phi_N - \Phi_2) + C_{N3}(\Phi_N - \Phi_3) + \cdots + C_{NN}\Phi_N$$
$$(3.52)$$

Of the total charge q_1 on body 1, a portion $C_{12}(\Phi_1 - \Phi_2)$ depends on the difference of potential with body 2. Correspondingly, on body 2, we note that since $C_{12} = C_{21}$, an equal but opposite charge is bound. But

of lines N_{12} terminates on an amount of charge $-q_2 = -Q_1 N_{12}/N_1$ on body 2, while N_{11} corresponds to a total charge

$$-q_1 = \frac{-Q_1 N_{11}}{N_1} = -(Q_1 - q_2)$$

residing on the ground plane. If the total charge on body 2 is Q_2, then since N_{12} lines of flux from body 1 terminate in a charge $-q_2$ on body 2, there must be $N(Q_2 + q_2) = N_{22}$ lines of flux leaving body 2 and terminating on the ground plane. If body 1 is at a potential Φ_1 and body 2 is at a potential Φ_2, then the capacitances C_{ij} are given by

$$C_{11} = \frac{N_{11}}{N\Phi_1} \qquad C_{12} = \frac{N_{12}}{N(\Phi_1 - \Phi_2)} \qquad C_{22} = \frac{N_{22}}{N\Phi_2}$$

and are just proportional to the fraction of the number of lines of flux leaving the body and terminating on adjacent bodies, divided by the potential difference. These results follow from (3.52) as follows. We have

$$Q_1 = C_{11}\Phi_1 + C_{12}(\Phi_1 - \Phi_2) = q_1 + q_2$$

and thus $q_1 = C_{11}\Phi_1 = N_{11}/N$, $q_2 = C_{12}(\Phi_1 - \Phi_2) = N_{12}/N$. Hence, $C_{11} = N_{11}/N\Phi_1$, etc., as stated above.

Electrostatic Shielding

The electric field within a closed conducting surface must be zero if no charge is placed within. For suppose that a field were present; then just inside the conducting surface, which is an equipotential, we could define another equipotential surface but at a different potential. Then over this surface $\mathbf{E}$ is everywhere outward (or inward). However, by Gauss' law, this requires the presence of an interior charge. But this contradicts the original assumption.

This characteristic leads to the use of hollow shells for electrostatic screening. If a body is placed within, it is uninfluenced by external fields. If the internal body is charged, the interior electric field is independent of external effects. The potential difference to the shell is also independent of external sources, but the potential with respect to ground depends on the potential of the shell with respect to ground. This variability is eliminated by grounding the shell.

Some of these remarks are more rigorously formulated through the use of (3.52). For example, based on Fig. 3.17 we have the following:

$$\begin{aligned} q_1 &= C_{11}\Phi_1 + C_{12}\Phi_1 + C_{13}(\Phi_1 - \Phi_3) \\ q_2 &= -C_{12}\Phi_1 - C_{23}\Phi_3 \\ q_3 &= C_{13}(\Phi_3 - \Phi_1) + C_{23}\Phi_3 + C_{33}\Phi_3 \end{aligned} \qquad (3.53)$$

since $\Phi_2 = 0$ because body 2 is grounded. If we choose $q_1 = 0$, then the

field inside body 2 is zero since there is no charge contained within. Therefore the potential Φ_1 must equal Φ_2, that is, equal zero.   From the first equation in the set (3.53), we see this requires that $C_{13} = 0$.   Thus in place of (3.53), we have

$$q_1 = (C_{11} + C_{12})\Phi_1$$
$$q_2 = -C_{12}\Phi_1 - C_{23}\Phi_3 \quad (3.54)$$
$$q_3 = (C_{23} + C_{33})\Phi_3$$

FIG. 3.17. Electrostatic shielding of bodies 1 and 3 by a grounded closed conducting shell 2.

Since body 2 is grounded, we may consider it as part of the ground plane, and then $C_{11} + C_{12}$ is the capacitance of body 1 with respect to ground while $C_{23} + C_{33}$ is the capacitance of body 3 with respect to ground.   Since the potentials of body 1 and 3 depend only on their own charges, there is no interaction between them.   Consequently, body 2 behaves as an electrostatic shield.
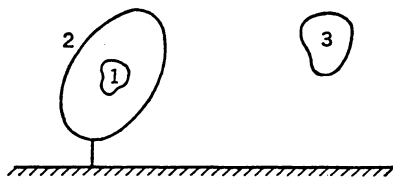
3.6. Electrostatic Energy

The absolute potential due to a single charge q has been found to be

$$\Phi = \frac{q}{4\pi\epsilon_0 R}$$

where R is the distance between the field point and the position of q. Consequently, the energy required to bring a charge q' from infinity to a distance R from q is

$$W_e = \frac{qq'}{4\pi\epsilon_0 R} \qquad (3.55)$$

This energy is what is meant by the potential energy of the charges and is a function of their final position only.

FIG. 3.18. An arbitrary assembly of charges.

Let us evaluate the energy required to assemble an arbitrary distribution of charges of arbitrary magnitudes, such as illustrated in Fig. 3.18. We can think of bringing each charge in from infinity in succession and evaluating the energy required.   Thus no energy is required to introduce the first charge.   The second charge brought in requires an energy as given by (3.55).   For the third charge we must consider the net energy arising from interaction with the first and second charges, and so on.
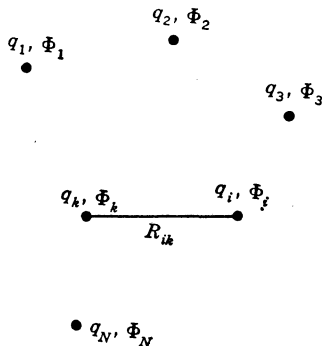
The total energy to establish N charges can be formulated in a relatively simple way. Consider that the kth charge is the last one to be brought in from infinity. Since all others are present, then by superposition the energy required would be

$$(W_e)_k = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N}{}' \frac{q_k q_i}{R_{ik}} \tag{3.56}$$

where R_{ik} is the distance from the ith to kth charge, and the primed summation symbolizes that $i = k$ is not included. The total energy required to assemble all N charges would not be correctly given if (3.56) were summed from $k = 1$ to $k = N$ since that equation is valid only for the last charge brought in; that is, no matter what order the charges are assembled in, the partial energy associated with any two charges, say, kth and jth, equals $q_k q_j / 4\pi\epsilon_0 R_{jk}$, and this term occurs only once during the assembly process. But if (3.56) is summed over k, each interaction occurs twice; for example, a partial energy contribution due to charges q_3 and q_5 is included not only when $k = 3$, $i = 5$, but also when $k = 5$, $i = 3$. This means that a summation of (3.56) over k yields exactly twice the desired value of energy. Then the energy W_e to assemble the charges is simply

$$W_e = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{N}{}' \frac{q_k q_i}{4\pi\epsilon_0 R_{ik}} \tag{3.57}$$

If Φ_k designates the potential at the kth charge due to all other charges, i.e.,

$$\Phi_k = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N}{}' \frac{q_i}{R_{ik}} \tag{3.58}$$

then (3.57) can be written as

$$W_e = \frac{1}{2} \sum_{k=1}^{N} q_k \Phi_k \tag{3.59}$$

We may inquire where the energy associated with a charge distribution is stored. An analogous question can be raised in mechanics if we are given two masses that are attached to the opposite ends of a compressed spring. The seat of the stored energy may be considered to be in the masses. We say that initially they lie in a region of higher potential energy; after expansion of the spring, potential energy is converted into kinetic energy. But one could also consider that the stored energy reposed in the stressed state of the spring. The first viewpoint coincides

with the expression (3.57), where the stored energy is linked to the charges and their positions.   This view is the counterpart of the action-at-a-distance concept, in that it ascribes physical reality to the charges and their spatial distribution alone.   On the other hand, the field concept should be capable of expressing the stored energy without recourse to a description of the charges causing the field, if the concept is to be fully complete.   The energy would then be described in terms of the "elastic" quality of the electric field, in similarity with the viewpoint that the spring stores the energy in the mechanical system considered.   It turns out that the energy associated with a charge distribution can be expressed in terms of the fields alone, and we now proceed to show this.

For a continuous charge distribution, (3.59) can be generalized to read

$$W_e = \tfrac{1}{2} \int_V \rho \Phi \, dV \tag{3.60}$$

where Φ is now a continuous function of position and is the potential distribution due to all charges.   It is no longer necessary to exclude the contribution due to $\rho \, \Delta V$ at the point where Φ is evaluated, since in the limit $\Delta V \to 0$ the contribution is zero anyway.   Since $\nabla \cdot \mathbf{D} = \rho$, we have

$$W_e = \frac{\epsilon_0}{2} \int_V (\nabla \cdot \mathbf{E}) \Phi \, dV$$

Now $\nabla \cdot (\Phi \mathbf{E}) = \Phi \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \Phi$, and hence

$$W_e = \frac{\epsilon_0}{2} \int_V \nabla \cdot (\Phi \mathbf{E}) \, dV - \frac{\epsilon_0}{2} \int_V \nabla \Phi \cdot \mathbf{E} \, dV$$

Using the divergence (Gauss') theorem and noting that $\mathbf{E} = -\nabla \Phi$,

$$W_e = \frac{\epsilon_0}{2} \oint_S \Phi \mathbf{E} \cdot d\mathbf{S} + \frac{\epsilon_0}{2} \int_V E^2 \, dV \tag{3.61}$$

If the original volume extends indefinitely, as it should to include all possible charge, then the surface S in (3.61) extends to infinity.   Considering S as the surface of a sphere of radius R, where $R \to \infty$, we note that $\Phi \propto 1/R$, $E \propto 1/R^2$, so that even though $S \propto R^2$, the integral is of order $1/R$ and vanishes in the limit $R \to \infty$.   We are then left with

$$W_e = \frac{\epsilon_0}{2} \int_V E^2 \, dV \tag{3.62}$$

We have consequently found a way of expressing the stored energy of a charge distribution in terms of the field alone.

The previous analysis, which applies to charges in free space, can be extended to regions involving dielectrics.   In this derivation we shall again consider the true charges as continuously distributed.   If an

amount of charge $\Delta\rho$ is introduced into an existing field Φ, then the energy is increased by an amount ΔW_e, where

$$\Delta W_e = \int_V \Phi \, \Delta\rho \, dV$$

This can also be written as

$$\Delta W_e = \int_V \Phi\Delta(\nabla \cdot \mathbf{D}) \, dV = \int_V \Phi(\nabla \cdot \Delta\mathbf{D}) \, dV \qquad (3.63)$$

Using the identity $\nabla \cdot \Phi \, \Delta\mathbf{D} = \Phi \, \nabla \cdot \Delta\mathbf{D} + \Delta\mathbf{D} \cdot \nabla\Phi$ and the divergence theorem gives

$$\Delta W_e = \int_V [\nabla \cdot (\Phi \, \Delta\mathbf{D}) - \Delta\mathbf{D} \cdot \nabla\Phi] \, dV = \oint_S \Phi \, \Delta\mathbf{D} \cdot d\mathbf{S} - \int_V \Delta\mathbf{D} \cdot \nabla\Phi \, dV$$

The surface integral term goes out for the same reasons as given in discussing (3.61); thus

$$\Delta W_e = \int_V \Delta\mathbf{D} \cdot \mathbf{E} \, dV$$

To obtain the total energy in the field, the above equation must be integrated. This involves the unknown functional relationship between $\mathbf{D}$ and $\mathbf{E}$. For the case where $\mathbf{D} = \epsilon\mathbf{E}$ and ϵ is a constant, we have

$$W_e = \int_0^D \Delta W_e = \int_0^E \int_V \mathbf{E} \cdot \epsilon \, \Delta\mathbf{E} \, dV = \int_V \int_0^E \frac{\epsilon \, dE^2}{2} \, dV$$

or $$W_e = \frac{\epsilon}{2} \int_V E^2 \, dV = \tfrac{1}{2} \int_V \mathbf{E} \cdot \mathbf{D} \, dV \qquad (3.64)$$

The above expression reduces to the earlier one if free-space conditions are supposed.

In (3.64), as in (3.62), we have shown that the potential energy of a charge distribution can be calculated from the field produced by the charges. In the expressions themselves, the quantity $U_e = \mathbf{E} \cdot \mathbf{D}/2$ has the dimensions of energy density. It is often referred to as if it were truly the energy density of the field. However, just as it was impossible to localize the energy as being associated with the charge or the field, it is impossible to relate the total energy of the field with any particular distribution of component energy. Thus $\mathbf{E} \cdot \mathbf{D}/2$ is an energy density only to the extent that its volume integral over all space correctly evaluates the total potential energy.

Example 3.3. Energy Storage in a Capacitor. The parallel-plate capacitor provides a simple example for the application of the results of this section. Conditions are as illustrated in Fig. 3.19, where the plates have an area A and spacing d. For simplicity, one plate may be chosen to have a potential zero; the other is then at potential V. By an exten-

sion of (3.60) we may write

$$W_e = \tfrac{1}{2} \int_S \Phi \rho_s \, dS \qquad (3.65)$$

This is zero over the lower surface, while for the upper surface we have $W_e = \tfrac{1}{2}VQ$.   Since $C = Q/V$, this may also be expressed as

$$W_e = \tfrac{1}{2}CV^2 \qquad (3.66)$$

From the method of derivation it is clear that (3.66) is valid for any two-body capacitor.

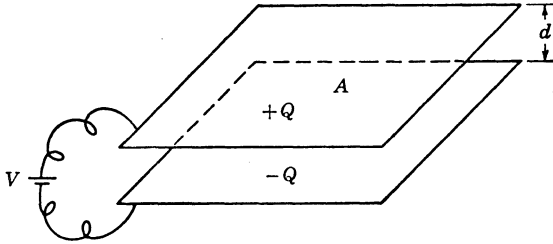We may also calculate this result from the fields.   We note that

FIG. 3.19. A parallel-plate capacitor.

$D = Q/A$ and is uniform within the capacitor and zero outside, neglecting fringing.   Hence,

$$W_e = \tfrac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} \, dV = \frac{1}{2\epsilon_0} \left(\frac{Q}{A}\right)^2 Ad = \frac{Q^2 d}{2\epsilon_0 A} \qquad (3.67)$$

But $C = \epsilon_0 A/d$; hence $W_e = Q^2/2C = CV^2/2$ as before.

This same result can also be obtained in a more familiar way.   The total charge Q on the upper plate is considered as having been transferred from the lower plate in differential increments.   At an intermediate stage the charge is q, and the potential is therefore $\Phi = q/C$.   The energy required to add a charge dq is simply $\Phi \, dq = q \, dq/C$.   Then the work to build up to a charge Q is

$$W_e = \frac{1}{C} \int_0^Q q \, dq = \frac{Q^2}{2C}$$

thus confirming (3.66) once again.

Energy in a Multicapacitor System

The method used to derive (3.59) is also valid for a system of N conducting bodies as illustrated in Fig. 3.14.   The energy stored in a system of N charged conducting bodies can be determined from (3.65).   We note

that over each separate conductor Φ is a constant, and hence the surface integral, if expressed as the sum of surface integrals over each body separately, becomes

$$W_e = \frac{1}{2} \int_S \rho_s \Phi \, dS = \frac{1}{2}\Phi_1 \int_{S_1} \rho_s \, dS + \frac{1}{2}\Phi_2 \int_{S_2} \rho_s \, dS$$
$$+ \cdots + \frac{1}{2}\Phi_j \int_{S_j} \rho_s \, dS + \cdots \quad (3.68)$$

But each integral is just the total charge on each conducting surface. Thus, if we let

$$Q_j = \int_{S_j} \rho_s \, dS \qquad (3.69)$$

we have

$$W_e = \frac{1}{2} \sum_j \Phi_j Q_j \qquad (3.70)$$

If (3.43) is used to express Q_i in terms of the potentials Φ_j, we can rewrite (3.70) as

$$W_e = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N c_{ij} \Phi_i \Phi_j \qquad (3.71)$$

This expression gives the energy stored in a system of N conducting bodies in terms of the potentials of the bodies.

For a two-body problem the capacitance C between the two bodies may be defined in terms of the stored energy in the electric field surrounding the bodies. From (3.66) we have

$$C = \frac{2W_e}{V^2} \qquad (3.72)$$

This alternative definition of capacitance C is often easier to evaluate than some of the other formulations.

3.7. Electrostatic Forces

Figure 3.20 shows a parallel-plate capacitor in a free-space medium with a total charge $+Q$ on one plate and $-Q$ on the other. Because of the presence of charge of opposite sign, one would expect that a force of attraction exists between the plates. Such a force does, of course, exist; our present purpose is to calculate its magnitude. One of the simplest ways of doing this makes use of the principle of virtual work. As applied to this problem one of the capacitor
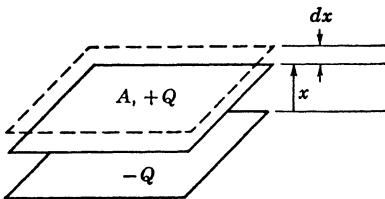

FIG. 3.20. Virtual displacement of one plate in a parallel-plate capacitor.

plates, say the upper one, is visualized as being displaced by an amount Δx away from the lower plate (x is the plate separation and is positive from lower to upper plate, as in Fig. 3.20). As a consequence, work must be performed, and the amount is simply $-F_x \Delta x$, where F_x is the electrostatic force that must be overcome by the external agent. From conservation of energy, this mechanical work must reappear as energy elsewhere, and in this case the stored energy of the field is the only other energy term that could possibly be involved. It must have increased, and the amount of increase is readily established from (3.67) to be

$$\Delta W_e = \frac{Q^2}{2\epsilon_0 A} \Delta x = \frac{\epsilon_0}{2} E^2 A \, \Delta x$$

where A is the area of the plates, and fringing of the field near the edges of the plates is neglected. Consequently,

$$-F_x \Delta x = \frac{\epsilon_0}{2} E^2 A \, \Delta x \tag{3.73a}$$

or in vector form,

$$\frac{\mathbf{F}}{A} = -\frac{\epsilon_0 E^2}{2} \mathbf{a}_x \tag{3.73b}$$

A generalization of this technique can be used to determine the force acting on any conducting body in the presence of an arbitrary distribution of charged conductors in free space. The x component of electrostatic force $\mathbf{F}_i \cdot \mathbf{a}_x$ on the ith body can be found by giving the ith body a virtual displacement $\Delta x_i \mathbf{a}_x$ and equating the work done, $(-\mathbf{F}_i \cdot \mathbf{a}_x) \Delta x_i$, to the net increase in the algebraic sum of all other energy changes that are produced. It is necessary that all energy terms that might be affected by the virtual displacement be included in the energy balance. In general, this includes a change in stored energy that results from the new position of the displaced body. Furthermore, if the potential between two or more bodies is held constant by the presence of a battery, then charge may be transferred by the battery in the virtual process, hence requiring the inclusion of a battery energy term also. If ΔW represents the work done by the batteries, then

$$-(F_i)_x \Delta x_i + \Delta W = \Delta W_e \tag{3.74}$$

or

$$(F_i)_x = \frac{\Delta W}{\Delta x_i} - \frac{\Delta W_e}{\Delta x_i} \tag{3.75}$$

The remaining force components are found by replacing x by y and z in turn.

Certain constraints are evident in a system of conducting charged bodies during the processes of displacing one or several of them. Those charged bodies that are isolated are constrained by the requirement that

the total charge be kept constant in the process; their potential may vary, however. Those bodies linked together by batteries will be constrained to maintain a constant relative potential; in this case the total charge on each body may change. If one or several bodies are actually moved through a finite distance and it is desired to calculate the total work done on the body by setting up an energy balance, then obviously the true constraints of the system must be known and utilized. However, if it is only required that the force, for a particular configuration, be evaluated, then the virtual work process is conceptual only. Consequently, we may set up hypothetical constraints in the knowledge that so long as the resulting energy balance is correctly described, the correct value of force will be determined. The force that exists under a given set of conditions has a definite value, and this value cannot be affected by the constraints that are assumed under a purely hypothetical displacement of one of the bodies. However, if a constraint is assumed, then the correct consequences of that constraint must be included in the energy calculations. We consider now two cases, that of constant charge and that of constant potential.

Constant Charge

Since the constraint requires that each conducting body be isolated, the possibility that energy be supplied by a battery source does not occur. In the limit $\Delta x_i \to 0$, (3.75) may be written as

$$(F_i)_x = -\left. \frac{\partial W_e}{\partial x_i} \right|_{\text{constant charge}} \tag{3.76}$$

where x_i is the x variable at the ith body. Equation (3.70) can be used to evaluate W_e. When this equation is substituted into (3.76) and the fact that Q_j is constant is made use of, we find that

$$(F_i)_x = -\tfrac{1}{2} \sum_j Q_j \frac{\partial \Phi_j}{\partial x_i} \tag{3.77}$$

Constant Potential

In order to maintain each body at a constant potential when one body suffers a virtual displacement, it may be necessary to transfer charge to any or all of the conducting bodies in the system. This is performed by a battery, and the work involved must be included in an energy balance. If dQ_j is added to the jth body as a consequence of the virtual displacement, the energy supplied is $\Phi_j \, dQ_j$, and the total of such energy is

$$dW = \sum_j \Phi_j \, dQ_j \tag{3.78}$$

Combining this equation with (3.70) and (3.75), we have in the limit

$$(F_i)_x = \sum_j \Phi_j \frac{\partial Q_j}{\partial x_i} - \frac{1}{2} \sum_j \Phi_j \frac{\partial Q_j}{\partial x_i}$$

$$= \frac{1}{2} \sum_j \Phi_j \frac{\partial Q_j}{\partial x_i} = \frac{\partial W_e}{\partial x_i} \bigg|_{\text{constant potential}} \qquad (3.79)$$

Note that the second summation makes use of the constant-potential constraint. Equation (3.79) expresses the very interesting fact that in a virtual displacement under a constant-potential constraint the work done by the field is equal to the increase in stored energy. The energy balance is explained by the fact that the battery supplies an energy equal to the sum of the mechanical and field terms.

The above results are applicable to the evaluation of torques also. It is only necessary to interpret F_i as a torque and dx_i as a small rotation $d\theta_i$.

Example 3.4.   Force on a Capacitor Plate. The force between plates of a parallel-plate capacitor is given by (3.73). The method used to obtain this result involved the principle of virtual work together with a constraint of constant charge. Note that the result can be obtained immediately from (3.77). As pointed out above, the constraint can also be chosen as that of constant potential. Such a constraint can be interpreted physically as being due to the connection of a battery across the capacitor. Let us calculate the force under such assumed conditions. For our present purpose Fig. 3.19 correctly describes the problem. We assume that the upper plate is pushed away from the lower plate by an amount dx, where x is the plate separation. Since $W_e = \frac{1}{2}CV^2$,

$$dW_e = \frac{1}{2}V^2 \, dC = \frac{1}{2}V^2 \, d\left(\frac{\epsilon_0 A}{x}\right) = -\frac{1}{2}V^2 \frac{\epsilon_0 A}{x^2}\bigg|_{x=d} dx$$

$$= -\frac{1}{2}\frac{V^2 C}{d} \, dx$$

Since V remains constant, Q must vary in view of the change in C. Thus

$$Q = CV$$

$$dQ = V \, dC = -\frac{V}{d} C \, dx$$

The change in Q is accomplished by the transfer of the charge dQ by the battery. The battery therefore does an amount of work $V \, dQ$. This plus the work done by an external force acting against the field, $-F_x \, dx$,

must equal the increase in stored energy. Consequently,

$$\frac{-V^2 C}{d} \, dx - F_x \, dx = -\frac{1}{2} \frac{V^2 C}{d} \, dx$$

or

$$F_x = -\frac{1}{2} \frac{V^2 C}{d} = -\frac{\epsilon_0 A E^2}{2}$$

The result confirms that given by (3.73). Note that it is obtained almost at once by using (3.79).

The above results are easily generalized to the case of an arbitrary two-body capacitor with capacitance C. For an arbitrary virtual displacement dr, the electrostatic force acting in the direction of dr is readily found if the rate of change $\partial C/\partial r$ can be found. For a constant-charge constraint we write $W_e = Q^2/2C$, and hence from (3.76),

$$F_r = -\frac{\partial W_e}{\partial r} = -\frac{Q^2}{2} \frac{\partial C^{-1}}{\partial r} = \frac{Q^2}{2C^2} \frac{\partial C}{\partial r} \tag{3.80}$$

For a constant-potential constraint we may write $W_e = CV^2/2$ in place of $Q^2/2C$, and since V is constant, we have, from (3.79),

$$F_r = \frac{\partial W_e}{\partial r} = \frac{V^2}{2} \frac{\partial C}{\partial r} = \frac{Q^2}{2C^2} \frac{\partial C}{\partial r} \tag{3.81}$$

since $V = Q/C$. The force F_r is, of course, the same in both cases, since for a given charge on the capacitor a unique field exists around the conductors, and hence the force acting on the conductors is also unique. The existence of the force does not depend on the virtual displacement or constraints that are assumed. The virtual displacement is conceptual and, together with assumed constraints and an energy balance, permits the force exerted by the field to be evaluated. Any change in the assumed constraints cannot change the force acting; it only changes the details of the energy-balance equation.

Forces in the Presence of Dielectrics

In the previous paragraphs we showed how the force on a charged body in an electrostatic field could be found. The equations that were derived assumed that the conducting bodies lie in a free-space medium. The results obtained, however, would still apply in the presence of dielectrics; that is, the process whereby a virtual displacement is given a particular body, and the consequent mechanical work equated to the increase in field energy less the work performed by the batteries, still applies. However, the change in energy stored in the field must now include energy storage in the dielectrics.

If a dielectric body which lies in an electric field is given a virtual displacement, then the energy stored in the field may be expected to change

in view of its dependence on the geometry.   As a consequence it follows that dielectric bodies in electrostatic fields will experience a net force. This force may be calculated from the principle of virtual work, just as in the case for conducting bodies.   The following example illustrates this technique.

Example 3.5.   Force on a Dielectric Slab.   A parallel-plate capacitor of width W, separation d, and length L is partially filled with a uniform
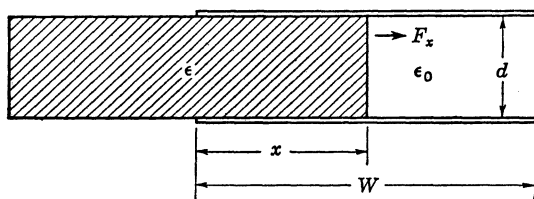

FIG. 3.21. Force on a dielectric slab in a parallel-plate capacitor.

dielectric slab, of permittivity ϵ, as illustrated in Fig. 3.21.   The force acting on the dielectric slab is desired.

With the geometry as given, the total stored energy is calculated to be

$$W_e = \tfrac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 (W - x)Ld + \tfrac{1}{2}\epsilon \left(\frac{V}{d}\right)^2 xLd + W_1$$

where W_1 is a correction term that takes account of the fringing of the field at the sides.   Let us postulate a virtual displacement Δx of the dielectric body in the positive x direction.   We shall arbitrarily consider this to occur under the condition of constant potential.   The work done by the field, $F_x \Delta x$, equals the increase in stored energy.   We have

$$F_x \Delta x = \Delta W_e = \tfrac{1}{2}\epsilon \left(\frac{V}{d}\right)^2 Ld\, \Delta x - \tfrac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 Ld\, \Delta x + \frac{\partial W_1}{\partial x} \Delta x$$

$$= \tfrac{1}{2}E^2(\epsilon - \epsilon_0)Ld\, \Delta x \qquad\qquad (3.82)$$

since the correction term W_1 does not change if the ends of the dielectric slab are not too close to the plate edges, i.e., if the fringing field remains constant.   The direction of the force is such as to draw the dielectric slab farther into the air gap of the capacitor since $\epsilon > \epsilon_0$ always.   The pressure P_x, in the x direction, is

$$P_x = \frac{F_x}{Ld} = \tfrac{1}{2}(\epsilon - \epsilon_0)E^2 \qquad\qquad (3.83)$$

and is the force per unit area exerted by the field on the plane end of the slab.

3.8. Electrostatic Forces (Maxwell Stress Tensor)

The total force acting on a volume distribution of charge ρ, which lies in an external electric field $\mathbf{E}$, is given by

$$\mathbf{F} = \int \mathbf{F}_v \, dV \tag{3.84}$$

where the volume density of force is

$$\mathbf{F}_v = \rho \mathbf{E} \tag{3.85}$$

As expressed by (3.84) and (3.85), the point of view that the force arises from an action at a distance is still manifest, even though the sources of the $\mathbf{E}$ field do not enter the calculation explicitly. The formulation is actually hybrid since it involves both fields and charge. According to the Faraday-Maxwell field theory, it should be possible to evaluate forces completely in terms of the field alone; that is, if the field is the means whereby forces are transmitted, one should be able to determine these forces by considering only the field. One can think of the transmission of forces via the field in the same way as one views that transmitted by a stretched rubber band. Indeed, one may view the flux lines as being elastic, the force action being a consequence of their state of stress. This implies that if a charge distribution in an external field is surrounded by an arbitrary surface area, then the total force acting on the charges contained must, in a sense, cross this area. But then it should be possible to calculate the total force by integrating over the arbitrary surface a "stress function" that depends, at each point, only on the field at that point and the direction of the surface area. Put mathematically, it suggests the existence of a force $\mathbf{T} \, dS$ which acts on an element of an arbitrary closed surface that contains charge and which satisfies the relation

$$\int_V \mathbf{F}_v \, dV = \int_S \mathbf{T} \, dS \tag{3.86}$$

The existence of a function $\mathbf{T}$ was asserted by Maxwell and forms an important part of the field concept.

We proceed now to show that the surface force $\mathbf{T}$ does exist and to derive an expression for it. This will require some vector manipulation so that the volume integral of (3.86) may be transformed into surface integrals. We begin by substituting for $\mathbf{F}_v$ the value given in (3.85) and note that $\nabla \cdot \mathbf{D} = \rho$; thus

$$\int_V \mathbf{F}_v \, dV = \int_V \rho \mathbf{E} \, dV = \int_V \mathbf{E} \nabla \cdot \mathbf{D} \, dV$$

If $\mathbf{E}$ is now expanded into rectangular components, we obtain

$$\int_V \mathbf{F}_v \, dV = \mathbf{a}_x \int_V E_x \nabla \cdot \mathbf{D} \, dV + \mathbf{a}_y \int_V E_y \nabla \cdot \mathbf{D} \, dV + \mathbf{a}_z \int_V E_z \nabla \cdot \mathbf{D} \, dV$$

Using vector identity (1.118), each term may be written as

$$\int_V \mathbf{F}_v \, dV = \mathbf{a}_x \int_V \nabla \cdot E_x \mathbf{D} \, dV + \mathbf{a}_y \int_V \nabla \cdot E_y \mathbf{D} \, dV + \mathbf{a}_z \int_V \nabla \cdot E_z \mathbf{D} \, dV$$

$$- \mathbf{a}_x \int_V \mathbf{D} \cdot \nabla E_x \, dV - \mathbf{a}_y \int_V \mathbf{D} \cdot \nabla E_y \, dV - \mathbf{a}_z \int_V \mathbf{D} \cdot \nabla E_z \, dV \tag{3.87}$$

The first three right-hand terms of (3.87) can be integrated by Gauss' theorem, and then summed, whereupon the sum becomes equal to $\oint \mathbf{E}(\mathbf{n} \cdot \mathbf{D}) \, dS$. Concerning the

remaining three terms, let us consider the coefficient of $\mathbf{a}_x$ in some detail. Expanding in rectangular coordinates we obtain

$$\mathbf{a}_x \int_V \mathbf{D} \cdot \nabla E_x \, dV = \mathbf{a}_x \int_V \left(D_x \frac{\partial E_x}{\partial x} + D_y \frac{\partial E_x}{\partial y} + D_z \frac{\partial E_x}{\partial z} \right) dV$$

But since $\nabla \times \mathbf{E} = 0$, it follows that $\partial E_x/\partial y = \partial E_y/\partial x$ and $\partial E_x/\partial z = \partial E_z/\partial x$. Accordingly,

$$\mathbf{a}_x \int_V \mathbf{D} \cdot \nabla E_x \, dV = \mathbf{a}_x \int_V \left(D_x \frac{\partial E_x}{\partial x} + D_y \frac{\partial E_y}{\partial x} + D_z \frac{\partial E_z}{\partial x} \right) dV$$

$$= \mathbf{a}_x \frac{\epsilon_0}{2} \int_V \frac{\partial}{\partial x} (E_x{}^2 + E_y{}^2 + E_z{}^2) \, dV \tag{3.88}$$

A similar expression is obtained for the $\mathbf{a}_y$ coefficient and the $\mathbf{a}_z$ coefficient, except that $\partial/\partial x$ is replaced by $\partial/\partial y$ and $\partial/\partial z$, respectively. Equation (3.87) may now be written as

$$\int_V \mathbf{F}_v \, dV = \oint_S \mathbf{E}(\mathbf{n} \cdot \mathbf{D}) \, dS - \frac{\epsilon_0}{2} \int_V \nabla(E^2) \, dV \tag{3.89}$$

The volume integral on the right-hand side can be converted to a surface integral using vector identity (1.127) (also see Prob. 3.10). As a result,

$$\int_V \mathbf{F}_v \, dV = \oint_S \epsilon_0 \left[(\mathbf{n} \cdot \mathbf{E})\mathbf{E} - \frac{E^2}{2} \mathbf{n} \right] dS \tag{3.90}$$

We can now identify (3.86) with (3.90), and therefore

$$\mathbf{T} = \epsilon_0 \left[(\mathbf{n} \cdot \mathbf{E})\mathbf{E} - \frac{E^2}{2} \mathbf{n} \right] \tag{3.91}$$

Some idea of the vector relationships involved between $\mathbf{T}$, $\mathbf{E}$, and $\mathbf{n}$ can be obtained from Fig. 3.22. An element of surface is illustrated, and this determines the direction of $\mathbf{n}$, of course. The field $\mathbf{E}$ is illustrated as making an arbitrary angle θ with $\mathbf{n}$. The surface force $\mathbf{T}$ is described by (3.91). From (3.91) we see that $\mathbf{T}$ has a component in the direction of $\mathbf{n}$ which is either a compression or tension force, while that in the direction $\mathbf{E}$ involves a shearing force, in general. Furthermore, $\mathbf{T}$ must lie in the plane determined by $\mathbf{E}$ and $\mathbf{n}$. If the components of $\mathbf{T}$ in the direction normal (T_n) and tangential (T_t) to the surface are computed, we find

$$T_n = \epsilon_0 \left(E^2 \cos^2 \theta - \frac{E^2}{2} \right) = \frac{\epsilon_0 E^2}{2} \left(2 \cos^2 \theta - 1 \right)$$

$$= \frac{\epsilon_0 E^2}{2} \cos 2\theta \tag{3.92a}$$

$$T_t = \epsilon_0 E^2 \cos \theta \sin \theta = \frac{\epsilon_0 E^2}{2} \sin 2\theta \tag{3.92b}$$

This clearly requires that $\mathbf{T}$ have a magnitude $\epsilon_0 E^2/2$ and make an angle 2θ with $\mathbf{n}$, as illustrated in Fig. 3.22.

To summarize, the force exerted on a charged region can be computed by integrating the surface force $\mathbf{T}$ over any bounding surface. At each surface element, $\mathbf{E}$ bisects the angle between $\mathbf{T}$ and the surface normal. Where $\mathbf{E}$ and $\mathbf{n}$ are in the same direction, the force is pure tension. For $\theta = 45°$, the stress transmitted is pure shearing stress. As
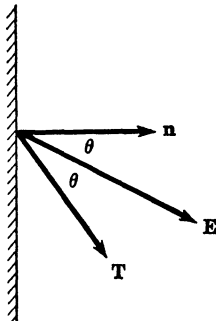
FIG. 3.22. Relation between $\mathbf{n}$, $\mathbf{E}$, and $\mathbf{T}$ at a plane surface.

θ increases further, the shearing stress diminishes until at $\theta = 90°$ it is pure pressure. The magnitude of **T** remains equal to $\epsilon_0 E^2/2$ and is independent of the angle between **E** and **n**.

It is also possible to write

$$\int_S \mathbf{T}\, dS = \int_S \boldsymbol{\tau} \cdot d\mathbf{S} \tag{3.93}$$

in which case $\boldsymbol{\tau}$ has the properties of a tensor. Maxwell actually expressed his results in this form; consequently $\boldsymbol{\tau}$ is referred to as the Maxwell stress tensor.

The result expressed by (3.91) has been derived only for forces involving charges in free space. If dielectric bodies are also present, then the volumetric force is no longer
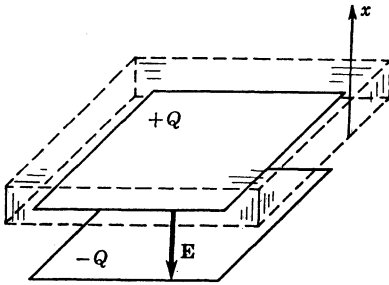

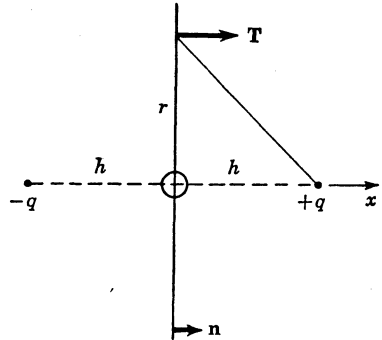FIG. 3.23. Closed surface for evaluating force on a capacitor plate.

FIG. 3.24. Evaluation of force between two point charges.

given by the simple expression (3.85) at every point. It turns out, however, that if the surface of integration lies completely in free space, then (3.90) correctly evaluates the net force exerted on enclosed dielectric bodies in addition to that exerted on enclosed true charge. This result is not surprising since we know that the external behavior of a dielectric body can be interpreted in terms of an equivalent true charge distribution (polarization charge).

The full meaning of the surface force **T** may be more apparent if we consider some examples. First let us note how it could have been used to determine the force between the charged parallel plates illustrated in Fig. 3.20. To find the force on the upper plate, a rectangular parallelepiped may be envisioned surrounding that plate, as shown in Fig. 3.23. Since **E** lies only between the plates, a nonzero value of **E** appears only at the lower surface of the parallelepiped. In view of the uniformity of **E**,

$$\int_S \mathbf{T}\, dS = \frac{-\epsilon_0 E^2 \mathbf{a}_x}{2} \int_S dS = \frac{-\epsilon_0 E^2}{2} A\, \mathbf{a}_x$$

and the result of (3.73) is given at once.

In Fig. 3.24 we have two equal and opposite charges separated by a distance $2h$. The force exerted on the negative charge will be found by integrating **T** dS over an infinite plane surface (closed at infinity). The line joining $-q$ and $+q$ is bisected by the surface and is perpendicular to it. The geometry is then precisely that in Fig. 2.18; consequently, the electric field over the plane is given by (2.66). Since **E** is always normal to the surface and in the negative x direction, the force exerted on $-q$

is in the positive x direction. Its magnitude is given by

$$
\begin{aligned}
F_{-q} &= \frac{\epsilon_0}{2}\left(\frac{q2h}{4\pi\epsilon_0}\right)^2 \int_0^\infty \int_0^{2\pi} \frac{r\,dr\,d\theta}{(h^2+r^2)^3} \\
&= -\pi\epsilon_0 \left(\frac{q2h}{4\pi\epsilon_0}\right)^2 \frac{1}{4(h^2+r^2)^2}\Big]_0^\infty \\
&= \frac{q^2}{4\pi\epsilon_0(2h)^2}
\end{aligned}
\tag{3.94}
$$

We note that the final result correctly corresponds to Coulomb's law of force.

3.9. Molecular Fields

In Sec. 3.1 some of the details of the polarization of dielectric materials were discussed. Basically, we noted that under the influence of an applied electric field the molecules of a dielectric would become polarized. Further, the dipole polarization was noted to be, in general, proportional to the applied field for field strengths well below the saturation level.

In order to actually calculate the polarization of a given material, it is necessary to know the polarizability of the material and the field. The latter must be the field that actually exists at the molecule. For gases, this is to a good approximation the applied field, i.e., the field that exists in the absence of the gas. Actually, the field at a particular molecule is the superposition of the applied field and that due to its neighboring polarized molecules. But for a gas, the separation of the molecules is sufficiently great so that the effect of adjacent dipoles on one another can be neglected.
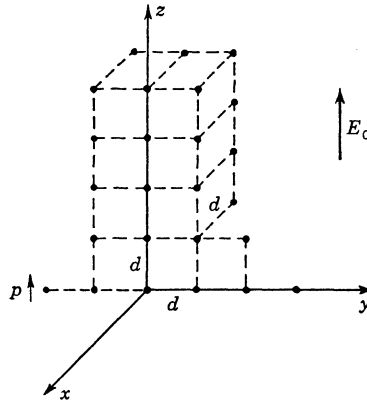

FIG. 3.25. An infinite cubical array of atoms.

For a solid dielectric, dipole-dipole interaction must be considered. In general, this may be fairly difficult to express analytically. We shall consider in some detail a specific, and common, configuration to illustrate the principle. We shall assume that the dielectric consists of a cubical array of atoms (or molecules) with spacing d between each atom along the x, y, z axis, as in Fig. 3.25. The medium is assumed infinite in extent, and a field $\mathbf{E}_0$ is applied in the z direction. Under the influence of the field $\mathbf{E}_0$, each atom becomes polarized with a dipole moment $\mathbf{p}$. Since there are $N = 1/d^3$ atoms per unit volume, the dipole moment per unit volume is $\mathbf{p}/d^3 = \mathbf{P}$. The field acting to polarize the atom at the origin (or any other atom) is the sum of the applied field $\mathbf{E}_0$ plus an interaction field $\mathbf{E}_i$. The interaction field is the field produced by all the neighboring dipoles. The polarizing field $\mathbf{E}_p$ may be written as

$$
\mathbf{E}_p = \mathbf{E}_0 + \mathbf{E}_i
\tag{3.95}
$$

and hence the induced dipole moment in each atom is

$$
\mathbf{p} = \alpha_e \mathbf{E}_p
\tag{3.96}
$$

where α_e is the polarizability of the atom. When the material behaves linearly, $\mathbf{p}$, $\mathbf{E}_i$, and $\mathbf{E}_p$ are all proportional to the applied field $\mathbf{E}_0$, and hence we may take $\mathbf{E}_i$ propor-

tional to **p**. Thus

$$\mathbf{E}_i = C\mathbf{p} \qquad (3.97)$$

where C is called the field interaction constant. Equation (3.96) now becomes

$$\mathbf{p} = \alpha_e(\mathbf{E}_0 + \mathbf{E}_i) = \alpha_e\mathbf{E}_0 + \alpha_e C\mathbf{p}$$

or

$$\mathbf{p} = \frac{\alpha_e\mathbf{E}_0}{1 - \alpha_e C} \qquad (3.98)$$

During the course of the analysis it will be shown that the average total electric field **E** in the dielectric is equal to the applied field $\mathbf{E}_0$. Hence the displacement flux density **D** is given by

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E}_0 + \frac{N\alpha_e\mathbf{E}_0}{1 - \alpha_e C} = \epsilon\mathbf{E} \qquad (3.99)$$

where $N = d^{-3}$ is the number of dipoles per unit volume. Solving for the permittivity ϵ gives

$$\epsilon = \epsilon_0 + \frac{N\alpha_e}{1 - \alpha_e C} \qquad (3.100)$$

a result known as the Clausius-Mossotti equation. Assuming that α_e is known and that the interaction constant C can be found, we can compute ϵ from (3.100). The interaction constant C is a function of the lattice geometry only.

Evaluation of Interaction Constant

The interaction constant C may be found by evaluating the field at the origin as produced by all the neighboring dipoles in the lattice illustrated in Fig. 3.25. From symmetry considerations it is readily seen that this field has a z component only. The scalar potential due to a dipole of moment p and located at (x',y',z') is

$$\Phi(x,y,z) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{R^2} = \frac{p(z - z')}{4\pi\epsilon_0 R^3}$$

since $z - z' = R\cos\theta$, and $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$. The z component of electric field is

$$\begin{aligned} E_z &= -\frac{\partial\Phi}{\partial z} = \frac{-p}{4\pi\epsilon_0}\left[\frac{1}{R^3} - \frac{3(z - z')^2}{R^5}\right] \\ &= \frac{p}{4\pi\epsilon_0}\frac{2(z - z')^2 - (x - x')^2 - (y - y')^2}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{5/2}} \end{aligned} \qquad (3.101)$$

Now the dipoles are located at $x' = nd$, $y' = md$, $z' = sd$ ($n, m, s = 0, \pm 1, \pm 2, \ldots$, but $n = m = s = 0$ excluded). Summing (3.101) over all the dipoles and placing $x = y = z = 0$, we obtain the following expression for the interaction field at the origin:

$$E_i = \frac{p}{4\pi\epsilon_0}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}{\sum_{s=-\infty}^{\infty}}'\frac{2(sd)^2 - (nd)^2 - (md)^2}{[(sd)^2 + (nd)^2 + (md)^2]^{5/2}} \qquad (3.102)$$

where the prime means omission of the term $n = m = s = 0$, which corresponds to the dipole at the origin.

The above series may be transformed into a rapidly converging series by use of the Poisson summation formula, and the sum may then be evaluated. For our purpose we shall approximate the sum by a triple integral. Let us introduce the variables

$x = n$, $y = m$, $z = s$ and the differentials $dx = dn$, $dy = dm$, $dz = ds$. The integration extends from $|x|$, $|y|$, $|z|$ equal to unity out to infinity. Since all quantities are squared, we may integrate over one octant only and multiply the result by 8. We now have

$$E_i = \frac{8p}{4\pi\epsilon_0 d^3} \int_1^\infty \int_1^\infty \int_1^\infty \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}\, dx\, dy\, dz$$

Since the integrand was originally obtained by differentiating with respect to z, we may integrate with respect to z at once to get

$$E_i = \frac{2p}{\pi\epsilon_0 d^3} \int_1^\infty \int_1^\infty \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \Big|_1^\infty dx\, dy$$
$$= \frac{2p}{\pi\epsilon_0 d^3} \int_1^\infty \int_1^\infty \frac{dx\, dy}{(x^2 + y^2 + 1)^{3/2}}$$

With the substitution $y = (x^2 + 1)^{1/2} \tan\theta$ and the limits $\sin^{-1}(2 + x^2)^{-1/2}$ to $\pi/2$ on θ, the integration over y is readily performed (the integrand becomes simply $\cos\theta\, d\theta$) to give

$$E_i = \frac{2p}{\pi\epsilon_0 d^3} \int_1^\infty \left[\frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)(x^2 + 2)^{1/2}} \right] dx$$

The first term gives

$$\int_1^\infty \frac{dx}{1 + x^2} = \tan^{-1} x \Big|_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

In the second term replace $x^2 + 1$ by u^2 to get

$$-\int_{\sqrt{2}}^\infty \frac{du}{u(u^4 - 1)^{1/2}} = -\tfrac{1}{2} \sec^{-1} u^2 \Big|_{\sqrt{2}}^\infty = -\frac{\pi}{4} + \frac{\pi}{6}$$

The final value for E_i is thus

$$E_i = \frac{2p}{\pi\epsilon_0 d^3} \left(\frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{4} \right) = \frac{p}{3\epsilon_0 d^3} \tag{3.103}$$

From (3.97) we find that the interaction constant C is

$$C = \frac{E_i}{p} = \frac{1}{3\epsilon_0 d^3} = \frac{N}{3\epsilon_0} \tag{3.104}$$

If the series (3.102) is summed, it is found that the interaction constant C is equal to $0.34/d^3\epsilon_0$, and hence we can conclude that the triple integral is a good approximation to the triple sum.

If we use the above value of C in (3.100), we find that the permittivity is given by

$$\epsilon = \epsilon_0 + \frac{N\alpha_e}{1 - N\alpha_e/3\epsilon_0}$$

or

$$\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} = \frac{N\alpha_e}{3\epsilon_0} \tag{3.105}$$

For a gas, ϵ is only slightly greater than ϵ_0, so that (3.105) becomes

$$\frac{\epsilon - \epsilon_0}{3\epsilon_0} = \frac{N\alpha_e}{3\epsilon_0}$$

or

$$\epsilon = \epsilon_0 + N\alpha_e \tag{3.106}$$

which, upon comparison with (3.100), is seen to be equivalent to neglecting the inter-action field. By combining (3.104) and (3.95) we find that the polarizing field is given by

$$\mathbf{E}_p = \mathbf{E}_0 + C\mathbf{p} = \mathbf{E}_0 + \frac{\mathbf{P}}{3\epsilon_0} \tag{3.107}$$

As a final step we shall show that the average electric field in the dielectric is equal to the applied field $\mathbf{E}_0$. To show this we must evaluate the average field produced by all the dipoles. The average is to be taken over the volume of a unit cell. We have already found a value for the interaction field, which is the field set up by all the dipoles except one. If we add to this the average field set up by the individual dipole, we shall obtain the average field produced by all the dipoles in the lattice. At this point it should be noted that replacing the triple sum in (3.102) by a triple integral is equivalent to determining an average value E_i for the interaction field.

The z component of the field produced by a single dipole, say the dipole at the origin, is given by (3.101) with x', y', and z' equal to zero. The average value of this field is E_{1a}, where

$$E_{1a} = \frac{8p}{4\pi\epsilon_0 d^3} \int_0^{d/2} \int_0^{d/2} \int_0^{d/2} \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \, dx \, dy \, dz \tag{3.108}$$

In (3.108) the integration is taken over one octant of the unit cell, which has a volume of $d^3/8$. This is equivalent to averaging over the unit cell because of the symmetry of the integrand. The above may be integrated in the same manner as the integral for E_i was performed. The reader may readily verify by carrying out the details that the end result is simply

$$E_{1a} = -\frac{p}{3\epsilon_0 d^3} \tag{3.109}$$

This result is interesting in that it shows that the average field produced by a single dipole in its own unit cell is the negative of the average field produced by all the neighboring dipoles. Consequently, the average field produced by all the dipoles in the lattice is $(E_d)_{av}$ and vanishes since

$$(E_d)_{av} = E_i + E_{1a} = \frac{p}{3\epsilon_0 d^3} - \frac{p}{3\epsilon_0 d^3} = 0$$

This completes the proof that the average field in the dielectric is just equal to the applied field E_0.

The above analysis clearly shows that the effective field acting to polarize a given molecule is not equal to E_0, for this is the average field in the dielectric and includes the contribution from the molecule itself. The field acting to polarize the molecule is equal to E_0 minus the contribution to the total average field from the molecule itself. This gives $E_0 - (-p/3\epsilon_0 d^3) = E_0 + p/3\epsilon_0 d^3$ for the polarizing field, a result in agreement with (3.107).

Chapter 3

3.1. (a) If an arbitrarily shaped dielectric of volume V is placed in an electric field, a dielectric polarization $\mathbf{P}$ results which is also equivalent to a charge density $-\nabla \cdot \mathbf{P}$ and a surface charge density $\mathbf{P} \cdot \mathbf{n}$. Since the dielectric is electrically neutral, the total induced charge must equal zero. Show this by making use of the divergence theorem.

(b) Consider a specific example where the body is a rectangular parallelepiped whose axis extends from $z = -l/2$ to $z = l/2$ and with a cross-sectional area A. Given that $\mathbf{P} = (Az^2 + B)\mathbf{a}_z$, determine the volume and surface charge density and show explicitly that the total charge is zero.

3.2. Consider a parallel-plate capacitor with sides a, b, and spacing d. The capacitor is half-filled with dielectric (0 to $a/2$, 0 to b) of relative dielectric constant κ. A potential V exists between the plates. Calculate the charge density on the plates and also the equivalent surface polarization charge on the dielectric surfaces. Neglect fringing effects.

3.3. A solid dielectric cylinder of length L and radius a is uniformly polarized with polarization **P**, where **P** is directed axially. Determine the electric field along the cylinder axis both within and outside the cylinder.

3.4. For a field strength of 3×10^6 volts per meter determine the relative displacement and nucleus and electron cloud for He and Ne. Compare with the radius of the atom.

3.5. An infinite dielectric slab of thickness t is placed in a uniform external field $\mathbf{E}_0$. The slab is inclined at an angle θ_1 to the field $\mathbf{E}_0$. Find the angle θ_1 such that the electric lines of flux in the slab make an angle $\theta_2 = \pi/4$ with the sides of the slab, i.e., so that $\theta_2 = \pi/4$. The dielectric constant is $\kappa = 4$. Find the density of surface polarization charge on the two faces of the slab.
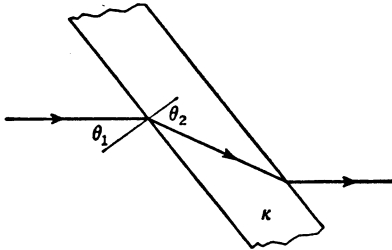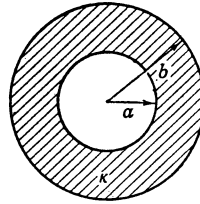

FIG. P 3.5   FIG. P 3.6

3.6. (a) Consider two coaxial cylinders of inner radius a and outer radius b, as illustrated. The space between cylinders is filled with a dielectric with permittivity $\epsilon = \kappa\epsilon_0$. Find the capacitance per meter length of cylinders.

(b) For a given difference of potential between the inner and outer conductors determine the magnitude of b/a that causes the greatest value of **E** to be a minimum under the restriction that the radius b is fixed.

3.7. Consider two coaxial cylinders, the inner having a radius a and the outer a radius c. The space $a < r < b$, where $b < c$, is filled with a dielectric with a dielectric constant κ. Find the capacitance per unit length. Show that the capacitance C is equal to the series capacitance of C_1 and C_2, where C_1 is given in Prob. 3.6 and C_2 is the capacitance of an air-filled coaxial cylinder capacitor with inner radius b and outer radius c.
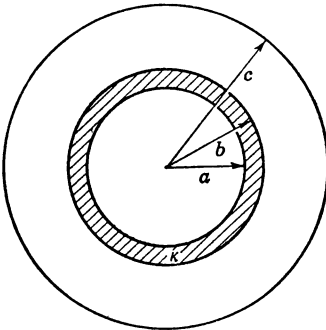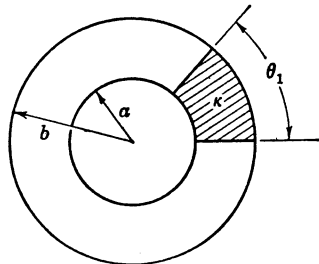

FIG. P 3.7   FIG. P 3.8

3.8. Consider two coaxial cylinders with the intervening space $0 < \theta < \theta_1$ filled with a dielectric with a dielectric constant κ. Find the capacitance per unit length,

and show that this capacitance is equal to the parallel combination of the capacitances of the air-filled section and the dielectric-filled section.

HINT: Note that the field E_r is independent of θ and depends only on the potential difference between the cylinders.

3.9. A coaxial cable consists of a copper inner conductor of 0.5 centimeter radius, a solid dielectric of permittivity 4.8 to a radius r_0, and a layer of oil ($\kappa = 2.4$) from r_0 to a radius of 2.0 centimeters. The dielectric strength of the solid dielectric is $E_d = 40 \times 10^3$ kilovolts per meter, while that for the oil is $E_0 = 30 \times 10^3$ kilovolts per meter. Find the value of r_0 and the value of potential that will result in maximum stress in both media.

3.10. Show that

$$\int_V \nabla E^2 \, dv = \oint_S E^2 \mathbf{n} \, dS$$

HINT: Consider $\int_V \nabla \cdot (\Psi \mathbf{A}) \, dv$, where $\mathbf{A}$ is a constant vector and $\Psi = E^2$, and utilize the divergence theorem.

3.11. Prove that the following relationship exists between the polarization $\mathbf{P}$ and equivalent volume and surface charge densities ρ_p and ρ_{sp} for an arbitrary volume V:

$$\int_V \mathbf{P} \, dv = \int_V \rho_p \mathbf{r} \, dv + \int_S \rho_{sp} \mathbf{r} \, dS$$

In the above, $\mathbf{r}$ is the position vector from an arbitrary origin, and $\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$.

HINT: Note that $\int_V x \nabla \cdot \mathbf{P} \, dv = \int_V \nabla \cdot \mathbf{P} x \, dv - \int_V \mathbf{P} \cdot \nabla x \, dv$, and similarly for y and z.

3.12. Consider an arbitrary distribution of point charges q_i $(i = 1, 2, \ldots, n)$, and let the potential at the ith charge due to all others be given by Φ_i; that is,

$$\Phi_i = \sum_{\substack{j=1 \\ j \neq i}}^{j=n} \frac{q_j}{4\pi\epsilon_0 r_{ij}}$$

where r_{ij} is the separation of the ith and jth charge. Consider another system of charges located at the same points but with magnitudes q_i' $(i = 1, 2, 3 \ldots, n)$ and corresponding to which the potential at the ith point is Φ_i', where

$$\Phi_i' = \sum_{\substack{j=1 \\ j \neq i}}^{j=n} \frac{q_j'}{4\pi\epsilon_0 r_{ij}}$$

Show that

$$\sum_{i=1}^{n} q_i \Phi_i' = \sum_{i=1}^{N} q_i' \Phi_i$$

This result is known as Green's reciprocation theorem.

3.13. (a) Given N conducting bodies where under one set of conditions the charge on the ith body is Q_i and for another it is Q_i'. Corresponding to each charge system

the potential of the ith body is Φ_i and Φ_i', respectively. Show that

$$\sum_{i=1}^{N} Q_i \Phi_i' = \sum_{i=1}^{N} Q_i' \Phi_i$$

HINT: Subdivide each body into infinitesimal elements, and apply Green's reciprocation theorem developed in Prob. 3.12.

(b) Using the result in (a) and with $Q_j' = 1$, $Q_i' = 0$ ($i \neq j$), confirm the formulation of (3.42). What physical significance can be attached to the coefficients of potential?

3.14. A spherical charge distribution of radius R is uniform and has a charge density ρ_0. Calculate the self-energy from (3.60) and from (3.64).

3.15. A dielectric spherical shell has an inner radius r_1, an outer radius r_2, and a permittivity ϵ. What net energy is required to move it from infinity to a point where it is concentric with a point charge Q?

3.16. A variable capacitor is constructed from two coaxial cylinders as illustrated. The inner cylinder is solid and free to slide in an insulating bushing. Find the magnitude and direction of the force acting to displace the center cylinder when a potential
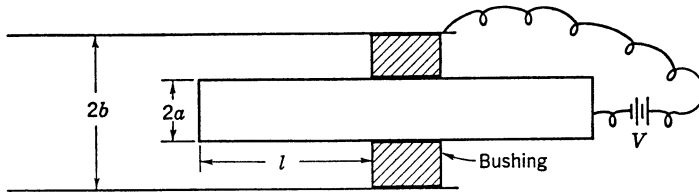

FIG. P 3.16

difference V is maintained between the cylinders. What is the magnitude of the force when $b = 2a = 1$ centimeter and $V = 1,000$ volts?

HINT: Find the rate of change of the energy stored in the capacitor when l is varied.

3.17. In Prob. 3.16 let the variable capacitor be charged so that the total charge on the inner conductor is Q. The battery is now disconnected. What are the magnitude and direction of the force in this case?

3.18. The capacitance of an air capacitor changes linearly from 25 to 350 micromicrofarads during a rotation from 0 to 180°. When set at θ ($0 < \theta < \pi$), what is the electrostatic torque if a voltage of 400 volts is applied between the plates?

3.19. A homogeneous field $\mathbf{E}$ exists within an infinite dielectric medium of permittivity ϵ. Find $\mathbf{E}$ and $\mathbf{D}$ in a hollow cavity within the dielectric if the cavity shape is

(a) A long thin cylinder parallel to $\mathbf{E}$.

(b) A thin flat plate whose broad dimension is perpendicular to $\mathbf{E}$.

(c) A sphere.

3.20. The dielectric constant of hydrogen gas at 0°C and atmospheric pressure is 1.000264.

(a) Compute the polarizability of the hydrogen molecule.

(b) Assuming ideal-gas laws apply, compute the dielectric constant for a pressure of 15 atmospheres and −150°C. (Neglect interaction effects.)

3.21. A dielectric sphere of radius a is placed in a uniform electric field. Find the resultant field inside and outside of the sphere.

HINT: The scheme outlined for the conducting sphere given for Prob. 2.15 may be utilized here. Continuity of Φ and normal $\mathbf{D}$ across the boundary will serve to determine the coefficients.

3.22. Consider two conducting bodies with an initial charge Q_1, difference of potential V, and capacitance C_1. If the two bodies are displaced so that the new capacitance is C_2, compute the work done on the bodies by the field if

(a) V is kept constant (i.e., consider a battery connected between the two bodies).

(b) Q_1 is kept constant (bodies isolated, no battery).

In each case account for the change in energy stored in the field. Note that the above process is not a virtual one, but involves finite displacements and finite-energy interchange.

3.23. Consider N conducting bodies at potential Φ_i, with total charge Q_i on the ith body, $i = 1, 2, \ldots, N$. Prove that the charge distributes itself in such a way that the energy stored in the electrostatic field is a minimum (Thomson's theorem).

HINT: Let Φ be the potential for the correct charge distribution and let $\Delta\Phi$ be the change in potential when the charge distribution is perturbed by a small amount on each body, and consider the integral

$$W_e = \frac{\epsilon_0}{2} \int_V [\nabla(\Phi + \Delta\Phi)]^2 \, dV$$

Show that

$$\int_V (\nabla\Phi) \cdot (\nabla\Delta\Phi) \, dV$$

vanishes since each body is an equipotential surface and the total charge is kept constant. Thus W_e is a minimum if the positive term

$$\int_V [\nabla(\Delta\Phi)]^2 \, dV$$

is zero, i.e., if $\Delta\Phi = 0$.

3.24. A uniform line charge of strength q_l coulombs per meter is placed a distance h above the plane surface of a semi-infinite dielectric of permittivity ϵ. Compute the field everywhere in space.

HINT: Confirm and utilize the following image technique. For field points in the free-space region the source is q_l, plus a line charge (image) q_l' at the image point, both taken as lying in an infinite free-space medium. For points within the dielectric we have q_l, plus q_l'' superposed, both taken as existing in an infinite medium of permittivity ϵ. The relation between q_l' and q_l'' can be found from the boundary conditions at the interface.