CHAPTER 2

# ELECTROSTATICS

This chapter develops the basic properties of the electrostatic field in vacuum. The law of Coulomb for the force between two point charges is the experimental basis for the work of this chapter. The electric field is defined as the force exerted on a unit positive charge and leads to its establishment as a fundamental entity. From Coulomb's law the electric field due to a point charge is readily evaluated. The principle of superposition is next used to establish the law for the field produced by a volume or surface distribution of charge. The total flux of the electrostatic field is then related to the charge by means of Gauss' law.

From an investigation of the electrostatic field its nature is discovered, which permits the field to be derived from the negative gradient of a scalar potential function. The scalar potential is related to the work done against the field in moving a unit charge to an arbitrary field point from a given reference. The differential form of Gauss' law is used to show that the potential function is a solution of Poisson's equation, the charge density being the source function.

With the concept of a potential established, the treatment leads naturally into a discussion of conductors, the behavior of the electric field at conductor surfaces, and the constant potential nature of conducting bodies. Several elementary boundary-value problems involving conductors are solved by application of Gauss' law and image techniques.

The last section introduces the electric dipole, the dipole potential, and the dipole field. The field from a volume distribution of dipoles is shown to be the same as that from an equivalent volume and surface charge distribution. This sets the stage for the theory of the behavior of insulating materials in electric fields, which is presented in Chap. 3.

## 2.1. Introduction to Electrostatics

The phenomenon that underlies the study of electrostatics was known to man since very early times. Thales of Miletus is credited with first observing that amber when rubbed attracted light objects to itself, a discovery that dates back to 600 B.C. Subsequent experimenters found that most substances when rubbed possessed this property to some extent. In particular, if a glass rod is rubbed with a silk cloth, both rod and cloth will attract small bits of paper. We say that they are electrified (a word derived from *ēlektron*, the Greek name for amber).

If an electrified glass rod is brought into contact with a gilded pith ball suspended by a silk string, the ball becomes electrified. Substances, such as the gold covering of the ball, which have the property of removing electrification from electrified objects are called conductors. Other substances, such as the silk thread, which remove the electrification very slowly are insulators. If two balls are electrified by the glass rod, they will be found to repel each other. But if one is electrified by the rod and the other by the silk cloth, they attract each other. A hypothesis to explain this assumed the existence of two kinds of electricity. Arbitrarily, that on the glass rod is taken as positive and that on the silk negative.

The connection between the static electricity noted above and electric currents such as are caused to flow by means of a battery was established by Faraday (1833) and Rowland (1876). They showed that electric current was the flow of electric charge, of the same nature as the charges of electrostatics. Developments in the field of atomic and nuclear physics have deepened our knowledge concerning the nature of positive and negative charged particles. We shall assume that the student is familiar with at least a qualitative picture of atomic structure.

## 2.2. Coulomb's Law

By means of a torsion balance which he developed, Coulomb, in 1785, investigated the nature of the force between charged bodies. The following conclusions were drawn from the results of his experiments as they relate to the force between two charged bodies which are very small compared with their separation, i.e., point charges:

1. The magnitude of the force is proportional to the product of the charge magnitudes.
2. The magnitude of the force is inversely proportional to the square of the distance between the charges.
3. The direction of the force is along the line connecting the charges.
4. The force is attractive if charges are unlike, repulsive if they are alike.
5. The force depends on the medium in which the charges are placed.

Coulomb's experiments have subsequently been repeated with much greater precision; the inverse-square-law behavior is known to be true to at least 1 part in $10^9$. It should be noted that the nature of Coulomb's and subsequent experiments is such as to provide a basis for a macroscopic theory. The above conclusions can be expected to hold only so long as the charged bodies are small compared with the distance separating them.

The information obtained by Coulomb can be formulated mathematically in what is known as Coulomb's law. Using vector notation, we have

$$\mathbf{F}_{12} = \frac{kq_1q_2}{\epsilon R^2}\,\mathbf{a}_R \tag{2.1}$$

In this equation $\mathbf{F}_{12}$ is the vector force acting on charge $q_2$ due to charge $q_1$. Its direction is governed by $\mathbf{a}_R$, a unit radius vector in the direction from $q_1$ to $q_2$. The symbols $q_1$ and $q_2$ specify both the magnitude and sign of the charges involved. The parameter $\epsilon$ is a property of the medium called the electric permittivity, $R$ is the distance between the charges, and $k$ is a constant of proportionality.

## 2.3. Units

In order to measure physical quantities, a standard of reference, or unit, must be defined so that the quantity can be expressed numerically. Because we deal with many physically related quantities, we seek a self-consistent system of units where every quantity can be defined in terms of a minimum number of basic, independent units consistent with a need for convenience and precision.† Many competing systems have been employed in the past for use in the area of electric and magnetic fields. A discussion of their development and conversion from one to another can be found elsewhere.‡

The system of units that has been almost universally adopted for use in applied electromagnetic theory is the meter-kilogram-second (mks, for short) system introduced by Giorgi in 1901. In this system length is measured in meters, mass in kilograms, time in seconds. For electric and magnetic quantities, a further basic unit must be defined, and this is usually chosen to be the coulomb for unit of charge or ampere for unit of current. Any electric or magnetic quantity can be expressed in terms of these fundamental units. The mks system has the advantage over systems used earlier in that primary electrical quantities are in the practical system; that is, potential is measured in volts, resistance in ohms, power in watts, etc.

We may illustrate how the unit of force is derived from the fundamental units by recalling that force equals mass times acceleration. Consequently, the unit of force in the mks system is kilogram-meters per

† For a discussion of optimization of the fundamental units, see A. G. McNish, The Basis of Our Measuring System, *Proc. IRE*, vol. 47, pp. 636–643, May, 1959.

‡ J. A. Stratton, "Electromagnetic Theory," sec. 1.8, Appendix I, McGraw-Hill Book Company, Inc., New York, 1941.

second squared and is called a newton. We may write

$$1 \text{ newton} = \frac{1 \text{ kilogram-meter}}{\text{second}^2} \tag{2.2}$$

The unit of energy is called the joule, and since it is given by the product of force times distance,

$$1 \text{ joule} = 1 \text{ newton-meter} \tag{2.3}$$

It is possible to determine the units of $\epsilon$ by inspection of the force equation (2.1). We easily verify that

$$\text{Units of } \epsilon = \frac{\text{coulombs}^2}{\text{newton–square meter}} = \frac{\text{coulombs}^2}{\text{joule-meter}} \tag{2.4}$$

We shall show in a later chapter that for a charged capacitor the energy in joules can be expressed in terms of the charge in coulombs and capacitance in farads as

$$\text{Energy in joules} = \frac{\frac{1}{2} \text{ coulombs}^2}{\text{farad}} \tag{2.5}$$

Consequently, (2.4) becomes

$$\text{Units of } \epsilon = \frac{\text{farads}}{\text{meter}} \tag{2.6}$$

The dimensionless constant $k$ in (2.1) is chosen as either $1/4\pi$ or unity, depending on whether a "rational" or "irrational" system of units is desired. The choice that is usually made is the rational system, and such a system will be used in this text. Thus the force equation may be written in rationalized mks units as

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon R^2} \mathbf{a}_R \tag{2.7}$$

The value of $\epsilon$ depends on the medium. For the balance of this chapter we assume the charges to lie in a free-space medium (vacuum). We adopt the notation $\epsilon_0$ as the electric permittivity of free space. This has been measured to be $8.854 \times 10^{-12}$ farad per meter, or very closely $(1/36\pi) \times 10^{-9}$ farad per meter. In (2.7) we may now remark that $\mathbf{F}_{12}$ is measured in newtons, $q_1$ and $q_2$ in coulombs, and $R$ in meters.

## 2.4. Electric Field

The force on a charge $q_0$ due to a system of point charges $q_1$, $q_2$, $q_3$, as illustrated in Fig. 2.1, can be found by successive application of Coulomb's law and the principle of superposition; that is, the force due to each charge is found as if it alone were present; then the vector sum of these forces is calculated to give the resultant force. A graphical solution is shown in Fig. 2.1.

Before giving the expression for the force caused by several charges we digress briefly to explain the notation that will be adopted. In field theory we must often consider simultaneously two sets of points, the source point $(x',y',z')$, which specifies the location of the source, and the field point $(x,y,z)$, which specifies the point at which the field is measured. The primes are used to clearly distinguish the source coordinates from the unprimed field coordinates. We shall find this distinction to be very

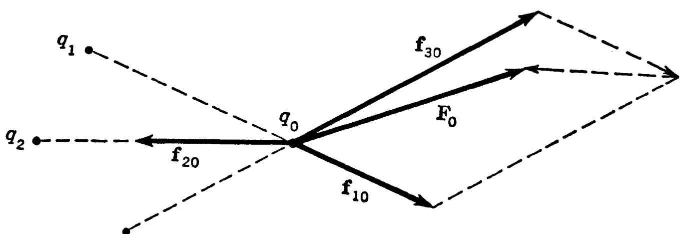


FIG. 2.1. Graphical solution for force on a single charge $q_0$ due to other charges.

helpful on many occasions. We shall follow the convention outlined in Sec. 1.17 and let $\mathbf{r}$ be a vector from the origin to the field point and let $\mathbf{r}'$ be a vector from the same origin to the source point. The vector $\mathbf{r} - \mathbf{r}'$ is then the vector distance from the source point to the field point. For brevity we denote the magnitude of $\mathbf{r} - \mathbf{r}'$ by the capital letter $R$ and a unit vector in the direction of $\mathbf{r} - \mathbf{r}'$ by $\mathbf{a}_R$; thus

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$$
$$\mathbf{a}_R = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

If we have several source points, we denote the vector distance from the origin to the $i$th source point $(x_i',y_i',z_i')$ by $\mathbf{r}_i'$ and let $R_i = |\mathbf{r} - \mathbf{r}_i'|$ and $\mathbf{a}_{R_i} = (\mathbf{r} - \mathbf{r}_i')/|\mathbf{r} - \mathbf{r}_i'|$.

For the force acting on a charge $q_0$ at $(x,y,z)$ due to point charges $q_1, q_2, \ldots, q_n$ at $(x_1',y_1',z_1'), \ldots, (x_n',y_n',z_n')$, we can now write

$$\mathbf{F} = \frac{q_0}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i'|^2} \frac{\mathbf{r} - \mathbf{r}_i'}{|\mathbf{r} - \mathbf{r}_i'|} = \frac{q_0}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i \mathbf{a}_{R_i}}{R_i^2} \tag{2.8}$$

In applying Coulomb's law, as in the above example, we have conceptually implied that the charge $q_i$ acts through the intervening medium, in some way, on charge $q_0$. The formulation of Coulomb is thus an "action-at-a-distance law." In other words, $q_i$ acts through the intervening space directly on charge $q_0$, causing the observable force, and vice versa, of course.

The same final result can be obtained on the basis of another principle, known as the field concept. Here the view is taken that the charge $q_i$ sets up a field which pervades all space and in particular the point at which $q_0$ lies. In this view the force exerted by $q_i$ on $q_0$ is communicated by means of this field. In other words, $q_i$ sets up a field; the field in turn has the property of causing a force to be exerted on a charge in the field. The force depends only on the strength of the field and not on the origin of the field; that is, the field has an independent existence; consequently, it does not even depend on whether a charge $q_0$ is present to detect it.

When a number of charges are present, each sets up a partial field and the total field is obtained by superposition. The net force on $q_0$ is the vector sum of the partial forces due to each component field or, equivalently, is the force due to the total field. The symbol **E** is used to represent the electric field.

The electrostatic field **E** that fulfills the property discussed above must consequently be defined by the force that is exerted on a unit charge in the field. It is a vector quantity in the same direction as the force. As we shall see, this definition is not quite satisfactory operationally, and so we modify it to read

$$\mathbf{E} = \lim_{\Delta q \to 0} \frac{\mathbf{F}}{\Delta q} \tag{2.9}$$

The formulation of (2.9) arises from an awareness that in making a measurement of **E** there is always the possibility that the measuring process itself may seriously disturb the conditions existing prior to measurement. The limiting process is introduced in (2.9) for the purpose of ensuring that the introduction of the exploratory charge $\Delta q$ into the field will not perturb the value of that field, i.e., will not affect the sources of that field. By letting $\Delta q$ go to zero, the value calculated by (2.9) should approach, as a limit, the field strength prior to measurement.

A fundamental difficulty arises in carrying out the limiting process $\Delta q \to 0$, because charge cannot be subdivided indefinitely. The smallest unit of charge, an electron or positron, is $1.60 \times 10^{-19}$ coulomb. In practice, so long as $\Delta q$ is very small compared with the sources producing the field, its introduction into the field may be presumed to have little influence on the behavior of the sources. In this case the ratio of force to charge $\Delta q$ will satisfactorily evaluate **E**. The definition embodied in (2.9) must thus be qualified by the afore-mentioned restriction.

Although the presence of an electrostatic field is revealed only if a force is exerted on a test charge, the field concept postulates the existence of the field anyway. In the study of electrostatics the field approach and the action-at-a-distance concept are indistinguishable. With time-varying sources, one is forced to ascribe a physical reality to the field because of the finite velocity of propagation of the interaction. Consequently, we shall emphasize the field concept under static conditions as well.

On the basis of the definition of **E** and by using Coulomb's law, we can calculate the electric field set up by a point charge $q$. In free space the electric field at the point $(x,y,z)$ caused by a point charge $q$ located at $(x',y',z')$ is given by

$$\mathbf{E}(x,y,z) = \frac{\mathbf{F}}{\Delta q} = \frac{q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^2}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\frac{\Delta q}{\Delta q} = \frac{q\mathbf{a}_R}{4\pi\epsilon_0 R^2} \tag{2.10}$$

For a series of point charges $q_1, q_2, \ldots, q_n$, the total electric field is found by superposing the field from each individual charge; thus

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i^2}\mathbf{a}_{R_i} \tag{2.11}$$

where $R_i^2 = (x - x_i')^2 + (y - y_i')^2 + (z - z_i')^2$. The sum indicated in (2.11) must be performed vectorially. The simplest way to proceed is to evaluate $R_i$ and $\mathbf{a}_{R_i} = (\mathbf{r} - \mathbf{r}_i')/R_i$ from their defining relations and combine the $x$, $y$, and $z$ components for all values of the summation index $i$.

When we encounter a large number of point charges in a finite volume, it is convenient to describe the source in terms of a charge density $\rho$. By conventional concepts we define the charge density as

$$\rho = \lim_{\Delta V \to 0}\frac{\Delta q}{\Delta V} \tag{2.12}$$

where $\Delta q$ is the algebraic sum of the charge in the volume $\Delta V$. The result of the limiting process is the charge density at the point in question. We have already noted that such a limiting process cannot be completely carried out because the charge cannot be subdivided indefinitely. However, so long as $\Delta V$ is small enough so that further decrease in $\Delta V$ does not substantially affect the value computed for $\rho$, yet $\Delta q$ is large compared with $1.60 \times 10^{-19}$ coulomb, there is no difficulty in defining $\rho$. We shall assume that as a result of this process the charge density $\rho$ can be represented by a continuous function of position. Clearly, the total charge $Q$ in a volume $V$ is given by

$$Q = \int_V \rho\, dV \tag{2.13}$$

The electric field set up by an arbitrary volume charge density can be found by superposition since each element $\rho\, dV$ behaves like a point source. Consequently, (2.11) generalizes to

$$\mathbf{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(x',y',z')}{R^2}\mathbf{a}_R\, dV' \tag{2.14}$$

where the integration is taken over the source coordinates $(x',y',z')$. In this equation $R$ is the distance from the source point $dV'$ to the field point and $\mathbf{a}_R$ is a unit vector in that direction.

Similar remarks can be made concerning the mathematically convenient concept of a surface charge density $\rho_s$ coulombs per square meter and a line charge density $\rho_l$ coulombs per meter as is presented above for a volume charge density. The calculation of electric field from a surface charge distribution $\rho_s$, for example, would be given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_s}{R^2} \mathbf{a}_R \, dS \tag{2.15}$$

with $R$ and $\mathbf{a}_R$ defined as in (2.14).

**Example 2.1. Field from an Infinite Charged Plane.** Through application of Coulomb's law and the principle of superposition, we wish to evaluate the field above a uniformly charged infinite plane surface. The charge density is $\rho_s$ coulombs per square meter. Let the point $P$ be an arbitrary field point at a fixed distance $h$ from the plane of charge. We first consider the contribution to the total field from an annular ring of charge centered about the foot of the perpendicular from $P$, as in Fig. 2.2. This ring contributes to the field in a direction normal to the plane ($z$ direction) only. This becomes clear if we divide the ring into a sum of pairs of charge elements on diametrically opposite sides of the ring, as illustrated. The partial field from each pair is in the $z$ direction. Consequently, the net contribution from the ring of area $dS = 2\pi r\, dr$ is

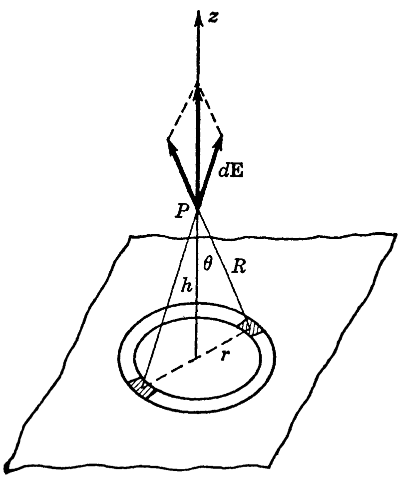

FIG. 2.2. Evaluation of field from an infinite charged plane.

$$dE_z = \frac{\rho_s 2\pi r\, dr}{4\pi\epsilon_0 R^2} \cos\theta \tag{2.16}$$

In this equation we made use of the obvious fact that each element of the ring is the same distance $R$ from $P$ and produces a partial field, making the same angle $\theta$ with the $z$ axis. From Fig. 2.2 it is possible to verify that $r = R \sin\theta = h \tan\theta$ and $dr = h \sec^2\theta\, d\theta$. Thus

$$E_z = \frac{\rho_s}{2\epsilon_0} \int_0^{\pi/2} \sin\theta \, d\theta = \frac{\rho_s}{2\epsilon_0} \tag{2.17}$$

Written vectorially, with **n** a unit vector normal to the charge surface and directed away from the surface,

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\mathbf{n} \tag{2.18}$$

Note that the magnitude of **E** is independent of the position of the point $P$. The field is uniform everywhere above the plane. Below the plane the field is also uniform but pointing in the negative $z$ direction.

## 2.5. Gauss' Flux Theorem

The electric field defined in (2.9) is an example of a vector field as discussed in Sec. 1.8. In particular, the field may be represented by means of the flux concept. The total flux of **E** from a point source $q$ may be readily calculated by integrating $\mathbf{E} \cdot d\mathbf{S}$ over a surface enclosing $q$.

Thus consider an element of surface $d\mathbf{S}$ as shown in Fig. 2.3 at a vector distance $r\mathbf{a}_r$ from a charge $q$, taken as the origin of a spherical coordinate system. The flux through an element of surface $d\mathbf{S}$ is

$$\mathbf{E} \cdot d\mathbf{S} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\mathbf{a}_r \cdot d\mathbf{S} \tag{2.19}$$

But the solid angle $d\Omega$ subtended by $d\mathbf{S}$ at $q$ is

$$d\Omega = \frac{\mathbf{a}_r \cdot d\mathbf{S}}{r^2}$$

and consequently,

$$\mathbf{E} \cdot d\mathbf{S} = \frac{q\, d\Omega}{4\pi\epsilon_0} \tag{2.20}$$

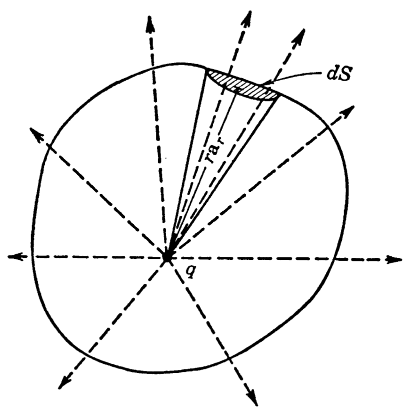

FIG. 2.3. Illustration of Gauss' law.

Let us integrate both sides of (2.20) over any closed surface $S$ containing $q$. In this case, since the total solid angle is $4\pi$,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \tag{2.21}$$

If the charge $q$ lies outside the surface $S$, then the surface integral vanishes since the total solid angle subtended at $q$ by the surface is zero. The physical interpretation of this result is that flux lines originating from an external charge $q$ and entering the surface $S$ must also leave this surface. If the enclosed volume contains a number of point charges $q_1, q_2, \ldots, q_n$, then (2.21) holds for the partial fields due to each charge separately.

By superposition,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \sum_{i=1}^{n} \frac{q_i}{\epsilon_0} = \frac{Q}{\epsilon_0} \tag{2.22}$$

where $\mathbf{E}$ is now the total field, and $Q$ is the algebraic sum of all charges contained in $S$. Equation (2.22) is a statement of Gauss' flux theorem. In words we say that the net outflow of flux of $\mathbf{E}$ through a closed surface

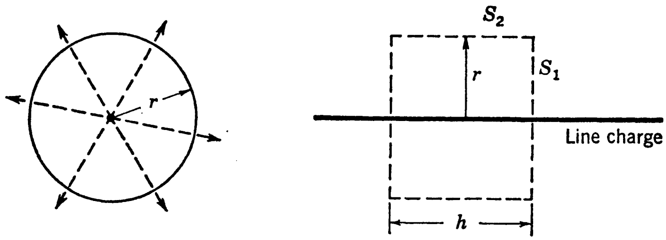


FIG. 2.4. Field from a line charge.

$S$ equals $1/\epsilon_0$ times the total charge (i.e., the algebraic sum of the sources) contained within $S$. Flux lines must therefore originate on positive charge and terminate on negative charge, the relative number of lines depending on the source strength. This theorem is particularly useful in simplifying the calculation of the electric field from symmetrical distributions of charge. We illustrate this point with the following examples.

**Example 2.2. Field from an Infinite Line Charge.** An infinite line charge has a strength of $\rho_l$ coulombs per meter. It is desired to find the electric field which it sets up. We note that because of symmetry, the electric field must be a function of $r$ only and, furthermore, can be in the $r$ direction only. Here $r$ is the distance from the line charge to any field point; i.e., it is the radial variable in cylindrical coordinates.

Let us surround the wire by a concentric cylindrical surface of axial length $h$ and radius $r$, as illustrated in Fig. 2.4. Application of Gauss' flux theorem to this surface leads to

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_{S_1} \mathbf{E} \cdot d\mathbf{S} + \int_{S_2} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \tag{2.23}$$

where $S_1$ is the end surfaces and $S_2$ the cylindrical surface. Since $\mathbf{E}$ has no component normal to the end surfaces, there is no contribution from that integral. Furthermore, because $\mathbf{E}$ is normal to the cylindrical surface and uniform over the fixed radius, we can integrate (2.23), giving

$$2\pi r h E_r = \frac{h}{\epsilon_0} \rho_l$$

and hence

$$E_r = \frac{\rho_l}{2\pi\epsilon_0 r} \tag{2.24}$$

In the flux concept, lines of flow of **E** are directed radially outward. Their divergent nature implies a continual reduction in the strength of the field. Indeed, this is borne out by the $1/r$ variation in (2.24).

For contrast, let us calculate the field by direct application of Coulomb's law. Let the line charge lie along the $x$ axis, and take the origin at the foot of the perpendicular from the field point. Consider the field due to

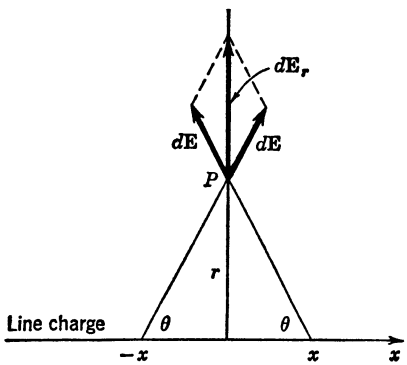


FIG. 2.5

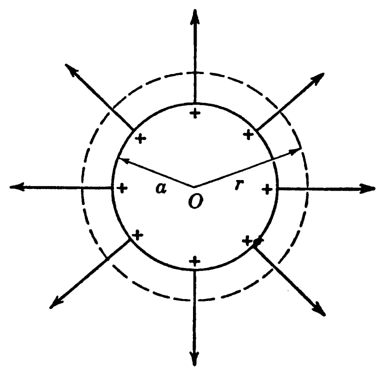


FIG. 2.6. Field from a charged sphere.

two differential charge elements of magnitude $\rho_l\,dx$, with one at $x$ and the other at $-x$. Using the geometry in Fig. 2.5, it is clear that the resultant is in the radial direction and of magnitude

$$dE_r = \frac{\rho_l\,dx}{4\pi\epsilon_0 r^2/\sin^2\theta}\,2\sin\theta$$

Hence

$$E_r = \frac{\rho_l}{2\pi\epsilon_0 r^2}\int_0^\infty \sin^3\theta\,dx \tag{2.25}$$

Since $x = r\cot\theta$, and $dx = -r\csc^2\theta\,d\theta$, the previous integral becomes

$$E_r = \frac{\rho_l}{2\pi\epsilon_0 r}\int_0^{\pi/2}\sin\theta\,d\theta = \frac{\rho_l}{2\pi\epsilon_0 r} \tag{2.26}$$

The result is the same as that obtained by the simpler Gauss-theorem technique.

**Example 2.3. Field from a Charged Sphere.** A spherical surface of radius $a$ carries a total charge $Q$ which is uniformly distributed over the surface. It is required to calculate the field from the charge.

From the symmetry of the charge distribution it is clear that the field depends only on the spherical coordinate $r$, where the origin is at the center of the spherical charge distribution. Furthermore, the electric field can have only a radial component; any other possibility violates either the symmetry or Gauss' flux theorem.

We choose the Gaussian surface to be that of a concentric sphere of radius $r$, as in Fig. 2.6. Applying Gauss' theorem and noting that $E_r$ is everywhere normal to the spherical surface and uniform thereon leads to

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi r^2 E_r = \begin{cases} \dfrac{Q}{\epsilon_0} & r > a \\ 0 & r < a \end{cases}$$

and hence

$$E_r = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2} & r > a \\ 0 & r < a \end{cases} \tag{2.27}$$

The field of a uniformly charged spherical surface is thus identical with that due to a point charge located at the center of the spherical surface and with the same total charge, provided the field point is external to the sphere. No field exists within the spherical surface. Note how much more difficult it would be to calculate the field by direct application of Coulomb's law.

## 2.6. Electrostatic Potential

In this section we shall show that the electrostatic field may be derived from the negative gradient of a scalar potential function. The scalar potential will be found to be equal to the work done against the field in moving a point charge from infinity up to its final position. The work done in moving a charge around a closed path turns out to be zero. This property classifies the electrostatic field as a conservative field (a field with zero rotation or curl).

Consider the electric field set up by a number of point charges $q_1$, $q_2$, . . . , $q_n$, as illustrated in Fig. 2.7. We desire to evaluate the integral $I$:

$$I = -\int_C \mathbf{E} \cdot d\mathbf{l} \tag{2.28}$$

This is a contour integral between the points $P_1$ and $P_2$ along some arbitrary path $C$, and $d\mathbf{l}$ is a displacement along this contour. Since $\mathbf{E}$ represents the force on a unit charge, the integral expressed by (2.28) evaluates the total external work (against the field, hence absorbing the minus sign) required to move a unit charge from $P_1$ to $P_2$ along $C$. The integral can be evaluated by using the principle of superposition. Let us first determine the result due to the single charge $q_i$. If the origin of coordinates is chosen at $q_i$, then the partial contribution becomes

$$I_i = -\int_C \frac{q_i \mathbf{a}_r \cdot d\mathbf{l}}{4\pi\epsilon_0 r^2} \tag{2.29}$$

An element of the path $C$ is illustrated in Fig. 2.8. From the geometry we note that $\mathbf{a}_r \cdot d\mathbf{l} = dl \sin\theta = dr$. Thus

$$I_i = -\int_C \frac{q_i\, dr}{4\pi\epsilon_0 r^2} = \frac{q_i}{4\pi\epsilon_0}\left(\frac{1}{R_{2i}} - \frac{1}{R_{1i}}\right) \tag{2.30}$$

where $R_{2i}$ is the magnitude of the distance from point $P_2$ to the position of the $i$th charge, and $R_{1i}$ is the magnitude of the distance from $P_1$ to the position of the $i$th charge. If the effect of all charges is now considered, the result is a summation over $i$ as follows:

$$I = -\int_C \mathbf{E} \cdot d\mathbf{l} = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0}\left(\frac{1}{R_{2i}} - \frac{1}{R_{1i}}\right) \tag{2.31}$$

A very important conclusion follows from the result expressed by (2.31): the work done in moving a test charge between any two points in an

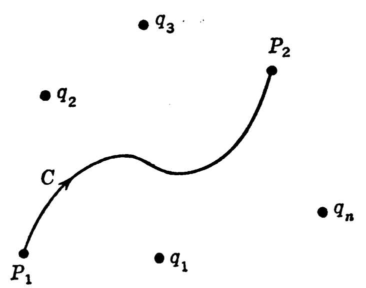


FIG. 2.7. Contour $C$ along which a unit charge is moved.

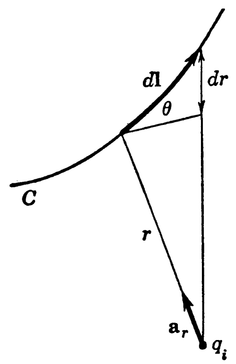


FIG. 2.8

electrostatic field depends only on the position of the end points and not on the path. It follows, for example, that the energy required to move a unit charge from some arbitrary reference point to some other point in the field is unique. Consequently, a relative potential may be assigned to that point, hence to every point, in the field. The "potential" at a point is nothing more than a scalar quantity which designates the energy required to move a unit charge from the reference point to the given point. The aggregate of potentials is a scalar field. Note the close analogy to potential energy in mechanics.

The difference in potential $\Phi_{12} = \Phi(P_2) - \Phi(P_1)$ between the points $P_2$ and $P_1$ may be defined as the work required to move a unit positive charge from $P_1$ to $P_2$ and is given by (2.31). The difference of potential is independent of the reference potential, of course. When the reference point is chosen as the point at infinity, then the relative potential of all other points under these conditions is known as the absolute potential. From (2.31) it is seen that the absolute potential $\Phi(P_2)$ at the point $P_2$ is given by

$$\Phi(P_2) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} \frac{q_i}{R_{2i}} \tag{2.32}$$

From what has been said so far it is clear that the line integral of electric field around any closed path must be zero. For if (see Fig. 2.9)

$$\Phi(P_2) - \Phi(P_1) = -\int_{C_1} \mathbf{E} \cdot d\mathbf{l} \qquad \text{and} \qquad \Phi(P_1) - \Phi(P_2) = -\int_{C_2} \mathbf{E} \cdot d\mathbf{l}$$

then

$$\oint_{C_1+C_2} \mathbf{E} \cdot d\mathbf{l} = 0$$

In this discussion $C_1$ and $C_2$ are completely arbitrary, as is the location of $P_1$ and $P_2$. But this means that the electric field is irrotational and consequently can be derived from a scalar potential. This fact has, however, already been noted. Actually, we could have proceeded along more analytic lines, as was done in Sec. 1.17. Specifically, (2.10) can be rewritten by inspection as

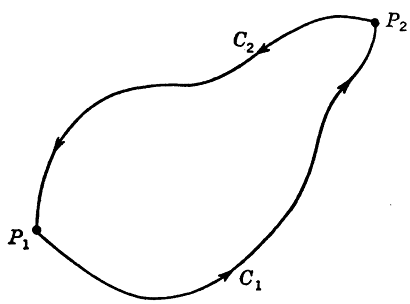


FIG. 2.9

$$\mathbf{E} = -\nabla \frac{q}{4\pi\epsilon_0 R} \tag{2.33}$$

This means that in general we can write

$$\mathbf{E} = -\nabla\Phi \tag{2.34}$$

The negative sign can be understood physically from the fact that **E** is in the direction that a positive charge moves, hence in the direction of *decreasing* potential. The scalar potential, by integration, is

$$\Phi = \frac{q}{4\pi\epsilon_0 R} + C \tag{2.35}$$

for a single point source $q$. Furthermore, for a volume source density, by superposition,

$$\Phi = \int_V \frac{\rho\, dV'}{4\pi\epsilon_0 R} + C \tag{2.36}$$

where $R$ is the distance from the source to the field point. The value of **E** is unaffected by the choice of the integration constant $C$, which is determined arbitrarily by assigning a potential to some reference point, as already noted. Since the curl of a gradient is zero, it follows immediately from (2.34) that

$$\nabla \times \mathbf{E} = 0 \tag{2.37}$$

which is further confirmation of the irrotational (conservative) nature of the electrostatic field.

The unit of potential is the volt. A difference of potential $\Phi(P_2) - \Phi(P_1)$ of 1 volt means that 1 joule of work is required to move a coulomb of charge from $P_1$ to $P_2$. From (2.34) we note that the electric field **E** may be expressed in units of volts per meter, as well as joules per coulomb.

We have shown that, corresponding to any electrostatic field **E**, there exists a scalar potential field $\Phi$ such that the electric field is equal to the negative gradient of the potential field. As a consequence, **E** will be normal to the equipotential surface $\Phi$ = constant; it points in the direction of the maximum rate of decrease of potential. If a charge is moved around any closed path, no net energy is required since the electrostatic field is conservative.

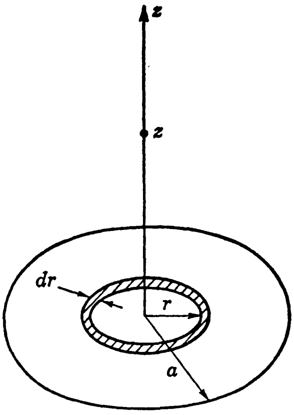

FIG. 2.10. A uniformly charged disk.

**Example 2.4. Potential on Axis of Charged Disk.** We wish to find the potential along the axis of a uniformly charged disk of radius $a$, as illustrated in Fig. 2.10. The surface charge density is $\rho_s$ coulombs per square meter. The polar axis is designated $z$, and the origin is its intersection with the disk.

For an annular ring of radius $r$ and width $dr$, the contribution to the potential at a point $z$ along the polar axis, using (2.35), is

$$d\Phi = \frac{\rho_s 2\pi r\, dr}{4\pi\epsilon_0 \sqrt{z^2 + r^2}}$$

If we now integrate over all the charge distribution, we obtain

$$\Phi_{\text{axis}} = \frac{\rho_s}{2\epsilon_0} \int_0^a \frac{r\, dr}{\sqrt{z^2 + r^2}} + C$$

$$= \frac{\rho_s}{2\epsilon_0} [(a^2 + z^2)^{1/2} - |z|] + C \tag{2.38}$$

The sign of $(z^2)^{1/2}$ has been chosen so that $\Phi(z) = \Phi(-z)$ and $\Phi$ decreases as $|z|$ becomes very large. This is necessitated by the symmetry in $z$ and the requirement that at great distances from the disk the potential behaves like that due to a point source. If the point $z$ recedes to infinity, then the potential can be made equal to zero by taking $C = 0$. Consequently, the absolute potential is

$$\Phi_{\text{axis}} = \frac{\rho_s}{2\epsilon_0} [(a^2 + z^2)^{1/2} - |z|] \tag{2.39}$$

Because of axial symmetry, the electric field on the axis will be in the $z$ direction. As a consequence, the electric field on the axis is related only to the variation in potential along the axis in accordance with (2.34). Note that for a calculation of the **E** field in general it is necessary to know $\Phi$ as a function of $r$, $\phi$, $z$. It is only because of the symmetry here that one can say, a priori, that $\partial\Phi/\partial r = \partial\Phi/\partial\phi = 0$ along the axis. Thus

$$(E_z)_{\text{axis}} = -\frac{\partial\Phi}{\partial z} = \frac{\rho_s}{2\epsilon_0}[1 - z(a^2 + z^2)^{-\frac{1}{2}}] \qquad \text{for } z > 0$$
$$(E_z)_{\text{axis}} = -\frac{\rho_s}{2\epsilon_0}[1 + z(a^2 + z^2)^{-\frac{1}{2}}] \qquad \text{for } z < 0 \tag{2.40}$$

We note that $E_z(z) = -E_z(-z)$, as is clear on physical grounds. The result for $z = 0$ is independent of $a$, hence corresponds to the earlier solution (2.18) for an infinite sheet of charge. The electric field is discontinuous by an amount $\rho_s/\epsilon_0$ in crossing the charged surface, a result which will be shown (Sec. 2.12) to be independent of the geometry of the surface.

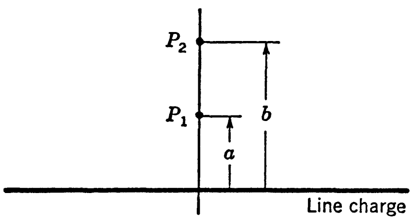

FIG. 2.11. An infinite line charge.

**Example 2.5. Potential from a Line Charge.** We wish to evaluate the difference in potential between the points $P_1$ and $P_2$ in the field of a line charge of infinite extent and of strength $\rho_l$ coulombs per meter. The point $P_1$ is at a radius $a$, and $P_2$ is at a radius $b$ along the same radial line from the line charge, as in Fig. 2.11. From (2.24) the electric field is $E_r = \rho_l/2\pi\epsilon_0 r$; consequently,

$$\Phi(b) - \Phi(a) = -\int_a^b \frac{\rho_l\,dr}{2\pi\epsilon_0 r} = \frac{\rho_l}{2\pi\epsilon_0}\ln\frac{a}{b} \tag{2.41}$$

Since the field is in the radial direction only, equipotential surfaces are concentric cylinders. Consequently, the result in (2.41) is true even if $P_1$ and $P_2$ are not on the same radial line. That is, for any $P'(a,\phi,z)$ then $\Phi(P_2) - \Phi(P') = \Phi(P_2) - \Phi(P_1) + [\Phi(P_1) - \Phi(P')]$. But

$$\Phi(P_1) - \Phi(P') = 0$$

since $P_1$ and $P'$ lie on an equipotential surface $r = a$.

If $b \to \infty$, then the potential relative to infinity as a reference results. Unfortunately, the expression given by (2.41) becomes infinite. The difficulty arises because in this case the infinite line charge itself extends to infinity. Consequently, we cannot express an absolute potential for this problem.

## 2.7. Conducting Boundaries

So far we have considered the problem of evaluating the electric field from given charge sources. To facilitate this process the electrostatic scalar potential was defined and related to the sources. This relationship is a scalar one and hence much simpler than the vector relationship. The electric field is readily found once the potential function is known.

Usually, however, the distribution of charge is not known. What is given are configurations of metallic bodies which are connected to primary sources such as a battery. The charge distribution will then be a consequence of these conditions, and in a sense, the problem will be to determine the resultant source distribution. We first need to know a little more about the characteristics of metal substances.

A property of metal bodies that has already been noted is that they are good "conductors of electricity"; that is, they readily permit a current flow or motion of charge. As a consequence, if charges are placed on or in conductors, they will move about as long as there is the slightest electric field producing a force on them. The charges move until they reach an equilibrium configuration. This is obviously characterized by the necessity that no field exists within or tangent to the surface of the conductor. This can happen only when all the charges reside on the surface of the conductor, for if any charge remained in the body, then by Gauss' flux theorem a nonzero field in the vicinity of such charges would have to exist. The surface charge at equilibrium must be so distributed that the total electric field inside the conductor and tangential to its surface is zero.

A finite time is required for equilibrium to be essentially achieved if some charge is suddenly placed in a body. This time is designated as the relaxation time and is the basis for distinguishing conductors from insulators. For a good conductor, such as copper, the relaxation time is of the order of $10^{-19}$ second, while for a good insulator, such as fused quartz, it is $10^6$ seconds. For all purposes we may assume the electrostatic field within a conductor to be identically zero.

The above discussion holds also in the case where a conductor is placed in an external electric field. In this case, current, consisting of free charge in the conductor, flows until a surface charge distribution is built up so that the field it produces within the conductor and tangent to its surface just cancels the external field. The body must remain electrically neutral, and hence the algebraic sum of the surface charge is zero in this case. Since the tangential component of electric field is zero on the surface of a conductor, this surface must be an equipotential one. Furthermore, the region within the body has no field; so it too is at the same electrostatic potential. The electrostatic field at the surface of a conductor has a direction normal to the surface.

By applying Gauss' flux theorem it is possible to determine a relationship between the surface charge density on a conductor at equilibrium and the electrostatic field at the surface. Consider a very small portion of any charged conducting surface, as illustrated in Fig. 2.12. An infinitesimal coin-shaped surface is visualized with one broad face parallel to and just outside the conducting surface and the opposite face inside. No flux crosses the lower surface which is within the conductor since **E** is zero in this region. No flux leaves through the sides since this would require that **E** have a component tangential to the surface. Furthermore, we can let $d \to 0$, so that the area of the sides is of lower order compared with that of the broad faces. Above the surface, however, a normal component of **E** exists. Thus the net outflow of flux of **E** from the closed coin surface is given by $\mathbf{E} \cdot d\mathbf{S}$, and this must equal the net charge within. If we let $\rho_s$ be the surface charge density and denote by $E_n$ the electric field which is in the direction of the outward normal at the surface, we have

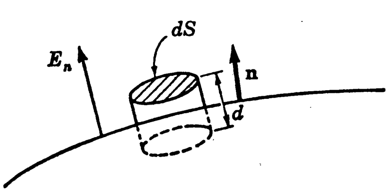


FIG. 2.12. Application of Gauss' law to relate normal **E** to surface charge on a conductor.

$$E_n\,dS = \frac{\rho_s}{\epsilon_0}\,dS$$

or

$$E_n = \mathbf{n} \cdot \mathbf{E} = \frac{\rho_s}{\epsilon_0} \tag{2.42}$$

If Fig. 2.12 represents a portion of an infinite plane conductor, then (2.42) seems, superficially, to contradict the result $E_n = \rho_s/2\epsilon_0$ obtained for a plane charge layer [Eq. (2.18)]. The reason for the difference is that (2.42) is a relationship that applies at a conducting surface where the total field on one side of the charge layer is zero. Thus all the lines of

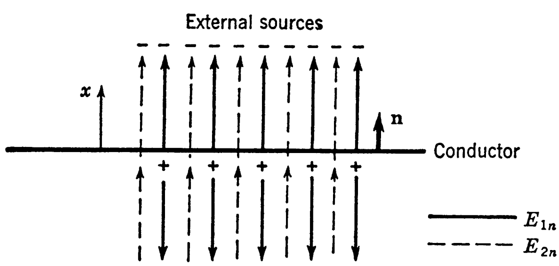


FIG. 2.13. Nature of field at a conductor surface.

flux are directed away from the surface in one direction only. Actually, the field at the surface arises from two systems of charges, the local surface charge $\rho_s$ and the charges that are remote from the conductor and which we could think of as lying uniformly on a distant parallel plane surface. The remote charges are of opposite sign and are physically required in order that the total charge be zero. (They may not be ignored on the basis of their remoteness since the magnitude of the charge is infinite. This is also clear when we note that the field of an infinite surface charge is independent of the distance from the surface.) We realize now that the problem solved in Example 2.1 resulted only in the partial field caused by the sources under consideration and that the total field can be found only if the opposite polarity sources are included. For

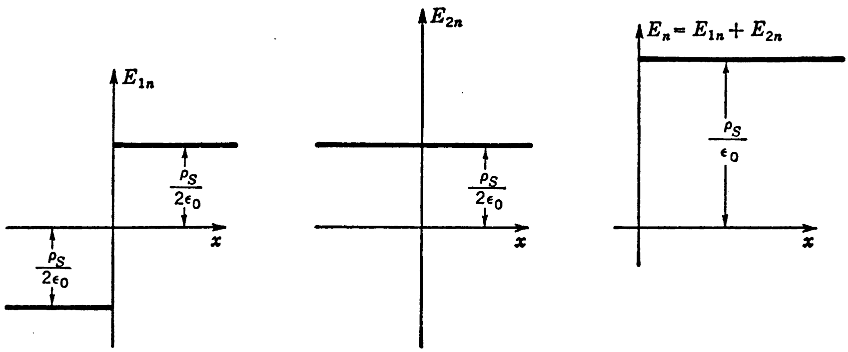


FIG. 2.14. Plot of normal fields as a function of distance $x$ away from conductor surface.

the conducting plane the induced surface charge can be maintained only if these other sources are present. The local surface charge will give a field $\rho_s/2\epsilon_0$ directed in both directions normal to the surface, as explained in Example 2.1. Let this field be denoted by $E_{1n}$, as in Fig. 2.13. The remote charges contribute a field $E_{2n}$, which is continuous across the surface charge layer. The combination of the fields $E_{1n}$ and $E_{2n}$ results in a zero field in the conductor and a field $E_n = \rho_s/\epsilon_0$ on the air side of the surface. In the case of both the single plane of charge and a surface charge layer on a conductor, the normal electric field changes discontinuously by an amount $\rho_s/\epsilon_0$ as the charge layer is crossed. A sketch of the fields $E_{1n}$, $E_{2n}$, and $E_n$ is given in Fig. 2.14.

**Example 2.6. Field between Coaxial Cylinders.** A typical example of a problem involving conducting boundaries and known potentials, where the charge and field distribution is to be determined, will now be considered. Figure 2.15 illustrates the cross section of a coaxial cable with air dielectric. The inner cylinder has a radius $a$; the outer cylindrical conductor has a radius $b$. By connecting a battery between inner and outer conductors a potential difference $V$ can be established. For definiteness, let the inner

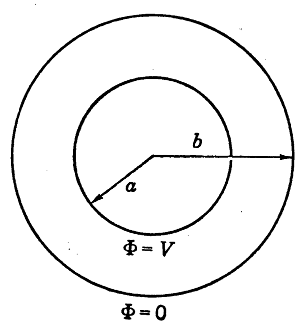


FIG. 2.15. Cross section of coaxial cylinders.

conductor be at a potential $V$ (the entire cylinder is at this potential since conducting surfaces are equipotential surfaces) and the outer conductor at zero potential. We wish to know the electric field everywhere. This is of practical importance since excessive values will cause dielectric breakdown, as will be explained in Chap. 3.

If we can determine the surface charge, then the field can be found. Let us assume $\rho_l$ coulombs per meter on the inner conductor and $-\rho_l$ coulombs per meter on the outer. From symmetry considerations the field between the inner and outer conductor is in the radial direction and by an application of Gauss' law is found to be

$$E_r = \frac{\rho_l}{2\pi\epsilon_0 r} \tag{2.43}$$

From the information given,

$$\Phi(a) - \Phi(b) = V = \int_a^b E_r\,dr = \frac{\rho_l}{2\pi\epsilon_0}\int_a^b \frac{dr}{r}$$

or

$$V = \frac{\rho_l}{2\pi\epsilon_0}\ln\frac{b}{a} \tag{2.44}$$

Consequently,

$$\rho_l = \frac{2\pi\epsilon_0 V}{\ln (b/a)} \tag{2.45}$$

and

$$E_r = \frac{V}{r\ln (b/a)} \qquad a \le r \le b \tag{2.46}$$

Application of Gauss' flux theorem to a concentric cylindrical surface whose radius is less than $a$ or greater than $b$ reveals that $E_r = 0$ for $r < a$ and $r > b$.

A flux plot of the electric field in this problem would show radial flux lines originating on the inner conductor and terminating at the outer conductor. Since the charge density on the inner conductor is $\rho_l/2\pi a$, the electric field, from (2.42), should be of magnitude $V/[a \ln (b/a)] = \rho_l/2\pi a\epsilon_0$. This is confirmed by (2.46) with $r = a$.

If the electric field were uniform in the region $a < r < b$, instead of varying with $r$, then it would be directly related to the difference of potential and the spacing $b - a$ and given by

$$E \approx \frac{V}{b-a} \tag{2.47}$$

We should expect this approximation to be very good if the spacing $b - a$ is very small compared with $b$ or $a$. Thus, if we let $b = a + \epsilon$, where $\epsilon \ll a$, then (2.46) can be approximated by

$$E_r = \frac{V}{r\ln (1 + \epsilon/a)} \approx \frac{V}{a\epsilon/a} = \frac{V}{\epsilon} = \frac{V}{b-a} \tag{2.48}$$

which is equivalent to (2.47).

## 2.8. Poisson's Equation

For a volume charge distribution Gauss' flux theorem as expressed in (2.21) can be written

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V \rho \, dV \tag{2.49}$$

In this equation the surface $S$ encloses the volume $V$. If, now, Gauss' theorem is used, (2.49) transforms to

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} \, dV = \frac{1}{\epsilon_0} \int_V \rho \, dV \tag{2.50}$$

By letting $V$ become very small, it is seen that the integrands in (2.50) must be equal, and a differential relationship between the field and source at the same point results. Thus, at any point,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.51}$$

As was noted in Sec. 1.10, the divergence of a vector field at a point $P$ is a measure of the strength of the source at $P$. In (2.51) we note the physically satisfying interpretation of $\rho/\epsilon_0$ as being the source of the electrostatic field.

If $\mathbf{E}$ is expressed as the negative of the gradient of a scalar potential $\Phi$, the following partial differential equation results:

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{2.52}$$

This is known as Poisson's equation. For a source-free region of space, Laplace's equation results; i.e.,

$$\nabla^2 \Phi = 0 \tag{2.53}$$

If the $\mathbf{E}$ field is known everywhere in a space in which conducting bodies are present, then by (2.42) the charges on the surfaces of the conductors are known. Conversely, given the surface charge on the conductors, the $\mathbf{E}$ field can be calculated by using (2.15). As we noted earlier, neither kind of information is likely to be available in the typical problem of electrostatics. The fundamental problem in a space free of charge (except for the charged conductors) is to solve for an electrostatic potential that satisfies Laplace's equation and also the boundary conditions on the conducting bodies, namely, $\Phi_i$ = constant, on the surface $S_i$. In addition, the interior of the conductors must have the same potential as the surface. The data that are available are the geometry of the conducting bodies and either the potential or the total charge on each. We shall show, in Sec. 2.9, that the electrostatic field is uniquely determined from such information, while the scalar potential is determined to within some arbitrary constant.

The fundamental problem of electrostatics can be solved in only a relatively few cases. For simple symmetrical geometry the technique using Gauss' law as in Example 2.6 can be employed. This works for parallel-plane and concentric spherical boundaries as well as concentric cylinders. Somewhat more elaborate boundary-value problems can be handled by methods of mathematical physics as exemplified by use of cylindrical and spherical harmonics. This approach will be considered in detail in Chap. 4. For the moment we shall content ourselves with the following example, which illustrates the method of direct solution of Laplace's equation. Its simplicity in the present case arises from the choice of boundaries such that $\Phi$ is a function of a single variable only.

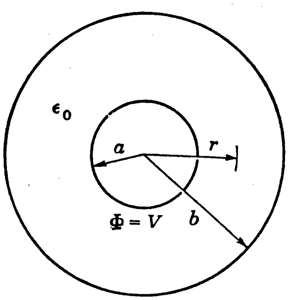


FIG. 2.16. Cross section of coaxial cable.

**Example 2.7. Solution of Laplace's Equation.** Let us consider here the problem presented in Example 2.6, which was solved by means of Gauss' flux theorem. The geometry is repeated, for convenience, in Fig. 2.16. We wish to determine the field within the coaxial cable subject to the boundary conditions that

$$\Phi(a) = V \qquad \Phi(b) = 0$$

Our purpose is to do this by direct solution of Laplace's equation, thereby illustrating that technique. The cylindrical nature of the boundaries suggests that Laplace's equation be written in cylindrical coordinates (Sec. 1.19).

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2.54}$$

If the cable is extremely long, then except near the ends, no variation with $z$ is to be expected. Furthermore, because of cylindrical symmetry, the potential cannot be a function of $\phi$. Accordingly, (2.54) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0 \tag{2.55}$$

It is possible to solve this differential equation by integrating twice; thus

$$\Phi = C_1 \ln r + C_2 \tag{2.56}$$

The integration constants may now be determined from the boundary conditions specified earlier. We have

$$V = C_1 \ln a + C_2 \tag{2.57a}$$
$$0 = C_1 \ln b + C_2 \tag{2.57b}$$

From this pair of equations it is easy to determine that

$$C_1 = \frac{V}{\ln (a/b)} \qquad C_2 = -\frac{V \ln b}{\ln (a/b)}$$

so that

$$\Phi = \frac{V}{\ln (a/b)} (\ln r - \ln b)$$
$$= \frac{V}{\ln (a/b)} \ln \frac{r}{b} \tag{2.58}$$

The electric field is obtained by taking the negative of the gradient of $\Phi$. In cylindrical coordinates, and because $\Phi$ is a function of $r$ only,

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{V}{r \ln (b/a)} \tag{2.59}$$

This result checks with that found in Example 2.6 and given by (2.46).

## 2.9. Uniqueness Theorem

Consider an arbitrary distribution of conducting bodies in a space free of charge, as in Fig. 2.17. According to the uniqueness theorem, the field is uniquely specified everywhere by giving the potential at the surface of each conductor, or by giving the total surface charge on each, or by giving the potential of some of the conductors and the total charge on the remainder. We proceed, now, to develop this theorem.

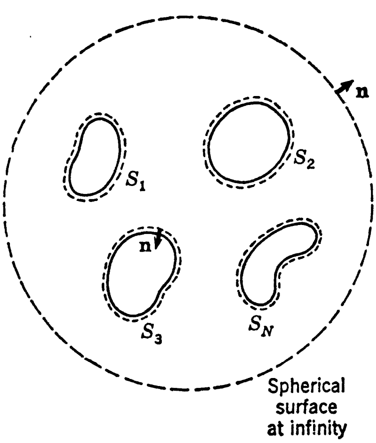


FIG. 2.17. Illustration of volume $V$ enclosed by the surface $S$ (dashed lines).

Let $\Psi$ represent any scalar function which is a solution of Laplace's equation $\nabla^2\Psi = 0$. Expanding $\nabla \cdot \Psi \nabla\Psi$ gives

$$\nabla \cdot \Psi \nabla\Psi = \nabla\Psi \cdot \nabla\Psi + \Psi \nabla^2\Psi = |\nabla\Psi|^2$$

since $\nabla^2\Psi$ is assumed equal to zero. If we integrate both sides throughout a volume $V$ and use Gauss' law (divergence theorem), we obtain, as in Sec. 1.14,

$$\int_V \nabla \cdot \Psi \nabla\Psi \, dV = \oint_S \Psi \nabla\Psi \cdot d\mathbf{S} = \oint_S \Psi \frac{\partial \Psi}{\partial n} dS = \int_V |\nabla\Psi|^2 \, dV \tag{2.60}$$

where $\mathbf{n}$ is the unit outward normal to the closed surface $S$ surrounding $V$. This result may be used to prove the uniqueness theorem. In (2.60) the surface $S$ must enclose the volume $V$. This requirement may be met if we surround our conductors by the surface of a sphere of infinite radius. The total closed surface so obtained is illustrated by the dashed lines in Fig. 2.17. On the surface of the infinite sphere of radius $r$ the potential $\Psi$ decreases at least as fast as $1/r$, $\nabla\Psi$ decreases as $1/r^2$ or faster, and the surface area increases as $r^2$ only so that

$$\lim_{r\to\infty} \oint_{\text{sphere}} \Psi\, \nabla\Psi \cdot d\mathbf{S} \to 0$$

With a closed surface constructed in the above manner we can apply (2.60) to obtain

$$\oint_{\text{sphere}} \Psi \frac{\partial \Psi}{\partial n}\, dS + \sum_{i=1}^{N} \oint_{S_i} \Psi \frac{\partial \Psi}{\partial n}\, dS = \sum_{i=1}^{N} \oint_{S_i} \Psi \frac{\partial \Psi}{\partial n}\, dS = \int_V |\nabla\Psi|^2\, dV \tag{2.61}$$

since the integral over the surface at infinity vanishes.

The requirement of the boundary-value problem is to find a potential function $\Psi$ which is a solution of Laplace's equation and which reduces to a specified constant value

$$\Psi = \Phi_i \text{ on conductor } S_i \qquad i = 1, 2, 3, \ldots, K-1$$

where $K-1$ is less than $N$. On the remaining conductors $S_j$ ($j = K$, $K+1, \ldots, N$) the potential $\Psi$ is to be compatible with the condition that on these conductors a total charge $Q_j$ ($j = K, \ldots, N$) exists. Let us suppose that you have found a solution $\Psi_1$ which satisfies the above requirements and your classmate has found a solution $\Psi_2$ that also satisfies the above requirements. The two solutions appear to be mathematically different since yours is in a closed form while your classmate's solution is in the form of a series. The question to be settled is whether the two solutions are identical or whether, perhaps, only one of the solutions is correct.

Let $\Psi$ in (2.61) be the difference between your solution and your classmate's solution; that is, $\Psi = \Psi_1 - \Psi_2$. In view of the fact that $\Psi_1$ and $\Psi_2$ are known to be solutions of Laplace's equation, then by superposition $\Psi = \Psi_1 - \Psi_2$ is also a solution, so that

$$\sum_{i=1}^{K-1} \oint_{S_i} (\Psi_1 - \Psi_2) \frac{\partial(\Psi_1 - \Psi_2)}{\partial n}\, dS + \sum_{j=K}^{N} \oint_{S_j} (\Psi_1 - \Psi_2) \frac{\partial(\Psi_1 - \Psi_2)}{\partial n}\, dS = \int_V |\nabla(\Psi_1 - \Psi_2)|^2\, dV \tag{2.62}$$

Now on $S_i$ ($i = 1, 2, \ldots, K-1$), both $\Psi_1$ and $\Psi_2$ reduce to the constant values $\Phi_i$. Hence the first set of terms on the left-hand side of

(2.62) vanish. On the remaining conductors $\Psi_1$ and $\Psi_2$ are, of course, constant (conductors always have constant potential surfaces) and $\partial\Psi_1/\partial n = \rho_{s1}/\epsilon_0$, $\partial\Psi_2/\partial n = \rho_{s2}/\epsilon_0$, where $\rho_{s1}$ and $\rho_{s2}$ are the surface density of charge for the two solutions; that is, $\partial\Psi/\partial n$ gives the normal electric field at the surface, noting that the outward normal to $S_i$ is the negative of the outward normal at the conducting surface. We may now replace (2.62) by

$$\sum_{j=K}^{N} (\Psi_1 - \Psi_2) \oint_{S_i} \frac{\rho_{s1} - \rho_{s2}}{\epsilon_0} \, dS = \int_V |\nabla(\Psi_1 - \Psi_2)|^2 \, dV \qquad (2.63)$$

But $\oint_{S_i} \rho_{s1} \, dS = \oint_{S_i} \rho_{s2} \, dS = Q_j$, since both solutions are compatible with the condition that the total charge on $S_j$ is $Q_j$. Hence the left-hand side of (2.63) vanishes, and we must have

$$\int_V |\nabla(\Psi_1 - \Psi_2)|^2 \, dV = 0$$

The integrand is always positive, and therefore the volume integral vanishes only if

$$\nabla(\Psi_1 - \Psi_2) = 0$$

which integrates to

$$\Psi_1 = \Psi_2 + C \qquad (2.64)$$

where $C$ is a constant. However, $\Psi_1 = \Psi_2$ on the surfaces $S_i$ ($i = 1, \ldots, K - 1$); so $C$ must equal zero. Thus the answer to the question posed earlier is that the two solutions are identical. In other words, if a solution can be found that satisfies all the conditions of the problem, then this solution is unique.

In the special case where the total charge on each conducting body is specified, we cannot say that the constant $C$ in (2.64) is zero. For this situation the potential is unique to within an arbitrary constant since no point has been chosen as a reference potential point. The electric field is, however, unique since its value does not depend on the constant $C$. The previous theorem obviously also includes the case where the potential on all conducting bodies is specified.

## 2.10. Method of Images

A certain class of boundary-value problems involving infinite conducting planes and wedges, conducting spheres, and conducting cylinders may be solved by a special technique known as the "method of images." When applicable, the technique is very powerful and leads to the solution in a direct manner. However, the technique is not a general one and applies only to a narrow class of problems. On the other hand, some of the concepts involved in the image technique are of a very fundamental nature and applicable to both static and time-varying fields.

Consider the boundary-value problem which consists of a point source $q$ placed a distance $h$ in front of an infinite conducting plane of negligible thickness, as in Fig. 2.18. Now in view of the uniqueness proof of the preceding section, if a potential field can be found such that it is an equipotential over the plane surface and such that its behavior in the immediate vicinity of $q$ is that appropriate to a point source of strength $q$, and

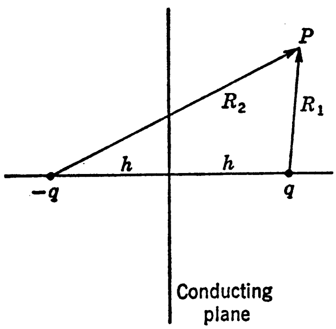


FIG. 2.18. Solution of the problem of a point charge and a conducting plane by the image technique.

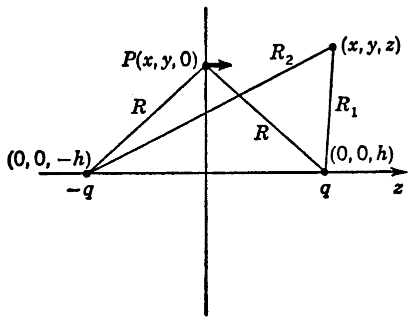


FIG. 2.19

also satisfies Laplace's equation, then it is unique.† This point is being emphasized because sometimes it is possible by intuition, inspection, and/or experience to construct a potential field which meets the necessary requirements. That we know the solution so obtained is unique is, of course, vital.

In the problem at hand all requirements can be met by a scalar potential field which arises from the charge $q$ and a charge $-q$ located at the mirror image of $q$. The potential at any point $P$ (see Fig. 2.18) is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2.65}$$

This function clearly satisfies Laplace's equation, since it arises from the superposition of the potentials due to point sources which, individually, are solutions. Furthermore, the behavior of the field in the neighborhood of $q$ corresponds to the field of a point source of strength $q$. Finally, the potential of points located on the conducting plane, for which $R_1 = R_2$, is constant (arbitrarily zero). Accordingly, (2.65) is the solution to the problem. The charge $-q$ is known as the image charge, and the technique involved is known as the method of images.

† This problem corresponds to the case where both charge and potential are specified. The point charge is a limiting form.

The electric field at the conducting plane can be calculated since the potential field is known. Consequently, we are in a position to determine the charge on the plane. It will be helpful to consider Fig. 2.19 where coordinate axes are set up. The field at $P(x,y,0)$ is desired. At any point $(x,y,z)$ the field in the $z$ direction is given by

$$E_z(x,y,z) = \frac{-q}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where $R_1 = [x^2 + y^2 + (z - h)^2]^{1/2}$ and $R_2 = [x^2 + y^2 + (z + h)^2]^{1/2}$. After differentiating, we obtain

$$E_z(x,y,z) = \frac{-q}{4\pi\epsilon_0}\left\{\frac{h - z}{[x^2 + y^2 + (z - h)^2]^{3/2}} + \frac{h + z}{[x^2 + y^2 + (z + h)^2]^{3/2}}\right\}$$

The normal field $E_n$ at the surface of the conducting plane is $E_z(x,y,0)$ and is given by

$$E_n = -\frac{qh}{2\pi\epsilon_0 R^3} = \frac{\rho_s}{\epsilon_0} \tag{2.66}$$

where $R^3 = (x^2 + y^2 + h^2)^{3/2}$. The surface charge density is, consequently, inversely proportional to the cube of the distance from the point charge. The total charge on the plane is evaluated by the following integral:

$$\int_S \rho_s \, dS = -\frac{qh}{2\pi}\int_S \frac{1}{R^3}\, dS \tag{2.67}$$

Using polar coordinates $r$, $\theta$, then $R^2 = h^2 + r^2$ and $dS = r\,dr\,d\theta$. Thus

$$\int_S \rho_s\, dS = -\frac{qh}{2\pi}\int_0^\infty \int_0^{2\pi} \frac{r\,dr\,d\theta}{(h^2 + r^2)^{3/2}} = -qh\int_0^\infty \frac{r\,dr}{(h^2 + r^2)^{3/2}}$$
$$= \frac{qh}{(h^2 + r^2)^{1/2}}\bigg|_0^\infty = -q \tag{2.68}$$

It is apparent that all lines of electric flux emanating from the point charge $q$ terminate on the plane conductor. An illustration of a portion of these flux lines is given in Fig. 2.20. For interest, equipotential lines are drawn in as well.

Although the combination of the image charge $-q$ and the point charge $q$ gives the correct field in the half space containing $q$, the image charge is, of course, fictitious. The component of field evaluated as due to the image charge is actually caused by the surface distribution $\rho_s$ given in (2.66). The fields due to either $-q$ or $\rho_s$ are fully equivalent in the half space containing $q$. For field points in the region $z < 0$, the sources continue to be $\rho_s$ on the plane and $q$ at $(0,0,h)$. Now if, for the points in the region $z > 0$, $\rho_s$ on the plane is equivalent to the point charge $-q$ at $(0,0,-h)$, then by symmetry the field due to $\rho_s$ in the region $z < 0$ is equivalent to $-q$ but at $(0,0,h)$. This means that the total field is due to $+q$ and $-q$, which are both at $(0,0,h)$, and hence the field in the space $z < 0$ is zero. The conducting sheet may be thought of as shielding the region $z < 0$ from electrostatic field sources in the region $z > 0$.

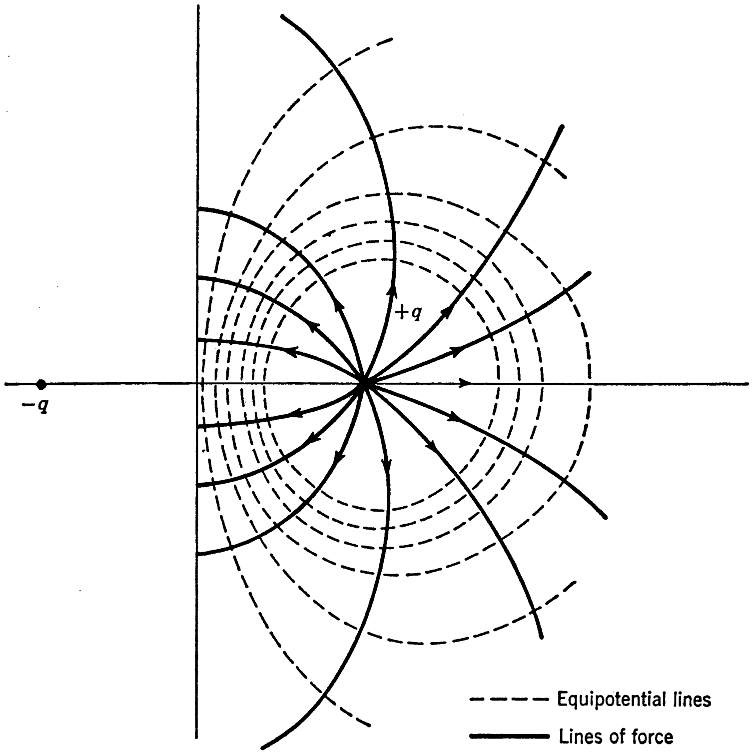


FIG. 2.20. Flux lines from a point charge in front of a conducting plane.

The problem considered here shows how charges may be induced in conducting bodies in the presence of other charges. This illustrates the reason why in measuring the electric field by taking the ratio $\mathbf{F}/\Delta q$ the test charge $\Delta q$ must be made very small if it is not to disturb a priori conditions. Thus suppose $q$ in Fig. 2.19 is actually introduced at $(0,0,h)$ to measure the field that exists prior to making the measurement (in this case zero, of course). Note that a nonzero force of strength $q^2/[4\pi\epsilon_0(2h)^2]$ is measured and an erroneous field of $q/16\pi\epsilon_0 h^2$ is presumed to have existed. The error in the measurement depends on the size of $q$.

A simple extension of the image technique involving a point source and a conducting plane is the case where a point source is located within two intersecting planes, such as $q$ in the right angle $AOB$ of Fig. 2.21. The image of $q$ in $OA$ is $-q$ at $P_1$, while the image of $q$ in $OB$ is $-q$ at $P_2$. But the combination of all three charges makes neither $OA$ nor $OB$ an equipotential surface. What has been neglected is imaging the images, a process that must be continued repeatedly. In this example, $q$ at $P_3$ satisfies both the requirement of $-q$ at $P_1$ and $-q$ at $P_2$ for proper imaging, and the group of four charges satisfies the boundary conditions and gives the correct solution for the field in $AOB$. The boundary requirements can be satisfied with a finite number of images only if $AOB$ is an exact submultiple of 180°, as in the 90° case illustrated. For the purpose of evaluating the position of the image charges it should be noted that they all lie on a circle.

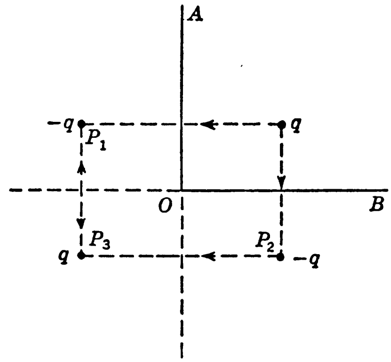


FIG. 2.21. A point charge located inside a 90° corner.

**Example 2.8. Inversion in a Sphere.** Another type of boundary-value problem that can be solved by the method of images is considered in this example. The charge $q_1$ is given as being located at a point $P_1$, a distance $R_1$ from the center of a conducting sphere of radius $a$, where $R_1 > a$. The field external to the sphere is desired.

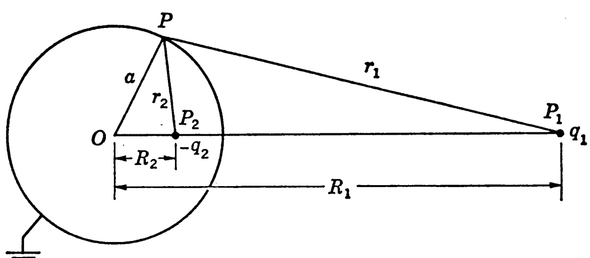


FIG. 2.22. Imaging of point charge $q_1$ in a sphere.

Let us consider an image charge $-q_2$ located at $P_2$ which is a distance $R_2$ from the center of the sphere along the line $OP_1$, as in Fig. 2.22. It is necessary that the combination of $q_1$ and $-q_2$ make the spherical surface a zero potential surface (the sphere is grounded). If we take any arbitrary point $P$ on the sphere, then it is required that

$$\frac{q_1}{4\pi\epsilon_0 r_1} - \frac{q_2}{4\pi\epsilon_0 r_2} = 0 \tag{2.69}$$

This will always be satisfied if we take

$$\frac{q_1}{q_2} = \frac{r_1}{r_2} \tag{2.70}$$

provided that an $R_2$ can be found so that $r_1/r_2$ is a constant, independent of the position $P$. Since any arbitrary point $P$ and the line $OP_1$ determine a plane, points in Fig. 2.22 can be thought of as located in that plane. If $OP_2 = R_2$ is chosen so that

$$\frac{OP_2}{a} = \frac{a}{OP_1} \tag{2.71}$$

then the triangle $OP_2P$ is similar to triangle $OP_1P$, and consequently

$$\frac{r_1}{r_2} = \frac{a}{OP_2} \tag{2.72}$$

will be a constant, as is necessary. To summarize, the field due to a charge $q$, a distance $R_1$ from the center of a grounded sphere of radius $a$, is found from the charge $q_1$ and an image charge $-q_2$ whose magnitude is

$$q_2 = \frac{R_2}{a} q_1 = \frac{a}{R_1} q_1 \tag{2.73}$$

and which is located on the line joining the center of the sphere and $q_1$ and at a distance

$$R_2 = \frac{a^2}{R_1} \tag{2.74}$$

from the center. Because of this relation the image technique in connection with a sphere is referred to as inversion in a sphere.

If the sphere were not grounded, then to maintain electrical neutrality an additional charge $+q_2$ must be placed inside the sphere.† (For the grounded sphere, $+q_2$ is essentially removed to infinity.) The location of $+q_2$ must be such as not to destroy the surface of the sphere as an equipotential. This is achieved by locating it at the center. Then the potential of the combination at any external field point is given as

$$\Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} - \frac{q_2}{r_2} + \frac{q_2}{r_0}\right) \tag{2.75}$$

† The total charge induced on the grounded sphere must be $-q_2$ because the combination $-q_2$ and $q_1$ correctly describes the electric field at the surface of the sphere according to the image theory just developed. Thus it would surely show the lines of flux as if they would terminate on $-q_2$. The location of a $+q_2$ within the sphere would lead to no net flux terminating on the sphere, as is necessary in the nongrounded case.

and the geometry is as given in Fig. 2.23. Actually, the charge $+q_2$ at the center is only an image charge that produces the same external effect as the uniform charge distribution $\rho_s = q_2/4\pi a^2$ residing on the outer surface of the sphere. The total surface charge on the sphere is zero

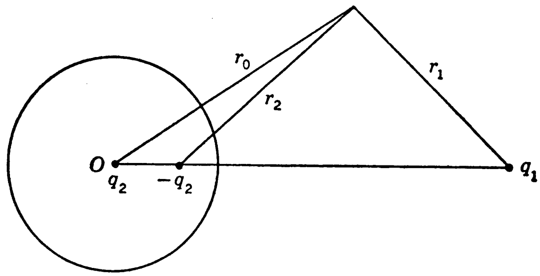


FIG. 2.23. Image charges $+q_2$ and $-q_2$ inside an ungrounded sphere.

since it equals the sum of the uniform distribution $q_2/4\pi a^2$ and a non-uniform distribution of total amount $-q_2$ which sets up the same external field as the $-q_2$ image charge. The total surface charge density could, of course, be found from (2.75) in the manner illustrated in connection with the infinite plane conductor.

**Example 2.9. Inversion in a Cylinder.** The problem of a line charge parallel with and outside a conducting cylinder of radius $a$ may be solved in a manner similar to the sphere problem considered in the previous

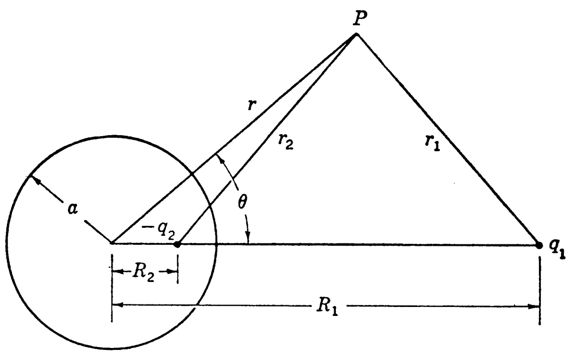


FIG. 2.24. Image line charge $-q_2$ inside a conducting cylinder of infinite length.

example. Consider a line charge of density $q_1$ coulombs per meter located a distance $R_1$ from the center of the cylinder and also an image line charge of density $-q_2$ at a distance $R_2$ from the center, as in Fig. 2.24.

The potential at any point $P$ is, from (2.41),

$$\Phi(P) = -\frac{q_1}{2\pi\epsilon_0}\ln r_1 + \frac{q_2}{2\pi\epsilon_0}\ln r_2 + C$$

where $C$ is an arbitrary constant, depending on the reference potential point. If we choose $q_2 = q_1$, we obtain

$$\Phi(P) = -\frac{q_1}{4\pi\epsilon_0} \ln \left(\frac{r_1}{r_2}\right)^2 + C \tag{2.76}$$

The constant potential surfaces are given by the condition

$$\frac{r_1}{r_2} = \text{constant} \tag{2.77}$$

The geometry here is the same as in the sphere problem, and condition

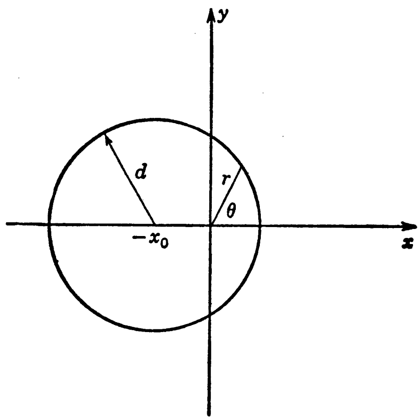


FIG. 2.25. Circle with center at $x = -x_0$, $y = 0$.

(2.77) is also the same; hence if we choose

$$R_2 = \frac{a^2}{R_1} \tag{2.78}$$

condition (2.77) will hold.

It will be of interest to derive equations for the entire family of equipotential surfaces specified by (2.77) and (2.78). From the law of cosines we get

$$r_2^2 = r^2 + R_2^2 - 2rR_2 \cos\theta$$
$$r_1^2 = r^2 + R_1^2 - 2rR_1 \cos\theta$$

To satisfy (2.77), let $r_1^2 = kr_2^2$, and then

$$r_1^2 - kr_2^2 = 0 = (1 - k)r^2 + R_1^2 - kR_2^2 - 2r(R_1 - kR_2) \cos\theta \tag{2.79}$$

A circle with center at $x = -x_0$, $y = 0$ and radius $d$, as in Fig. 2.25, has the equation

$$r^2 + 2rx_0 \cos\theta + x_0^2 - d^2 = 0 \tag{2.80}$$

Comparing with (2.79) shows that

$$x_0 = \frac{R_1 - kR_2}{k - 1} \tag{2.81a}$$

$$d^2 - x_0^2 = \frac{R_1^2 - kR_2^2}{k - 1} \tag{2.81b}$$

In the example we are considering, the radius of the cylinder is $a$, and hence $d = a$. Also the center of the cylinder is located at the origin of coordinates so that $x_0 = 0$. Thus from (2.81$a$) we find that

$$k = \frac{R_1}{R_2} \tag{2.82}$$

and from (2.81$b$) we confirm the relationship given in (2.78). The family of equipotential surfaces depends on the parameter $k$, and when $k$ is given

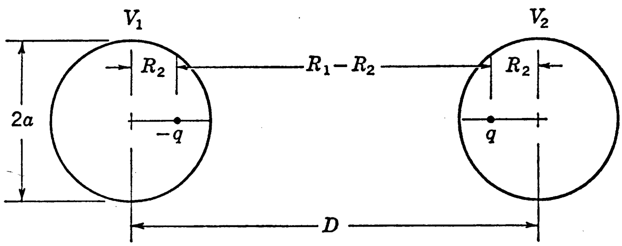


FIG. 2.26. Two-conductor transmission line.

by (2.82), the equipotential surface that coincides with the conducting cylinder is specified. The potential of the cylindrical conducting surface is

$$\Phi(a) = -\frac{q_1}{4\pi\epsilon_0}\ln k + C = -\frac{q_1}{4\pi\epsilon_0}\ln\frac{R_1}{R_2} + C \tag{2.83}$$

If the cylinder is grounded, then (2.83) determines $C$, since $\Phi(a)$ will be zero. The potential at any other point is then given by

$$\Phi(P) = -\frac{q_1}{4\pi\epsilon_0}\ln\left(\frac{r_1}{r_2}\right)^2 + \frac{q_1}{4\pi\epsilon_0}\ln\frac{R_1}{R_2} = -\frac{q_1}{4\pi\epsilon_0}\ln\frac{r_1^2R_2}{r_2^2R_1} \tag{2.84}$$

The above results may be used to solve the two-wire transmission-line problem. Consider two infinitely long parallel cylinders of radius $a$ and center-to-center separation $D$, as in Fig. 2.26. If we choose the constant $C$ to be zero in (2.83) and hypothesize that the distributed surface charge on the cylinders is equivalent to the line charges $-q$ and $+q$, as illustrated in Fig. 2.26, then the potential of the left-side cylindrical surface will be, as for Fig. 2.24,

$$V_1 = \frac{-q}{4\pi\epsilon_0}\ln\frac{R_1}{R_2} \tag{2.85a}$$

By symmetry, a similar constant potential surface surrounds the positive line charge and the potential of the right-side cylinder is

$$V_2 = \frac{q}{4\pi\epsilon_0} \ln \frac{R_1}{R_2} = -V_1 \tag{2.85b}$$

The fact that the cylindrical surfaces can be made equipotential ones is the basis of the original hypothesis. We must still determine $R_1$ and $R_2$. From the geometry of Fig. 2.26 it is seen that $D = R_1 + R_2$. Also $R_1R_2 = a^2$, and hence

$$R_1 = \frac{D}{2} + \left(\frac{D^2}{4} - a^2\right)^{1/2} \tag{2.86a}$$

$$R_2 = \frac{D}{2} - \left(\frac{D^2}{4} - a^2\right)^{1/2} \tag{2.86b}$$

Since $R_1$ and $R_2$ are uniquely specified, satisfaction of (2.86) will always lead to a valid solution of the boundary-value problem. The potential difference between the cylinders is

$$V_2 - V_1 = \frac{q}{2\pi\epsilon_0} \ln \frac{D + (D^2 - 4a^2)^{1/2}}{D - (D^2 - 4a^2)^{1/2}} \tag{2.87}$$

and may be adjusted to any given value by a proper choice of the line charge density $q$.

The equipotential surfaces are circles with centers located a distance $x_0$ from the cylinder center and away from the other cylinder and having a radius $d$. Equations (2.81) and (2.83) relate the parameters $x_0$ and $d$ to the particular constant potential surface (value of $k$) being considered. The magnitude of the total charge on each cylinder is $q$ coulombs per meter and could be determined from (2.87) if the difference in potential between the conductors were given.

## 2.11. Dipoles and Volume Discontinuities

The field due to two point charges of equal magnitude but opposite sign at distances which are large compared with the separation of the charges is called a dipole field. The product of the charge times the separation is the dipole moment. The dipole field is of particular importance, as we shall see, in the discussion of dielectrics.

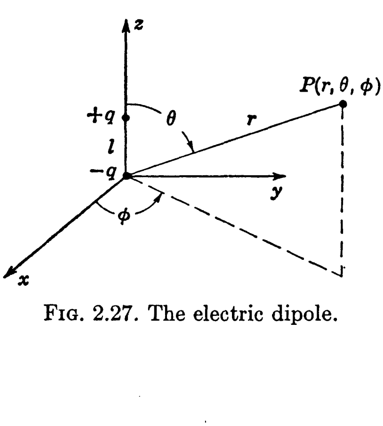


FIG. 2.27. The electric dipole.

Consider Fig. 2.27, where a dipole of charge magnitude $q$ and separation $l$ is illustrated. The origin has been chosen at the charge $-q$, and the polar axis is defined by the line joining $-q$ and $q$. An obvious and correct way of finding the field at an arbitrary point $P$ is to superpose the field due to each point charge separately. We are interested in the potential field as well as the electric field and will determine the former first.

Referring to Fig. 2.27, the potential at $P$ due to $-q$ alone is

$$\Phi(P)_- = \frac{-q}{4\pi\epsilon_0 r} \tag{2.88}$$

Since the charge $+q$ is displaced a distance $l$ along the positive $z$ axis, its contribution to the potential at $P$ is the same as what would be found by displacing $P$ a distance $l$ in the negative $z$ direction with $+q$ located at the origin. By Taylor's theorem, the latter comes out to be

$$\Phi(P)_+ = \frac{q}{4\pi\epsilon_0 r} + \frac{\partial}{\partial z}\left(\frac{q}{4\pi\epsilon_0 r}\right)\Bigg|_P (-l) + \frac{1}{2!}\frac{\partial^2}{\partial z^2}\left(\frac{q}{4\pi\epsilon_0 r}\right)\Bigg|_P (-l)^2 + \cdots \tag{2.89}$$

The total dipole potential at $P$, by superposition, is then

$$\Phi(P) = -\frac{\partial}{\partial z}\left(\frac{q}{4\pi\epsilon_0 r}\right)\Bigg|_P l + \frac{1}{2!}\frac{\partial^2}{\partial z^2}\left(\frac{q}{4\pi\epsilon_0 r}\right)\Bigg|_P l^2 + \cdots \tag{2.90}$$

Since

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{r^3}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3}\left(1 - \frac{3z^2}{r^2}\right) \qquad \text{etc.}$$

For $l \ll r$, as originally specified, all terms except the first can be neglected and the dipole potential reduces to

$$\Phi(P) = \frac{qzl}{4\pi\epsilon_0 r^3} \tag{2.91}$$

where $z$ and $r$ correspond to the point $P$. Now

$$\cos\theta = \frac{z}{r}$$

and hence

$$\Phi(P) = \frac{ql\cos\theta}{4\pi\epsilon_0 r^2} \tag{2.92}$$

The product $ql$ is the dipole moment $p$. It can be written as a vector, with a direction defined as from $-q$ to $+q$. In this case (2.92) has the simple form

$$\Phi(P) = \frac{\mathbf{p}\cdot\mathbf{a}_r}{4\pi\epsilon_0 r^2} \tag{2.93}$$

The electric field of the dipole can be readily found by taking the negative gradient of (2.93), using spherical coordinates. We get

$$\mathbf{E} = \frac{ql}{4\pi\epsilon_0 r^3}(\mathbf{a}_r 2\cos\theta + \mathbf{a}_\theta \sin\theta) \tag{2.94}$$

A sketch of the lines of force in a dipole field is given in Fig. 2.28. The three-dimensional picture of the lines of force is obtained by revolving the pattern in Fig. 2.28 about the dipole axis. In the $r$, $\theta$ plane the differential equation for the lines of force is

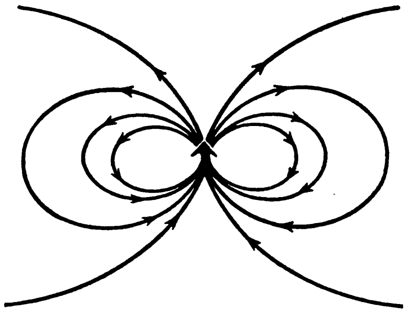

FIG. 2.28. Lines of force in an electric dipole field.

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta} \qquad \text{or} \qquad \frac{dr}{r} = 2\cot\theta\,d\theta$$

after substituting from (2.94). This equation is readily integrated to give

$$\ln r = 2\ln\sin\theta + \ln C$$

or

$$r = C\sin^2\theta$$

For each value of the integration constant $C$, a particular line of force is obtained. The constant potential contours are orthogonal to the lines of force and hence have a slope which is the negative reciprocal of the slope of the lines of force. The differential equation is therefore

$$\frac{dr}{r\,d\theta} = -\tfrac{1}{2}\tan\theta$$

This may be integrated to give

$$\ln r = \tfrac{1}{2}\ln\cos\theta + \tfrac{1}{2}\ln C$$

or

$$r^2 = C\cos\theta$$

That this equation is the equation for a constant potential contour is readily seen from (2.92) by equating $\Phi$ to a constant. Each value of the constant $C$ determines a particular constant potential surface.

For a volume distribution of charge it was convenient to specify a charge density $\rho$ as a mathematically well-defined function. Its definition and the restrictions required were discussed in Sec. 2.4. In the same way a volume distribution of dipoles can be represented by a vector function $\mathbf{P}$, which gives the dipole moment per unit volume at a point. Specifically,

$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{\Sigma \mathbf{p}_i}{\Delta V} \tag{2.95}$$

where $\Sigma\mathbf{p}_i$ is the vector sum of the dipole moments in the volume $\Delta V$. It is again necessary that $\Delta V$ be sufficiently large so that individual dipole characteristics do not affect the result but small enough to get a true limit. We shall assume that the **P** that results is a continuous function.

The potential set up by an arbitrary volume distribution of dipoles will now be computed by using the above definition of **P**. Let $\mathbf{P}(x',y',z')$ be the dipole density at the point $(x',y',z')$, and let $dV'$ be an element of volume. Since the distance to a field point is inherently large compared with the extent of the differential volume element $dV'$, (2.93) applies at any field point. Consequently, by superposition,

$$\Phi(x,y,z) = \int_V \frac{\mathbf{P}(x',y',z') \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\, dV' \tag{2.96}$$

where $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ and is the distance between the volume element $dV'$ and the field point $(x,y,z)$. This result can be transformed into one that has an interesting physical interpretation. Noting that

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}$$

we have

$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) = \nabla' \cdot \frac{\mathbf{P}}{R} - \frac{\nabla' \cdot \mathbf{P}}{R}$$

Thus (2.96) becomes

$$\begin{aligned}\Phi(x,y,z) &= \frac{1}{4\pi\epsilon_0}\int_V \nabla' \cdot \frac{\mathbf{P}}{R}\, dV' - \frac{1}{4\pi\epsilon_0}\int_V \frac{\nabla' \cdot \mathbf{P}}{R}\, dV' \\ &= \oint_S \frac{\mathbf{P} \cdot \mathbf{n}}{4\pi\epsilon_0 R}\, dS' + \int_V \frac{-\nabla' \cdot \mathbf{P}}{4\pi\epsilon_0 R}\, dV' \end{aligned} \tag{2.97}$$

after using Gauss' theorem on the first volume integral. From a comparison of (2.97) with (2.36) we deduce that the field due to a volume dipole distribution **P** is the same as that from an equivalent volume and surface charge distribution such that

$$\rho_p = -\nabla' \cdot \mathbf{P} \qquad \text{in } V \tag{2.98a}$$

and

$$\rho_{sp} = \mathbf{P} \cdot \mathbf{n} \qquad \text{on } S \tag{2.98b}$$

Actually, if the volume be taken to include the surface, then the surface charge distribution is included in the expression for $\rho_p$; that is, just as we can let $\rho$ contain $\rho_s$ as a limiting case, we let $-\nabla' \cdot \mathbf{P}$ include $\mathbf{P} \cdot \mathbf{n}$ as a limiting case and

$$\Phi = \int_V \frac{-\nabla' \cdot \mathbf{P}}{4\pi\epsilon_0 R}\, dV' \tag{2.99}$$

The results expressed by (2.98) can be understood on a purely physical basis. Where $\nabla' \cdot \mathbf{P} \neq 0$, this means a net creation of dipole moment per unit volume, hence an incomplete cancellation of charge density from adjacent dipoles. Similarly, the surface charge density occurs because the dipoles ending on the surface cannot be neutralized for want of an adjacent dipole layer, as illustrated in Fig. 2.29.

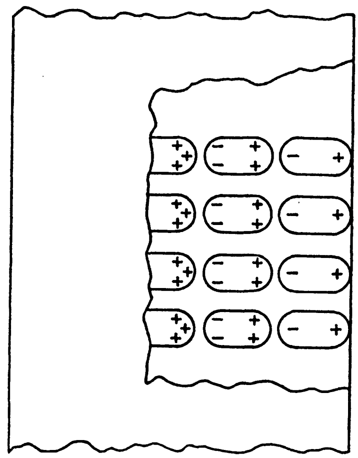

FIG. 2.29. Illustration of creation of volume and surface polarization charge because of incomplete cancellation of charge between adjacent dipoles.

**2.12. Field Behavior at a Charged Surface**

A general shaped open surface on which an arbitrary surface charge density $\rho_s(x',y',z')$ exists is illustrated in Fig. 2.30. In solving certain types of problems in electrostatics it is necessary to know the behavior of the scalar potential and the electric field in crossing such a surface. This can be determined from the fundamental relationships. Thus the scalar potential $\Phi$ due to the surface charge distribution is given by

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s}{R}\,dS' \tag{2.100}$$

where $R$ is the distance between the charge element $\rho_s\,dS'$ and the field point; that is, $R = |\mathbf{r} - \mathbf{r}'|$. Consider the variation in $\Phi$ as the field point moves along a path $\Gamma$ that crosses the charged surface. In the vicinity of the surface the path followed may be considered linear and a small element of surface ($\Delta S_0$) surrounding the intersection of the path with the surface may be considered plane. Figure 2.30 illustrates the over-all geometry, and Fig. 2.31 is an enlargement of the region near the surface. We can resolve (2.100) into the following components:

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_{S-\Delta S_0} \frac{\rho_s\,dS'}{R} + \frac{1}{4\pi\epsilon_0}\int_{\Delta S_0} \frac{\rho_s\,dS'}{R} \tag{2.101}$$

In the first integral of (2.101), $R$ is always finite and the contribution to $\Phi$ that is produced is clearly continuous as the field point crosses the surface. It will be easier to evaluate the second integral if we consider $\Delta S_0$ to be a very small circular area of radius $r_0$ centered about the point of crossing the surface, that is, $O$ of Fig. 2.31. If the field point is within a very small distance $\Delta d$ of the surface, and recognizing that over $\Delta S_0$, $\rho_s$ is essentially constant, we have

$$\frac{1}{4\pi\epsilon_0}\int_{\Delta S_0}\frac{\rho_s\,dS'}{R} = \frac{\rho_s}{2\epsilon_0}\int_0^{r_0}\frac{r'\,dr'}{\sqrt{r'^2+(\Delta d)^2}} = \frac{\rho_s}{2\epsilon_0}\left[\sqrt{r_0^2+(\Delta d)^2} - |\Delta d|\right] \quad (2.102)$$

For a point passing through the surface, $\Delta d$ decreases to zero and then increases negatively. But under these conditions the contribution to the total potential from (2.102) will also be continuous in crossing the surface; consequently, the total electrostatic potential is continuous across an arbitrary charged surface.

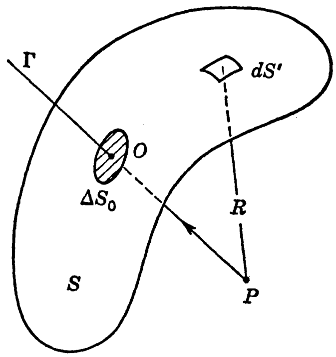


FIG. 2.30. An arbitrary surface charge distribution.

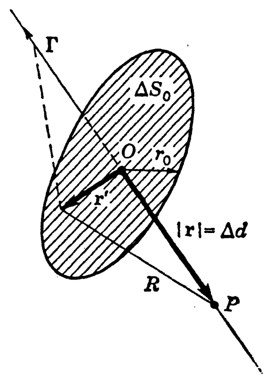


FIG. 2.31. Geometry of surface charge layer close to the surface.

A somewhat different situation arises if we consider the behavior of the electric field. In this case the normal component of **E** suffers a discontinuity equal to $\rho_s/\epsilon_0$ in crossing a charged surface. This can be established very easily by application of Gauss' flux theorem, and this will be presented in Sec. 3.3. This same result can also be verified in a direct way as follows. We have

$$E_n = -\nabla\Phi\cdot\mathbf{n} = \frac{-1}{4\pi\epsilon_0}\int_S \rho_s\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{n}\,dS' \quad (2.103)$$

The total surface can be broken up into two parts as before. Then

$$E_n = \frac{-1}{4\pi\epsilon_0}\int_{S-\Delta S_0}\rho_s\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{n}\,dS' - \frac{1}{4\pi\epsilon_0}\int_{\Delta S_0}\rho_s\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{n}\,dS' \quad (2.104)$$

The field contributed by the first integral in (2.104) must be continuous along a path passing through the charge surface since the only variable, $|\mathbf{r}-\mathbf{r}'|$, is finite and well-behaved. In this case, however, a discontinuity is introduced by the second integral of (2.104). Again, for simplicity, $\Delta S_0$ is chosen sufficiently small so that it may be considered plane and so that $\rho_s$ may be taken out as a constant. If we let the origin be located at the field point, then with reference to Fig. 2.32,

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}\int_{\Delta S_0}\rho_s\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\mathbf{n}\,dS' \\ = -\frac{\rho_s}{4\pi\epsilon_0}\int_{\Delta S_0}\nabla'\left(\frac{1}{R}\right)\cdot\mathbf{n}\,dS' = \frac{\rho_s}{4\pi\epsilon_0}\int_{\Delta S_0}\frac{\mathbf{a}_R\cdot\mathbf{n}}{R^2}\,dS' \\ = \frac{\rho_s}{4\pi\epsilon_0}\int_{\Delta S_0}d\Omega = \frac{\rho_s\Omega_0}{4\pi\epsilon_0} \quad (2.105)\end{aligned}$$

where $\Omega_0$ is the solid angle subtended at the field point by $\Delta S_0$. As the field point moves across $\Delta S_0$, as illustrated in Fig. 2.32, $\Omega_0$ increases to a value of $2\pi$, and then in

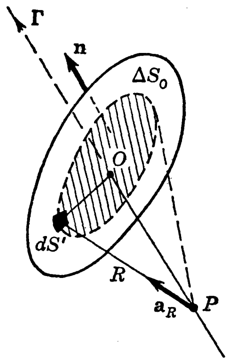


FIG. 2.32

crossing the surface it suddenly changes to $-2\pi$. Consequently, the normal component of **E** suffers a discontinuity, in crossing a charged surface, which we have found to be

$$E_{n1} - E_{n2} = \frac{\rho_s}{\epsilon_0} \tag{2.106}$$

where $\rho_s$ is the surface charge density at the point of discontinuity. This result corresponds to what was found in Example 2.4, except that we have now assured it for any surface geometry.

We show later, in Sec. 3.3, that the tangential component of **E** is continuous across a charge surface.

## Chapter 2

**2.1.** Find the electric field at the point $x = 4$, $y = z = 0$, due to point charges $Q_1 = 8$ coulombs, $Q_2 = -4$ coulombs, and located at $z = 4$ on the $z$ axis and $y = 4$ on the $y$ axis, respectively.

**2.2.** Positive point charges of magnitude 4, 2, and 2 coulombs are located in the $yz$ plane at $y = 0$, $z = 0$; $y = 1$, $z = 1$; and $y = -1$, $z = -1$; respectively. Find the force acting on a unit negative point charge located at $x = 6$ on the $x$ axis.

Hint: Evaluate the vector force from each charge first, and then add up the partial forces vectorially.

**2.3.** Find the potential at an arbitrary point $(x,y,z)$ from the three positive charges specified in Prob. 2.2. From the potential function find the electric field and the force exerted on a unit negative charge at $x = 6$ on the $x$ axis. This problem is an example of evaluating the force on a charge by means of the field concept.

**2.4.** Consider two infinite positive line charges of density $q$ coulombs per meter and located at $y = \pm 1$, $x = 0$, $-\infty \leq z \leq \infty$, together with two similar negative line charges located at $x = \pm 1$, $y = 0$. Derive an analytical expression for the constant potential curves in the $xy$ plane. Sketch the constant potential curves and the electric-field flux lines.

**2.5.** Find the force per meter exerted on the positive line charges by the negative line charges of Prob. 2.4. Use both the field approach and Coulomb's law. Note that the use of the field approach is much simpler.

HINT: When Coulomb's law is used, it is necessary to integrate over the total length of the negative line charges. Begin by considering the force exerted on a length $dl$ of the positive line charge centered on $z = 0$ by charge elements $-q_l\,dz$ at $\pm z$, $x = \pm 1$. By symmetry this force is in the $y$ direction only.

**2.6.** Consider two infinite line charges of density $q_l$ coulombs per meter, parallel to the $z$ axis, and located at $x = \pm x_0$, $y = 0$. Consider two particular lines of flux, with an angle $\theta$ between them, leaving one line charge, and show that at infinity the angle between these same two flux lines is $\theta/2$.

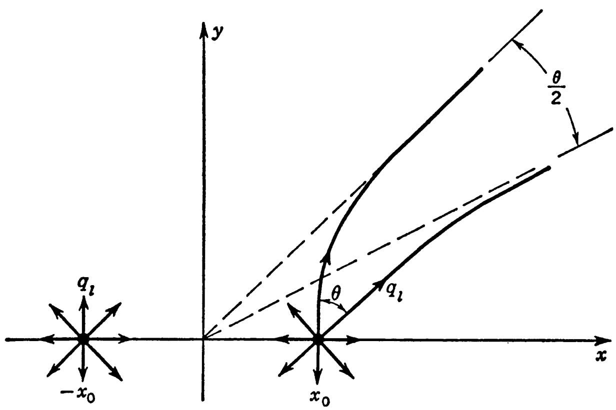


FIG. P 2.6

HINT: Note that very near the line source the flux lines are radial and equally spaced around the line source. (The equal spacing represents the uniformity of the field with azimuth, a condition that must hold very close to the line source.) The number of lines is equal to the charge contained within a cylinder surrounding the line source. At infinity the lines of flux from the two line charges must appear to arise from the sum of the line charges as if concentrated along a line which is their center of gravity. Consequently, they are radial and equispaced. The total lines of flux at infinity is equal to the total charge contained within a cylinder surrounding both line charges. By noting that the total flux in any given flux tube does not change, the relation called for above can be established by setting up a proportionality between the angular spacing of the flux lines near the line charge and at infinity and the charge contained within cylinders surrounding one line source and both line sources, respectively.

**2.7.** What relation must be satisfied, analogous to that developed in Prob. 2.6, if the line charges are replaced by two positive point charges located at $x = \pm x_0$, $y = z = 0$?

HINT: Note that the flux tubes have rotational symmetry about the $x$ axis.

**2.8.** A positive line charge $Nq_l$ coulombs per meter and a negative line charge $-q_l$ coulombs per meter, parallel to the $z$ axis, are located at $x = \pm x_0$, $y = 0$. With reference to Fig. P 2.8, determine the maximum angle $\theta_m$ for which the lines of flux from the positive line charge will extend out to infinity and not terminate on the negative line charge.

HINT: Apply arguments similar to those needed for Prob. 2.6.

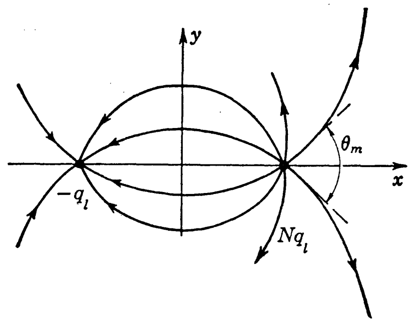


FIG. P 2.8

**2.9.** Consider two infinitely long concentric coaxial cylinders of radius $a$ and $b$, as illustrated. The inner cylinder carries a uniform surface charge of density $\sigma_1$ coulombs per square meter, while the outer cylinder carries a uniform charge of density $\sigma_2$ coulombs per square meter. Use Gauss' law to find the electric field in the three regions $r < a$, $a < r < b$, and $r > b$. What is the potential difference between the two cylinders?

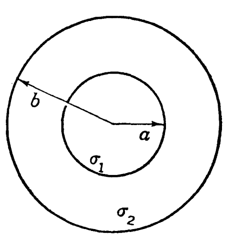


FIG. P 2.9

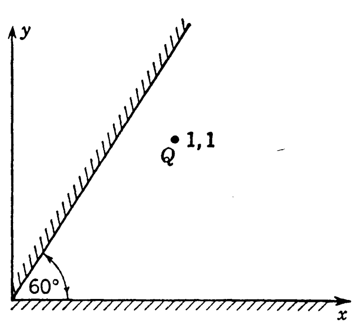


FIG. P 2.10

**2.10.** A point charge of $Q$ coulombs is located at $x = 1$, $y = 1$ in the space between an infinite 60° conducting wedge. Find the location and sign of all the image charges. What is the potential at the point $x = 2$, $y = 1$?

**2.11.** A small body of mass $m$ carries a charge $Q$. The body is placed $h$ meters below an infinite conducting plane. Use image theory to find the required charge $Q$ in order that the electrostatic force may be just sufficient to overcome the force of gravity. Assume $m = 1$ gram, $h = 2$ centimeters.

**2.12.** The electric field produced by a sphere of charge with a density $\rho(r)$ is given by

$$E_r = \begin{cases} r^3 + Ar^2 & r \le a \\ (a^5 + Aa^4)r^{-2} & r \ge a \end{cases}$$

Find the charge distribution $\rho(r)$.

**2.13.** A charge distribution $\rho(r)$ is placed inside a conducting sphere of radius $a$. The electric field is given by

$$E_r = \begin{cases} Ar^4 & r \leq a \\ Ar^{-2} & r > a \end{cases}$$

Find the charge distribution $\rho(r)$ within the sphere and the surface charge $\rho_s$ on the surface of the sphere.

**2.14.** Two concentric spheres of radii $a$ and $b$ ($a < b$) are uniformly charged with charge densities $\rho_{s1}$ and $\rho_{s2}$ per square meter. Use Gauss' law to find the electric field for all values of $r$. If $\rho_{s1} = -\rho_{s2}$, find the potential difference between the spheres.

**2.15.** (*a*) A conducting sphere of radius $a$ is placed in a uniform field $\mathbf{E}_0$ directed along the $z$ axis. Positive and negative charges are induced on the sphere, which in turn sets up an induced field $\mathbf{E}_i$ such that the total field $\mathbf{E}_0 + \mathbf{E}_i$ vanishes in the interior of the sphere and has a zero tangential component along the surface of the sphere. Find the induced field $\mathbf{E}_i$ inside and outside the sphere. Show that outside the sphere the induced field is the same as that produced by an electric dipole of moment $P = 4\pi a^3\epsilon_0 E_0$ located at the origin.

Hint: The field $\mathbf{E}_0$ may be found from the function $-\nabla\Phi_0$, where $\Phi_0 = -zE_0 = -E_0 r\cos\theta$. Let the induced potential be $\Phi_1$ for $r < a$ and $\Phi_2$ for $r > a$. Both $\Phi_1$ and $\Phi_2$ are solutions of Laplace's equation and must vary with $\theta$ according to $\cos\theta$ since $\Phi_0$ does. From Prob. 1.15 appropriate solutions for the induced potentials are found to be $\Phi_1 = Ar\cos\theta$ ($r < a$), $\Phi_2 = Br^{-2}\cos\theta$ ($r > a$), since $\Phi_1$ must remain finite at $r = 0$ and $\Phi_2$ must vanish at $r = \infty$. To find the coefficients $A$ and $B$, impose the boundary conditions that at $r = a$ the total potential is continuous across the surface $r = a$ and inside the sphere the total field vanishes.

(*b*) Find the charge distribution on the surface of the sphere.

**2.16.** A point charge $Q$ is located a distance $a < b$ from the center of a conducting sphere of radius $b$. Find the charge distribution on the inner surface of the sphere. Obtain an expression for the force exerted on $Q$. The sphere is initially uncharged. Does the force depend on whether the sphere is grounded or not?

**2.17.** A point charge $Q$ is located at a distance $R$ from the center of an insulated conducting sphere of radius $b < R$. The sphere is ungrounded and initially uncharged. Show that the force attracting $Q$ to the sphere is

$$F = \frac{Q^2b^3}{4\pi\epsilon_0 R^3}\frac{2R^2 - b^2}{(R^2 - b^2)^2}$$

If the sphere is grounded, what is the force on $Q$?

**2.18.** A conducting sphere carries a total charge $Q_0$. A point charge $q$ is brought into the vicinity of the ungrounded charged sphere. Obtain an expression for the distance from the center of the sphere for which the force on $q$ is zero.

**2.19.** The entire $xz$ plane is charged with a charge distribution $\rho_s(x,z)$. There is no charge in the region $|y| > 0$. Which of the following potential functions are a valid solution, in the half space $y > 0$, for the problem, and what is the corresponding charge distribution $\rho_s(x,z)$ on the $xz$ plane?

$$\Phi_1 = e^{-y}\cosh x$$
$$\Phi_2 = e^{-y}\cos x$$
$$\Phi_3 = e^{-\sqrt{2}y}\cos x\sin z$$
$$\Phi_4 = \sin x\sin y\sin z$$

**2.20.** (*a*) A small electric dipole of moment $\mathbf{P}$ is placed in a uniform electric field $\mathbf{E}_0$. Show that the torque acting on the dipole is given by

$$\mathbf{T} = \mathbf{P} \times \mathbf{E}_0$$

(b) If the field $\mathbf{E}_0$ varies throughout space, show that the dipole is also subjected to a force given by

$$\mathbf{F} = (\mathbf{P} \cdot \nabla)\mathbf{E}_0$$

and that the torque about an arbitrary origin is then $\mathbf{r} \times (\mathbf{P} \cdot \nabla)\mathbf{E}_0 + \mathbf{P} \times \mathbf{E}_0$.

HINT: Let $d\mathbf{l}$ be the dipole vector length, and expand $\mathbf{E}_0$ in a Taylor series about the negative charge to find the first-order change in $\mathbf{E}_0$ at the positive charge, i.e., to obtain

$$\Delta\mathbf{E}_0 = \mathbf{a}_x \frac{\partial E_{0x}}{\partial l} dl + \mathbf{a}_y \frac{\partial E_{0y}}{\partial l} dl + \mathbf{a}_z \frac{\partial E_{0z}}{\partial l} dl = \frac{\partial \mathbf{E}_0}{\partial l} dl$$

Next note that the expression $\boldsymbol{\tau} \cdot \nabla$ gives the derivative in the direction of $\boldsymbol{\tau}$. The net force is the sum of the forces acting on the positive and negative charges.

**2.21.** For the four point charges located in the $xz$ plane, as in Fig. P 2.21, show that for $r \gg d$ the potential $\Phi$ is given by

$$\Phi = \frac{-3Qd^2xz}{4\pi\epsilon_0 r^5} = \frac{-3Qd^2}{4\pi\epsilon_0 r^3} \sin\theta \cos\theta \cos\phi$$

The four charges constitute an electric quadrupole (double dipole).

HINT: Superpose the potential from each charge, and expand the radial distance from each point charge to the field point in a binomial series, retaining the first three terms in the expansion; e.g.,

$$[x^2 + y^2 + (z - d)^2]^{-1/2} \approx r^{-1}\left(1 - \frac{2zd}{r^2}\right)^{-1/2} = r^{-1}\left[1 + \frac{zd}{r^2} + \frac{3}{2}\left(\frac{zd}{r^2}\right)^2 + \cdots\right]$$

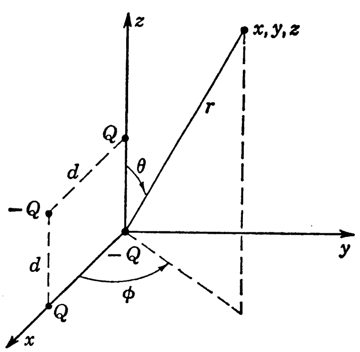


FIG. P 2.21

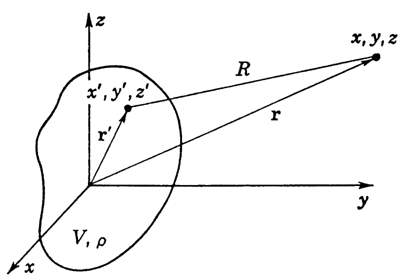


FIG. P 2.22

**2.22.** Consider a charge distribution $\rho(x',y',z')$ located within a volume $V$ near the origin of an $xyz$ coordinate system, as in Fig. P 2.22. Expand $R^{-1}$ in a Taylor series with respect to $x'$, $y'$, $z'$ about the origin to show that the first three terms of the multipole expansion for the potential $\Phi$ is given by

$$\Phi(x,y,z) = \int_V \frac{\rho(x',y',z')}{4\pi\epsilon_0 R} dV' = \Phi_1 + \Phi_2 + \Phi_3 + \cdots$$

where $$\Phi_1 = \frac{1}{4\pi\epsilon_0 r}\int_V \rho(x',y',z')\, dV' \qquad \text{coulomb potential}$$

$$\Phi_2 = \frac{-1}{4\pi\epsilon_0}\left(r_x^{-1}\int_V \rho x'\,dV' + r_y^{-1}\int_V \rho y'\,dV' + r_z^{-1}\int_V \rho z'\,dV'\right)$$

$$= \frac{-1}{4\pi\epsilon_0}\nabla\left(\frac{1}{r}\right)\cdot\int_V \mathbf{r}'\rho\,dV' \qquad \text{dipole potential}$$

$$\Phi_3 = \frac{1}{8\pi\epsilon_0}\left(r_{xx}^{-1}\int_V x'^2\rho\,dV' + r_{yy}^{-1}\int_V y'^2\rho\,dV' + r_{zz}^{-1}\int_V z'^2\rho\,dV' + 2r_{xy}^{-1}\int_V x'y'\rho\,dV'\right.$$

$$\left. + 2r_{xz}^{-1}\int_V x'z'\rho\,dV' + 2r_{yz}^{-1}\int_V y'z'\rho\,dV'\right) \qquad \text{quadrupole potential}$$

and $r_x^{-1} = \partial r^{-1}/\partial x$, $r_{xy}^{-1} = \partial^2 r^{-1}/\partial x\,\partial y$, etc.

Use this expansion to verify the results of Prob. 2.21.

HINT: Taylor's expansion may be written symbolically as

$$\frac{1}{R} = \frac{1}{r} + \left[\left(x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right) + \tfrac{1}{2}\left(x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'} + z'\frac{\partial}{\partial z'}\right)^2 + \cdots\right]\frac{1}{R}$$

where all derivatives are evaluated at the origin. Note that $\partial R^{-1}/\partial x' = -\partial r^{-1}/\partial x$, $\partial^2 R^{-1}/\partial x'\,\partial y' = \partial^2 r^{-1}/\partial x\,\partial y$, etc., for $x' = y' = z' = 0$, that is, at the origin.

**2.23.** Consider a ring of charge of radius $a$, center at the origin, and lying in the $xy$ plane. The charge distribution around the ring is given by

$$q_l = q_1 \cos\phi + q_2 \sin 2\phi \qquad \text{coulombs per meter}$$

Use the multipole expansion of Prob. 2.22 to obtain the first three terms in the expression for $\Phi$ for $r \gg a$.

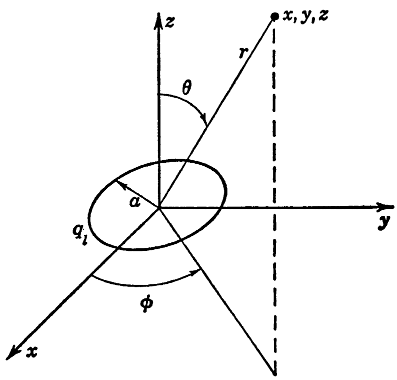


FIG. P 2.23