CHAPTER 11

RADIATION AND ANTENNAS

We have noted that under time-varying conditions Maxwell's equations predict the radiation of electromagnetic energy from current sources. While such a phenomenon takes place at all frequencies, its relative magnitude is insignificant until the size of the source region is comparable to wavelength. In constructing circuits to operate at higher and higher frequencies, this means that a point is reached where radiation from the circuit will interfere with the desired circuit characteristics and the use of other techniques and devices, such as waveguides and resonators, is necessary. In this chapter, however, radiation is the desired end product. We shall, consequently, be interested in some of the characteristics of radiators, such as their efficiency and the resultant radiation patterns. We shall examine the transmitting properties of the dipole antenna and the array of dipoles and conclude with a discussion of the receiving antenna and reciprocity.

11.1. Radiation from a Linear Current Element

The simplest radiating structure is that of an infinitesimal current element. An understanding of the properties of such an antenna is of great use since, in principle at least, all radiating structures can be considered as a sum of small radiating elements. Furthermore, many practical antennas at low frequencies are very short compared with wavelength, and the results we obtain here will be sufficiently accurate to describe their behavior.

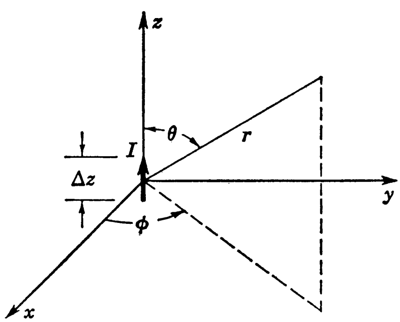

FIG. 11.1. An infinitesimal linear-current radiator.

Thus, consider a linear current element $I = I_0e^{j\omega t}$ of length Δz, oriented in the z direction and located at the origin, as in Fig. 11.1. For convenience, we assume that I_0 is a real amplitude factor. The charge associated with this current element may be obtained by noting that the current flowing into the upper end must equal the time rate of increase of charge at the upper end. Thus $j\omega Q = I_0$ or $Q = -jI_0/\omega$ at the upper end and $-Q = jI_0/\omega$ at the lower end of the current element. The small linear current element may be viewed as two charges Q and $-Q$ oscillating back and forth.

From Sec. 9.9, the vector and scalar potentials from general volume distributions of current and charge are

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{R} e^{-jk_0R}\, dV' \tag{11.1a}$$

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{R} e^{-jk_0R}\, dV' \tag{11.1b}$$

In the present case we are dealing with a differential current element only, so that

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0 I_0\,\Delta z}{4\pi r}\,\mathbf{a}_z e^{-jk_0r} \\ &= \frac{\mu_0 I_0\,\Delta z}{4\pi r}(\mathbf{a}_r\cos\theta - \mathbf{a}_\theta \sin\theta)e^{-jk_0r} \end{aligned} \tag{11.2}$$

The oscillating charge is equivalent to a small electric dipole of moment $Q\,\Delta z = -jI_0\,\Delta z/\omega$. (This accounts for the antenna being also referred to as an elementary dipole, or doublet.) The scalar potential Φ is readily seen to be given by

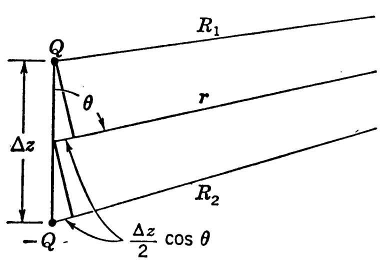

FIG. 11.2. Evaluation of scalar potential.

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2}\right)$$

where R_1 and R_2 are the distances specified in Fig. 11.2. From this figure it is seen that $R_1 \approx r - (\Delta z/2)\cos\theta$ and $R_2 \approx r + (\Delta z/2)\cos\theta$, since the paths from the ends of the dipole to the field point are essentially parallel. The expression for Φ becomes

$$\Phi = \frac{Qe^{-jk_0r}}{4\pi\epsilon_0 r}\left[e^{jk_0\frac{\Delta z}{2}\cos\theta}\left(1 + \frac{\Delta z\cos\theta}{2r}\right) - e^{-jk_0\frac{\Delta z}{2}\cos\theta}\left(1 - \frac{\Delta z\cos\theta}{2r}\right)\right]$$

after replacing R_1^{-1} by

$$r^{-1}\left(1 - \frac{\Delta z\cos\theta}{2r}\right)^{-1} \approx r^{-1}\left(1 + \frac{\Delta z\cos\theta}{2r}\right)$$

and analogously for R_2^{-1}. Using the expansion $e^x \approx 1 + x$ for x small, we obtain

$$\Phi = \frac{Q\,\Delta z}{4\pi\epsilon_0}\left(\frac{\cos\theta}{r^2} + \frac{jk_0\cos\theta}{r}\right)e^{-jk_0r} \tag{11.3}$$

since $k_0\,(\Delta z/2)\cos\theta$ is small.

The electric and magnetic fields are given by (see Sec. 9.9)

$$\mathbf{H} = \mu_0^{-1}\nabla \times \mathbf{A} \tag{11.4a}$$
$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi \tag{11.4b}$$
$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla(\nabla \cdot \mathbf{A})}{j\omega\mu_0\epsilon_0} \tag{11.4c}$$

From (11.4b) we get, for **E**,

$$\mathbf{E} = -\frac{j\omega\mu_0 I_0\,\Delta z}{4\pi r}(\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta)e^{-jk_0r} + \frac{jI_0\,\Delta z}{4\pi\omega\epsilon_0}\nabla\left(\frac{\cos\theta}{r^2} + \frac{jk_0\cos\theta}{r}\right)e^{-jk_0r} \tag{11.5}$$

after replacing Q by $-jI_0/\omega$ in (11.3). We may readily show that the Lorentz condition $(j\omega\mu_0\epsilon_0)^{-1}\,\nabla\nabla\cdot\mathbf{A} = -\nabla\Phi$ is satisfied by **A** and Φ of (11.2) and (11.3). We have

$$\nabla\cdot\mathbf{A} = \frac{\mu_0 I_0\,\Delta z}{4\pi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cos\theta\,\frac{e^{-jk_0r}}{r}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\,\frac{e^{-jk_0r}}{r}\right)\right]$$

after expressing the divergence in spherical coordinates. Carrying out the differentiation gives

$$\nabla\cdot\mathbf{A} = -\frac{\mu_0 I_0\,\Delta z}{4\pi}\left(\frac{jk_0\cos\theta}{r} + \frac{\cos\theta}{r^2}\right)e^{-jk_0r} \tag{11.6}$$

We see from this expression that $(j\omega\mu_0\epsilon_0)^{-1}\,\nabla\nabla\cdot\mathbf{A}$ is equal to $-\nabla\Phi$, and hence we may compute the fields from the vector potential **A** alone, as in (11.4c). Again we point out that this possibility arises for time-varying fields since current and charge are not independent; i.e., they satisfy the continuity equation. In this specific case the choice of $j\omega Q = I_0$ (the continuity condition) is the necessary relationship for Φ and **A** to satisfy the Lorentz condition.

After carrying out the operations indicated in (11.4) we find that

$$\mathbf{H} = \frac{I_0\,\Delta z}{4\pi}\sin\theta\left(\frac{jk_0}{r} + \frac{1}{r^2}\right)e^{-jk_0r}\mathbf{a}_\phi \tag{11.7a}$$

$$\mathbf{E} = -\frac{I_0\,\Delta z}{2\pi}\frac{jZ_0}{k_0}\cos\theta\left(\frac{jk_0}{r^2} + \frac{1}{r^3}\right)e^{-jk_0r}\mathbf{a}_r - \frac{I_0\,\Delta z}{4\pi}\frac{jZ_0}{k_0}\sin\theta\left(\frac{-k_0^2}{r} + \frac{jk_0}{r^2} + \frac{1}{r^3}\right)e^{-jk_0r}\mathbf{a}_\theta \tag{11.7b}$$

The radial component of the complex Poynting vector is

$$\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_r = E_\theta H_\phi^* = \left(\frac{I_0\,\Delta z}{4\pi}\sin\theta\right)^2\frac{Z_0}{k_0}\left(\frac{k_0^3}{r^2} - \frac{j}{r^5}\right)$$

The integral of one-half of the real part of this expression over a sphere of radius r gives the total average power radiated into space by the current element. From the above we see that this radiated power is given by

$$P_r = \left(\frac{k_0 I_0 \, \Delta z}{4\pi}\right)^2 \frac{Z_0}{2} \int_0^{2\pi} \int_0^{\pi} \sin^3 \theta \, d\theta \, d\phi = \frac{(k_0 I_0 \, \Delta z)^2}{12\pi} Z_0 \qquad (11.8)$$

The only part of the fields entering into this expression for the radiated power is the part consisting of the terms varying as r^{-1}, that is, the part

$$H_\phi = \frac{jk_0 I_0 \, \Delta z}{4\pi r} \sin \theta \, e^{-jk_0 r} \qquad (11.9a)$$

$$E_\theta = \frac{jk_0 I_0 \, \Delta z \, Z_0}{4\pi r} \sin \theta \, e^{-jk_0 r} \qquad (11.9b)$$

This part of the field is therefore called the far-zone, or radiation, field. For large values of r it is the only part of the total field which has a significant amplitude. Note in particular that E_r vanishes as r^{-2} for large r. The far-zone, or radiation, field is a spherical TEM wave since the constant-phase surfaces are spheres and E_θ, H_ϕ lie in a surface perpendicular to the direction of propagation (radial direction). From (11.9) it is seen that $E_\theta = Z_0 H_\phi$, which is the same relation that holds for plane TEM waves.

The part of the field varying as r^{-2} and r^{-3} is called the near-zone, or induction, field. It is similar in nature to the static fields surrounding a small linear-current element and an electric dipole. This induction field predominates in the region $r \ll \lambda_0$, where λ_0 is the wavelength. The induction field does not represent an outward flow of power, but instead gives rise to a storage of reactive energy in the vicinity of the radiating current element. This energy oscillates back and forth between the source and the region of space surrounding the source. The complex Poynting vector involving the near-zone-field components is a pure imaginary quantity.

The total power radiated by an antenna is conveniently expressed in terms of the power absorbed in an equivalent resistance called the radiation resistance. For the current element the radiation resistance R_0 is defined by the relation

$$\tfrac{1}{2} R_0 I_0^2 = P_r \qquad (11.10)$$

From (11.8) we find that

$$R_0 = \frac{(k_0 \, \Delta z)^2}{6\pi} Z_0 = 80\pi^2 \left(\frac{\Delta z}{\lambda_0}\right)^2 \qquad (11.11)$$

after replacing k_0 by $2\pi/\lambda_0$ and Z_0 by 120π ohms. As an example, if $\Delta z = \lambda_0/100$, we find that $R_0 = 0.079$ ohm. This example shows that for a current element which is 1 per cent of a wavelength long, the radiation resistance is very small. Appreciable power would be radiated only if the current amplitude I_0 were very large. A large current, on the other hand, would lead to large amounts of power dissipation in the conductor, and hence a very low efficiency. We can conclude from this analysis of the radiating properties of a short linear current element that current-carrying systems that have linear dimensions small compared with the wavelength radiate negligible power. An efficient radiator or antenna must have dimensions comparable to or greater than the wavelength.

A further property of the linear current radiator that is worthy of consideration is the directional property or relative amount of power radiated in different directions. The power density radiated in the direction specified by the polar angle θ and azimuth angle ϕ is

$$\begin{aligned} dP &= \tfrac{1}{2}r^2 \operatorname{Re} (\mathbf{E} \times \mathbf{H}^* \cdot \mathbf{a}_r) \\ &= \frac{(I_0 k_0 \, \Delta z)^2}{32\pi^2} Z_0 \sin^2 \theta \qquad \text{watts/unit solid angle} \end{aligned} \tag{11.12}$$

The radiated power per unit solid angle is independent of the azimuth angle, as expected, because of the symmetry involved. As a function of θ, the power radiated per unit solid angle varies as $\sin^2 \theta$, and hence the radiation is most intense in the $\theta = \pi/2$ direction and zero in the direction $\theta = 0, \pi$. The directivity function $D(\theta,\phi)$ in the direction θ, ϕ is defined as the ratio of the power radiated per unit solid angle in the direction θ, ϕ divided by the total average power radiated per unit solid angle. From (11.8) the total power radiated is P_r. Since there are 4π steradians in a sphere, the average power radiated per unit solid angle is

$$\frac{P_r}{4\pi} = \frac{(I_0 k_0 \, \Delta z)^2 Z_0}{48\pi^2} \tag{11.13}$$

A fictitious isotropic radiator radiating a total power P_r uniformly in all directions would radiate an amount of power, per unit solid angle, given by (11.13). For the linear current element the directivity D is a function of θ only. Combining (11.12) and (11.13) shows that

$$D(\theta) = \tfrac{48}{32} \sin^2 \theta = 1.5 \sin^2 \theta \tag{11.14}$$

The maximum value of $D(\theta)$ is $D(\pi/2) = 1.5$, and this is commonly called the directivity of the radiator. The directivity is a measure of how effective the antenna is in concentrating the radiated power in a given direction.

The directivity function $D(\theta,\phi)$ defines a three-dimensional surface called the polar radiation pattern of the antenna. Figure 11.3 illustrates the polar radiation pattern for the short linear current radiator. In a plane ϕ = constant, the beamwidth between the half-power points is 90° [determined by solving the equation $D(\theta) = 0.5D(\pi/2)$ or $\sin^2 \theta = 0.5$].

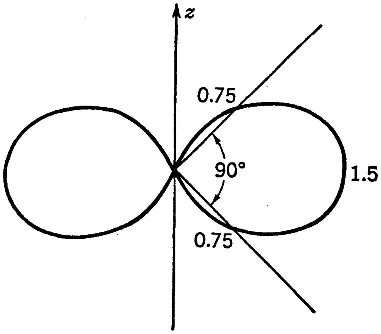

FIG. 11.3. Cross section of polar radiation pattern for infinitesimal current radiator (complete pattern is obtained by revolving cross section around polar axis).

The directivity function has been defined as the ratio of the power density in a given direction compared with the power density of a fictitious isotropic radiator with the same total radiated power. The antenna efficiency can be included if a function is defined as the ratio of power density in a given direction to the power density of an isotropic radiator with the same input power. The latter function is called the gain function $G(\theta,\phi)$, and it includes the losses in the antenna. The maximum value of G is often referred to as the "gain" of the antenna. The distinction between directivity and gain is not always carefully adhered to, in practice.

11.2. The Half-wave Antenna

The short linear current element considered above constitutes a mathematical ideal radiator. The results of the analysis, together with the principle of superposition, may be used to find the fields radiated by an antenna structure on which the current distribution is known. An antenna which is often used in practice is the half-wave dipole antenna illustrated in Fig. 11.4. This antenna consists of two thin linear con-

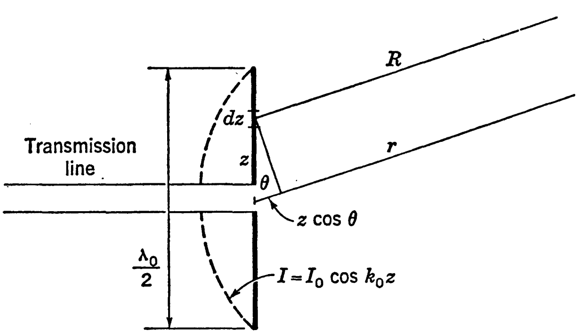

FIG. 11.4. Half-wave dipole antenna.

ductors each of length $\lambda_0/4$ and connected to a two-wire transmission line at the center. The radiation resistance of this antenna will be shown to be 73.13 ohms. This is a practical value of radiation resistance for which it is possible to obtain a good efficiency, i.e., a large amount of radiated power compared with the power loss in the conductors.

On a thin half-wave antenna it is found experimentally that the current along the antenna has a sinusoidal variation of the form

$$Ie^{j\omega t} = I_0 \cos k_0 z \, e^{j\omega t}$$

where I_0 is the current amplitude at the feeding point, assumed real for convenience. Each current element $I\,dz$ may be considered as a short linear current radiator and the total field obtained by summing up the fields radiated by each element. If we confine our attention to the far-zone or radiation field only, then, by using (11.9), we see that the current element $I_0 \cos k_0 z \, dz$ at z radiates a partial field

$$\begin{aligned} dE_\theta &= \frac{jk_0 I_0 Z_0 \sin\theta}{4\pi R} \cos k_0 z \, e^{-jk_0 R} \, dz \\ dH_\phi &= Y_0 \, dE_\theta \end{aligned}$$

From the law of cosines

$$R = (r^2 + z^2 - 2rz\cos\theta)^{1/2} = r\left[1 - \left(\frac{2z}{r}\right)\cos\theta + \frac{z^2}{r^2}\right]^{1/2}$$

The latter expression can be expanded in powers of z/r by the binomial theorem. Since $r \gg \lambda_0$ is assumed, we may retain the leading term only and get $R = r - z\cos\theta$. This result may be interpreted geometrically as equivalent to the assumption that the paths from each differential element to the distant field point are parallel. In the denominator of dE_θ we may replace R by r, but in the exponential we must use the more accurate expression $r - z\cos\theta$. The total radiated electric field is thus

$$\begin{aligned} E_\theta &= \frac{jk_0 I_0 Z_0 \sin\theta}{4\pi r} e^{-jk_0 r} \int_{-\lambda_0/4}^{\lambda_0/4} \cos k_0 z \, e^{jk_0 z\cos\theta} \, dz \\ &= \frac{jk_0 I_0 Z_0 \sin\theta}{2\pi r} e^{-jk_0 r} \int_0^{\lambda_0/4} \cos k_0 z \cos(k_0 z \cos\theta) \, dz \end{aligned}$$

after replacing the exponential by $\cos(k_0 z\cos\theta) + j\sin(k_0 z\cos\theta)$ and noting that the term involving the sine is an odd function and integrates to zero. The integration is readily performed by using the identity

$$\cos k_0 z \cos(k_0 z\cos\theta) = \tfrac{1}{2}\{\cos[k_0 z(1+\cos\theta)] + \cos[k_0 z(1-\cos\theta)]\}$$

The final result is

$$\begin{aligned} E_\theta &= \frac{jI_0 Z_0 \sin\theta}{4\pi r} e^{-jk_0 r} \left[\frac{\sin\frac{\pi}{2}(1+\cos\theta)}{1+\cos\theta} + \frac{\sin\frac{\pi}{2}(1-\cos\theta)}{1-\cos\theta}\right] \\ &= \frac{jI_0 Z_0}{2\pi r} e^{-jk_0 r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \end{aligned} \quad (11.15)$$

The total power radiated is obtained by integrating one-half of the real part of the complex Poynting vector $E_\theta H_\phi^* = Y_0|E_\theta|^2$ over a sphere of radius r. We have

$$P_r = \frac{I_0^2 Z_0}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta\, d\phi$$
$$= \frac{I_0^2 Z_0}{4\pi} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta \qquad (11.16)$$

By an appropriate change of variables the integral is transformed to†

$$P_r = \frac{I_0^2 Z_0}{8\pi} \int_0^{2\pi} \frac{1 - \cos u}{u}\, du$$

The latter integral is

$$\int_0^{2\pi} \frac{1 - \cos u}{u}\, du = \ln 1.781 - \mathrm{Ci}\,(2\pi) + \ln 2\pi$$

where Ci x is the cosine integral

$$\mathrm{Ci}\, x = -\int_x^{\infty} \frac{\cos u}{u}\, du$$

and is tabulated. In particular, Ci $(2\pi) = -0.0226$. Thus we have

$$P_r = \frac{I_0^2 Z_0}{8\pi} [\ln 2\pi(1.781) - \mathrm{Ci}\,(2\pi)]$$
$$= \frac{I_0^2 Z_0}{8\pi} (2.4151 + 0.0226) = 36.57 I_0^2 \qquad (11.17)$$

The current at the feeding point is I_0, and hence from the relation $\frac{1}{2} I_0^2 R_0 = P_r$, the radiation resistance is found to be 73.13 ohms.

The near-zone field for the half-wave dipole does not contribute to the radiated power. In actual fact, the near-zone field represents a storage of reactive energy in the immediate space surrounding the antenna. This reactive energy gives rise to a reactive term in the input impedance presented by the antenna to the transmission-line feeder. By choosing a proper antenna length, the average electric and magnetic energy stored in the near-zone field can be made equal and the input reactive term will vanish. This is equivalent to adjusting a tuned circuit to a resonant condition. For a thin half-wave dipole antenna this resonant length is found to be a few per cent shorter than a half wavelength.

† J. Stratton, "Electromagnetic Theory," sec. 8.7, McGraw-Hill Book Company, Inc., New York, 1941.

The directivity of the half-wave dipole is given by

$$D(\theta) = \frac{60}{36.57}\left[\frac{\cos\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2 \tag{11.18}$$

The maximum value of D is 1.64, which is only slightly larger than that for the short linear current radiator. A plot of the radiation pattern is given in Fig. 11.5. This pattern is similar to that of the short current radiator except that the half-power beamwidth is 78° instead of 90°.

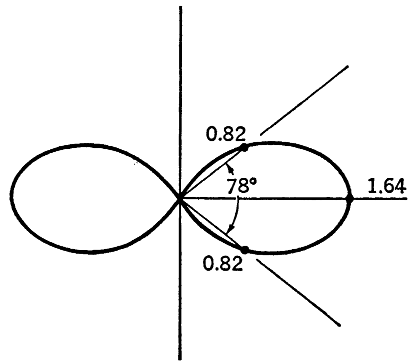

FIG. 11.5. Radiation pattern of a half-wave dipole antenna.

11.3. Introduction to Arrays

In examining the radiation pattern of an elementary dipole, we note that very little directivity is achieved. Maximum radiation takes place at right angles to the axis; however, this falls off relatively slowly as the polar angle decreases toward zero, and furthermore the pattern is uniform with respect to azimuth. Although the half-wave dipole achieves a somewhat greater concentration of energy in the direction normal to the axis, its pattern does not differ substantially from that of the elementary dipole. We should find that other-length resonant-wire antennas, while producing more complicated patterns, do not result in highly directive patterns. One way in which greater control of the radiation pattern may be achieved is by the use of an array of dipole (or other) antennas. Such arrays are capable of producing directional patterns or special characteristics of other sorts.

If we visualize the total primary current source as made up of differential radiators, then the resultant pattern is the superposition of the field contributions from each elementary source. This means that a highly directive antenna will result if the amplitude and phase of each element can be suitably chosen so that cancellation of the fields in all but the desired direction is essentially achieved. The resonant-wire antenna does not permit sufficient flexibility of assignment of phase and amplitude since all elements are in the same phase while the amplitude variation is sinusoidal. The desired freedom can be achieved by arranging together a number of separately driven antennas whose spacing and excitation are at our disposal. In general, it is found that desired results can be achieved through the use of identical elements equally spaced, and this case only will be considered. As a consequence of using identical array elements, it turns out that the total pattern can be formulated as the product of the pattern of the element times the pattern of the array, as if each element were an isotropic radiator. In this way the characteristics of an array can be discussed independently of the characteristics of its elements.

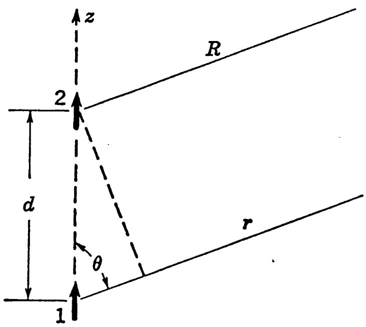

FIG. 11.6. Array of two collinear infinitesimal current elements.

As a very simple example we consider two identical infinitesimal linear current elements which are collinear and along the z axis with a spacing d. The geometric arrangement is illustrated in Fig. 11.6. Antenna 1 is chosen at the origin, and it produces an electric field at a far-zone point P that, according to (11.9b), equals

$$E_{\theta 1} = \frac{jk_0 Z_0 I_0 \,\Delta z}{4\pi r} \sin\theta \, e^{-jk_0 r} \qquad (11.19)$$

The field due to antenna 2 is similar except that the distance to P is R instead of r. Using the law of cosines we have

$$R = (r^2 + d^2 - 2rd\cos\theta)^{1/2} = r\left(1 - \frac{2d\cos\theta}{r} + \frac{d^2}{r^2}\right)^{1/2}$$

The latter may be expanded by means of the binomial theorem in powers of d/r. The statement that P is in the far zone of the array requires that r be sufficiently greater than d so that for phase calculations we can take the two leading terms $R = r - d\cos\theta$, with little error.† This procedure is completely equivalent to that followed in the analysis of the half-wave dipole and similarly permits the geometrical interpretation that the paths from each array element to the field point may be considered to be parallel. It is sufficient to take $R = r$ when only magnitudes are concerned. With these "far-zone approximations," the field due to antenna 2 is

$$E_{\theta 2} = \frac{jk_0 Z_0 I_0 \,\Delta z}{4\pi r} \sin\theta \, e^{-jk_0(r - d\cos\theta)} \qquad (11.20)$$

and the total field is

$$E_\theta = \frac{jk_0 Z_0 I_0 \,\Delta z}{4\pi} \sin\theta \, \frac{e^{-jk_0 r}}{r} (1 + e^{jk_0 d\cos\theta}) \qquad (11.21)$$

† The criterion is usually specified as

$$r > \frac{2d^2}{\lambda_0}$$

although the numerical coefficient is sometimes taken as unity or 4.

If instead of infinitesimal current elements we had assumed arbitrary antennas 1 and 2, which were identical geometrically and were excited by identical currents but were displaced from each other a distance d along the polar axis, then the field of antenna 1 could be written

$$\mathbf{E}_1 = \mathbf{f}(\theta,\phi) \frac{e^{-jk_0r}}{r} \tag{11.22a}$$

while that of antenna 2 would be

$$\mathbf{E}_2 = \mathbf{f}(\theta,\phi) \frac{e^{-jk_0r}}{r} e^{jk_0d\cos\theta} \tag{11.22b}$$

Equation (11.22a) expresses the fact that any antenna, including arrays, may be considered as made up of a large number of elementary sources, where, provided the field point is in the far zone, the resultant field due to each element contains a common factor e^{-jk_0r}/r just as in the previous example. Apart from this factor we have a complex-phasor-vector summation, $\mathbf{f}(\theta,\phi)$, which in general is a function of the direction (θ,ϕ) to the field point. Antenna 2 being identical with antenna 1 results in a superposition of partial fields from elements that correspond to those of antenna 1, except for a displacement d along the z axis, hence resulting in an additional common phase factor $e^{jk_0d\cos\theta}$ for the current elements of antenna 2. The total field due to the two-element array may be written

$$\mathbf{E} = \mathbf{f}(\theta,\phi) \frac{e^{-jk_0r}}{r} (1 + e^{jk_0d\cos\theta}) \tag{11.23}$$

Note that the specific example leading to (11.21) conforms to this general result.

For N radiators spaced a distance d apart along the z axis and equally excited, the previous result can be readily generalized to give

$$\mathbf{E} = \mathbf{f}(\theta,\phi) \frac{e^{-jk_0r}}{r} \left[1 + \sum_{n=1}^{N-1} e^{jk_0nd\cos\theta}\right] \tag{11.24}$$

and it is now necessary that $r \gg Nd$ for the far-zone approximation to hold. The form of (11.24) displays the total field as the product of the pattern of the element antenna and what we call the array factor. The latter, in this case, is

$$1 + \sum_{n=1}^{N-1} e^{jk_0nd\cos\theta}$$

and may also be thought of as the pattern of an array of isotropic radiators excited in the same way as the actual antennas. More generally, if the nth antenna has a relative amplitude C_n and phase $e^{j\alpha_n}$, then the array factor A becomes

$$A = 1 + \sum_{n=1}^{N-1} C_n e^{j(k_0 nd \cos\theta + \alpha_n)} \tag{11.25}$$

Normally, only the absolute value of the field is required. The array factor in (11.23), for example, becomes

$$|A| = |1 + e^{jk_0 d\cos\theta}| = \left|\cos\frac{k_0 d\cos\theta}{2}\right| \tag{11.26}$$

Example 11.1. A Two-element Array. Let us calculate the pattern due to two infinitesimal dipoles whose axes are "horizontal" but which are spaced "vertically," as shown in Fig. 11.7. They are identical, and the magnitude of their excitation is the same, but a relative phase shift of $e^{-jk_0 d}$ is imposed on antenna 2, as noted in Fig. 11.7. Since this geometry no longer has the axial symmetry of earlier cases, a pattern that is a function of ϕ in addition to θ must be expected.

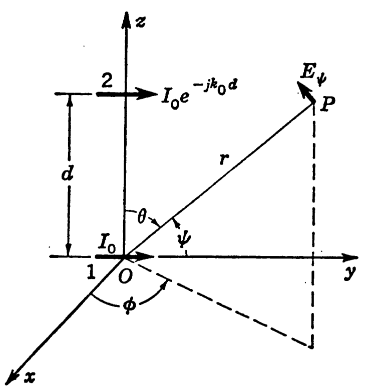

FIG. 11.7. Vertical array of two horizontal elementary dipoles.

The electric field of antenna 1 at point P is found from (11.9b) to be

$$E_{\psi 1} = \frac{jk_0 Z_0 I_0\,\Delta z}{4\pi r}\sin\psi\, e^{-jk_0 r} \tag{11.27a}$$

where ψ is measured from the axis of the current element and is the polar angle relative to the dipole axis. The direction of E_ψ is normal to r in the OP-OY plane, as illustrated in Fig. 11 7. For the field of element 2 we get

$$E_{\psi 2} = \frac{jk_0 Z_0 I_0\,\Delta z}{4\pi r}\sin\psi\, e^{jk_0 d\cos\theta} e^{-jk_0 d} e^{-jk_0 r} \tag{11.27b}$$

where the factor $e^{-jk_0 d}$ is due to the relative phase of excitation of the second element. The total field E_ψ is then

$$E_\psi = \frac{jk_0 Z_0 I_0\,\Delta z}{4\pi r} e^{-jk_0 r}\sin\psi\,[1 + e^{jk_0 d(\cos\theta - 1)}] \tag{11.28}$$

and the quantity in the bracket is readily identified as the array factor. The absolute value of this array factor is found to be

$$|A| = |1 + e^{jk_0 d(\cos\theta - 1)}| = 2\left|\cos\frac{k_0 d(\cos\theta - 1)}{2}\right| \tag{11.29}$$

Equation (11.28) can be put into a more useful form by expressing ψ in terms of ϕ and θ. From the geometry of Fig. 11.7 the unit vector in the r direction, $\mathbf{a}_r$, is

$$\mathbf{a}_r = \mathbf{a}_x \sin\theta\cos\phi + \mathbf{a}_y \sin\theta\sin\phi + \mathbf{a}_z\cos\theta$$

so that $$\sin\psi = |\mathbf{a}_r \times \mathbf{a}_y| = (1 - \sin^2\phi\sin^2\theta)^{1/2}$$

Furthermore, $$\mathbf{E}_\psi = E_\theta\mathbf{a}_\theta + E_\phi\mathbf{a}_\phi$$

where $$E_\theta = \frac{-E_\psi\cos\theta\sin\phi}{\sqrt{1-\sin^2\phi\sin^2\theta}}$$

$$E_\phi = \frac{-E_\psi\cos\phi}{\sqrt{1-\sin^2\phi\sin^2\theta}}$$

since $$\begin{aligned}\mathbf{a}_\psi &= -\mathbf{a}_y\csc\psi + \mathbf{a}_r\cot\psi \\ &= -(\mathbf{a}_\theta\cos\theta\sin\phi + \mathbf{a}_\phi\cos\phi)\csc\psi + \mathbf{a}_r(\cot\psi + \sin\theta\sin\phi)\end{aligned}$$

We may now write the total field $\mathbf{E}$ as

$$\mathbf{E} = \frac{-jk_0Z_0I_0\,\Delta z\,e^{-jk_0r}}{2\pi r}\cos\frac{k_0d(\cos\theta-1)}{2}\,e^{jk_0d(\cos\theta-1)/2} \times (\mathbf{a}_\theta\cos\theta\sin\phi + \mathbf{a}_\phi\cos\phi) \qquad (11.30)$$

The full three-dimensional pattern of the antenna is given by (11.30). However, instead of treating the pattern as a whole, it is (usually) sufficient to describe the antenna pattern in the principal coordinate planes. We obtain, then, the following:

yz-plane pattern ($\phi = \pi/2$)

$$|\mathbf{E}| = E_\theta = \frac{k_0Z_0I_0\,\Delta z}{2\pi r}|\cos\theta|\left|\cos\frac{k_0d(\cos\theta-1)}{2}\right| \qquad (11.31)$$

xz-plane pattern ($\phi = 0$)

$$|\mathbf{E}| = E_\phi = \frac{k_0Z_0I_0\,\Delta z}{2\pi r}\left|\cos\frac{k_0d(\cos\theta-1)}{2}\right| \qquad (11.32)$$

xy-plane pattern ($\theta = \pi/2$)

$$|\mathbf{E}| = E_\phi = \frac{k_0Z_0I_0\,\Delta z}{2\pi r}|\cos\phi|\left|\cos\frac{k_0d}{2}\right| \qquad (11.33)$$

For the case where $d = \lambda_0/4$, hence $k_0d = \pi/2$, the pattern in the xz plane is proportional to $\cos[(\pi/4)(\cos\theta - 1)]$, while the pattern in the yz plane is obtained by multiplying the former pattern by $\cos\theta$. The results are plotted in Fig. 11.8. For the case where $d = 3\lambda_0/4$, the results are plotted in Fig. 11.9. Note that the pattern in the xy plane is independent of d and is simply a sinusoid in ϕ.

The patterns of Figs. 11.8 and 11.9 show a maximum of radiation in the direction of the positive z axis. This could have been foreseen from the nature of the excitation. We note that antenna 2 is excited with a lagging phase that corresponds precisely to the phase delay of a wave

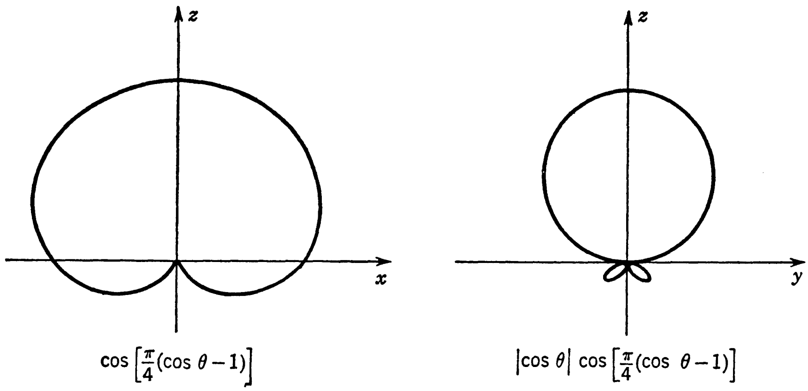

FIG. 11.8. Normalized patterns in xz and yz planes for vertically stacked horizontal dipoles separated by $\lambda_0/4$.

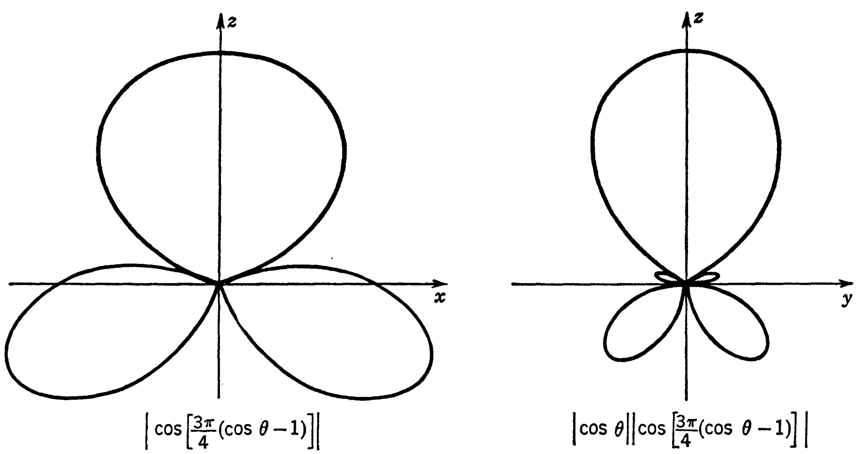

FIG. 11.9. Normalized patterns in xz and yz planes for vertically stacked horizontal dipoles separated by $3\lambda_0/4$.

leaving antenna 1 and propagating in the positive z direction. Consequently, the partial fields contributed by antennas 1 and 2 arrive in phase and therefore add together in the $+z$ direction. For other directions there will usually be only partial addition. If the number of elements is increased and a progressive phase delay given each successive element, then the partial fields of each element can be made to add in the $+z$ direction. However, when a large number of elements are involved, then usually, in other than the "forward" direction, the phase of each contribution will tend to cause cancellation of the field from each element, with a net result that the field strength in these directions is relatively small. An array of the type just described is called an end-fire array since the maximum occurs along the line of the array. Let us consider long arrays of this type analytically.

11.4. Linear Arrays

We consider now a linear array of N elements equally spaced a distance d apart. If we choose the line of the array to be the polar axis, then the geometry is as illustrated in Fig. 11.10 and the array factor is given by (11.25). The latter can be easily rederived if the array of Fig. 11.10 is considered to be composed of isotropic elements where the amplitude and phase of the nth element, relative to the reference element at the origin, are $C_n e^{j\alpha_n}$.

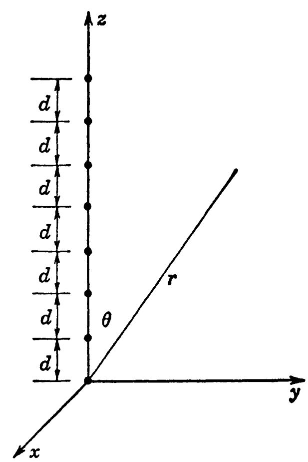

FIG. 11.10. Linear array of many elements along z axis.

A case of considerable interest will be considered where the amplitude of excitation of each element is the same and where the relative phase shift of the excitation is of the form

$$\alpha_n = -nk_0 d \cos \theta_0 \tag{11.34}$$

where θ_0 is a constant. In this form, by choosing $\theta_0 = 0$, the end-fire case previously discussed results. If $\theta_0 = \pi/2$, then all α_n's are zero and each element is excited in phase. In this case we see by inspection that the maximum radiation occurs in a direction normal to the line of the array, since in this direction the partial contribution from each element adds directly. An array for which this condition holds is designated a broadside array. By considering the excitation phase according to (11.34), we see that the special cases of end-fire and broadside arrays are included.

Putting (11.34) into (11.25) and setting $C_n = 1$, we obtain

$$|A| = |1 + e^{jk_0 d(\cos\theta - \cos\theta_0)} + e^{j2k_0 d(\cos\theta - \cos\theta_0)} + \cdots + e^{j(N-1)k_0 d(\cos\theta - \cos\theta_0)}| \tag{11.35}$$

for an array of N elements. Since (11.35) is a geometric progression with a ratio $e^{jk_0 d(\cos\theta - \cos\theta_0)}$, the sum can be expressed as

$$|A| = \left| \frac{e^{jNk_0 d(\cos\theta - \cos\theta_0)} - 1}{e^{jk_0 d(\cos\theta - \cos\theta_0)} - 1} \right| = \left| \frac{\sin \dfrac{Nk_0 d(\cos\theta - \cos\theta_0)}{2}}{\sin \dfrac{k_0 d(\cos\theta - \cos\theta_0)}{2}} \right| \tag{11.36}$$

This pattern, considered as a function of $x = k_0d(\cos\theta - \cos\theta_0)$, is of the form

$$|A| = f(x) = \left|\frac{\sin\frac{Nx}{2}}{\sin\frac{x}{2}}\right| \qquad x = k_0d(\cos\theta - \cos\theta_0) \qquad (11.37)$$

A typical curve of $f(x)$ for large N is shown in Fig. 11.11. Note that for $x = 0$, $f(x)$ approaches the value N. Furthermore, the nulls will occur

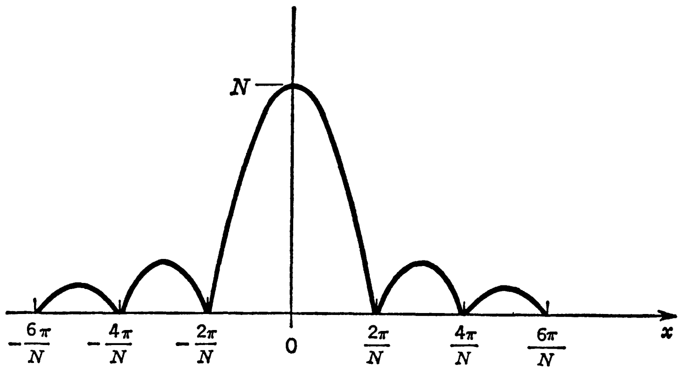

FIG. 11.11. Universal pattern for a linear array with uniform amplitude and progressive phase shift.

for $Nx/2 = \pi, 2\pi, 3\pi, \ldots, (N-1)\pi$ (larger values correspond to pattern repetition). Also if N is large, the subsidiary maxima correspond approximately to maxima of the numerator; that is, $x = 3\pi/N, 5\pi/N, \ldots$. At these points the magnitude of $f(x)$ is given by

$$f\left(\frac{3\pi}{N}\right) \approx \frac{2N}{3\pi} \qquad f\left(\frac{5\pi}{N}\right) \approx \frac{2N}{5\pi} \qquad f\left(\frac{7\pi}{N}\right) \approx \frac{2N}{7\pi} \qquad \cdots \qquad (11.38)$$

a result that depends on N being sufficiently large so that $\sin(3\pi/N) \approx 3\pi/N$, etc. Under these conditions we see that the sidelobe level, i.e., the ratio of the peak amplitude of the main lobe to the subsidiary-lobe peak amplitude, is independent of N, for large N. Its value is $3\pi/2$, or about 13.5 decibels.

Since the maximum value of $f(x)$ occurs for $x = 0$, then in the actual pattern the main beam is in the direction $\theta = \theta_0$. We note, then, that the direction of the main beam can be shifted by altering the progressive phase shift. The position of the first nulls of the main beam is important since it characterizes the beam width. We shall designate the "upper" null by the value of $\theta = \theta_0 + \theta^+$ and the "lower" null by $\theta = \theta_0 - \theta^-$. Then θ^+ satisfies the equation

$$k_0d[\cos(\theta_0 + \theta^+) - \cos\theta_0] = \frac{2\pi}{N} \qquad (11.39a)$$

while θ^- is given by

$$k_0 d[\cos(\theta_0 - \theta^-) - \cos\theta_0] = -\frac{2\pi}{N} \tag{11.39b}$$

If N is very large, it seems likely (we shall confirm this later) that θ^+ and θ^- will be very small. In this case we can approximate $\cos\theta$ by $(1 - \theta^2/2)$ and $\sin\theta$ by θ, hence obtaining

$$\cos(\theta_0 + \theta^+) - \cos\theta_0 \approx -\theta^+ \sin\theta_0 - \frac{(\theta^+)^2}{2}\cos\theta_0 \tag{11.40a}$$

$$\cos(\theta_0 - \theta^-) - \cos\theta_0 \approx \theta^- \sin\theta_0 - \frac{(\theta^-)^2}{2}\cos\theta_0 \tag{11.40b}$$

Provided that θ_0 is sufficiently greater than $\theta^\pm$, the quadratic term in (11.40) can be dropped. With the resultant expression, (11.39) can be solved for θ^+ and θ^-, yielding

$$\theta^+ = \theta^- = \frac{2\pi}{N}\frac{1}{k_0 d \sin\theta_0} = \frac{\lambda_0}{Nd\sin\theta_0} \tag{11.41}$$

The assumption that θ^+ and θ^- are small is seen to be justified provided Nd/λ_0 is large and θ_0 not too close to zero, as we have already required. The beamwidth Δ may be defined by the total angle between nulls, a quantity which is simply $2\theta^+$. Assuming that N is large, then the total array length $L = (N - 1)d \approx Nd$, and

$$\Delta = \frac{2\lambda_0}{L\sin\theta_0} \tag{11.42}$$

For a given array length in wavelengths, the minimum beamwidth occurs for the broadside array, where $\theta_0 = \pi/2$, in which case

$$\Delta_{bs} = \frac{2\lambda_0}{L} \tag{11.43}$$

For the end-fire case $\theta_0 = 0$, and (11.41) does not hold. However, we can return to (11.40) and utilize the quadratic term (the linear terms go out) to establish $\theta^+ = \theta^- = \sqrt{2\lambda_0/Nd}$. Consequently,

$$\Delta_{ef} = 2\sqrt{\frac{2\lambda_0}{L}} \tag{11.44}$$

for large N.

11.5. Two-dimensional Arrays

The linear arrays discussed above produce patterns which are axially symmetric. If, for example, a highly directive pencil beam is desired, then the linear array by itself cannot be used to produce such a pattern; that is, we can construct an array that is long compared with wavelength and thereby achieve a narrow beam but the array pattern will be the same in each longitudinal plane. Further shaping is required in this case. We recall, however, that the array pattern must be multiplied by the directivity of the element of the array to obtain the over-all pattern. The element can itself be an array with its elements spaced at right angles to the original array, as suggested in Fig. 11.12a. In this case the array element can produce a narrow beam symmetric about its own axis. The product of the two patterns, in this case, yields a maximum only over a

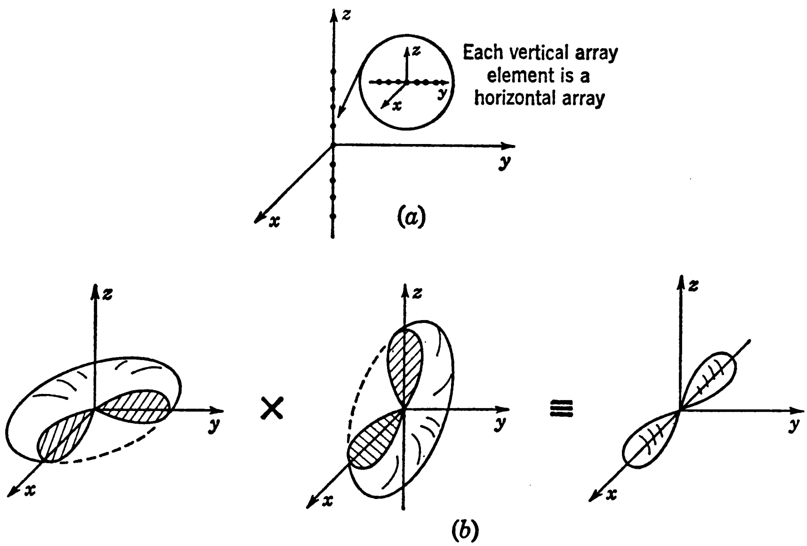

FIG. 11.12. (a) A vertical array of horizontal arrays; (b) three-dimensional pattern of vertical array, horizontal array, and resultant pattern obtained by multiplication (sidelobes neglected).

small range of θ and ϕ for which the component patterns are maximum. The over-all result is an array factor that corresponds to two pencil beams back to back. This is illustrated in Fig. 11.12b. It is easy to eliminate one of the pencil beams by choosing the individual elements of the array to have a null in the direction of one of the two beams.

We note in the example discussed above that the final physical arrangement is that of a two-dimensional array. The pattern capabilities of such an array are much more versatile than those of the linear array. For a class of problems the pattern of a two-dimensional array, such as the one discussed above, can be reduced to that of determining the pattern of two linear arrays. Let us formulate this more specifically.

Figure 11.13 illustrates a two-dimensional array with uniform horizontal and vertical spacing h and v, where h and v are not necessarily equal. The total number of horizontal rows is N, and the total number of vertical columns is M. We designate by the double subscript mn the element in the mth column and nth row. Then if the excitation of the mnth element can be written in the form

$$e^{j(m\alpha+n\beta)}$$

where α and β are constants, we have a uniform (progressive) phase shift from one element to the next along either the horizontal or vertical direction. Furthermore, we are considering the case where the amplitude of excitation is constant, the so-called uniform array. Under these conditions it is possible to think of the array as a linear array of vertical elements, the latter being linear arrays of horizontal elements, or vice versa. In either case the array factor A is the product of the array factor of the horizontal array A_h and the vertical array A_v. This result can be substantiated analytically, and we turn now to this task.

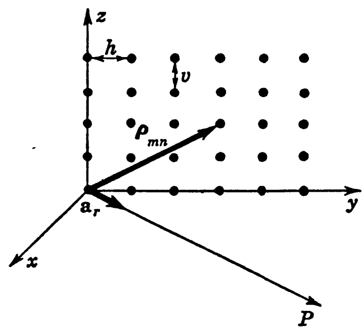

FIG. 11.13. Two-dimensional array.

Let the direction to the field point $P(\theta,\phi)$ be given by the unit vector $\mathbf{a}_r$, where

$$\mathbf{a}_r = \mathbf{a}_x \sin\theta\cos\phi + \mathbf{a}_y \sin\theta\sin\phi + \mathbf{a}_z\cos\theta$$

The mnth element can be described by a vector $\boldsymbol{\varrho}_{mn}$ from the origin to its location; that is,

$$\boldsymbol{\varrho}_{mn} = mh\mathbf{a}_y + nv\mathbf{a}_z$$

In accordance with the far-zone assumptions, the contribution from the mnth element arrives at P with a phase, relative to that from the reference element at the origin, which arises owing to the difference in the respective path lengths, where the latter is simply the projection of $\boldsymbol{\varrho}_{mn}$ on $\mathbf{a}_r$. The total pattern or array factor is thus

$$A = \sum_{n=0}^{N}\sum_{m=0}^{M} e^{-jk_0\boldsymbol{\varrho}_{mn}\cdot\mathbf{a}_r}e^{j(m\alpha+n\beta)} \tag{11.45}$$

Expanding the dot product and combining terms allow us to write

$$A = \sum_{m=0}^{M} e^{-jm(k_0h\sin\theta\sin\phi-\alpha)}\sum_{n=0}^{N} e^{-jn(k_0v\cos\theta-\beta)} \tag{11.46}$$

and it is clear that the total array factor is the product of the separate linear array factors; that is,

$$A = A_hA_v \tag{11.47}$$

where
$$A_h = \sum_{m=0}^{M} e^{-jm(k_0 h \sin\theta \sin\phi - \alpha)} \tag{11.48a}$$
$$A_v = \sum_{n=0}^{N} e^{-jn(k_0 v \cos\theta - \beta)} \tag{11.48b}$$

If the electric field of an array element is $\mathbf{f}(\theta,\phi)e^{-jk_0r}/r$, then the over-all field produced by the array is

$$\mathbf{E} = A_h A_v \mathbf{f}(\theta,\phi)\frac{e^{-jk_0r}}{r} \tag{11.49}$$

One of the conclusions reached concerning the linear array was that the sidelobe level, for a large number of elements, is a constant. For the two-dimensional array of the type considered, this same conclusion must hold, since the total array factor is simply the product of the individual linear array factors. In some cases it is of importance to adjust the sidelobe level below the values achieved by the class of arrays described here. In this instance the restriction to uniform amplitude excitation, such as has been assumed, must be removed. Even greater generality may be desired where both amplitude and phase of each element are independently specified. A fairly elaborate theory for optimizing array patterns with respect to certain desired parameters exists, and a means for synthesizing such arrays is available. A number of references are given at the end of the chapter for the reader who wishes to pursue this topic further.

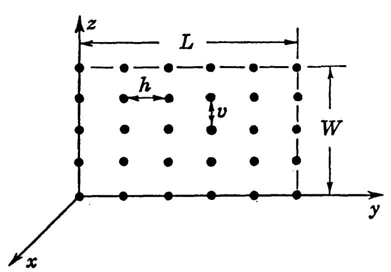

FIG. 11.14. A two-dimensional array of length L and width W.

11.6. Continuous Distributions

Figure 11.14 again illustrates a two-dimensional array where the over-all length is L and the width is W. In terms of numbers of elements and spacing, $L \approx hM$ and $W \approx vN$. If all elements are excited in phase, then a pencil beam in the directions ($\theta = \pi/2$, $\phi = 0, \pi$) is produced. Considered in the xz plane ($\phi = 0$), we see from (11.47) and (11.48) that the pattern is that of the vertical array (the horizontal array factor is a constant in this case), and hence the beam width is simply $2\lambda_0/W$. In the xy plane ($\theta = \pi/2$), Eqs. (11.48) reveal that the pattern is due to the horizontal array factor alone, so that the beam width is $2\lambda_0/L$. This result does not depend on the actual spacing of the elements so long as the total number is large. One can increase the number of elements and decrease their spacing, until in the limiting case a continuous distribution is obtained.

A broad class of radiating antennas, particularly useful in the microwave region, are horn-type radiators, paraboloidal reflectors, and lenses, some of which are illustrated in Fig. 11.15. In each case the energy must pass through a physically distinct aperture. It is possible to show that the field in the aperture behaves like an equivalent source. This fact is related to Huygens' principle, which states that each element of a wavefront may be considered as a secondary source. One may therefore use the results just obtained for the uniform array to predict the beam width

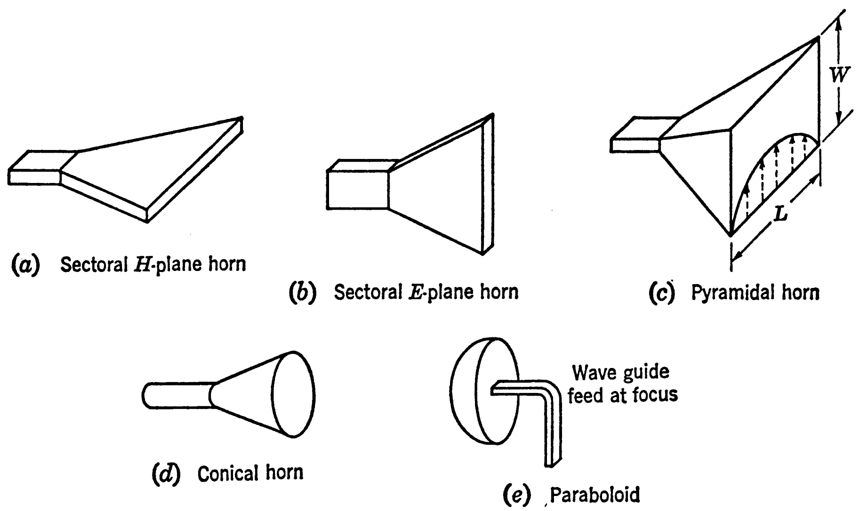

FIG. 11.15. Aperture-type antennas.

due to an aperture-type antenna for the case of uniform-aperture field intensity.

In Table 11.1 the theoretical results for several types of radiators are given. In addition to the case of uniform "illumination," tapered intensities and waveguide distributions are also shown. For the TE_{10} aperture distribution, the equivalent vertical array is uniform but the horizontal array is tapered sinusoidally. Results are also given for circular apertures, and although the numerical factor cannot be checked from the array theory, the behavior as a function of size follows the expected form. The results in Table 11.1 assume that the aperture dimensions are at least several wavelengths.

11.7. Network Formulation for Transmitting-Receiving System

So far we have discussed the antenna under transmitting conditions only. In this and the following section we shall show that the behavior of the receiving antenna can be determined from its characteristics when

TABLE 11.1. BEAM WIDTH FOR SEVERAL APERTURE-TYPE ANTENNAS
Dimensions large compared with wavelength

Type	Field distribution across aperture	Width of major lobe between nulls	
		xy plane	xz plane
Rectangular paraboloid	Uniform	$\frac{115°}{L/\lambda_0}$	$\frac{115°}{W/\lambda_0}$
Rectangular paraboloid	Sinusoidal	$\frac{172°}{L/\lambda_0}$	$\frac{172°}{W/\lambda_0}$
Circular paraboloid (diameter D)	Uniform	$\frac{140°}{D/\lambda_0}$	$\frac{140°}{D/\lambda_0}$
Pyramidal horn†	TE_{10}	$\frac{172°}{L/\lambda_0}$	$\frac{115°}{W/\lambda_0}$
Conical horn† (diameter D)	TE_{11}	$\frac{194°}{D/\lambda_0}$	$\frac{140°}{D/\lambda_0}$

† Assumes small flare angle.

transmitting. To assist in this analysis we first derive the Lorentz reciprocity theorem.

Let $\mathbf{E}_a$, $\mathbf{H}_a$ and $\mathbf{E}_b$, $\mathbf{H}_b$ represent solutions to Maxwell's equations, in a source-free region of space, which arise from different sources outside the region under consideration. Then we may form

$$\nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b) = \mathbf{H}_b \cdot \nabla \times \mathbf{E}_a - \mathbf{E}_a \cdot \nabla \times \mathbf{H}_b$$

and if we utilize the fact that the fields satisfy the homogeneous Maxwell equations, this may be written

$$\nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b) = -j\omega\mu\mathbf{H}_a \cdot \mathbf{H}_b - j\omega\epsilon\mathbf{E}_a \cdot \mathbf{E}_b$$

By interchanging the subscripts a and b, we also have

$$\nabla \cdot (\mathbf{E}_b \times \mathbf{H}_a) = -j\omega\mu\mathbf{H}_a \cdot \mathbf{H}_b - j\omega\epsilon\mathbf{E}_a \cdot \mathbf{E}_b$$

Consequently,
$$\nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b) - \nabla \cdot (\mathbf{E}_b \times \mathbf{H}_a) = 0 \tag{11.50}$$

If (11.50) is integrated throughout the given source-free region and use is made of the divergence theorem, we have

$$\oint_S (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot d\mathbf{S} = 0 \tag{11.51}$$

which is the desired form of the reciprocity theorem.

Figure 11.16 illustrates two antennas and their associated equipment, and it is understood that either antenna may be transmitting with the other receiving. For simplicity in the analysis, the feed line to each antenna is assumed to be a coaxial line, and it is further assumed that the electromagnetic sources (the vacuum tubes) may be entirely enclosed by a conducting surface that forms an extension to the outer conductor of the coaxial line. The dipole antenna shown represents any of the wide variety of actual practical antennas that might be used. Also, of course,

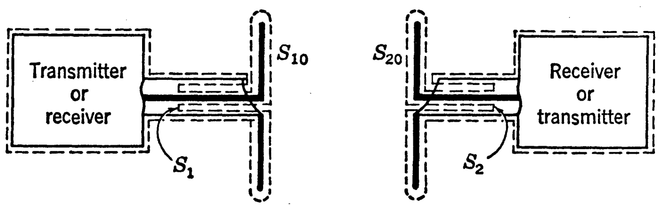

FIG. 11.16. Transmitting and receiving antennas (arbitrarily located).

the relative locations of the transmitter and receiver are completely arbitrary.

At antenna 1 we specify input conditions at an arbitrary transverse reference surface S_1 in the coaxial transmission line. The only restriction on the location of S_1 is that it must be sufficiently far from the antenna or other discontinuity so that only the dominant TEM mode exists. Similar considerations apply to the specification of reference surface S_2. At each of these surfaces both voltage and current are defined in terms of conventional transmission-line concepts. If we consider the current I_1 that flows at S_1, then because of the linearity of Maxwell's equations, it must be linearly related to the voltage at S_1 and at S_2. (In terms of the fields, the magnetic field at S_1, which is proportional to the current, is linearly related to the electric field at S_1 and S_2.) We may thus write

$$I_1 = Y_{11}V_1 + Y_{12}V_2 \qquad (11.52a)$$

Similarly,

$$I_2 = Y_{21}V_1 + Y_{22}V_2 \qquad (11.52b)$$

Equations (11.52) are in the same form as that for a standard four-terminal network composed of linear bilateral lumped elements. The similarity should not be surprising since in both cases the linearity of Maxwell's equations leads to the formulation. The fact that coupling in the antenna problem involves radiation fields whereas in circuit analysis only induction fields are involved is only a matter of detail. That I_1 is uniquely related to V_1 and V_2 has not been shown, but is a consequence of the general uniqueness theorem in electromagnetic theory. Equations

(11.52) can be solved simultaneously for V_1 and V_2, leading to an alternative form

$$V_1 = I_1Z_{11} + I_2Z_{12} \quad (11.53a)$$
$$V_2 = I_1Z_{21} + I_2Z_{22} \quad (11.53b)$$

Just as in network theory, we can show that $Y_{12} = Y_{21}$, that is, reciprocity. For this proof we choose a volume of space bounded by a fairly complicated surface which consists of the surface at infinity and the doubly connected surfaces of the two antenna installations, as illustrated by dashed lines in Fig. 11.16, that is, $S_1 + S_{10}$ and $S_2 + S_{20}$. The surface S_{10} (and also S_{20}) is taken along the antenna surface and runs along the outer conductor of the coaxial line and includes the conducting surface that excludes the energy sources, used to generate the high-frequency currents, from the volume being considered. This surface then connects with S_1 (S_2) along the inner surface of the outer conductor and the outer surface of the inner conductor of the coaxial line. Surfaces S_1 and S_2 are the transverse planar surfaces in the coaxial line that has already been described. Let $\mathbf{E}_a$, $\mathbf{H}_a$ be the fields set up by a source inside the surface $S_1 + S_{10}$, while $\mathbf{E}_b$, $\mathbf{H}_b$ are the fields set up by a source inside the surface $S_2 + S_{20}$. If we apply (11.51) to the region bounded by the aforementioned surfaces, then since $\mathbf{E} \times d\mathbf{S}$ is zero along the conducting surfaces S_{10} and S_{20}, these integrals vanish. Furthermore, the contribution f om the surface at infinity can be set equal to zero if we assume a vanishingly small amount of dissipation in the medium, so that $\mathbf{E}$ and $\mathbf{H}$ decrease slightly faster than $1/R$.† As a result the following relationship is arrived at, namely,

$$\int_{S_1} (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot d\mathbf{S} + \int_{S_2} (\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a) \cdot d\mathbf{S} = 0 \quad (11.54)$$

since $S_1 + S_2$ is the only portion of the total surface for which the integral does not vanish. Now on either S_1 or S_2 we have, for a coaxial line, that

$$\mathbf{H} = \mathbf{a}_\phi H_\phi = \frac{I}{2\pi r}\mathbf{a}_\phi \quad (11.55a)$$
$$\mathbf{E} = \mathbf{a}_r E_r \quad (11.55b)$$

and

$$V = \int_{r_i}^{r_o} E_r\, dr \quad (11.55c)$$

† Actually, if the nature of the radiation fields from finite sources is considered, then in absence of dissipation,

$$\oint_{S_\infty} \mathbf{E}_a \times \mathbf{H}_b \cdot d\mathbf{S} = \oint_{S_\infty} \mathbf{E}_b \times \mathbf{H}_a \cdot d\mathbf{S}$$

and the result of (11.54) is also obtained.

This means, for example, that

$$\int_{S_1} (\mathbf{E}_a \times \mathbf{H}_b) \cdot d\mathbf{S} = 2\pi \int_{r_i}^{r_0} E_{ra} H_{\phi b} r \, dr = I_{1b} \int_{r_i}^{r_0} E_{ra} \, dr = I_{1b} V_{1a} \qquad (11.56)$$

The double subscripts identify the reference plane and the source condition, that is, I_{1b} is the current at S_1 caused by the field radiated by the source inside $S_2 + S_{20}$, while V_{1a} is the voltage at S_1 as produced by the field from the source inside $S_1 + S_{10}$. Carrying out the integration of (11.54) as in the example above leads to

$$V_{1a} I_{1b} + V_{2a} I_{2b} - V_{1b} I_{1a} - V_{2b} I_{2a} = 0 \qquad (11.57)$$

When only the field $\mathbf{E}_a$, $\mathbf{H}_a$ is present, (11.52) gives

$$I_{1a} = Y_{11} V_{1a} + Y_{12} V_{2a}$$
$$I_{2a} = Y_{21} V_{1a} + Y_{22} V_{2a}$$

while if $\mathbf{E}_b$, $\mathbf{H}_b$ is the only field present,

$$I_{1b} = Y_{11} V_{1b} + Y_{12} V_{2b}$$

and

$$I_{2b} = Y_{21} V_{1b} + Y_{22} V_{2b}$$

Using these results in (11.57) yields

$$(V_{1a} V_{2b} - V_{2a} V_{1b})(Y_{12} - Y_{21}) = 0 \qquad (11.58)$$

Since the a and b conditions are completely arbitrary, (11.58) can be satisfied, in general, only if

$$Y_{12} = Y_{21} \qquad (11.59)$$

as we wished to show.

The formulation of (11.56) and the reciprocity of (11.59) were facilitated by choosing the transmission lines to be coaxial and the reference planes at a point such that only TEM waves need be considered. However, these results will also apply to waveguide feed systems for each mode separately, provided appropriate definitions of voltage and current are made. A detailed discussion of this situation may be found in Collin and other references, given at the end of the chapter.

11.8. Antenna Equivalent Circuits

Ordinarily, the separation of transmitting and receiving antennas is very great, so that if antenna 1 is transmitting, (11.53*a*) may be written as

$$V_1 = I_1 Z_{11} \qquad (11.60)$$

and Z_{11} is the internal impedance of the antenna at the chosen reference plane S_1; that is, for large separation, the coupling, represented by Z_{12}, can be neglected, reflecting the fact that the transmitting antenna will hardly be affected by a very distant receiving antenna. The equivalent circuit of antenna 1 is shown in Fig. 11.17, where $\mathcal{E}_1$ is the effective emf of the source and Z_{i1} its internal impedance. Under matched conditions $Z_{i1} = Z_{11} = R_{11}$ and $V_1 = \mathcal{E}_1/2$.

When antenna 1 is operating under receiving conditions, the equivalent circuit that follows from (11.53*a*) is represented as shown in Fig. 11.18.

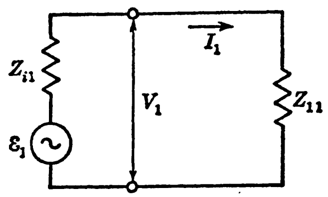

FIG. 11.17. Equivalent circuit for transmitting antenna.

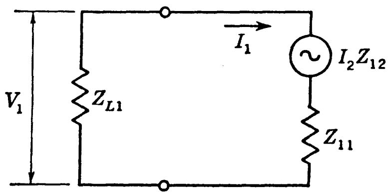

FIG. 11.18. Equivalent circuit for receiving antenna.

Here Z_{L1} is the impedance of the load. In this case we may no longer ignore Z_{12} even though it is very small, since it represents the equivalent source of energy, and in its absence no current flows in the receiving circuit. We can interpret $-I_2Z_{12}$ as the effective emf due to transmitter 2, and Z_{11} as the internal impedance of this equivalent source. In general, I_2 is a function of Z_{L1}, but since it is assumed that the antennas are weakly coupled, we can consider I_2Z_{12} as a true voltage source independent of load conditions at the receiver. Note that the impedance of an antenna as measured when transmitting is the same as its effective internal impedance when acting as a receiver and equals Z_{11} when the antennas are sufficiently separated to justify the circuit of Fig. 11.17. The received voltage V_1 can be readily calculated from the circuit of Fig. 11.18:

$$V_1 = \frac{I_2Z_{12}}{Z_{L1} + Z_{11}} Z_{L1} \tag{11.61}$$

Under matched conditions $Z_{L1} = Z_{11} = R_{11}$, and then

$$V_1 = \frac{I_2Z_{12}}{2} \tag{11.62}$$

In (11.62) the received signal depends on the current I_2 and on the parameter Z_{12}, which in turn is a function of the type of antennas and their location. But the actual mechanism of reception involves, usually, a plane wave incident on antenna 1 from a particular direction and with a certain polarization. We should be able to develop a formula that yields the received signal, given only the properties of antenna 1 and the incident wave, thereby removing specific reference to the source of the wave, i.e., antenna 2. The following discussion is devoted to this objective.

11.9. Receiving Antennas

In the last section the operation of a receiving antenna was described in terms of direct coupling to a transmitting source. A description of this interaction was formulated in terms of a four-terminal network which is capable of transforming the voltage and current at the input to the transmitting antenna into voltage and current conditions in the receiving-antenna feed line. Another point of view is possible where we consider the antenna as an element that transforms the incident electromagnetic wave into voltage and current in its transmission line. In place of the direct interaction of transmitting and receiving antennas we now view the transmitting antenna as setting up a field which the receiving antenna in turn transforms into the received signal. Since the latter method of analysis requires only a description of the incident wave on a receiving antenna, the source of that wave is unimportant.

In this section we shall determine how the voltage at the receiving-antenna reference plane S_1 is related to an incident field $\mathbf{E}_i$ whose magnitude, direction, and polarization are known. In the following discussion we shall assume that the transmitter is sufficiently far from the receiver so that the incident field may be considered to be a plane wave. Such conditions usually are obtained in practice. Since the actual source of $\mathbf{E}_i$ is irrelevant, we may suppose it to arise from an equivalent elementary current source; that is, we can always define a dipole of length Δl and current I_e such that if properly oriented and located, a field $\mathbf{E}_i$ is produced at the receiving antenna. In order to determine the desired relation between $\mathbf{E}_i$ and the received voltage, we first need to develop a slightly modified form of the reciprocity theorem.

Let us begin with Maxwell's equations, which we apply to a given region containing fixed material bodies and impressed source $\mathbf{J}_1$ or $\mathbf{J}_2$. Then

$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1 \qquad (11.63a)$$

$$\nabla \times \mathbf{H}_1 = j\omega\epsilon\mathbf{E}_1 + \mathbf{J}_1 \qquad (11.63b)$$

$$\nabla \times \mathbf{E}_2 = -j\omega\mu\mathbf{H}_2 \qquad (11.64a)$$

$$\nabla \times \mathbf{H}_2 = j\omega\epsilon\mathbf{E}_2 + \mathbf{J}_2 \qquad (11.64b)$$

If we dot-multiply (11.63a) by $\mathbf{H}_2$ and (11.64b) by $\mathbf{E}_1$ and subtract (11.64b) from (11.63a), we obtain

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 = -j\omega\mu\mathbf{H}_1 \cdot \mathbf{H}_2 - j\omega\epsilon\mathbf{E}_1 \cdot \mathbf{E}_2 - \mathbf{E}_1 \cdot \mathbf{J}_2 \qquad (11.65)$$

Similarly, by dot-multiplying (11.63b) by $\mathbf{E}_2$ and (11.64a) by $\mathbf{H}_1$ and then subtracting, we obtain

$$\mathbf{E}_2 \cdot \nabla \times \mathbf{H}_1 - \mathbf{H}_1 \cdot \nabla \times \mathbf{E}_2 = j\omega\epsilon\mathbf{E}_1 \cdot \mathbf{E}_2 + j\omega\mu\mathbf{H}_1 \cdot \mathbf{H}_2 + \mathbf{E}_2 \cdot \mathbf{J}_1 \qquad (11.66)$$

The left side of (11.65) can be identified as $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2)$, while the left side of (11.66) is $\nabla \cdot (\mathbf{H}_1 \times \mathbf{E}_2)$. If (11.65) and (11.66) are integrated over a given volume V and the divergence theorem applied to the left-hand sides and the results added, we obtain

$$\int_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2)\, dV = \oint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{S} \quad (11.67)$$

If we now let the volume under consideration be all of space, then S is

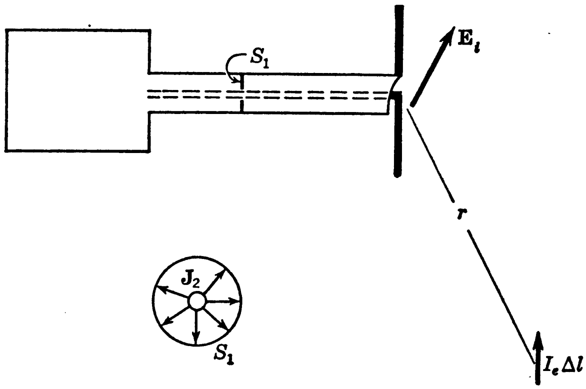

FIG. 11.19. Current source $\mathbf{J}_2$ on surface S_1.

the surface at infinity. In this case the surface integral vanishes, as noted earlier. Consequently

$$\int_V \mathbf{E}_2 \cdot \mathbf{J}_1\, dV = \int_V \mathbf{E}_1 \cdot \mathbf{J}_2\, dV \quad (11.68)$$

Note that (11.68) involves impressed currents only since induced currents are included by making ϵ complex.

Figure 11.19 illustrates the problem at hand where the antenna represents an arbitrary receiving antenna whose output voltage and current are referred to plane S_1 in the (assumed) coaxial feed line. The incident field $\mathbf{E}_i$ is taken as arising from the equivalent current element $I_e\, \Delta l$. The current element represents the source $\mathbf{J}_1$ in utilizing (11.68). For the source $\mathbf{J}_2$ in (11.68) we take a radial current source over the interconductor space on S_1. We choose the surface current density of this source to be $\mathbf{J}_s$, where

$$\mathbf{J}_s = \frac{I_0}{2\pi r}\, \mathbf{a}_r$$

This choice is a completely arbitrary one and has been made only to facilitate the use of (11.68) to arrive at an expression for the received signal.

The current source will cause a discontinuity in H_ϕ, but because of the chosen functional form, this boundary condition can be satisfied by a TEM mode alone. Consequently, the current source may be thought of as launching TEM waves propagating toward the antenna and toward the receiving load. We shall assume matched conditions so that there are no reflected waves and, further, assume that the load impedance and the radiation resistance R_{11} are both equal to the characteristic impedance of the coaxial line. Note that the terminating impedance designated as the receiving load impedance above may also be considered as the internal impedance of the current generator that maintains the source $\mathbf{J}_2$. The field radiated by $\mathbf{J}_s$ will be designated $\mathbf{E}_t$. With this notation the left side of (11.68) becomes

$$E_t I_e \,\Delta l \cos \alpha \tag{11.69}$$

since $\mathbf{J}_1$ is zero at all points in the volume except over the differential current element where $J_1\, dV = I_e\, \Delta l$. The angle α in (11.69) is the angle between the direction of $\mathbf{E}_t$ and the polar axis of the dipole. In terms of the geometry given in Fig. 11.20 we have $\cos \alpha = \cos \psi \sin \theta$,

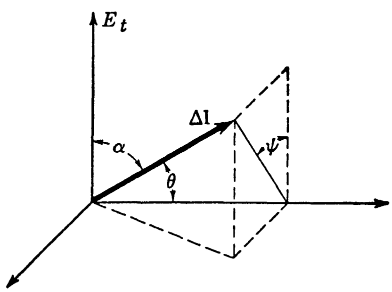

FIG. 11.20. Dipole geometry.

where ψ is the angle between the plane of polarization (that formed by $\mathbf{E}_t$ and the direction of the wave normal) and the plane of the dipole, and θ is the polar angle referred to the dipole axis. The entire expression (11.68) reduces to

$$E_t I_e \,\Delta l \cos \psi \sin \theta = \int_{S_1} \frac{E_r I_0}{2\pi r}\, dS = I_0 \int_{r_1}^{r_2} E_r\, dr = I_0 V_r \tag{11.70}$$

where E_r is the TEM received field at S_1 due to the incident field $\mathbf{E}_i$, while V_r is the received signal voltage.

Equation (11.70) can be put into a more useful form if the following energy relationships are incorporated. When the source is $\mathbf{J}_1$ (the elementary current radiator), then the radiated power is

$$P_r = \tfrac{1}{2} I_e^2 R_{rad} = \tfrac{1}{2} I_e^2 80\pi^2 \left(\frac{\Delta l}{\lambda_0}\right)^2 \tag{11.71}$$

from (11.11). But if $\mathbf{E}_i$ is the field set up at the receiving antenna, then

$$\frac{E_i^2}{2Z_0} = \frac{P_r G_e}{4\pi r^2} \tag{11.72}$$

where G_e is the gain of the current element radiator in the direction of the receiving antenna, that is, $G_e = 1.5 \sin^2 \theta$, and r is the separation between antennas. Consequently,

$$E_i = \frac{60\pi I_e \, \Delta l \sin \theta}{r\lambda_0} \tag{11.73}$$

When $\mathbf{J}_s$ is active as the source (condition 2), then it equals the discontinuity in H_ϕ across S_1. From the assumption that the antenna and load are matched to the transmission line, symmetry requires that the input current to the antenna be $I_0/2$. Consequently, the radiated power is

$$P_t = \frac{1}{2}\left(\frac{I_0}{2}\right)^2 R_{11} \tag{11.74}$$

The field set up by the antenna at the location of the current element is then such that

$$\frac{1}{2}\frac{E_t^2}{Z_0} = \frac{P_t G}{4\pi r^2} \tag{11.75}$$

where $G = G(\phi,\theta)$ is the gain of the antenna in the direction of the current element. Solving for E_t from (11.74) and (11.75) gives

$$E_t = \frac{(30GR_{11})^{1/2}}{2r} I_0 \tag{11.76}$$

Finally, by substituting (11.76) and (11.73) into (11.70), we obtain the desired result:

$$V_r = \frac{E_t I_e \, \Delta l}{I_0} \cos \psi \sin \theta = \frac{\lambda_0}{2\pi}\left(\frac{R_{11}G}{120}\right)^{1/2} E_i \cos \psi \tag{11.77}$$

When the receiving antenna is oriented for maximum reception,† $\cos \psi = 1$. In this case,

$$V_r = \frac{1}{2}\frac{\lambda_0 E_i}{\pi}\left(\frac{R_{11}G}{120}\right)^{1/2} \tag{11.78}$$

This result is perfectly general, except for the assumptions of matched conditions, linear polarization, and orientation for maximum signal reception. It relates the received voltage to the incident wave. The fact that

† This supposes that the antenna on transmission produces a linearly polarized wave, as has been tacitly assumed. If this is not the case, the problem must be reformulated in terms of the sum of two linearly polarized waves of appropriate relative magnitude and phase.

we chose to think of the incident wave as arising from a dipole source is completely irrelevant to the relationship described by (11.78). Note also that (11.78) is independent of the choice of the assumed source $\mathbf{J}_s$. In fact, (11.78) can be derived without introducing the source $\mathbf{J}_s$. This alternative approach is left as a problem.

From the above result we see that for an incident wave of a given magnitude the maximum received signal depends only on the direction from which the wave comes. This functional dependence is described by the gain $G(\phi,\theta)$ of the receiving antenna in the direction of the incident wave. We may now understand why receiving antennas are not treated as such in the general literature. This is because the characteristics of the transmitting antenna, namely, R_{11} and $G(\phi,\theta)$, are precisely those needed to analyze the same antenna on reception. The transmitting pattern is, as noted, identical with the receiving pattern.

For the matched system the total power absorbed in the load is given by

$$P_{abs} = \frac{V_r^2}{2R_{11}} = \frac{1}{8}\frac{\lambda_0^2}{\pi^2}\frac{E_i^2}{120}G(\phi,\theta)$$

The absorption cross section of an antenna is the effective area of interception of an incident plane wave; that is,

$$P_{abs} = \tfrac{1}{2}A\,\frac{E^2}{120\pi}$$

where A is the absorption cross section. For any matched antenna we have

$$A(\theta,\phi) = \frac{G(\theta,\phi)\lambda_0^2}{4\pi} \tag{11.79}$$

In the treatment of antennas in this chapter we tacitly assumed that both the transmitting and receiving antennas were in free space. In a practical communications system additional (parasitic) sources must be considered. For example, the effect of the ground must be considered if substantial energy is directed earthward. This may often be disposed of by assuming perfect conductivity and using the method of images. Greater accuracy can be obtained if the earth is considered flat and homogeneous but a complex dielectric constant used to describe its properties. The formula of Chap. 10 for reflection from a dielectric interface may then be used.

For certain ranges of frequency, propagation effects such as those due to the ionosphere or the atmosphere must be considered. Furthermore, diffraction effects due to the earth itself or to features on the earth may have to be taken into account. For a consideration of these problems the student is referred to the suggested references below.

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Chapter 11

11.1. Calculate the far-zone field of a linear antenna whose length is one wavelength under the following two conditions:

(*a*) The antenna is parasitically excited, and the current distribution may be assumed to be $I = I_0 \sin k_0 z$. (The center of the antenna is at $z = 0$.)

(*b*) The antenna is driven at the center so that the current distribution is given by $I = I_0|\sin k_0 z|$.

11.2. The current distribution on a particular half-wave linear antenna is found to be more accurately given by $I = I_0 \cos^2 k_0 z$. Write an expression for the far-zone field, and compute the radiation resistance.

11.3. A two-wire transmission line of length l and separation d is terminated in its characteristic impedance and carries a traveling TEM wave. Calculate the far-zone field, and plot as a function of polar angle, taking the direction of the transmission line along the polar axis with conductors at $x = \pm d/2$. (Neglect radiation from the load itself.)

11.4. (*a*) Show that a linear current radiator normal to and above a perfectly conducting ground plane radiates a field which can be calculated by removing the ground and replacing its effect by a mirror-image current element directed in the same direction (as illustrated).

(*b*) Repeat (*a*) but with a horizontal current element. The image in this case is oppositely directed (see illustration).

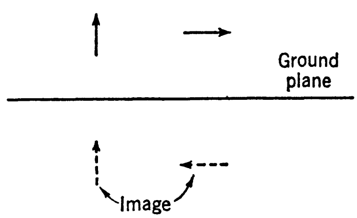

FIG. P 11.4

HINT: It is sufficient to show that the total field due to antenna and image satisfies Maxwell's equations and that tangential **E** is zero over the surface of the conductor. That this is sufficient follows from a uniqueness proof analogous to the one developed for Laplace's equation.†

11.5. Find the field radiated from a small current loop of radius a assumed to carry a current $I = I_0 e^{j\omega t}$. The loop is located at the origin and in the xy plane. Show that the radiation resistance of the loop is $R = 320\pi^4(\pi a^2/\lambda_0^2)^2$. See Prob. 9.17 for fields set up by a magnetic dipole; i.e., consider the loop as a small magnetic dipole.

11.6. At high frequencies the magnetic flux linking a circular current loop C assumed to carry a current $I_0 e^{j\omega t}$ is given by

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \oint_C \oint_C \frac{I_0}{4\pi\mu_0} \frac{e^{-jk_0 r}}{r} \, d\mathbf{l} \cdot d\mathbf{l}'$$

where $d\mathbf{l}$ and $d\mathbf{l}'$ are two elements of arc length along C separated by a distance r. Expand $e^{-jk_0 r}$ in the form $1 - jk_0 r - \frac{1}{2}k_0^2 r^2 + jk_0^3 r^3/6 + \cdots$, and show that the term involving k_0^3 gives rise to an induced voltage in the loop which is 180° out of phase with the current. The applied voltage must overcome this induced voltage and in so doing does work on the current. Thus this term gives rise to an input resistance to the loop. Show that the applied voltage required to overcome the induced voltage due to this part of the flux linkage is $\frac{1}{6}Z_0 k_0^4 \pi a^4 I_0$ and that the input resistance computed by this method is equal to the radiation resistance of the loop, as found in Prob. 11.5.

11.7. In general, the directivity of an array of elements does not equal the array directivity times the element directivity. Prove this. Under what conditions is the directivity of the array approximately that of the antenna?

11.8. A three-dimensional array of isotropic radiators has sources of equal amplitude

† See J. A. Stratton, "Electromagnetic Theory," p. 486, McGraw-Hill Book Company, Inc., New York, 1941.

and phase located at $\mathbf{r}_0 = na\mathbf{a}_x + mb\mathbf{a}_y + lc\mathbf{a}_z$, where

$$n = 0, 1, 2, \ldots, N$$
$$m = 0, 1, 2, \ldots, M$$
$$l = 0, 1, 2, \ldots, L$$

Find the resultant radiation pattern, and show that it is the product of three array factors and the factor $e^{-jk_0r}/4\pi r$. How many main beams are produced if a and b are equal to $\lambda_0/2$ and $c = \lambda_0$?

11.9. An interferometer consists of two identical antennas, A_1 and A_2, spaced by an amount $L = N\lambda_0$. (See Fig. P 11.9.)

(*a*) Show that the resultant radiation pattern is a multilobed pattern. How many lobes are there?

(*b*) If the relative phase angle of the signals received by antennas A_1 and A_2 can be measured to within an accuracy of 1°, what is the resultant angular accuracy that is obtained in measuring the position of the source that emits the signal? Assume that it is always possible to know which lobe the source is located in and that the source is located in the xy plane.

(*c*) For $N = 1{,}000$ and the source making a true angle of $\theta = 0°, 30°, 60°, 90°$ with the x axis, compute the angular error, under conditions of (*b*).

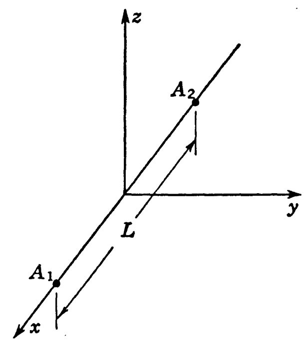

FIG. P 11.9

Interferometers of this type are widely used in radio astronomy and in missile tracking systems. Usually several closer-spaced antennas are also used in order to determine which lobe the source of the received signal is located in. An additional interferometer with base line along the y axis would also be required in order to obtain the direction to an arbitrarily located source.

11.10. An N-element, uniformly spaced, arbitrarily excited linear array has an array factor given by

$$A = \sum_{n=1}^{N} a_n e^{j\psi_n} e^{jk_0 nd \cos\theta}$$

where the line of the array is along the polar axis, θ is the polar angle, and d the element separation. The nth element has an excitation magnitude a_n and relative phase ψ_n.

(*a*) Show that by letting the complex variable $Z = e^{jk_0 d \cos\theta}$ and the complex number $A_n = a_n e^{j\psi_n}$ such an array can be represented by a polynomial, and vice versa.

(*b*) What is the locus of the complex variable Z in part *a*, and what is the range of variation of Z that corresponds to a real pattern (i.e., that corresponds to $0 < \theta < \pi$)?

(*c*) By factoring the polynomial in (*a*), the array factor can be given by the product of line segments from each root to the value of Z corresponding to a given θ. Confirm and describe this in greater detail. Where must at least one polynomial root lie if the pattern is to have a null?

(*d*) The pattern of a two-element array spaced $\lambda_0/4$ apart and with a 90° relative phase is given by $|1 + e^{j\pi(\cos\theta - 1)/2}|$. This pattern is a cardioid, and note that it has no sidelobes, although its main lobe is extremely broad.

If we form the pattern $|1 + e^{j\pi(\cos\theta-1)/2}|^N$ for large N, then the resulting beamwidth should be relatively narrow, as might be visualized by the process of successive multiplication of a cardioid pattern. Interestingly, this pattern remains without any sidelobes. Making use of the polynomial representation, determine the array size and excitation that produce a pattern corresponding to $N = 5$. Plot the pattern. This array is known as the binomial array. (Why?)

11.11. Making use of Prob. 9.22 and Green's second theorem (1.68) with Φ a solution of $\nabla^2\Phi + k^2\Phi = -g(x,y,z)$, and $\psi = (1/4\pi R)e^{-jkR}$, show that the field at any point due to the sources $g(x,y,z)$ may be written

$$\Phi = \frac{1}{4\pi}\int_V g(x,y,z)\,\frac{e^{-jkR}}{R}\,dV + \frac{1}{4\pi}\oint_S \left[\frac{e^{-jkR}}{R}\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial}{\partial n}\left(\frac{e^{-jkR}}{R}\right)\right] dS$$

where n is the outward normal to S. (The surface integrals take into account the presence of sources which lie outside the chosen finite region V. If $V \to \infty$, then all sources are included in the volume integral and the surface integral contribution can be shown to vanish.)

11.12. If the vector wave equation for the electric field in a finite charge-free region is written out, we have

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

Note that each component satisfies an equation of the form denoted by the scalar Φ in Prob. 11.11. Show that by summing the components, a vector relationship is obtained of the form

$$\mathbf{E} = \frac{1}{4\pi}\oint_S \left[\frac{e^{-jkR}}{R}\frac{\partial\mathbf{E}}{\partial n} - \mathbf{E}\frac{\partial}{\partial n}\left(\frac{e^{-jkR}}{R}\right)\right] dS$$

11.13. The equation developed in Prob. 11.12 shows how the field in a source-free region of space is obtainable from given values on a bounding surface. This is an electromagnetic formulation of Huygens' principle.

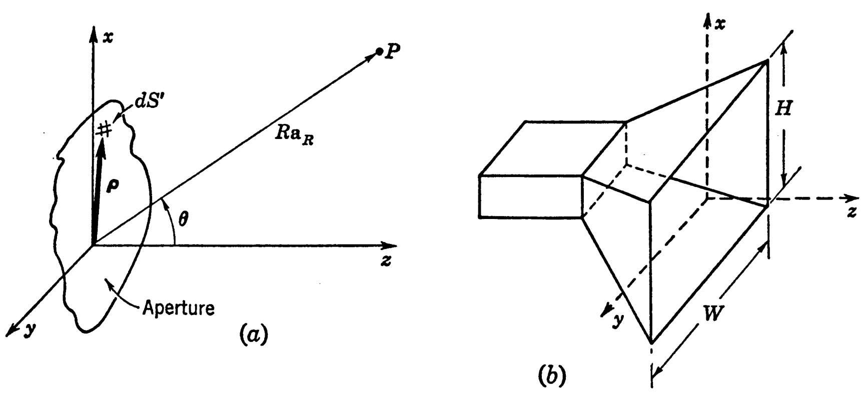

FIG. P 11.13

The equation in Prob. 11.12 can be used to determine the field radiated from a plane aperture since the region beyond the aperture is source-free, and the fields in the aperture may be thought of as a secondary source. Using this as a starting point, Silver† shows that the radiated field may be approximated by the scalar relationship

$$u_p = \frac{jk_0}{4\pi R}\,e^{-jk_0R}\,(1 + \cos\theta)\int_{\text{aperture}} u^{-jk_0\mathbf{r}'\cdot\mathbf{a}_R}\,dS'$$

† S. Silver, "Microwave Antenna Theory and Design," sec. 5.14, McGraw-Hill Book Company, Inc., New York, 1949.

the relative magnitude of the two signals depends on their relative phase of arrival and on the ground-reflection coefficient; that is, assuming $h_1 \ll r$, $h_2 \ll r$, let the direct distance be r_1 and the ground-reflected path be r_2. The received signal is

$$E_{rec} = E_0|1 + \zeta e^{jk_0(r_2-r_1)}|$$

and ζ is the ground-reflection coefficient, while E_0 is the free-space field magnitude. (Note that the assumed conditions allow for the approximation that the electric field from the direct and the reflected paths is in the same direction.)

(*a*) Calculate ζ for earth constants $\sigma = 10^{-5}$, $\kappa = 6$, by utilizing (10.42*a*) and replacing ϵ_0 by complex $\epsilon = \kappa\epsilon_0 - j\sigma/\omega$. Plot ζ as a function of θ for $0 < \theta < 90°$. The frequency is 1.0 megacycle per second.

(*b*) For the geometry of Prob. 11.15, repeat, using the more accurate representation in terms of earth conditions and with the specific ζ in (*a*).

11.18. Repeat Prob. 11.17, but assume parallel polarization. For part *b* the antenna may be visualized as being a vertical dipole. The reflection coefficient may be obtained from (10.44).

11.19. Calculate the absorption cross section of an elementary dipole of length l and a half-wave antenna, both under conditions for maximum absorption of power.

11.20. A half-wave receiving antenna provides the input to a receiver at a frequency of 500 kilocycles and with a receiver bandwidth of 10 kilocycles. The receiver has adequate gain; the limitation on its ability to detect a usable signal arises mainly from the presence of noise in the input. The input noise power is given by $P_n = kTB$, and it accompanies the input signal power under matched conditions. In this formula k is Boltzmann's constant (1.38×10^{-23}), T is the "antenna temperature" and may be taken as ambient (290°K), and B is the bandwidth in cycles per second.

It is desired that the output signal-to-noise power ratio equal 20 decibels, and in view of the additional noise introduced within the receiver itself (noise figure of 6 decibels), an input signal-to-noise power ratio of 26 decibels is required. Assuming antenna orientation for maximum signal, what maximum field strength is required? If the transmitter is 200 miles away and the transmitting antenna has a gain of 5 in the direction of the receiver, what transmitting power is required? (Assume free-space conditions. Neglect the effect of the ground, and neglect antenna losses.)

11.21. A communication system is to be designed which will allow for reception of signals from a satellite. A description of its parameters follows:

Satellite transmitter

1. Power radiated, 100 watts effective
2. Antenna pattern omnidirectional (this is necessitated by probable inability to ensure proper orientation of satellite such as would be necessary with a high-gain antenna)

Receiving system

1. Paraboloidal antenna of diameter, 80 feet (the receiving cross section may be approximated as 0.5 times the aperture area)
2. Output signal-to-noise ratio, 10 decibels
3. Noise figure of receiver, 2 decibels (this means that an input signal-to-noise ratio of 12 decibels is required)
4. Losses from antenna to receiver (polarization, transmission lines, etc.), 6 decibels
5. Bandwidth, 10 cycles per second
6. Noise temperature, 290°K [under an assumed matched condition the input noise power P_n is given by $P_n = kTB = 4 \times 10^{-21}B$ (see Prob. 11.20)]
7. Frequency, 400 megacycles per second

(a) What is the range of the above system?

(b) How does the range depend on the receiving-antenna diameter? What is the range if the diameter is 600 feet?

(c) How does range depend on frequency? What is the range for $f = 1{,}000$ megacycles per second. (Note that at higher frequencies satisfactory operation of equipment becomes more difficult.)

11.22. The relation (11.78) may be derived in an alternative way to that used in the text. In (11.67) let the surface S be the sphere at infinity, and a surface enclosing the receiving antenna similar to that in Fig. 11.16. Now note that (11.67) becomes

$$\int_V \mathbf{E}_2 \cdot \mathbf{J}_1 \, dV = \int_{S_1} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot d\mathbf{S}$$

since the source $\mathbf{J}_2$ is not present in the volume under consideration. When the antenna is used as a transmitting antenna (source $\mathbf{J}_2$ inside $S_1 + S_{10}$), then the field $\mathbf{E}_2$, $\mathbf{H}_2$ in the coaxial line is a TEM wave, $H_\phi = I_0/2\pi r$, $E_r = Z_0 H_\phi$. The current element $I_e \, \Delta l$ also sets up a TEM wave ($\mathbf{E}_1$, $\mathbf{H}_1$) in the coaxial line, given by $H_\phi = -I_r/2\pi r$, $E_r = -Z_0 H_\phi = Z_0 I_r/2\pi r$, where I_r is the received current. The change in sign for H_ϕ arises because of the difference in the direction of propagation of the two TEM waves. Show that (11.67) gives

$$E_t I_e \, \Delta l \cos \alpha = 2 Z_c I_0 I_r = 2 I_0 V_r$$

in place of (11.70). In place of (11.74) show that the appropriate relation to use now is $P_t = \frac{1}{2} V_0 I_0 = \frac{1}{2} I_0^2 Z_c = \frac{1}{2} I_0^2 R_{11}$ for $Z_c = R_{11}$. Using these relations together with (11.71) to (11.73) and (11.75), show that (11.78) follows.