CHAPTER 10

PLANE WAVES, WAVEGUIDES, AND RESONATORS

We have now reached a point in the development of electromagnetic field theory where it is possible to consider a wide variety of important applications. It is, however, outside the scope of the present book to do much more than examine a small number of these. The topics to be considered in this chapter are those of plane waves in free space, reflection of plane waves from a dielectric interface and a conducting plane, the transmission line, rectangular and circular waveguides, and the cavity resonator. These particular topics are chosen because of their great importance at microwave frequencies (frequencies from about 1,000 megacycles up to and beyond 100,000 megacycles) in practical communication systems and also because the solutions are quite readily obtained and provide an elegant demonstration of the validity of Maxwell's field equations.

We shall be dealing entirely with steady-state sinusoidal fields with angular frequency ω. Thus all field vectors are represented by complex phasor vector quantities. Also, we shall assume that ϵ and μ for material bodies are real and constant, unless otherwise stated.

10.1. Classification of Wave Solutions

For most of the topics in this chapter, as well as for a large number of other problems of practical importance, it is possible to separate the solutions of Maxwell's equations in a source-free region into three basic types of fields. These three classifications are:

1. Transverse electromagnetic waves (TEM waves). The transverse electromagnetic wave is characterized by the condition that both the electric and magnetic field vectors lie in a plane perpendicular to the axis of propagation, i.e., have no components in the direction of propagation. The electric field for TEM waves may be derived from the transverse gradient (gradient in the plane transverse to the axis of propagation) of a scalar potential which satisfies the two-dimensional Laplace equation.

2. Transverse electric waves (TE, or H, waves). Transverse electric waves are characterized by having an electric field which is entirely in a plane transverse to the (assumed) direction of propagation. Only the magnetic field $\mathbf{H}$ has a component in the direction of propagation, and

hence this wave type is also known as an H wave. For TE waves it is possible to express all field components in terms of the axial-magnetic-field component.

3. Transverse magnetic waves (TM, or E, waves). Transverse magnetic waves are waves whose magnetic field vector is entirely in a plane transverse to the (assumed) axis of propagation. Only the electric field **E** has a component in the direction of propagation. For TM waves all field components may be expressed in terms of the axial electric field.

The above three types of solutions are sufficiently general so that any arbitrary field solution can be built up by superposing appropriate amounts of each wave type. The only basis for the above classification or division is that the solutions of many practical wave problems fall naturally into one or another of the above types. It is at times more convenient to choose other forms of solutions, but these are just suitable linear combinations of TE and TM wave types and so will not be considered here.

We shall apply the afore-mentioned classification of waves to a broad class of problems characterized by the fact that the geometry is uniform along a given direction; that is, if any material bodies are involved, they are assumed to be cylindrical, and we take the axis to be the z axis. Under these conditions the field solutions can vary axially only by a phase factor $e^{-j\beta z}$. The nature of the field variation in z depends on the propagation constant; if this is real, then unattenuated wave propagation exists. With this type of z dependence all derivatives with respect to z may be replaced by the factor $-j\beta$. The vector operator ∇ becomes

$$\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} - j\beta \mathbf{a}_z = \nabla_t - j\beta \mathbf{a}_z \tag{10.1}$$

where ∇_t signifies the transverse part (x and y part) of the ∇ operator.

It is convenient to separate out the z dependence and decompose all the fields into transverse and axial components. Thus we shall write

$$\begin{aligned}\mathbf{E}(x,y,z) &= \mathbf{E}_t(x,y,z) + \mathbf{E}_z(x,y,z)\\ &= \mathbf{e}(x,y)e^{-j\beta z} + \mathbf{e}_z(x,y)e^{-j\beta z} \qquad (10.2a)\\ \mathbf{H}(x,y,z) &= \mathbf{H}_t(x,y,z) + \mathbf{H}_z(x,y,z)\\ &= \mathbf{h}(x,y)e^{-j\beta z} + \mathbf{h}_z(x,y)e^{-j\beta z} \qquad (10.2b)\end{aligned}$$

where $\mathbf{e}$, $\mathbf{h}$ are transverse vector (x and y components only) functions of the transverse coordinates; $\mathbf{e}_z$, $\mathbf{h}_z$ are z-directed vector functions of x and y; and $\mathbf{E}_t$, $\mathbf{H}_t$ represent the transverse fields including the z dependence while $\mathbf{E}_z$, $\mathbf{H}_z$ represent the axial fields.

Maxwell's equations in a source-free region may now be written in the following form:

$$\nabla_t \times \mathbf{e} = -j\omega\mu\mathbf{h}_z \tag{10.3a}$$
$$\mathbf{a}_z \times \nabla_t e_z + j\beta\mathbf{a}_z \times \mathbf{e} = j\omega\mu\mathbf{h} \tag{10.3b}$$
$$\nabla_t \times \mathbf{h} = j\omega\epsilon\mathbf{e}_z \tag{10.4a}$$
$$\mathbf{a}_z \times \nabla_t h_z + j\beta\mathbf{a}_z \times \mathbf{h} = -j\omega\epsilon\mathbf{e} \tag{10.4b}$$
$$\nabla_t \cdot \mathbf{e} = j\beta e_z \tag{10.5a}$$
$$\nabla_t \cdot \mathbf{h} = j\beta h_z \tag{10.5b}$$

For example, we shall derive (10.3). The curl equation for $\mathbf{E}$ is

$$(\nabla_t - j\beta\mathbf{a}_z) \times (\mathbf{e} + \mathbf{e}_z)e^{-j\beta z} = -j\omega\mu(\mathbf{h} + \mathbf{h}_z)e^{-j\beta z}$$

Deleting the factor $e^{-j\beta z}$ and expanding give

$$\nabla_t \times \mathbf{e} + \nabla_t \times \mathbf{e}_z - j\beta\mathbf{a}_z \times \mathbf{e} - j\beta\mathbf{a}_z \times \mathbf{e}_z = -j\omega\mu(\mathbf{h} + \mathbf{h}_z) \tag{10.6}$$

Now $\mathbf{a}_z \times \mathbf{e}_z = 0$, $\nabla_t \times \mathbf{e}_z = \nabla_t \times \mathbf{a}_z e_z = -\mathbf{a}_z \times \nabla_t e_z$, and furthermore $\nabla_t \times \mathbf{e}$ is a z-directed vector function while $\mathbf{a}_z \times \nabla_t e_z$ and $\mathbf{a}_z \times \mathbf{e}$ are transverse vectors. Equating the transverse and axial parts of both sides of (10.6) now gives (10.3*a*) and (10.3*b*). In a similar way (10.4) follows from the curl equation for $\mathbf{H}$. Equations (10.5*a*) and (10.5*b*) are the divergence equations $\nabla \cdot \epsilon\mathbf{E} = 0$, $\nabla \cdot \mu\mathbf{H} = 0$, with ϵ, μ considered as constant and with $\partial/\partial z$ replaced by $-j\beta$ and the z dependence deleted.

The remainder of this section is devoted to a derivation of the basic equations relating the field components for the three wave types. The following sections will make use of these results for constructing solutions to a variety of practical problems. A fuller appreciation of the properties of the various wave types will be obtained from a study of these examples.

TEM Waves

For TEM waves $\mathbf{e}_z = \mathbf{h}_z = 0$, and (10.3) reduces to the following:

$$\nabla_t \times \mathbf{e} = 0 \tag{10.7a}$$
$$\mathbf{h} = \frac{\beta}{\omega\mu}\mathbf{a}_z \times \mathbf{e} \tag{10.7b}$$

Equation (10.7*a*) is just the condition that $\mathbf{e}$ may be derived from the transverse gradient of a scalar potential $\Phi(x,y)$; thus

$$\mathbf{e}(x,y) = -\nabla_t\Phi(x,y) = -\mathbf{a}_x\frac{\partial\Phi}{\partial x} - \mathbf{a}_y\frac{\partial\Phi}{\partial y} \tag{10.8}$$

Since $\mathbf{e}_z = 0$, the divergence equation (10.5*a*) gives $\nabla_t \cdot \mathbf{e} = 0$, and hence

$$\nabla_t^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{10.9}$$

The relation (10.8) is physically understandable since, with $\mathbf{h}_z = 0$, the line integral of $\mathbf{e}$ around any closed contour C in the xy plane is zero, because there is no magnetic flux linking this contour. Thus there is associated with the electric field of a TEM wave a unique scalar potential (apart from a constant). Furthermore, since Φ is a solution of Laplace's equation in the transverse plane, the field $\mathbf{e}(x,y)$ has the same properties as a static electric field. This is a very interesting result in that, even though the fields may vary with time at a rate of thousands of megacycles per second, the field distribution in the transverse plane is a static field distribution.

The field $\mathbf{e}(x,y)e^{-j\beta z}$ is also a solution of the Helmholtz equation:

$$\nabla^2(\mathbf{e}e^{-j\beta z}) + k^2\mathbf{e}e^{-j\beta z} = 0$$

or

$$\nabla_t^2\mathbf{e} + (k^2 - \beta^2)\mathbf{e} = 0 \tag{10.10}$$

Expanding the relation $\nabla_t \times (\nabla_t \times \mathbf{e})$ gives

$$\nabla_t \times (\nabla_t \times \mathbf{e}) = \nabla_t\nabla_t \cdot \mathbf{e} - \nabla_t^2\mathbf{e} = 0$$

since $\nabla_t \times \mathbf{e} = 0$. The divergence of $\mathbf{e}$, that is, $\nabla_t \cdot \mathbf{e}$, is also zero, and hence $\nabla_t^2\mathbf{e} = 0$. Therefore (10.10) can be satisfied only if

$$\beta = \pm k = \pm\omega(\mu\epsilon)^{1/2} \tag{10.11}$$

From (10.7b) the magnetic field $\mathbf{h}$ is found to be given by

$$\mathbf{h} = \frac{k}{\omega\mu}\mathbf{a}_z \times \mathbf{e} = Y\mathbf{a}_z \times \mathbf{e} \tag{10.12}$$

where $Y = (\epsilon/\mu)^{1/2}$ is the intrinsic admittance of the medium.

The solution for TEM waves may be summarized as follows. First find a scalar potential Φ which satisfies the two-dimensional Laplace equation and any imposed boundary conditions. The electric and magnetic fields are then given by

$$\mathbf{E}_t = \mathbf{e}e^{-jkz} = -\nabla_t\Phi e^{-jkz} \tag{10.13a}$$

$$\mathbf{H}_t = Y\mathbf{a}_z \times \mathbf{e}e^{-jkz} \tag{10.13b}$$

For a wave propagating in the $-z$ direction, replace k by $-k$ and Y by $-Y$.

TE Waves

For transverse electric (TE) waves, $\mathbf{e}_z = 0$ but $\mathbf{h}_z \neq 0$. For these waves all field components may be expressed in terms of the axial magnetic field $\mathbf{h}_z$, as we shall presently establish.

The magnetic field $\mathbf{H} = (\mathbf{h} + \mathbf{h}_z)e^{-j\beta z}$ must be a solution of the Helmholtz equation $\nabla^2\mathbf{H} + k^2\mathbf{H} = 0$, and hence

$$\nabla_t^2 h_z + k_c^2 h_z = 0 \tag{10.14a}$$

$$\nabla_t^2\mathbf{h} + k_c^2\mathbf{h} = 0 \tag{10.14b}$$

where $k_c^2 = k^2 - \beta^2$. From (10.4a) $\nabla_t \times \mathbf{h} = 0$ since $\mathbf{e}_z = 0$, and hence

$$\nabla_t \times (\nabla_t \times \mathbf{h}) = \nabla_t \nabla_t \cdot \mathbf{h} - \nabla_t^2 \mathbf{h} = 0$$

Replacing $\nabla_t^2 \mathbf{h}$ from (10.14b) and using (10.5b) to replace $\nabla_t \cdot \mathbf{h}$ by $j\beta h_z$, we obtain

$$\nabla_t \nabla_t \cdot \mathbf{h} = j\beta \nabla_t h_z = \nabla_t^2 \mathbf{h} = -k_c^2 \mathbf{h}$$

or

$$\mathbf{h} = -\frac{j\beta}{k_c^2} \nabla_t h_z \tag{10.15}$$

This relation expresses the transverse-magnetic-field vector function in terms of h_z. The function $h_z(x,y)$ is seen to play the role of a scalar potential function from which $\mathbf{h}$ may be derived.

In order to express $\mathbf{e}$ in terms of $\mathbf{h}$, we take the vector product of (10.3b) by $\mathbf{a}_z$ to obtain

$$j\beta \mathbf{a}_z \times (\mathbf{a}_z \times \mathbf{e}) = j\beta[(\mathbf{a}_z \cdot \mathbf{e})\mathbf{a}_z - (\mathbf{a}_z \cdot \mathbf{a}_z)\mathbf{e}] = -j\beta \mathbf{e} = j\omega\mu \mathbf{a}_z \times \mathbf{h}$$

since $\mathbf{a}_z \cdot \mathbf{e} = 0$ and $\mathbf{e}_z = 0$. Replacing $\omega\mu$ by kZ, where $Z = (\mu/\epsilon)^{1/2}$, now gives

$$\mathbf{e} = -\frac{k}{\beta} Z \mathbf{a}_z \times \mathbf{h} \tag{10.16}$$

The factor kZ/β has the dimensions of an impedance and is called the wave impedance for TE or H waves. It will be designated by the symbol Z_h; that is,

$$Z_h = \frac{k}{\beta} Z \tag{10.17}$$

In component form (10.16) gives

$$\frac{e_x}{h_y} = -\frac{e_y}{h_x} = Z_h \tag{10.18}$$

Thus the ratio of the transverse electric field to the mutually perpendicular transverse magnetic field is equal to the wave impedance (apart from a minus sign in one case).

The solution for TE waves may be summarized as follows. First find a solution for h_z; that is, obtain a solution of

$$\nabla_t^2 h_z + k_c^2 h_z = 0$$

The parameter k_c is usually determined by boundary conditions that the field must satisfy. This will be elaborated on when specific examples are considered. Once k_c is determined, β may be obtained from the relation $\beta^2 = k^2 - k_c^2$. The vector function $\mathbf{h}$ is obtained from h_z by means of (10.15), and $\mathbf{e}$ is found from $\mathbf{h}$ by means of (10.16). The electric and magnetic fields are then given by

$$\mathbf{E} = \mathbf{e}e^{-j\beta z}$$
$$\mathbf{H} = (\mathbf{h} + \mathbf{h}_z)e^{-j\beta z}$$

For a wave propagating in the $-z$ direction, β is replaced by $-\beta$ in (10.15) and (10.16). The sign of $\mathbf{h}$ changes, but not the sign of $\mathbf{e}$. A reversal of the sign of either $\mathbf{h}$ or $\mathbf{e}$ is required in order to obtain a reversal in the direction of power flow (Poynting vector).

TM Waves

For TM waves $\mathbf{h}_z = 0$ but $\mathbf{e}_z \neq 0$. For this wave type all field components may be expressed in terms of e_z. The required equations may be derived in a manner similar to that used for TE waves but with the role of electric and magnetic fields interchanged. This possibility is a direct consequence of the symmetry of Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$ in a source-free region. In actual fact this symmetry forms the basis of the "principle of duality" for electromagnetic fields.

The duality principle states that if $\mathbf{E}_1$, $\mathbf{H}_1$ are solutions of the equations

$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1 \qquad \nabla \times \mathbf{H}_1 = j\omega\epsilon\mathbf{E}_1$$
$$\nabla \cdot \mathbf{E}_1 = 0 \qquad \nabla \cdot \mathbf{H}_1 = 0$$

then a second field $\mathbf{E}_2$, $\mathbf{H}_2$, where

$$\mathbf{E}_2 = \pm Z\mathbf{H}_1 \tag{10.19a}$$
$$\mathbf{H}_2 = \mp Y\mathbf{E}_1 \tag{10.19b}$$

is also a solution. Substitution into Maxwell's equations verifies the result at once. For example,

$$\nabla \times \mathbf{E}_2 = \pm\nabla \times Z\mathbf{H}_1 = \pm j\omega\epsilon Z\mathbf{E}_1$$

But $Z\epsilon = Y\mu$ and $Y\mathbf{E}_1 = \mp\mathbf{H}_2$, and hence

$$\nabla \times \mathbf{E}_2 = \pm j\omega\mu(\mp\mathbf{H}_2) = -j\omega\mu\mathbf{H}_2$$

In a similar way it is readily verified that $\nabla \times \mathbf{H}_2 = j\omega\epsilon\mathbf{E}_2$, and hence $\mathbf{E}_2$, $\mathbf{H}_2$ as given by (10.19) is a solution if $\mathbf{E}_1$, $\mathbf{H}_1$ is a solution. This principle is very useful in practice for constructing solutions for TE waves from those for TM waves, and vice versa.

For TM waves we have, analogous to (10.14),

$$\nabla_t^2 e_z + k_c^2 e_z = 0 \tag{10.20a}$$
$$\nabla_t^2 \mathbf{e} + k_c^2 \mathbf{e} = 0 \tag{10.20b}$$

The equations analogous to (10.15) and (10.16) are obtained by using the duality principle, i.e., replacing $\mathbf{h}$, h_z, and $\mathbf{e}$ in these equations by $Y\mathbf{e}$, Ye_z, and $-Z\mathbf{h}$, respectively. Thus (10.15) becomes

$$Y\mathbf{e} = -\frac{j\beta}{k_c^2}\nabla_t Y e_z$$

or

$$\mathbf{e} = -\frac{j\beta}{k_c^2}\nabla_t e_z \tag{10.21}$$

and (10.16) becomes

$$\mathbf{h} = \frac{k}{\beta} Y \mathbf{a}_z \times \mathbf{e} \tag{10.22}$$

Equations (10.20), (10.21), and (10.22) are the required relations expressing the field components for TM waves in terms of the axial-electric-field function e_z. The relation (10.22) may be written in component form as

$$-\frac{h_x}{e_y} = \frac{h_y}{e_x} = \frac{k}{\beta} Y = Y_e = Z_e^{-1} \tag{10.23}$$

where $Z_e = Z\beta/k$ is the wave impedance for TM waves. The combination of (10.23) with (10.17) shows that

$$Z_h Z_e = Z^2 \tag{10.24}$$

a result which expresses the dual relationship between TE and TM waves.

The complete solution for TM waves is

$$\mathbf{E} = (\mathbf{e} + \mathbf{e}_z)e^{-j\beta z} \tag{10.25a}$$
$$\mathbf{H} = \mathbf{h}e^{-j\beta z} \tag{10.25b}$$

For a wave propagating in the $-z$ direction the sign of $\mathbf{e}_z$ and β is reversed. This changes the sign of $\mathbf{h}$ but leaves the sign of $\mathbf{e}$ unchanged (the sign of $\mathbf{e}_z$ is changed only to keep the sign of $\mathbf{e}$ unchanged).

10.2. Plane Waves

In Chap. 9 we considered the solution for a plane wave with components E_x, H_y and propagating in the z direction according to $e^{-jk_0 z}$. We should now like to reformulate the properties of a plane wave for an arbitrary direction of propagation, $\mathbf{n}$. We note that if $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the radius vector from the origin, then

$$\mathbf{n} \cdot \mathbf{r} = \text{constant} \tag{10.26}$$

is the equation of a plane which is perpendicular to the unit vector $\mathbf{n}$, as in Fig. 10.1. Consequently, if we want to consider a wave propagating in a direction given by $\mathbf{n}$, then the appropriate propagation factor to use is $e^{-jk_0\mathbf{n}\cdot\mathbf{r}}$.

The mathematical formulation for the electric field **E** of a uniform plane wave propagating in a direction **n** can be written

$$\mathbf{E} = \mathbf{E}_0 e^{-jk_0\mathbf{n}\cdot\mathbf{r}} \tag{10.27}$$

where $\mathbf{E}_0$ is a constant vector. The restrictions on $\mathbf{E}_0$ can be found from the requirement that (10.27) be a solution of Maxwell's field equations in a

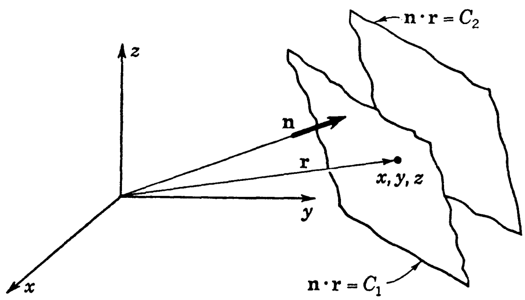

FIG. 10.1. Illustration of planes specified by $\mathbf{n}\cdot\mathbf{r}$ = constant.

source-free region of free space. The charge density is assumed to be zero; consequently, the divergence of **E** must be zero, and hence

$$\nabla\cdot\mathbf{E} = 0 = \nabla\cdot\mathbf{E}_0 e^{-jk_0\mathbf{n}\cdot\mathbf{r}} = \mathbf{E}_0\cdot\nabla e^{-jk_0\mathbf{n}\cdot\mathbf{r}} \tag{10.28}$$

since $\mathbf{E}_0$ is a constant vector. Now $\mathbf{n}\cdot\mathbf{r} = n_x x + n_y y + n_z z$, where n_x, n_y, n_z are the components of **n**, and so

$$\mathbf{a}_x \frac{\partial}{\partial x} e^{-jk_0\mathbf{n}\cdot\mathbf{r}} = -jk_0 n_x \mathbf{a}_x e^{-jk_0\mathbf{n}\cdot\mathbf{r}}, \text{ etc.}$$

so that

$$\nabla e^{-jk_0\mathbf{n}\cdot\mathbf{r}} = -jk_0\mathbf{n}e^{-jk_0\mathbf{n}\cdot\mathbf{r}}$$

Therefore (10.28) gives

$$-jk_0(\mathbf{E}_0\cdot\mathbf{n})e^{-jk_0\mathbf{n}\cdot\mathbf{r}} = 0$$

or

$$\mathbf{n}\cdot\mathbf{E}_0 = 0 \tag{10.29}$$

Thus (10.27) is a possible solution only if $\mathbf{E}_0$ lies in a plane that is perpendicular to the direction of propagation specified by the unit vector **n**.

The magnetic field may be found from the curl equation for **E** as follows:

$$-j\omega\mu_0\mathbf{H} = \nabla\times\mathbf{E} = \nabla\times\mathbf{E}_0 e^{-jk_0\mathbf{n}\cdot\mathbf{r}} = -\mathbf{E}_0\times\nabla e^{-jk_0\mathbf{n}\cdot\mathbf{r}}$$

or

$$\mathbf{H} = (\mathbf{E}_0\times\mathbf{n})\frac{-jk_0}{j\omega\mu_0} e^{-jk_0\mathbf{n}\cdot\mathbf{r}} = Y_0(\mathbf{n}\times\mathbf{E}_0)e^{-jk_0\mathbf{n}\cdot\mathbf{r}} \tag{10.30}$$

The magnetic field associated with the electric field given by (10.27) also lies in a plane transverse to the direction of propagation and furthermore is also perpendicular to $\mathbf{E}_0$, as in Fig. 10.2. Equations (10.27), (10.29), and (10.30) define a general plane transverse electromagnetic wave propagating in the direction **n**. The wave is called a plane wave since the constant phase surfaces given by $k_0\mathbf{n} \cdot \mathbf{r}$ = constant are planes.

If, for example, we wish to call the z axis the axis of propagation, then

$$\mathbf{E} = (\mathbf{E}_0 e^{-jk_0(xn_x+yn_y)})e^{-jk_0n_z z} \tag{10.31}$$

and $\beta_0 = k_0 n_z$. Since $\mathbf{E}_0$ does not lie in the xy plane (excluding $n_x = n_y = 0$), the wave would not be classified as a TEM wave with respect to the z axis. Depending on the direction of $\mathbf{E}_0$, it could be a TE,

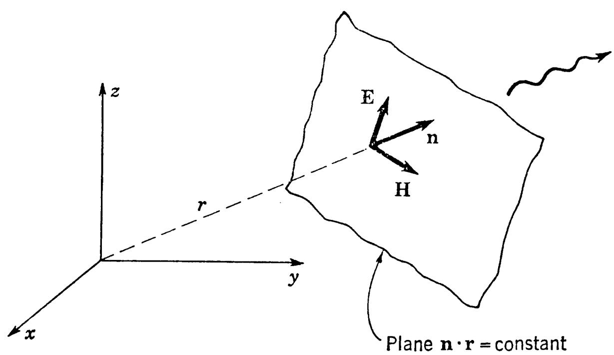

FIG. 10.2. Space relation between the field components and direction of propagation for a plane TEM wave.

TM, or a combination of a TE and a TM wave. In this respect the classification of a wave solution as a TEM, TE, or TM wave does depend on our choice of a preferred direction to be considered as the direction of propagation. In actual fact, the wave specified by (10.31) propagates in the direction **n**, and not in the z direction.

The classification into TE or TM categories is more meaningful for a wave of the type to be discussed now. Let us superpose on the solution (10.31) a similar wave solution with the direction of propagation given by

$$\mathbf{n}_1 = -\mathbf{a}_x n_x - \mathbf{a}_y n_y + \mathbf{a}_z n_z$$

Then

$$\mathbf{E} = \mathbf{E}_0(e^{-jk_0(xn_x+yn_y)} + e^{jk_0(xn_x+yn_y)})e^{-j\beta_0 z} = 2\mathbf{E}_0 \cos [k_0(xn_x + yn_y)]e^{-j\beta_0 z} \tag{10.32}$$

where $\beta_0 = k_0 n_z$. This solution represents a wave propagating in the z direction only. In the transverse plane the solution is that of a standing wave. If $n_y = 0$, then **n** is a vector in the xz plane and it is not inconsistent to choose an $\mathbf{E}_0$ that lies in the y direction; that is, $\mathbf{E}_0 = E_0\mathbf{a}_y$.

The corresponding solution for the magnetic field is

$$-j\omega\mu_0\mathbf{H} = \nabla \times \mathbf{E} = -\mathbf{a}_y \times \nabla E$$
$$= 2E_0(j\beta_0\mathbf{a}_x \cos k_0n_xx - k_0n_x\mathbf{a}_z \sin k_0n_xx)e^{-j\beta_0z} \quad (10.33)$$

and is seen to have x and z components. This solution is clearly a TE wave, and under no circumstances could it be considered as a TEM wave since $H_z \neq 0$. In general, then, we are able to state that the combination of TEM waves propagating in different directions gives wave solutions of the TE and TM types. Nevertheless, it is convenient at times to classify obliquely propagating TEM waves as TE or TM waves also.

Reflection from a Dielectric Interface, Perpendicular Polarization

In this discussion we shall examine the problem of the reflection of an obliquely incident plane wave from a dielectric interface. With reference

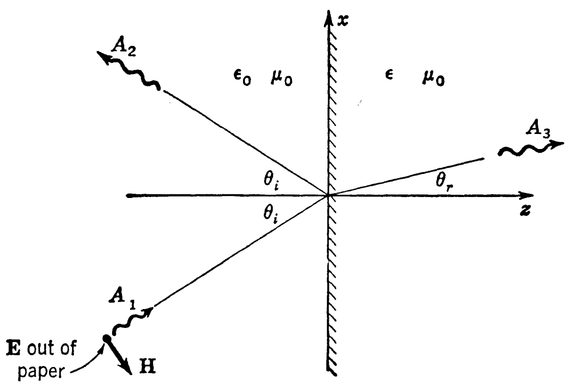

FIG. 10.3. Plane wave incident on a dielectric interface.

to Fig. 10.3, let the half space $z > 0$ be filled with a homogeneous, isotropic, lossless dielectric with a permittivity ϵ. The dielectric constant is $\kappa = \epsilon/\epsilon_0$, and the index of refraction is $\eta = \kappa^{1/2}$.

Without loss in generality we may choose the plane of incidence as the xz plane, and then

$$\mathbf{n} = \mathbf{a}_x \sin \theta_i + \mathbf{a}_z \cos \theta_i$$

where θ_i is the angle of incidence measured relative to the interface normal. Rather than consider an arbitrary polarized incident wave (i.e., we have yet to specify the orientation of the electric field), it is more convenient to treat two special cases separately. For one we choose a wave with the electric field in the y direction. This wave is called perpendicular-polarized since the electric field is perpendicular to the plane of incidence, where the latter is defined by the interface normal and the unit vector $\mathbf{n}$, that is, the xz plane. The corresponding magnetic field has both x and z components, and consequently the wave is a TE wave with respect to the z axis. In the other case the roles of the electric and magnetic fields are interchanged; i.e., the electric field lies in the xz plane while the magnetic field is directed along the y axis. This wave is a TM wave and is referred to as a parallel-polarized wave since the electric field is parallel to the plane of incidence. A superposition of these two cases gives the solution for an arbitrary-polarized incident wave. The two cases are treated separately because of the existence of certain basic differences, as we shall discover.

For the perpendicular-polarized incident wave let the electric field be

$$\mathbf{E}_i = \mathbf{a}_y A_1 e^{-jl_0x - j\beta_0 z} \tag{10.34}$$

where $l_0 = n_x k_0 = k_0 \sin \theta_i$ and $\beta_0 = k_0 \cos \theta_i$. The corresponding magnetic field may be found from the curl of $\mathbf{E}_i$ and is, from (10.30),

$$\mathbf{H}_i = A_1(-Y_0 \cos \theta_i \mathbf{a}_x + Y_0 \sin \theta_i \mathbf{a}_z) e^{-jl_0x - j\beta_0 z} \tag{10.35}$$

In order to satisfy the boundary conditions at the interface $z = 0$ when a plane wave is incident, it is necessary to assume that a part of it is reflected from the dielectric and a part of it is transmitted into the dielectric. At the interface the total tangential electric and magnetic fields must be equal on adjacent sides of the interface. This is possible only if all field components have the same variation with x on either side of the interface. Consequently, the form of the reflected and transmitted electric fields must be

$$\mathbf{E}_r = \mathbf{a}_y A_2 e^{-jl_0x + j\beta_0 z} \tag{10.36}$$

$$\mathbf{E}_t = \mathbf{a}_y A_3 e^{-jlx - j\beta z} \tag{10.37}$$

where $l = k \sin \theta_r$, $\beta = k \cos \theta_r$, and θ_r is the angle of refraction, i.e., specifies the direction of propagation in the dielectric, as illustrated in Fig. 10.3. As noted above, l must equal l_0 in order to satisfy the boundary conditions for all values of x, and hence

$$k_0 \sin \theta_i = k \sin \theta_r$$

or

$$\sin \theta_i = \kappa^{1/2} \sin \theta_r = \eta \sin \theta_r \tag{10.38}$$

Equation (10.38) is the well-known Snell's law of refraction.

The magnetic fields for the reflected and transmitted waves are found from the corresponding electric fields using (10.30) and are

$$\mathbf{H}_r = A_2(Y_0 \cos \theta_i \mathbf{a}_x + Y_0 \sin \theta_i \mathbf{a}_z) e^{-jl_0x + j\beta_0 z} \tag{10.39}$$

$$\mathbf{H}_t = A_3(-Y \cos \theta_r \mathbf{a}_x + Y \sin \theta_r \mathbf{a}_z) e^{-jl_0x - j\beta z} \tag{10.40}$$

where $Y = (\epsilon/\mu_0)^{1/2} = \eta Y_0$. For the reflected wave the x component of magnetic field is reversed in sign corresponding to the use of the exponential function $e^{j\beta_0 z}$.

The amplitude coefficients A_2 and A_3 are determined by making the total tangential electric and magnetic field components continuous across the interface. The following two equations result from these two conditions:

$$A_1 + A_2 = A_3$$
$$(A_1 - A_2)Y_0 \cos\theta_i = A_3 Y \cos\theta_r$$

As noted in Sec. 9.8, matching the tangential fields at the boundary automatically ensures the proper behavior of the normal field components. The reflection coefficient ζ_1 is defined as the ratio of the reflected electric field amplitude A_2 to the incident electric field amplitude A_1. Similarly, the transmission coefficient T_1 is defined as the ratio of the transmitted electric field amplitude to the incident wave amplitude. We have

$$A_2 = \zeta_1 A_1 \qquad A_3 = T_1 A_1$$

and the boundary conditions become

$$1 + \zeta_1 = T_1 \tag{10.41a}$$
$$(1 - \zeta_1)Y_0 \cos\theta_i = T_1 Y \cos\theta_r = T_1 \eta Y_0 \cos\theta_r \tag{10.41b}$$

Solving these equations for ζ_1 and T_1 gives

$$\zeta_1 = \frac{\cos\theta_i - \eta\cos\theta_r}{\cos\theta_i + \eta\cos\theta_r} \tag{10.42a}$$
$$T_1 = 1 + \zeta_1 = \frac{2\cos\theta_i}{\cos\theta_i + \eta\cos\theta_r} \tag{10.42b}$$

These latter equations are called the Fresnel reflection and transmission equations for a perpendicular-polarized incident wave.

The student familiar with transmission-line circuit theory† will recognize the close analogy between the present problem and that of a junction of two transmission lines of different characteristic impedance. The transverse electric field is analogous to the voltage wave, while the transverse magnetic field is analogous to the current wave on a transmission line. The wave impedance is the counterpart of the characteristic impedance of a transmission line. The basis for the analogy is that the continuity at an interface of the total tangential fields $\mathbf{E}_t$ and $\mathbf{H}_t$ in the field analysis corresponds to the continuity at the transmission-line junction of the total V and I in the equivalent transmission-line analysis. Thus for the TE wave on the air side of the interface the wave impedance is

$$Z_{h0} = -\frac{E_y}{H_x} = Z_0 \sec\theta_i = Z_0 \frac{k_0}{\beta_0} \tag{10.43a}$$

† A discussion of the transmission line will be given in Sec. 10.4.

while for the dielectric region

$$Z_h = -\frac{E_y}{H_x} = Z \sec \theta_r = Z \frac{k}{\beta} \tag{10.43b}$$

where $Z_0 = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}$ and $Z = \left(\frac{\mu_0}{\epsilon}\right)^{1/2}$

The transmission-line circuit illustrated in Fig. 10.4 is formally equivalent to the problem being considered here. According to transmission-

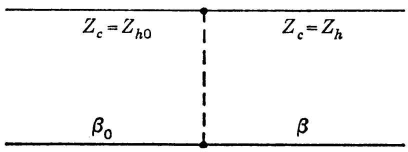

FIG 10.4. Transmission-line equivalent circuit for Fig. 10.3, perpendicular polarization.

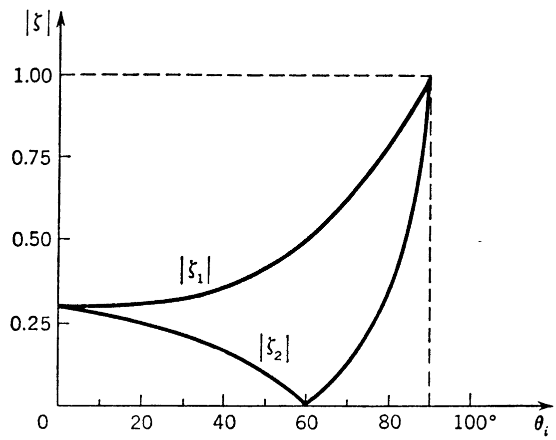

FIG. 10.5. Modulus of reflection coefficient for $\kappa = 3$.

line theory the reflection coefficient of the junction is

$$\zeta_1 = \frac{Z_h - Z_{h0}}{Z_h + Z_{h0}} = \frac{Z \sec \theta_r - Z_0 \sec \theta_i}{Z \sec \theta_r + Z_0 \sec \theta_i} = \frac{\cos \theta_i - \eta \cos \theta_r}{\cos \theta_i + \eta \cos \theta_r}$$

which is the same as (10.42*a*). Using Snell's law, we have

$$\eta \cos \theta_r = (\kappa - \sin^2 \theta_i)^{1/2}$$

and hence

$$\zeta_1 = -\frac{(\kappa - \sin^2 \theta_i)^{1/2} - \cos \theta_i}{(\kappa - \sin^2 \theta_i)^{1/2} + \cos \theta_i} \tag{10.44}$$

A plot of $|\zeta_1|$ as a function of θ_i for $\kappa = 3$ is given in Fig. 10.5. It is seen that $|\zeta_1|$ continually increases with increasing values of the angle θ_i.

Minimum reflection occurs at normal incidence, and the value of $|\zeta_1|$ for this condition depends on κ.

Reflection from a Dielectric Interface, Parallel Polarization

The solution for the case of a parallel-polarized incident wave is similar but with the role of electric and magnetic fields interchanged. The details are left as a problem. The reflection and transmission coefficients ζ_2, T_2 may be readily found from the equivalent transmission-line circuit

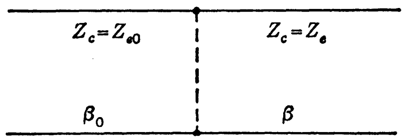

FIG. 10.6. Equivalent transmission-line circuit of dielectric interface, parallel polarization.

illustrated in Fig. 10.6. From (10.23) the wave impedances in the free-space and dielectric regions are

$$Z_{e0} = \frac{E_x}{H_y} = \frac{\beta_0}{k_0} Z_0 = Z_0 \cos \theta_i \qquad (10.45a)$$

$$Z_e = \frac{\beta}{k} Z = \frac{Z_0}{\eta} \cos \theta_r \qquad (10.45b)$$

Snell's law again holds; so $\kappa Z_e = Z_0(\kappa - \sin^2 \theta_i)^{1/2}$. Thus the reflection and transmission coefficients for the *tangential* electric field are

$$\zeta_2 = \frac{Z_e - Z_{e0}}{Z_e + Z_{e0}} = \frac{(\kappa - \sin^2 \theta_i)^{1/2} - \kappa \cos \theta_i}{(\kappa - \sin^2 \theta_i)^{1/2} + \kappa \cos \theta_i} \qquad (10.46a)$$

$$T_2 = 1 + \zeta_2 = \frac{2(\kappa - \sin^2 \theta_i)^{1/2}}{(\kappa - \sin^2 \theta_i)^{1/2} + \kappa \cos \theta_i} \qquad (10.46b)$$

An interesting property of ζ_2 is that for some particular value of θ_i it vanishes. From (10.46a), $\zeta_2 = 0$ when

$$\kappa - \sin^2 \theta_i = \kappa^2 \cos^2 \theta_i = \kappa^2 - \kappa^2 \sin^2 \theta_i$$

Denoting the solution for θ_i by θ_B, we have

$$\sin \theta_B = \left(\frac{\kappa}{\kappa + 1}\right)^{1/2} \qquad (10.47)$$

This particular angle is called the Brewster angle. For a parallel-polarized wave incident at the angle θ_B, no reflection takes place and all the incident power is transmitted into the dielectric. A similar phenomenon does not occur for a perpendicular-polarized wave unless the dielectric medium has a permeability greater than unity. A plot of $|\zeta_2|$ is given in Fig. 10.5 for $\kappa = 3$. Up to an angle $\theta_i = \theta_B$ the reflection coefficient continually decreases. Beyond the angle θ_B the reflection coefficient increases rapidly up to a value of unity at grazing incidence when $\theta_i = 90°$.

The concept of a wave impedance is of great practical importance since it provides a formal analogy between wave problems and transmission-line problems. In transmission-line circuit problems the total line voltage and current are made continuous at the load termination. Similar boundary conditions are imposed on the tangential electric and magnetic fields at a discontinuity interface. For this reason the same formulas for reflection and transmission coefficients are applicable to wave problems. It is important to note, however, that the analogy holds only if the axis of propagation is chosen normal to the discontinuity interface. For obliquely incident TEM waves the use of a transmission-line equivalent circuit leads naturally to a classification of the incident wave as a TE or TM wave.

10.3. Reflection from a Conducting Plane

From an analysis of the problem of reflection of a plane wave from a conducting plane, the behavior of the electromagnetic field at the surface of a conductor may be deduced. We shall be able to show that the total current per unit width flowing in the conducting plane is essentially independent of the conductivity. As the conductivity is made to approach infinity, the current is squeezed into a narrower and narrower layer, until in the limit a true surface current is obtained. The conductor will be shown to be characterized as a boundary surface exhibiting a surface impedance

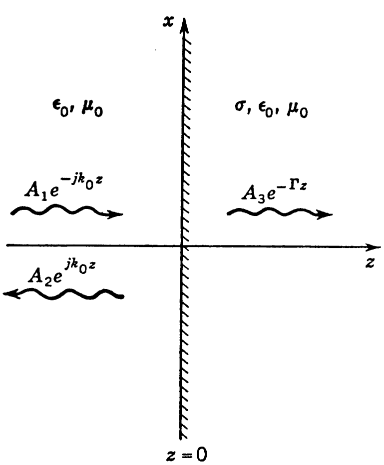

FIG. 10.7. Plane wave incident on a conducting plane.

$$Z_m = R_m + jX_m$$

where $R_m = X_m = (\sigma\delta)^{-1}$ and δ is the skin depth. The power loss in the conductor is then readily shown to be given by

$$P_l = \tfrac{1}{2}R_m\mathbf{J}_s \cdot \mathbf{J}_s^* \qquad \text{per unit area}$$

where $\mathbf{J}_s$ is the surface current density. Since $\mathbf{J}_s$ is also equal to $\mathbf{n} \times \mathbf{H}$, we have a very convenient method of evaluating the power loss in a conductor from a knowledge of the tangential magnetic field at the surface.

Let a plane TEM wave be incident on a conducting interface located at $z = 0$; that is, the half space $z \geq 0$ is filled with a conducting medium, as in Fig. 10.7. The incident field is chosen as follows:

$$\mathbf{E}_i = A_1\mathbf{a}_x e^{-jk_0z} \tag{10.48a}$$
$$\mathbf{H}_i = A_1 Y_0 \mathbf{a}_y e^{-jk_0z} \tag{10.48b}$$

At the interface there will be a reflected wave

$$\mathbf{E}_r = A_2\mathbf{a}_x e^{jk_0z} \tag{10.49a}$$
$$\mathbf{H}_r = -A_2 Y_0 \mathbf{a}_y e^{jk_0z} \tag{10.49b}$$

and a transmitted wave of the form

$$\mathbf{E}_t = A_3\mathbf{a}_x e^{-\Gamma z} \tag{10.50a}$$
$$\mathbf{H}_t = A_3 Y_m \mathbf{a}_y e^{-\Gamma z} \tag{10.50b}$$

where Γ and Y_m are yet to be determined. In a conducting medium the curl equation for $\mathbf{H}$ is $\nabla \times \mathbf{H} = j\omega\epsilon_0\mathbf{E} + \sigma\mathbf{E} \approx \sigma\mathbf{E}$, since the conduction current is much greater than the displacement current. If we rewrite this equation as $\nabla \times \mathbf{H} = j\omega(\sigma/j\omega)\mathbf{E}$, we see that $\sigma/j\omega$ may be considered as the permittivity in Maxwell's equations. Using this analogy, we may construct the solution for the plane wave in the conductor from the solution for the incident wave. Thus by analogy we have

$$\Gamma = j\omega\left(\frac{\mu_0\sigma}{j\omega}\right)^{1/2} = (j\omega\mu_0\sigma)^{1/2}$$
$$Z_m = Y_m^{-1} = \left(\frac{j\omega\mu_0}{\sigma}\right)^{1/2} = \frac{\Gamma}{\sigma}$$

since $\qquad jk_0 = j\omega(\mu_0\epsilon_0)^{1/2} \qquad$ and $\qquad Z_0 = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}$

The square root of j equals $(1 + j)/\sqrt{2}$, and hence

$$\Gamma = \frac{1+j}{\delta} \tag{10.51}$$
$$Z_m = \frac{1+j}{\sigma\delta} = R_m + jX_m \tag{10.52}$$

where δ is the skin depth and is given by

$$\delta = \left(\frac{2}{\omega\mu_0\sigma}\right)^{1/2} \tag{10.53}$$

It is seen that the conductor exhibits an impedance with equal resistive and inductive parts. Furthermore, the resistive part is just the d-c resistance of a sheet of metal 1 meter square and of thickness δ. (Actually, the resistance is independent of the area of the square plate.) Thus with reference to Fig. 10.8, the d-c resistance between the two faces 1 and 2 is given by

$$R_m = \frac{L}{L\delta\sigma} = \frac{1}{\delta\sigma} \qquad \text{ohms/square}$$

Since the resistance is independent of the linear dimension L, it is called a surface resistance and is measured in ohms per square. The impedance Z_m is called the intrinsic impedance of the conductor. For the present case of normal incidence the ratio of the tangential electric and magnetic fields at the interface, called the surface impedance, equals the intrinsic impedance.

At the interface the tangential fields must be continuous; hence

$$A_1 + A_2 = A_3$$
$$(A_1 - A_2)Y_0 = A_3 Y_m$$

If we let $A_2 = \zeta A_1$, $A_3 = TA_1$, where ζ and T are the reflection and transmission coefficients, we have

$$1 + \zeta = T \tag{10.54a}$$
$$1 - \zeta = \frac{Z_0}{Z_m} T \tag{10.54b}$$

Solving for ζ and T gives

$$\zeta = \frac{Z_m - Z_0}{Z_m + Z_0} \tag{10.55a}$$
$$T = \frac{2Z_m}{Z_m + Z_0} \tag{10.55b}$$

For any reasonably good conductor, Z_m is very small compared with Z_0. For example, for copper ($\sigma = 5.8 \times 10^7$ mhos per meter) at a frequency of 1,000 megacycles, $\delta = 2 \times 10^{-6}$ meter, $R_m = 0.0086$ ohm, while $Z_0 = 377$ ohms. For all practical purposes the field in front of the conductor ($z < 0$) is the same as would exist if σ were infinite, since ζ differs from -1 by a negligible amount. For the same reason the amount of power transmitted into the conductor is very small; that is, T is very small. We cannot, however, neglect the power transmitted into conducting surfaces in the case of transmission lines and waveguides since any loss that is present is important in determining the attenuation constant of a wave. While the attenuation due to conductor losses could be expected to be negligible for short laboratory connections, it would enter significantly in long transmission lines. A method for calculating the attenuation constant will be developed in the following discussion.

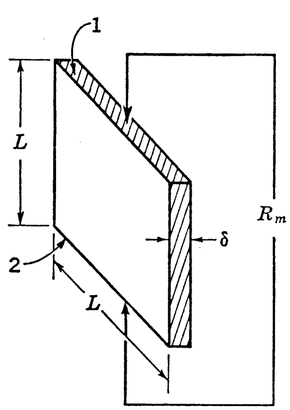

FIG. 10.8. Equivalent low-frequency resistance problem.

The current flowing in the conductor is $J_x = \sigma E_x = \sigma T A_1 e^{-\Gamma z}$. The total current per unit width of conductor is

$$J_s = \sigma T A_1 \int_0^\infty e^{-\Gamma z}\, dz = \frac{\sigma T A_1}{\Gamma} \qquad \text{amp/m}$$

Replacing Γ by $(1 + j)/\delta$ and noting that at the conductor surface $H_y = T A_1 Y_m$, we get

$$J_s = \frac{\delta \sigma Z_m}{1 + j} H_y = H_y \tag{10.56}$$

since $Z_m = (1 + j)/\sigma\delta$. This result also can be obtained by applying $\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$ (displacement current being neglected) to a rectangular contour C whose long dimension runs from $z = 0$ to $z = +\infty$ and whose short dimension is a unit length parallel to the x axis. The details are left to the student.

If we now let σ tend to infinity, we find that $\delta \to 0$, $\zeta \to -1$, and $H_y \to 2H_i$. The total current J_s does not vanish since, from (10.56), it clearly approaches the value $2H_i$. However, it is squeezed into a narrower and narrower layer and in the limit becomes a true surface current measured in amperes per meter.

The power loss per unit area in the xy plane may be evaluated from the complex Poynting vector at the surface. We have

$$P_l = \tfrac{1}{2} \operatorname{Re} (E_x H_y^*) = \tfrac{1}{2} A_1 A_1^* T T^* \operatorname{Re} Y_m^* = \tfrac{1}{4} |T A_1|^2 \sigma\delta \tag{10.57}$$

We may also evaluate P_l by means of the following volume integral whose integrand expresses the joule heating loss per unit volume. We have

$$P_l = \tfrac{1}{2} \int_0^1 \int_0^1 \int_0^\infty \sigma E_x E_x^*\, dx\, dy\, dz = \frac{\sigma}{2} |A_1 T|^2 \int_0^\infty e^{-2z/\delta}\, dz$$
$$= \tfrac{1}{4} |A_1 T|^2 \sigma\delta \tag{10.58}$$

The two methods, of course, give the same results. This result can be put into another useful form if we replace $|A_1 T|$ by $|J_s \Gamma/\sigma|$. We then obtain

$$P_l = \tfrac{1}{4} |J_s \Gamma|^2 \frac{\delta}{\sigma} = \tfrac{1}{2} |J_s|^2 R_m$$

In practice, the following approximate method is generally used to evaluate the power loss per unit area. The tangential magnetic field is first found using the assumption that σ is infinite. The surface current density is then determined from the boundary condition

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}$$

where **n** is the outward normal to the conductor surface. Next it is recalled that the surface of a conductor exhibits a surface impedance Z_m, and hence the power loss per unit area is

$$P_l = \tfrac{1}{2}|H_t|^2 R_m \tag{10.59}$$

where H_t is the tangential magnetic field at the surface, evaluated for infinite σ. For the present problem $H_t = H_y = 2A_1Y_0$, and hence

$$P_l = 2|A_1|^2 Y_0^2 R_m$$

To compare this result with (10.57), note that

$$T \approx \frac{2Z_m}{Z_0} = \frac{2(1+j)}{\sigma\delta Z_0}$$

and hence (10.57) gives approximately

$$\tfrac{1}{4}|A_1|^2 \frac{4}{(\sigma\delta Z_0)^2}|1+j|^2\sigma\delta = 2|A_1|^2 Y_0^2 R_m$$

which is the result (10.59).

The approximation involved in (10.59) is that we take for H_t its value when σ is infinite. This is, however, a very good approximation, since $|Z_m| \ll Z_0$. Therefore in practice we are justified in determining the surface current density $\mathbf{J}_s$ by using the boundary condition $\mathbf{n} \times \mathbf{H} = \mathbf{J}_s$ and computing **H** as though the conductivity were infinite.

In the case of infinite conductivity the tangential electric field at the conductor surface is zero. For finite conductivity there has to be a finite value of tangential electric field in order to obtain a component of the Poynting vector directed into the conductor. The tangential electric field at the surface is given by $\mathbf{E}_t = \mathbf{J}/\sigma$, where **J** is the current density at the surface. The student may verify that $\mathbf{E}_t$ is also given by

$$\mathbf{E}_t = \mathbf{J}_s Z_m \tag{10.60}$$

The above results were derived for a plane wave incident along the surface normal. For an obliquely incident wave the previous work can be utilized provided that (10.48) to (10.50) are interpreted as applying to the transverse fields and that the intrinsic admittances are replaced by the appropriate wave admittances. The wave impedance in the free-space region is, from (10.43*b*) and (10.45*a*),

$$Z_{h0} = Z_0 \sec\theta_i \qquad \text{perpendicular polarization}$$
$$Z_{e0} = Z_0 \cos\theta_i \qquad \text{parallel polarization}$$

The corresponding wave impedances in the conductor are

$$Z_h = Z_m \sec \theta_r$$
$$Z_e = Z_m \cos \theta_r$$

Since Snell's law must hold, we have

$$jk_0 \sin \theta_i = \Gamma \sin \theta_r = \frac{1+j}{\delta} \sin \theta_r$$

Now δ is very small compared with k_0^{-1}, and therefore $\sin \theta_r$ is very small and also complex. If $\sin \theta_r$ is very small, it follows that

$$\cos \theta_r = (1 - \sin^2 \theta_r)^{1/2}$$

is very nearly equal to unity. This shows that even for oblique incidence the conductor may be assumed to exhibit a surface impedance Z_m because Z_e and Z_h differ from Z_m by a negligible amount. Thus the procedure outlined earlier for the evaluation of the power loss in a conductor is valid for oblique incidence as well. The method breaks down only when the conductor is curved and has a radius of curvature not much greater than the skin depth. Conductors with such small radii of curvature are rarely encountered, except perhaps as small-diameter wires at the lower frequencies. In the majority of cases the use of the surface impedance Z_m and (10.59) is entirely valid for computing power loss at arbitrary conducting surfaces with arbitrary electromagnetic fields.

10.4. Transmission Lines

A transmission line consists of two or more uniform and parallel conductors. It is used to transmit high-frequency electromagnetic energy from a given source (generator) to a load, e.g., antenna. The cross sections of several typical transmission lines are illustrated in Fig. 10.9.

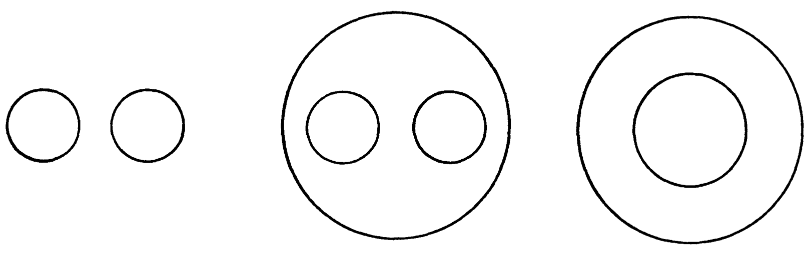

FIG. 10.9. Cross sections of typical transmission lines. (a) Two-wire line; (b) shielded two-wire line; (c) coaxial line.

The principal type of wave that may propagate along an ideal ($\sigma = \infty$) transmission line is a TEM wave. Thus the field surrounding the conductors of a transmission line is governed by the equations for TEM waves given in Sec. 10.1. With reference to Fig. 10.10, which illustrates a general two-conductor line, we can write

$$\mathbf{E} = \mathbf{E}_t = \mathbf{e}e^{-jkz} \tag{10.61a}$$
$$\mathbf{H} = \mathbf{H}_t = \mathbf{h}e^{-jkz} = Y\mathbf{a}_z \times \mathbf{e}e^{-jkz} \tag{10.61b}$$

where it is assumed that the medium surrounding the conductors has electrical parameters ϵ and μ_0. The field $\mathbf{e}$ is equal to the negative transverse

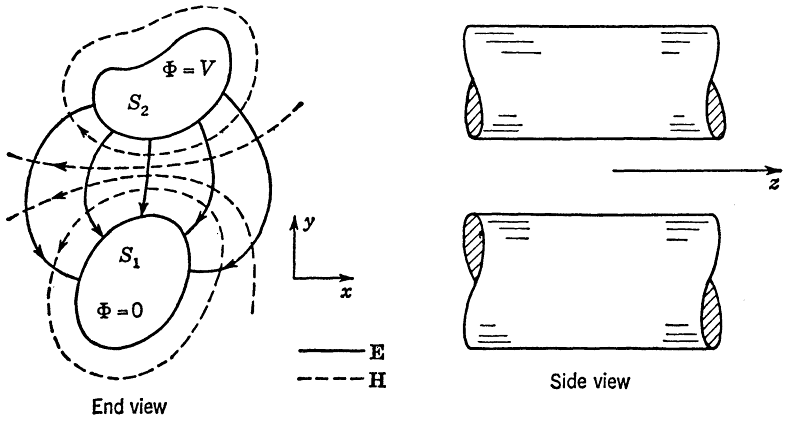

FIG. 10.10. A general two-conductor transmission line.

gradient of a scalar potential Φ, and Φ is a solution of the two-dimensional Laplace equation; that is,

$$\mathbf{e} = -\nabla_t\Phi \tag{10.62a}$$
$$\nabla_t^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{10.62b}$$

A nontrivial solution for Φ exists only if there is a potential difference V between the conductors. Thus associated with the electric field (10.61a) there is a unique voltage wave Ve^{-jkz}. The line integral

$$-\int_{S_1}^{S_2} \mathbf{e} \cdot d\mathbf{l} = \int_{S_1}^{S_2} d\Phi = V$$

is independent of the path by virtue of (10.62a), where the integration is taken from an arbitrary point on S_1 to an arbitrary point on S_2.

The boundary conditions on Φ, namely, that it equal a constant, say zero, on one conductor and V on the other, are independent of frequency. Since Φ must be a solution of Laplace's equation as well, then the uniqueness theorem requires that Φ be independent of frequency. In other words, the transverse field distribution of a transmission line is independent of frequency and, as a matter of fact, is precisely the distribution under static conditions. This is, of course, a general property of TEM waves, as was noted earlier.

From (10.61*b*) it is seen that the magnetic lines of flux are perpendicular to the electric lines of flux and hence must coincide with the constant-potential contours in the xy plane. The line integral of **h** around the conductor S_2 in Fig. 10.10 gives

$$\oint_{S_2} \mathbf{h} \cdot d\mathbf{l} = I \tag{10.63}$$

where I is the total z-directed current on S_2. The result follows from Ampère's circuital law since there is no z-directed displacement current but only a z-directed conduction current density $\mathbf{J}_s = \mathbf{n} \times \mathbf{h}$ on each conductor. On the conductor S_1 the total current flowing is $-I$. Thus associated with the magnetic field (10.61*b*) there is a unique current wave Ie^{-jkz}.

From the fact that the potential plot is independent of frequency, the direction and relative magnitude of the magnetic lines of flux must then be the same at all frequencies. Furthermore, since (10.63) holds at all frequencies, then if I remains the same, the absolute magnitude of **h** does not depend on the frequency as well. Thus the field distribution of **h** is nothing more than that under time-stationary conditions.

In view of the unique relationship between **e** and V on one hand and **h** and I on the other, it follows that the properties of a transmission line may be described in terms of the fields existing around the conductors or in terms of the associated voltage and current waves. In a field description the parameters of interest are the propagation constant k and the intrinsic impedance Z of the medium surrounding the conductors. In a voltage-current description the ratio V/I defines the characteristic impedance Z_c of the line. The characteristic impedance is the counterpart of the intrinsic impedance Z and in actual fact differs from Z by a factor which is a function of the line geometry only. The circuit parameters of a transmission line are discussed in the next section along with their method of derivation and will shed further light on the interrelationship between the field and circuit descriptions.

10.5. Transmission-line Parameters

In a circuit analysis of a transmission line it is postulated that the line can be characterized by the following distributed circuit parameters:

R = series resistance per meter
L = series inductance per meter
G = shunt conductance per meter
C = shunt capacitance per meter

By considering a differential section dz of line, as in Fig. 10.11, an equivalent circuit involving the above-mentioned parameters may be constructed as shown. The following circuit equations then arise from conventional circuit theory:

$$\mathcal{V}(z) - \left[\mathcal{V}(z) + \frac{d\mathcal{V}}{dz}\,dz\right] = -\frac{d\mathcal{V}}{dz}\,dz = (j\omega L + R)\,dz\,\mathcal{I} \tag{10.64}$$

$$\mathcal{I}(z) - \left[\mathcal{I}(z) + \frac{d\mathcal{I}}{dz}\,dz\right] = -\frac{d\mathcal{I}}{dz}\,dz = (j\omega C + G)\,dz\,\mathcal{V} \tag{10.65}$$

Equation (10.64) states that the decrease in voltage along a length dz of the line is equal to the voltage drop in the series impedance $(j\omega L + R)\,dz$, while (10.65) gives the decrease in current because of the shunt current

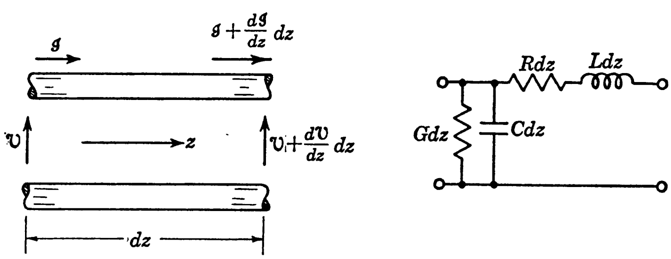

FIG. 10.11. A differential section of transmission line and its equivalent circuit.

flowing through the shunt admittance $(j\omega C + G)\,dz$. Differentiating (10.64) with respect to z and substituting into (10.65) give

$$\frac{d^2\mathcal{V}}{dz^2} - (j\omega L + R)(j\omega C + G)\mathcal{V} = 0 \tag{10.66a}$$

Similarly, we may obtain

$$\frac{d^2\mathcal{I}}{dz^2} - (j\omega L + R)(j\omega C + G)\mathcal{I} = 0 \tag{10.66b}$$

A solution to (10.66a) is

$$\mathcal{V} = Ve^{-\Gamma z} \tag{10.67}$$

where V is an amplitude constant and

$$\Gamma = [(j\omega L + R)(j\omega C + G)]^{1/2} \tag{10.68}$$

For any good line $\omega L \gg R$ and $\omega C \gg G$, so that (10.68) gives

$$\Gamma = j\beta + \alpha \approx j\omega\sqrt{LC} + \tfrac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) \tag{10.69}$$

From (10.64) the solution for $\mathcal{I}$ is

$$\mathcal{I} = \frac{\Gamma}{j\omega L + R} V e^{-\Gamma z} = V \left(\frac{j\omega C + G}{j\omega L + R}\right)^{1/2} e^{-\Gamma z} \approx V \sqrt{\frac{C}{L}} e^{-\Gamma z} \tag{10.70}$$

for a line with small losses. The ratio $\mathcal{V}/\mathcal{I}$ defines the characteristic impedance Z_c; thus

$$Z_c = \sqrt{\frac{L}{C}} \tag{10.71}$$

Lossless Line

In this section we shall establish the validity of the above approach as well as obtaining methods for the evaluation of L, C, R, and G. We consider a lossless line first with ϵ real and the conductivity σ of the conductors infinite. With reference to Fig. 10.10 and using the boundary condition $\epsilon \mathbf{n} \cdot \mathbf{e} = \rho_s$, where ρ_s is the surface charge density per meter on the conductor S_2, we have

$$Q = \epsilon \oint_{S_2} \mathbf{n} \cdot \mathbf{e} \, dl \tag{10.72}$$

for the total charge per meter on S_2. Since the potential of S_2 is V, the capacitance per meter between S_2 and S_1 may be defined, as under static conditions, to be

$$C = \frac{Q}{V} \tag{10.73}$$

At the surface S_2 the electric field has a normal component only while the magnetic field has only a tangential component. Furthermore, from (10.61*b*) we find that $|\mathbf{h}| = Y|\mathbf{e}| = Y\mathbf{n} \cdot \mathbf{e}$ on the surface of S_2. Hence the current flowing on S_2 is given by

$$I = \oint_{S_2} \mathbf{h} \cdot d\mathbf{l} = \oint_{S_2} Y\mathbf{n} \cdot \mathbf{e} \, dl = \frac{YQ}{\epsilon} \tag{10.74}$$

Thus the characteristic impedance of the line is given by

$$Z_c = \frac{V}{I} = \frac{V}{Q} \epsilon Z = \frac{\epsilon Z}{C} \tag{10.75}$$

Since C/ϵ is a function of the line geometry only, Z_c differs from Z only by a factor which is a function of the line geometry.

An alternative expression for Z_c may be derived from an energy definition of C and L, such as was introduced in electrostatics and magnetostatics. Thus the energy stored in the electric field per unit length of line is

$$W_e = \tfrac{1}{4} \epsilon \iint_{x,y} \mathbf{e} \cdot \mathbf{e} \, dx \, dy$$

and we define C by the following:

$$W_e = \tfrac{1}{4}CV^2 \tag{10.76}$$

The energy stored in the magnetic field per unit length of line is

$$\begin{aligned} W_m &= \tfrac{1}{4}\mu_0 \iint_{x,y} \mathbf{h} \cdot \mathbf{h} \, dx \, dy \\ &= \tfrac{1}{4}\mu_0 Y^2 \iint_{x,y} (\mathbf{a}_z \times \mathbf{e}) \cdot (\mathbf{a}_z \times \mathbf{e}) \, dx \, dy \end{aligned}$$

But

$$\begin{aligned} (\mathbf{a}_z \times \mathbf{e}) \cdot (\mathbf{a}_z \times \mathbf{e}) &= \mathbf{a}_z \cdot [\mathbf{e} \times (\mathbf{a}_z \times \mathbf{e})] \\ &= \mathbf{a}_z \cdot [(\mathbf{e} \cdot \mathbf{e})\mathbf{a}_z - (\mathbf{e} \cdot \mathbf{a}_z)\mathbf{e}] = \mathbf{e} \cdot \mathbf{e} \end{aligned}$$

because $\mathbf{e} \cdot \mathbf{a}_z = 0$. Hence, since $Y^2 = \epsilon/\mu_0$,

$$W_m = \tfrac{1}{4}\epsilon \iint_{x,y} \mathbf{e} \cdot \mathbf{e} \, dx \, dy = W_e \tag{10.77}$$

The inductance L per meter may be defined from the energy relation $W_m = \tfrac{1}{4}LI^2$, and we see that

$$LI^2 = CV^2$$

or

$$\frac{V}{I} = \sqrt{\frac{L}{C}} \tag{10.78}$$

Consequently, we must have

$$\frac{\epsilon Z}{C} = \sqrt{\frac{L}{C}} = Z_c \tag{10.79}$$

by equating (10.75) and (10.78). From (10.79) we obtain the relation $\sqrt{LC} = \epsilon Z = \sqrt{\epsilon\mu_0}$. Thus for a line with no losses, $\alpha = 0$ and $\beta = \omega\sqrt{LC} = \omega\sqrt{\epsilon\mu_0} = k$. For the ideal line we can therefore conclude that the circuit approach is consistent with the rigorous analysis based on Maxwell's equations if L and C are defined as above. Since these definitions are in terms of time-stationary energy formulas and since we know that the electric and magnetic field distributions are precisely those under static conditions, then the static values of L and C are appropriate and correct at any frequency.

Line with Lossy Dielectric

Let us now consider the case where the dielectric surrounding the conductors is lossy. The dielectric loss may be due to a finite conductivity or polarization damping forces, or both. In all cases the effect of losses may be accounted for by a complex permittivity $\epsilon = \epsilon' - j\epsilon''$. The imaginary part ϵ'' is directly responsible for the loss. Substitution of a complex ϵ into the equations for TEM waves does not modify the form of these equations, so that a TEM wave solution is still possible. With ϵ complex, a shunt current of density $\mathbf{J} = \omega\epsilon''\mathbf{e}$ will flow from conductor S_2 to S_1 (note in the curl equation for $\mathbf{H}$ that $\omega\epsilon''$ is the equivalent conductivity). The total shunt conduction current I_s per meter is given by

$$I_s = \omega\epsilon'' \oint_{S_2} \mathbf{n} \cdot \mathbf{e}\, dl = \frac{\omega\epsilon'' Q}{\epsilon'} \qquad (10.80)$$

since this is the total current flowing away from S_2. A shunt conductance G may be defined through the relation

$$I_s = VG \qquad (10.81)$$

From (10.80) and the relation $C = Q/V$, we now have

$$G = \frac{I_s}{V} = \frac{\omega\epsilon''}{\epsilon'}\frac{Q}{V} = \frac{\omega\epsilon''}{\epsilon'} C \qquad (10.82)$$

Thus the shunt conductance, since it is directly related to the capacitance C, depends on the geometry in precisely the same way as C, a result already established in Chap. 5. An alternative definition of G is in terms of the power loss in the dielectric per unit length of line. This definition is $P_{ld} = \frac{1}{2}V^2G = \frac{1}{2}\int_{xy\text{ plane}} \omega\epsilon''|\mathbf{e}|^2\, dS$ and is readily shown to be equivalent to that in (10.82) by a technique similar to that used in Sec. 5.8.

From the circuit equation for Γ, we have

$$\Gamma = [j\omega L(j\omega C + G)]^{1/2} \qquad (10.83a)$$

while from the field equations

$$k = \omega(\mu_0\epsilon)^{1/2} = \omega[\mu_0(\epsilon' - j\epsilon'')]^{1/2} \qquad (10.83b)$$

Since $LC = \mu_0\epsilon'$ and $G/\omega C = \epsilon''/\epsilon'$, (10.83a) becomes

$$\Gamma = j\omega(LC)^{1/2}\left(1 - \frac{jG}{\omega C}\right)^{1/2} = j\omega(\mu_0\epsilon')^{1/2}\left(1 - \frac{j\epsilon''}{\epsilon'}\right)^{1/2} = jk \qquad (10.84)$$

Therefore again we find that the circuit analysis and field analysis are equivalent.

The General Lossy Line

When the conductors have a finite conductivity σ, a TEM wave solution is no longer possible. With a z-directed current along the conductors, there must be a z component of electric field $\mathbf{e}_z = Z_m\mathbf{J}$. The TEM wave is perturbed into a wave having at least a z component of electric field. However, for any good conductor, Z_m is so small that the solution is still essentially a TEM wave. We may find the power loss in the conductors by using the approximate technique outlined in Sec. 10.3.

The surface current density in the case where $\sigma = \infty$ is given by

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{h}$$

The power loss in the conductors per meter for finite conductivity is thus approximated by

$$P_{lc} = \tfrac{1}{2} R_m \oint_{S_1+S_2} |\mathbf{n} \times \mathbf{h}|^2 \, dl$$

and this may be written as

$$P_{lc} = \tfrac{1}{2} R_m \oint_{S_1+S_2} |\mathbf{h}|^2 \, dl \tag{10.85}$$

since at the conductor surface $\mathbf{h}$ has a tangential component only. The series resistance R per meter may be defined by the relation

$$\tfrac{1}{2} R I^2 = P_{lc}$$

and hence

$$R = \frac{R_m}{I^2} \oint_{S_1+S_2} |\mathbf{h}|^2 \, dl = R_m \frac{\oint_{S_1+S_2} |\mathbf{h}|^2 \, dl}{\left(\oint_{S_2} |\mathbf{h}| \, dl\right)^2} \tag{10.86}$$

To compute the attenuation constant α, we note first that the power flow along the line will be of the form

$$P = P_0 e^{-2\alpha z}$$

where α is the attenuation constant for the electric and magnetic field waves and P_0 is the power flow at $z = 0$. The rate of decrease of P with z must equal the power loss per unit length arising from the dielectric and the conductors. Expressed mathematically,

$$-\frac{dP}{dz} = 2\alpha P = P_{lc} + P_{ld} \tag{10.87}$$

Since we define $\tfrac{1}{2} I^2 R = P_{lc}$ and $\tfrac{1}{2} V^2 G = P_{ld}$, then (10.87) can be expressed in terms of these line constants as well. This gives us

$$2\alpha P = \tfrac{1}{2} V^2 G + \tfrac{1}{2} I^2 R$$

For a low-loss line $P = \tfrac{1}{2} VI = \tfrac{1}{2} V^2 Y_c = \tfrac{1}{2} I^2 Z_c$, where

$$Z_c = Y_c^{-1} = \left(\frac{L}{C}\right)^{\frac{1}{2}}$$

Therefore (10.87) gives

$$\alpha = \tfrac{1}{2} \left[R \left(\frac{C}{L}\right)^{\frac{1}{2}} + G \left(\frac{L}{C}\right)^{\frac{1}{2}} \right] \tag{10.88}$$

which is the same result as given by (10.69) for the voltage and current wave of a low-loss line, provided R and G are defined from the above energy relations.

As long as the losses are small, we can again justify the circuit approach to transmission lines by an analysis based on Maxwell's equations and the approximate method of evaluating power loss in a conductor. In the preceding analysis we neglected the small increase in the inductance of the line when σ is finite. This increase arises from a penetration of the magnetic field into the conductor. Since the effective depth of penetration is the skin depth, the internal inductance is very small compared with the external inductance and may usually be neglected.

Terminated Transmission Line

To complete the picture of the transmission line we shall consider a lossless line terminated in a load impedance Z_L at $z = 0$, as in Fig. 10.12.

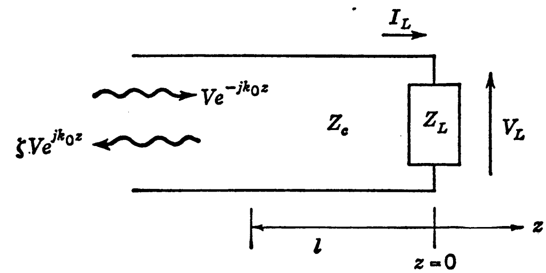

FIG. 10.12. A transmission line terminated in a load impedance.

Let a voltage wave Ve^{-jk_0z} be incident from the left. In general, a reflected wave ΓVe^{jk_0z} will be produced by the load, where Γ is the reflection coefficient. The current waves associated with the incident and reflected voltage waves are $Y_cVe^{-jk_0z}$ and $-\Gamma Y_cVe^{jk_0z}$, where Y_c is the characteristic admittance of the line. At $z = 0$ the total voltage across the load impedance Z_L is V_L, and the current flowing through the load is I_L, where

$$V_L = (1 + \Gamma)V$$
$$I_L = Y_c(1 - \Gamma)V$$

Since $V_L/I_L = Z_L$, we obtain

$$\frac{1+\Gamma}{1-\Gamma} = \frac{Z_L}{Z_c} \tag{10.89}$$

Solving for Γ gives

$$\Gamma = \frac{Z_L - Z_c}{Z_L + Z_c} \tag{10.90}$$

In general, Γ is complex since Z_L may be complex. Only when $Z_L = Z_c$ will the reflection coefficient be zero. The maximum voltage amplitude on the line will be $(1 + |\Gamma|)V$, while the minimum voltage amplitude will be $(1 - |\Gamma|)V$. The ratio of the two is called the voltage standing-wave be derivable as the transverse gradient of a scalar potential Φ that satisfies Laplace's equation. However, since Φ must also be constant on the trace of the conductors in the cross-sectional plane, the theory of harmonic functions demands that Φ be a constant when the conductor boundary is simply connected, as would arise with hollow cylindrical waveguides. In this event, a null **E** and **H** field results. The two basic types of waves that may propagate in a waveguide are the TE and TM waves (or modes).

It turns out that for a given wave or mode solution there exists a lower frequency, called the cutoff frequency, below which the mode will not propagate. Above the cutoff frequency the mode propagates and both the phase velocity v_p and the guide wavelength are greater than the corresponding quantities for plane waves in free space. If λ_c is the cutoff wavelength, corresponding to the cutoff frequency f_c, the guide wavelength λ_g is found to be given by

$$\lambda_g = \frac{\lambda_0}{[1 - (\lambda_0/\lambda_c)^2]^{1/2}} \tag{10.95}$$

and the phase velocity v_p is given by

$$v_p = \frac{\lambda_g}{\lambda_0} c \tag{10.96}$$

where λ_0 is the free-space wavelength and c is the velocity of light in free space. The velocity of energy propagation and the velocity with which a signal propagates (group velocity) are equal and given by

$$v_g = \frac{\lambda_0}{\lambda_g} c$$

or

$$v_g v_p = c^2 \tag{10.97}$$

Thus the group velocity is always less than c, as it must be since, according to the theory of relativity, energy or a signal cannot be propagated with a velocity exceeding c.

Equations (10.95) to (10.97) hold for any empty cylindrical waveguide. The only difference from one guide to the next is the specific value of the cutoff wavelength λ_c, which depends on the geometry of the guide cross section. We proceed now to a detailed derivation of the above results for both the rectangular and circular guides. For a more general treatment the reader is referred to one of the specialized texts in this field.

TE Waves

Figure 10.13 illustrates a rectangular waveguide of cross-sectional dimensions a and b. The conductivity of the walls will be assumed to be infinite, and the interior of the guide, free space. For the TE waves the equations derived in Sec. 10.1 are applicable. The fields are given by

$$\mathbf{H} = [\mathbf{h}(x,y) + \mathbf{h}_z(x,y)]e^{-j\beta z} \tag{10.98a}$$
$$\mathbf{E} = \mathbf{e}(x,y)e^{-j\beta z} \tag{10.98b}$$
$$\mathbf{h} = -\frac{j\beta}{k_c^2}\nabla_t h_z \tag{10.98c}$$
$$\mathbf{e} = -Z_h \mathbf{a}_z \times \mathbf{h} \tag{10.98d}$$

and h_z is a solution of

$$\nabla_t^2 h_z + k_c^2 h_z = 0 \tag{10.98e}$$

where $k_c^2 = k_0^2 - \beta^2$ and the wave impedance $Z_h = k_0 Z_0/\beta$. The first step is to find a solution for $h_z(x,y)$.

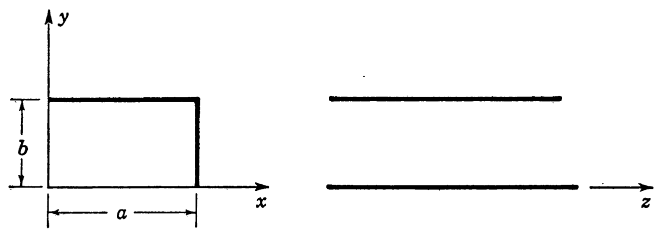

FIG. 10.13. The rectangular waveguide.

The standard method of solving a partial differential equation such as (10.98e) is the method of separation of variables. We assume that $h_z(x,y)$ can be expressed as a product of a function of x alone and a function of y alone; that is, $h_z(x,y) = f(x)g(y)$. Substitution of this type of solution into (10.98e) gives

$$\frac{\partial^2 h_z}{\partial x^2} + \frac{\partial^2 h_z}{\partial y^2} + k_c^2 h_z = g\frac{d^2 f}{dx^2} + f\frac{d^2 g}{dy^2} + k_c^2 fg = 0$$

Dividing by fg gives

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + k_c^2 = 0 \tag{10.99}$$

If we vary x, only the term $\frac{1}{f}\frac{d^2 f}{dx^2}$ in (10.99) can vary since the other terms are not functions of x. However, since (10.99) must hold for all values of x and y, it is necessary that the term involving f and also the term involving g be constant. Hence we can write

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -k_x^2$$

or

$$\frac{d^2 f}{dx^2} + k_x^2 f = 0 \tag{10.100}$$

and

$$\frac{d^2 g}{dy^2} + k_y^2 g = 0 \tag{10.101}$$

where k_x^2 and k_y^2 are called separation constants. In order for (10.99) to hold, we must have

$$-k_x^2 - k_y^2 + k_c^2 = 0$$

or

$$k_c = (k_x^2 + k_y^2)^{1/2} \tag{10.102}$$

The separation-of-variables technique reduces a partial differential equation into two or more separate ordinary differential equations.

The solutions to (10.100) and (10.101) are (apart from an additional arbitrary amplitude constant)

$$f(x) = \cos k_x x + A \sin k_x x$$
$$g(y) = \cos k_y y + B \sin k_y y$$

and hence

$$h_z(x,y) = (\cos k_x x + A \sin k_x x)(\cos k_y y + B \sin k_y y)$$

The constants A, B, k_x, k_y will be determined by the boundary conditions that h_z must satisfy on the guide walls. Since the normal component of **h** is zero at a perfect conducting surface and **h** is proportional to $\nabla_t h_z$, it follows that

$$\frac{\partial h_z}{\partial x} = 0 \qquad x = 0, a$$
$$\frac{\partial h_z}{\partial y} = 0 \qquad y = 0, b$$

At $x = y = 0$, these conditions are satisfied if $A = B = 0$. At $x = a$ we must have $\sin k_x a = 0$, while at $y = b$, $\sin k_y b = 0$. Therefore the possible solutions for k_x and k_y are

$$k_x = \frac{n\pi}{a} \qquad n = 0, 1, 2, \ldots \tag{10.103a}$$

$$k_y = \frac{m\pi}{b} \qquad m = 0, 1, 2, \ldots \tag{10.103b}$$

Both n and m cannot be zero or the gradient of h_z will be zero. There is a double infinity of possible solutions. Each particular solution is called a mode and designated as a TE_{nm} or H_{nm} mode for the nmth solution. Thus for the nmth solution

$$h_{z,nm} = \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \tag{10.104}$$

From this solution for h_z all the other field components may be found by means of (10.98).

For the nmth solution the corresponding values of k_c and β will be written as $k_{c,nm}$ and β_{nm}. Similar subscripts will be used on the field components. From (10.102) and (10.103) we have

$$k_{c,nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \tag{10.105a}$$

$$\beta_{nm} = \left[k_0^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2\right]^{1/2} \tag{10.105b}$$

For propagation in the z direction, β_{nm} must be real. This is possible only if

$$\frac{2\pi}{\lambda_0} = k_0 > \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2}$$

or

$$\lambda_0 < \frac{2}{[(n/a)^2 + (m/b)^2]^{1/2}} = \frac{2ab}{(n^2b^2 + m^2a^2)^{1/2}} \tag{10.106}$$

The particular value of λ_0 for which the left-hand side of (10.106) equals the right-hand side, i.e.,

$$\lambda_0 = \lambda_{c,nm} = \frac{2ab}{(n^2b^2 + m^2a^2)^{1/2}} \tag{10.107}$$

is called the cutoff wavelength for the nmth mode. For all values of $\lambda_0 < \lambda_{c,nm}$, β_{nm} is real, while for $\lambda_0 > \lambda_{c,nm}$, the propagation constant β_{nm} is imaginary. For the latter case the corresponding fields are exponentially damped in the z direction since $e^{-j\beta_{nm}z} = e^{-|\beta_{nm}|z}$. Such solutions are known as evanescent waves.

In practice, the waveguide dimensions a and b are chosen so that for the frequency band of interest only a single mode can propagate. Usually a is chosen as approximately equal to $2b$. If $a = 2b$, (10.107) gives

$$\lambda_{c,nm} = \frac{2a}{(n^2 + 4m^2)^{1/2}} \tag{10.108}$$

The largest cutoff wavelength occurs for the TE_{10} mode, that is, $n = 1$, $m = 0$. For this mode $\lambda_{c,10} = 2a$. The next modes to propagate are the TE_{20} and TE_{01} modes with $\lambda_{c,20} = \lambda_{c,01} = a$, followed by the TE_{11} mode with $\lambda_{c,11} = 2a/\sqrt{5}$. Provided we restrict the frequency to be in the range where $a < \lambda_0 < 2a$, only the TE_{10} mode will be able to propagate. This is the dominant mode of propagation in a rectangular guide (the first TM mode to propagate has $\lambda_c = \lambda_{c,11}$).

Multimode propagation is avoided in practice because each mode that could propagate has a different phase and group velocity and a different field configuration. The first difference means that the phase relation between the portions of the signal power carried by each mode continually varies along the guide and makes it difficult to extract all the energy from the guide at the receiving end. The second difference means that a different arrangement of coupling probes or loops must be used to excite each mode in the guide as well as to couple the energy out of the guide.

The TE_{10} Mode

For the dominant TE_{10} mode,

$$h_{z,10} = A \cos \frac{\pi x}{a}$$

where A is an amplitude constant. From (10.98) the fields are found to be

$$H_z = A \cos \frac{\pi x}{a} e^{-j\beta_{10}z}$$

$$H_x = j\beta_{10} \frac{a}{\pi} A \sin \frac{\pi x}{a} e^{-j\beta_{10}z}$$

$$E_y = -\frac{k_0}{\beta_{10}} Z_0 H_x$$

$$E_x = E_z = H_y = 0$$

where $k_{c,10} = 2\pi/\lambda_{c,10} = \pi/a$, and $\beta_{10} = [(2\pi/\lambda_0)^2 - (\pi/a)^2]^{1/2}$. The guide wavelength λ_g is defined as the distance the wave must propagate in order to undergo a phase change of 2π radians. Thus $\beta_{10}\lambda_g = 2\pi$, or

$$\lambda_g = \frac{2\pi}{\beta_{10}} = \frac{\lambda_0}{[1 - (\lambda_0/2a)^2]^{1/2}}$$

For this mode E_y is analogous to the voltage and $-H_x$ is analogous to the current on a transmission line. The wave impedance $Z_h = k_0 Z_0/\beta_{10}$ is the counterpart of the characteristic impedance. A sketch of the field configuration is given in Fig. 10.14. The density of lines is a measure of the relative amplitude or strength of the field.

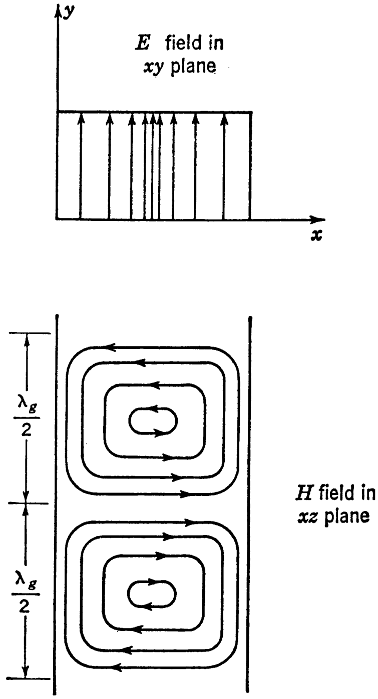

FIG. 10.14. Field configuration for TE_{10} mode at a particular instant of time.

The time-average power flow in the z direction is given by one-half of the real part of the integral of the complex Poynting vector over the guide cross section. We have

$$\begin{aligned} P &= \tfrac{1}{2} \operatorname{Re} \int_0^b \int_0^a \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{a}_z \, dx \, dy \\ &= \tfrac{1}{2} \operatorname{Re} \int_0^b \int_0^a - E_y H_x^* \, dx \, dy \\ &= \frac{a^2}{2\pi^2} k_0 \beta_{10} Z_0 A A^* \int_0^b \int_0^a \sin^2 \frac{\pi x}{a} \, dx \, dy \\ &= \frac{a^3 b}{4\pi^2} k_0 \beta_{10} Z_0 A A^* \end{aligned} \qquad (10.109)$$

The losses in the waveguide walls due to good but not perfect conductors may be found by the scheme previously used in discussing the transmission line. In this case the currents that flow in each of the four walls must be considered separately. Thus

$$\mathbf{J}_s\Big|_{x=0} = \mathbf{n} \times H_z\mathbf{a}_z\Big|_{x=0} = -Ae^{-j\beta_{10}z}\mathbf{a}_y$$

$$\mathbf{J}_s\Big|_{x=a} = -Ae^{-j\beta_{10}z}\mathbf{a}_y$$

$$\mathbf{J}_s\Big|_{y=0} = \mathbf{n} \times (H_z\mathbf{a}_z + H_x\mathbf{a}_x)\Big|_{y=0} = \left(\mathbf{a}_x \cos\frac{\pi x}{a} - \mathbf{a}_z j\beta_{10}\frac{a}{\pi}\sin\frac{\pi x}{a}\right)Ae^{-j\beta_{10}z}$$

$$\mathbf{J}_s\Big|_{y=b} = \left(-\mathbf{a}_x \cos\frac{\pi x}{a} + \mathbf{a}_z j\beta_{10}\frac{a}{\pi}\sin\frac{\pi x}{a}\right)Ae^{-j\beta_{10}z}$$

The losses per unit length are found by integrating $(R_m/2)\oint \mathbf{J}_s \cdot \mathbf{J}_s^*\, dl$ around the waveguide walls. In detail, we obtain

$$P_{lc} = R_m\left[\int_0^b AA^*\,dy + \int_0^a AA^*\left(\cos^2\frac{\pi x}{a} + \frac{\beta_{10}^2 a^2}{\pi^2}\sin^2\frac{\pi x}{a}\right)dx\right]$$
$$= AA^*R_m\left(b + \frac{a}{2} + \frac{\beta_{10}^2 a^3}{2\pi^2}\right)$$

The attenuation α may be found from the relation noted in (10.87); that is,

$$\alpha = \frac{1}{2}\frac{P_{lc}}{P} = \frac{R_m(b + \beta_{10}^2 a^3/2\pi^2 + a/2)}{(2a^3b/4\pi^2)k_0\beta_{10}Z_0}$$
$$= \frac{R_m k_0}{bZ_0\beta_{10}}\left[\frac{2b}{a}\left(\frac{\lambda_0}{2a}\right)^2 + 1\right] \tag{10.110}$$

The time-average electric energy stored in a unit length of guide is

$$W_e = \frac{\epsilon_0}{4}\int_0^b\int_0^a\int_0^1 E_y E_y^*\,dx\,dy\,dz$$
$$= \frac{\epsilon_0 a^2}{4\pi^2}k_0{}^2Z_0{}^2AA^*\int_0^b\int_0^a\int_0^1 \sin^2\frac{\pi x}{a}\,dx\,dy\,dz$$
$$= \frac{\epsilon_0 a^3 b}{8\pi^2}k_0{}^2Z_0{}^2AA^* \tag{10.111a}$$

The time-average magnetic energy stored in a unit length of guide is

$$W_m = \frac{\mu_0}{4}\int_0^b\int_0^a\int_0^1 (H_xH_x^* + H_zH_z^*)\,dx\,dy\,dz$$
$$= \frac{\mu_0}{4}b\int_0^a\left(\frac{a^2}{\pi^2}\beta_{10}^2 AA^*\sin^2\frac{\pi x}{a} + AA^*\cos^2\frac{\pi x}{a}\right)dx$$
$$= \frac{\mu_0 a^3 b}{8\pi^2}AA^*\left(\beta_{10}^2 + \frac{\pi^2}{a^2}\right) = W_e \tag{10.111b}$$

since $\beta_{10}^2 + \pi^2/a^2 = k_0^2$ and $\mu_0 = Z_0^2\epsilon_0$. Since power flow is a rate of flow of energy, the velocity of energy transport v_g may be found by multiplying $W_e + W_m$ by v_g and equating the result to (10.109) for power flow. Thus

$$v_g = \frac{P}{2W_e} = \frac{\beta_{10}}{k_0\epsilon_0 Z_0} = \frac{\beta_{10}}{k_0}\frac{1}{(\mu_0\epsilon_0)^{1/2}} = \frac{\lambda_0}{\lambda_g}c \tag{10.112}$$

For the TE_{10} mode we have for the z and t dependence

$$\exp(-j\beta_{10}z + j\omega t) = \exp j\omega\left[t - \left(\frac{\beta_{10}}{\omega}\right)z\right]$$

Thus the phase velocity v_p is given by $\omega/\beta_{10} = k_0c/\beta_{10}$, and hence $v_pv_g = c^2$, as stated earlier in (10.97).

We shall now show that the velocity with which a signal is propagated is equal to the velocity v_g. An amplitude-modulated signal

$$S = (1 + M\cos\omega_m t)\cos\omega t$$

may be expressed as

$$\begin{aligned} S &= \operatorname{Re}\left(1 + \frac{M}{2}e^{j\omega_m t} + \frac{M}{2}e^{-j\omega_m t}\right)e^{j\omega t} \\ &= \operatorname{Re}\left[e^{j\omega t} + \frac{M}{2}\left(e^{j(\omega+\omega_m)t} + e^{j(\omega-\omega_m)t}\right)\right] \end{aligned} \tag{10.113}$$

The term $M\cos\omega_m t$ is the signal being transmitted.

The modulation frequency ω_m is assumed to be very small compared with the carrier frequency ω. From (10.113) we see that we must consider the propagation of three components with frequencies ω, $\omega + \omega_m$, and $\omega - \omega_m$ along the guide. Since β_{10} is a function of frequency, each component will propagate with a different phase velocity. We may expand β_{10} in a Taylor series about the point ω to get

$$\beta_{10}(\omega + \Delta\omega) = \beta_{10}(\omega) + \beta_{10}'\,\Delta\omega + \cdots$$

where $\beta_{10}' = d\beta_{10}/d\omega$ at ω. Provided ω_m is small enough, we have

$$\beta_{10}(\omega \pm \omega_m) = \beta_{10}(\omega) \pm \omega_m\beta_{10}'$$

Thus as the signal S propagates along the guide, its z and t dependence will be

$$\begin{aligned} S &= \operatorname{Re}\left[e^{j\omega t - j\beta_{10}(\omega)z} + \frac{M}{2}\left(e^{j(\omega+\omega_m)t}e^{-j(\beta_{10}+\omega_m\beta_{10}')z} + e^{j(\omega-\omega_m)t}e^{-j(\beta_{10}-\omega_m\beta_{10}')z}\right)\right] \\ &= \operatorname{Re} e^{j\omega t - j\beta_{10}(\omega)z}\left[1 + \frac{M}{2}\left(e^{j\omega_m(t-\beta_{10}'z)} + e^{-j\omega_m(t-\beta_{10}'z)}\right)\right] \\ &= [1 + M\cos\omega_m(t - \beta_{10}'z)][\cos(\omega t - \beta_{10}z)] \end{aligned} \tag{10.114}$$

Thus the signal appears at z in an undistorted form but delayed in time by an amount $\tau = \beta_{10}' z$. The distance z traveled divided by the time delay τ defines the group velocity or signal velocity and is

$$v_g = \frac{1}{\beta_{10}'} = \frac{d\omega}{d\beta_{10}} = c\,\frac{dk_0}{d\beta_{10}}$$

since $ck_0 = \omega$. Now $\beta_{10}^2 + (\pi/a)^2 = k_0^2$; so $2k_0\,dk_0 = 2\beta_{10}\,d\beta_{10}$, and hence

$$v_g = \frac{\beta_{10}}{k_0}c \tag{10.115}$$

which is the same velocity as derived for the energy transport.

The above analysis is based on the assumption that ω_m is small enough so that only the first two terms in the Taylor series expansion are required. If more terms are required, it is then found that signal distortion takes place because of the phase dispersion between the various frequency components.

TE$_{10}$ *Mode as a Superposition of Plane Waves*

A physical understanding of why the guide wavelength and phase velocity are greater than the corresponding quantities for plane waves may be obtained by decomposing the TE_{10} solution into two obliquely propagating plane waves. For the electric field E_y we may write

$$\begin{aligned} E_y &= -2jA_1 \sin\frac{\pi x}{a}\, e^{-j\beta_{10}z} \\ &= A_1(e^{-j\pi x/a} - e^{j\pi x/a})e^{-j\beta_{10}z} \end{aligned}$$

where $2jA_1 = jk_0Z_0Aa/\pi$. If we now write $\pi/a = k_0 \sin\theta_i$, $\beta_{10} = k_0\cos\theta_i$, the relation $\beta_{10}^2 + (\pi/a)^2 = k_0^2$ is satisfied. The solution for E_y becomes

$$E_y = A_1(e^{-jk_0(x\sin\theta_i + z\cos\theta_i)} - e^{-jk_0(-x\sin\theta_i+z\cos\theta_i)})$$

which represents two plane waves propagating at angles θ_i and $-\theta_i$ relative to the z axis. One of these component waves is illustrated in Fig. 10.15. From this figure it is clear that when the plane wave has progressed a distance c in 1 second, the intersection of the phase front with the z axis has progressed a distance $c/\cos\theta_i = k_0c/\beta_{10} = v_p$. Similarly, it is clear that the spacing between adjacent wave crests in the z direction is greater by a factor $\sec\theta_i$ than the spacing in a direction normal to the phase front. Hence λ_g is greater than λ_0. Energy in a plane wave propagates in a direction normal to the phase front with a velocity c. The projection or component of this velocity along the z direction is $c\cos\theta_i = v_g$. It is because of the zigzag path the TEM waves follow as they reflect back and forth between the side walls of the guide, while progressing along the guide, that the properties of the guide as noted above arise. The cutoff condition, for example, corresponds to the case where $\theta_i \to \pi/2$. Under these conditions we obtain a picture of the wave

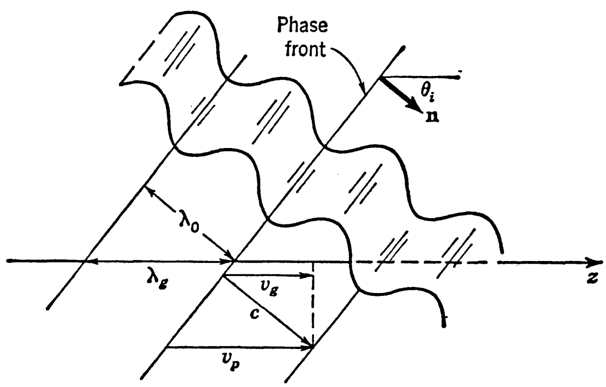

FIG. 10.15. Obliquely propagating plane wave.

propagating back and forth in the transverse plane with no component in the axial direction.

TM Waves

For TM waves all the field components may be expressed in terms of the axial electric field function $e_z(x,y)$. The solution for e_z is similar to that for h_z, with the exception that e_z must be of the form

$$e_z = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

in order that it will vanish on the guide walls. A notable feature of this solution is that neither n nor m can be zero or e_z will vanish. As a result, the lowest-order TM mode is the TM_{11} mode. For $a = 2b$, this mode, as well as all the other TM_{nm} modes, will not propagate when $a < \lambda_0 < 2a$, hence confirming that the TE_{10} mode is dominant under those conditions. Apart from this difference the TM_{nm} modes are the duals of the TE_{nm} modes. The field components for TE_{nm} and TM_{nm} modes are listed in Table 10.1 along with other important information.

The attenuation constant α, measured in nepers per meter, for the TE and TM modes is given below. For the TE modes,

$$\alpha = \frac{2R_m}{bZ_0(1 - k_{c,nm}^2/k_0^2)^{1/2}}\left[\left(1+\frac{b}{a}\right)\frac{k_{c,nm}^2}{k_0^2} + \frac{b}{a}\left(\frac{\epsilon_{0m}}{2} - \frac{k_{c,nm}^2}{k_0^2}\right)\frac{n^2ab + m^2a^2}{n^2b^2 + m^2a^2}\right] \qquad (10.116a)$$

TABLE 10.1. PROPERTIES OF MODES IN A RECTANGULAR WAVEGUIDE

	TE modes	TM modes
H_z	$\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}e^{-j\beta_{nm}}$	0
E_z	0	$\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}e^{-j\beta_{nm}z}$
H_x	$\frac{j\beta_{nm}n\pi}{ak_{c,nm}^2}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}e^{-j\beta_{nm}z}$	$-\frac{E_y}{Z_{e,nm}}$
H_y	$\frac{j\beta_{nm}m\pi}{bk_{c,nm}^2}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}e^{-j\beta_{nm}z}$	$\frac{E_x}{Z_{e,nm}}$
E_x	$Z_{h,nm}H_y$	$-\frac{j\beta_{nm}n\pi}{ak_{c,nm}^2}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}e^{-j\beta_{nm}z}$
E_y	$-Z_{h,nm}H_x$	$-\frac{j\beta_{nm}m\pi}{bk_{c,nm}^2}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}e^{-j\beta_{nm}z}$
$Z_{h,nm}$	$Z_0\frac{k_0}{\beta_{nm}} = Z_0\left[1-\left(\frac{f_{c,nm}}{f}\right)^2\right]^{-1/2}$	
$Z_{e,nm}$	....	$Z_0\frac{\beta_{nm}}{k_0} = Z_0\left[1-\left(\frac{f_{c,nm}}{f}\right)^2\right]^{1/2}$
$k_{c,nm}$	$\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^{1/2}$	$\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^{1/2}$
β_{nm}	$\left[k_0^2-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2\right]^{1/2}$	$\left[k_0^2-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2\right]^{1/2}$
$\lambda_{c,nm}$	$\frac{2ab}{[(nb)^2+(ma)^2]^{1/2}}$	$\frac{2ab}{[(nb)^2+(ma)^2]^{1/2}}$
$f_{c,nm}$	$(\lambda_{c,nm})^{-1}(\mu_0\epsilon_0)^{-1/2}$	$(\lambda_{c,nm})^{-1}(\mu_0\epsilon_0)^{-1/2}$

while for the TM modes,

$$\alpha = \frac{2R_m}{bZ_0(1-k_{c,nm}^2/k_0^2)^{1/2}}\frac{n^2b^3+m^2a^3}{n^2b^2a+m^2a^3} \tag{10.116b}$$

where $R_m = (\omega\mu_0/2\sigma)^{1/2}$ and $\epsilon_{0m} = 1$ for $m = 0$ and $\epsilon_{0m} = 2$ for $m > 0$.

10.7. Circular Waveguides

The propagation of waves through a hollow waveguide of circular cross section can be readily considered on the basis of the general theory developed in Sec. 10.1. Following the scheme outlined in that section, we consider separately TEM, TM, and TE modes. In this case no TEM wave propagation is possible because the conducting-boundary contour in the transverse plane is simply connected. Let us then start with a consideration of TM waves.

For TM waves in cylindrical waveguides, all field components may be derived from the axial electric field e_z. This field, furthermore, satisfies the reduced Helmholtz equation

$$\nabla_t^2 e_z + k_c^2 e_z = 0$$

where

$$k_c^2 = k_0^2 - \beta^2$$

and $e^{-j\beta z}$ variation with z is assumed. In view of the circular cross-sectional geometry, as illustrated in Fig. 10.16, it is appropriate to expand

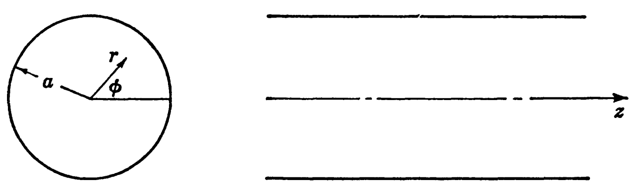

FIG. 10.16. The circular cylindrical waveguide.

the Laplacian in the above equation in circular cylindrical coordinates, so that we get

$$\frac{\partial^2 e_z}{\partial r^2} + \frac{1}{r}\frac{\partial e_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 e_z}{\partial \phi^2} = -k_c^2 e_z \tag{10.117}$$

We seek a solution of (10.117) by the method of separation of variables and therefore assume that we may express

$$e_z(r,\phi) = f(r)g(\phi) \tag{10.118}$$

Substituting (10.118) into (10.117) and then dividing by fg results in the following equation:

$$\frac{f''}{f} + \frac{1}{r}\frac{f'}{f} + \frac{1}{r^2}\frac{g''}{g} = -k_c^2 \tag{10.119}$$

where the primes represent derivatives with respect to the argument. If we multiply (10.119) by r^2 and rearrange, we can get

$$\frac{r^2 f''}{f} + \frac{rf'}{f} + k_c^2 r^2 = -\frac{g''}{g} \tag{10.120}$$

The left-hand side of (10.120) is a function only of r, while the right-hand side is a function only of ϕ. Since (10.120) must be identically correct for all r and ϕ, this could be true only if both sides were equal to the same constant ν^2. Consequently, we have been able to reduce the solution of (10.117) to the solution of the following two ordinary differential equations:

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(k_c{}^2 - \frac{\nu^2}{r^2}\right)f = 0 \tag{10.121a}$$

$$\frac{d^2g}{d\phi^2} + \nu^2 g = 0 \tag{10.121b}$$

In the problem at hand the field must be periodic in ϕ with periodicity 2π. Consequently, it is necessary to choose $\nu = n$, an integer, and the solution to (10.121*b*) will be of the form

$$g(\phi) = A \cos n\phi + B \sin n\phi \tag{10.122a}$$

The differential equation given in (10.121*a*) may be recognized as Bessel's equation, the solution for which may be written

$$f(r) = CJ_n(k_cr) + DY_n(k_cr) \tag{10.122b}$$

The Bessel function of the second kind, $Y_n(r)$, has a singularity at the origin. Since such a singularity is inconsistent with the physical fields that are expected in the waveguide, we must choose $D = 0$. We finally get for e_z:

$$e_z = J_n(k_cr)(A \cos n\phi + B \sin n\phi) \tag{10.123}$$

Let us now consider the requirement that $e_z = 0$ when $r = a$; that is, the tangential electric field along the conducting boundary must vanish. A nontrivial solution is obtained only if certain values of k_c, the eigenvalues, are chosen. These values must be such that $J_n(k_ca) = 0$; that is, k_ca must be a root of the nth-order Bessel function. An infinite number of roots exist, and we shall designate the mth root of the nth-order Bessel function as p_{nm}. This means that

$$J_n(p_{nm}) = 0$$

Values of p_{nm} for the first few modes are given in Table 10.2. We may

TABLE 10.2. VALUES OF p_{nm} FOR TM MODES

n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

now specify by the double-subscript notation $k_{c,nm}$ the doubly infinite set of eigenvalues for the TM modes of the circular cylindrical waveguide. Thus

$$k_{c,nm} = \frac{p_{nm}}{a} \tag{10.124}$$

Each choice of m and n specifies an eigenfunction solution or mode of the problem, which we designate TM_{nm}. The quantity n is related to the number of circumferential field variations, while m describes the number of radial variations of the field. The complete problem requires matching prescribed boundary conditions at some plane $z = z_1$ and $z = z_2$. In order to do this, a summation of different modes (including the TE type yet to be discussed) will be necessary, in general. The boundary conditions serve to specify the constants A and B for each mode.

The propagation constant for the nmth mode is given by

$$\beta_{nm} = \left[k_0^2 - \left(\frac{p_{nm}}{a}\right)^2\right]^{1/2} \tag{10.125}$$

and we note the same cutoff property that was found for the rectangular guide. Thus the eigenvalue $k_{c,nm}$ given by (10.124) is also the cutoff wave number for that mode. The cutoff wavelength is simply

$$\lambda_{c,nm} = \frac{2\pi a}{p_{nm}} \tag{10.126}$$

The remaining field components for the TM_{nm} wave are given in Table 10.3. They were derived from (10.123), with $k_{c,nm} = p_{nm}/a$, using the formulas developed in Sec. 10.1. The expression for the wave impedance is written in the form

$$Z_e = Z\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{1/2}$$

which is similar to that given for the rectangular guide. Actually, we could show that this form is correct for any cylindrical waveguide; variations in cross-sectional shape affect the value of Z_e through the value of f_c. The lowest value of p_{nm} is the first root of the zero-order Bessel function for which $p_{01} = 2.405$. The cutoff wavelength, from (10.126), is then $2.61a$.

To explore the TE modes that exist in the circular cylindrical guide we must consider h_z as our potential function. It satisfies an analogous equation to (10.117); so we may write, immediately,

$$h_z = J_n(k_c r)(A' \cos n\phi + B' \sin n\phi) \tag{10.127}$$

The boundary conditions in this case require that $\partial h_z/\partial r = 0$ when $r = a$; i.e., the normal component of magnetic field at a conducting boundary vanishes. Let us designate by p'_{nm} the mth root of the following equation:

$$J'_n(x) \equiv \frac{dJ_n(x)}{dx} = 0$$

TABLE 10.3. PROPERTIES OF MODES IN A CIRCULAR CYLINDRICAL WAVEGUIDE

	TE modes	TM modes
H_z	$J_n\left(\frac{p'_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{cases}\cos n\phi\\ \sin n\phi\end{cases}$	0
E_z	0	$J_n\left(\frac{p_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{cases}\cos n\phi\\ \sin n\phi\end{cases}$
H_r	$-\frac{j\beta_{nm}p'_{nm}}{ak^2_{c,nm}}J'_n\left(\frac{p'_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{cases}\cos n\phi\\ \sin n\phi\end{cases}$	$-\frac{E_\phi}{Z_{e,nm}}$
H_ϕ	$-\frac{jn\beta_{nm}}{rk^2_{c,nm}}J_n\left(\frac{p'_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{cases}-\sin n\phi\\ \cos n\phi\end{cases}$	$\frac{E_r}{Z_{e,nm}}$
E_r	$Z_{h,nm}H_\phi$	$-\frac{j\beta_{nm}p_{nm}}{ak^2_{c,nm}}J'_n\left(\frac{p_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{cases}\cos n\phi\\ \sin n\phi\end{cases}$
E_ϕ	$-Z_{h,nm}H_r$	$-\frac{jn\beta_{nm}}{rk^2_{c,nm}}J_n\left(\frac{p_{nm}r}{a}\right)e^{-j\beta_{nm}z}\begin{cases}-\sin n\phi\\ \cos n\phi\end{cases}$
β_{nm}	$\left[k_0^2-\left(\frac{p'_{nm}}{a}\right)^2\right]^{1/2}$	$\left[k_0^2-\left(\frac{p_{nm}}{a}\right)^2\right]^{1/2}$
$Z_{h,nm}$	$Z_0\left[1-\left(\frac{f_{c,nm}}{f}\right)^2\right]^{-1/2}=\frac{k_0}{\beta_{nm}}Z_0$	
$Z_{e,nm}$	....	$Z_0\left[1-\left(\frac{f_{c,nm}}{f}\right)^2\right]^{1/2}=\frac{\beta_{nm}}{k_0}Z_0$
$k_{c,nm}$	$\frac{p'_{nm}}{a}$	$\frac{p_{nm}}{a}$
$\lambda_{c,nm}$	$\frac{2\pi a}{p'_{nm}}$	$\frac{2\pi a}{p_{nm}}$
$f_{c,nm}$	$(\lambda_{c,nm})^{-1}(\mu_0\epsilon_0)^{-1/2}$	$(\lambda_{c,nm})^{-1}(\mu_0\epsilon_0)^{-1/2}$

Then the eigenvalues for the problem are

$$k_{c,nm}=\frac{p'_{nm}}{a} \tag{10.128}$$

Values for p'_{nm} for the first few modes are given in Table 10.4. In other ways the results for the TE case follow analogously to the TM case already discussed. Table 10.3 summarizes the fields that exist for the TE_{nm} modes. The latter are obtained from (10.127), with $k_{c,nm} = p'_{nm}/a$, using the general equations developed in Sec. 10.1.

TABLE 10.4. VALUES OF p'_{nm} FOR TE MODES

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

The lowest value of p'_{nm} is p'_{11}, which equals 1.84 and for which the cutoff wavelength is $3.41a$. Consequently, the TE_{11} mode is the dominant mode in the circular cylindrical waveguide. A sketch of the field lines for this mode is given in Fig. 10.17. For $2.61a < \lambda_0 < 3.41a$, only the TE_{11} mode can propagate in a circular cylindrical waveguide.

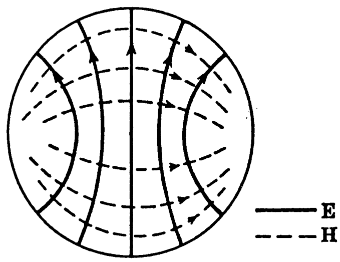

FIG. 10.17. TE_{11} field distribution in transverse plane.

The attenuation produced by imperfectly conducting walls may be calculated by means of the same technique that was used in the case of the transmission line and the rectangular waveguide. The results are, for TM modes,

$$\alpha = \frac{R_m}{aZ_0}\left[1 - \left(\frac{f_{c,nm}}{f}\right)^2\right]^{-1/2}$$

while for TE modes

$$\alpha = \frac{R_m}{aZ_0}\left[1 - \left(\frac{f_{c,nm}}{f}\right)^2\right]^{-1/2}\left[\left(\frac{f_{c,nm}}{f}\right)^2 + \frac{n^2}{p'^2_{nm} - n^2}\right]$$

where α is measured in nepers per meter, and $R_m = (\omega\mu_0/2\sigma)^{1/2} = 1/\sigma\delta$ and is the surface resistivity of the metallic walls. The attenuation constant for the TM waves as a function of increasing frequency decreases to a minimum at $f = \sqrt{3}\, f_{c,nm}$ and then increases indefinitely. The same general behavior occurs for the TE modes with the exception of the TE_{0m} waves. The latter are of very great interest because the corresponding attenuation constant decreases indefinitely with frequency and hence gives rise to the possibility of long-distance communication links.

10.8. Electromagnetic Cavities

At high frequencies the electromagnetic-cavity resonator replaces the lumped-parameter LC resonant circuit. Virtually any metallic enclosure, when properly excited, will function as an electromagnetic resonator; that is, for certain specific frequencies, electromagnetic field oscillations can be sustained within the enclosure with a very small expenditure of power. The only power that needs to be supplied is just that needed to compensate for the power loss in the cavity walls. Electromagnetic cavities are used as the resonant circuit in high-frequency tubes such as the klystron, for bandpass filters, and for wave meters to measure frequency, as well as for a number of other applications.

In this section we shall examine the basic properties of a rectangular cavity of the type illustrated in Fig. 10.18. Again, as was the case for

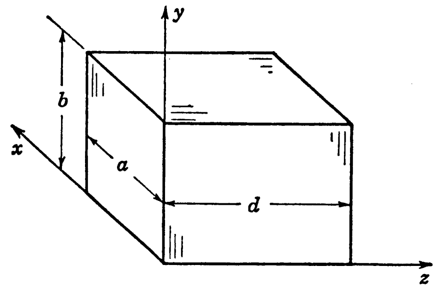

FIG. 10.18. A rectangular cavity resonator.

the rectangular guide, an understanding of the behavior of the rectangular cavity provides an understanding of other shapes of cavities also. The only essential difference between cavities of different shapes is the detailed structure of the interior fields, since this depends on the geometry or shape of the cavity.

The field solutions in a rectangular cavity are readily constructed from the corresponding solutions for the waveguide modes. For example, we may consider the cavity in Fig. 10.18 as being a part of a rectangular waveguide. Let a TE_{10} mode propagate in the positive z direction, and let a short-circuiting conducting plate be placed at $z = d$. Complete reflection will take place, and the electric field will be of the form

$$\begin{aligned} E_y &= A_1(e^{-j\beta_{10}z} - e^{-j\beta_{10}(2d-z)}) \sin \frac{\pi x}{a} \\ &= A_1 e^{-j\beta_{10}d} \sin \frac{\pi x}{a} (e^{-j\beta_{10}(z-d)} - e^{j\beta_{10}(z-d)}) \\ &= -2jA_1 e^{-j\beta_{10}d} \sin \frac{\pi x}{a} \sin \beta_{10}(z-d) \\ &= A \sin \frac{\pi x}{a} \sin \beta_{10}(z-d) \end{aligned}$$

where $A = -2jA_1e^{-j\beta_{10}d}$. At any plane where $\sin \beta_{10}(z - d)$ vanishes, we can place another conducting plate and thus obtain a rectangular enclosure. On the other hand, if the dimension d is given and the ends of the cavity are at $z = 0$ and $z = d$, we must have

$$\sin \beta_{10} d = 0$$

or

$$\beta_{10} = \frac{s\pi}{d} \qquad s = 1, 2, 3, \ldots$$

With β_{10}, k_x, and k_y all fixed by the cavity dimensions, it follows that only certain discrete values of k_0 will yield a possible solution. Since $\beta^2 = k_0^2 - k_x^2 - k_y^2$, we have

$$k_0 = \frac{2\pi f}{c} = \left[\left(\frac{s\pi}{d}\right)^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2} \tag{10.129}$$

in general, or for the TE_{10} mode with a single sinusoidal variation along z,

$$f = c\left(\frac{1}{4d^2} + \frac{1}{4a^2}\right)^{1/2} \tag{10.130}$$

The mode of oscillation, whose frequency of oscillation is given by (10.130), is designated as the TE_{101} mode, since there is only a single standing-wave-pattern loop in the x and z directions and none in the y direction. For the higher-order TE_{n0s} modes there will be n loops along the x direction and s loops along the z direction. The corresponding resonant frequencies are given by (10.129) with $m = 0$.

In addition to the TE_{n0s} modes there are the TE_{nms} modes and their duals, the TM_{nms} modes. The TE and TM modes may be derived from the z component of the magnetic and electric field, respectively, by equations similar to those given in Sec. 10.1 (replace $-j\beta$ by $\partial/\partial z$). The solutions for the z components of the fields are readily found to be

$$H_z = \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\sin\frac{s\pi z}{d} \qquad \text{TE modes}$$

$$E_z = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\cos\frac{s\pi z}{d} \qquad \text{TM modes}$$

and from these the remaining components may be found. Although there are an infinite number of discrete modes of oscillations in a rectangular cavity, we shall study only the TE_{101} mode in detail.

Since the TE_{101} mode is a TE_{10} waveguide standing-wave field, the only field components present are E_y, H_x, and H_z. For the electric field we have

$$E_y = A\sin\frac{\pi x}{a}\sin\frac{\pi z}{d} \tag{10.131a}$$

The magnetic field is readily determined from the curl equation

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}$$

and is

$$H_x = \frac{-j}{\omega\mu_0}\frac{\partial E_y}{\partial z} = -\frac{j\pi A}{\omega\mu_0 d}\sin\frac{\pi x}{a}\cos\frac{\pi z}{d} \tag{10.131b}$$

$$H_z = \frac{j}{\omega\mu_0}\frac{\partial E_y}{\partial x} = \frac{j\pi A}{\omega\mu_0 a}\cos\frac{\pi x}{a}\sin\frac{\pi z}{d} \tag{10.131c}$$

The resonant frequency of the cavity for this mode of oscillation is, from (10.130),

$$\frac{\omega}{2\pi} = f = \frac{c}{2}(a^{-2} + d^{-2})^{1/2} \tag{10.132}$$

The electric and magnetic fields in a cavity are in phase quadrature, as is readily seen from (10.131), since a factor j multiplies the expressions for H_x and H_z.

The total time-average electric energy stored within the cavity is

$$\begin{aligned} W_e &= \frac{\epsilon_0}{4}\int_0^a\int_0^b\int_0^d AA^*\sin^2\frac{\pi x}{a}\sin^2\frac{\pi z}{d}\,dx\,dy\,dz \\ &= \frac{abd}{16}\epsilon_0 AA^* \end{aligned} \tag{10.133}$$

The total time-average magnetic energy stored in the cavity is

$$\begin{aligned} W_m &= \frac{\mu_0}{4}\int_0^a\int_0^b\int_0^d \frac{AA^*}{\omega^2\mu_0^2}\left(\frac{\pi^2}{d^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi z}{d} + \frac{\pi^2}{a^2}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi z}{d}\right)dx\,dy\,dz \\ &= \frac{AA^*}{\omega^2\mu_0}\frac{abd}{16}\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{d^2}\right) = W_e \end{aligned} \tag{10.134}$$

since $k_0^2 = \omega^2\mu_0\epsilon_0 = (\pi/d)^2 + (\pi/a)^2$. Thus at the resonant frequency, $W_e = W_m$, a property similar to that for a resonant LC circuit at low frequencies.

The expressions given by (10.131) are the complex-phasor representations for the real physical field. The real-physical-field components, denoted by a prime, are obtained by multiplying by $e^{j\omega t}$ and taking the real part; thus

$$E_y' = A\sin\frac{\pi x}{a}\sin\frac{\pi z}{d}\cos\omega t \tag{10.135a}$$

$$H_x' = \frac{\pi A}{\omega\mu_0 d}\sin\frac{\pi x}{a}\cos\frac{\pi z}{d}\sin\omega t \tag{10.135b}$$

$$H_z' = -\frac{\pi A}{\omega\mu_0 a}\cos\frac{\pi x}{a}\sin\frac{\pi z}{d}\sin\omega t \tag{10.135c}$$

where ω is given by (10.132) and the amplitude constant A is assumed real.

In any practical cavity the walls have finite conductivity, and hence any mode of oscillation that has been excited, say by an impulse, must decay exponentially.† Thus the time behavior of the oscillations must be of the form $e^{-\alpha t}\cos\omega t$ and $e^{-\alpha t}\sin\omega t$, rather than the form given in (10.135), which is the steady-state solution for the ideal cavity. To determine the damping constant α, we must evaluate the power loss in the cavity walls. This may be done by assuming that the current flowing on the walls is the same as for the ideal cavity. We then have, from Sec. 10.3,

$$P_{lc} = \tfrac{1}{2}R_m \oint_{\text{walls}} \mathbf{H}_t \cdot \mathbf{H}_t^* \, dS \tag{10.136}$$

where $\mathbf{H}_t$ is the tangential magnetic field at the cavity wall, and $R_m = 1/\sigma\delta$. The total time-average energy in the cavity is

$$W = W_e + W_m$$

and must decay with time as follows:

$$W = W_0 e^{-2\alpha t}$$

where W_0 is the time-average energy in the cavity at $t = 0$. The negative rate of change of W with time must equal the power loss in the walls, and hence

$$-\frac{dW}{dt} = 2\alpha W_0 e^{-2\alpha t} = 2\alpha W = P_{lc}$$

or

$$\alpha = \frac{P_{lc}}{2W} \tag{10.137}$$

This relation permits the damping constant α to be determined.

The quality factor, or Q, of a resonant circuit may be defined as

$$Q = 2\pi \frac{\text{time-average energy stored}}{\text{energy loss per cycle of oscillation}} \tag{10.138}$$

Since P_{lc} is the energy dissipated in the cavity walls per second, the energy loss in one cycle, or a time interval $\tau = 1/f$, is P_{lc}/f. From (10.137) and (10.138), we now see that

$$\alpha = \frac{P_{lc}/f}{2W/f} = \frac{f}{2}\frac{P_{lc}/f}{W} = \frac{f}{2}\frac{2\pi}{Q} = \frac{\omega}{2Q} \tag{10.139}$$

† This type of behavior is characteristic of any low-loss oscillatory physical system where the power loss is directly proportional to the energy present at any instant of time. In this case $dW/dt = -kW$, so that $W = W_0e^{-kt}$, where W_0 is the energy present at $t = 0$.

In order to determine the Q of the cavity for the TE_{101} mode, we must evaluate (10.136) for the power loss in the walls. On the end walls at $z = 0, d$, we have, from (10.131),

$$|H_t| = |H_x| = \frac{\pi A}{\omega\mu_0 d} \sin \frac{\pi x}{a}$$

and the power loss in these two walls is

$$\begin{aligned} P_{lc1} &= \frac{R_m}{2} \left(\frac{\pi A}{\omega\mu_0 d}\right)^2 2 \int_0^a \int_0^b \sin^2 \frac{\pi x}{a}\, dx\, dy \\ &= \frac{ab}{2} R_m \left(\frac{\pi A}{\omega\mu_0 d}\right)^2 \end{aligned}$$

where A is again assumed to be real. On the upper and lower walls at $y = 0, b$, we have

$$|H_t|^2 = |H_x|^2 + |H_z|^2 = \left(\frac{\pi A}{\omega\mu_0}\right)^2 \left(\frac{1}{d^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi z}{d} + \frac{1}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi z}{d}\right)$$

and the power loss is

$$\begin{aligned} P_{lc2} = \frac{R_m}{2} \left(\frac{\pi A}{\omega\mu_0}\right)^2 2 \int_0^a \int_0^d \bigg(\frac{1}{d^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi z}{d} \\ + \frac{1}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi z}{d}\bigg) dx\, dz = \frac{R_m}{4} \left(\frac{\pi A}{\omega\mu_0}\right)^2 \left(\frac{a}{d} + \frac{d}{a}\right) \end{aligned}$$

On the remaining two walls at $x = 0, a$,

$$|H_t| = |H_z| = \frac{\pi A}{\omega\mu_0} \sin \frac{\pi z}{d}$$

with a corresponding power loss

$$\begin{aligned} P_{lc3} &= \frac{R_m}{2} \left(\frac{\pi A}{\omega\mu_0 a}\right)^2 2 \int_0^b \int_0^d \sin^2 \frac{\pi z}{d}\, dy\, dz \\ &= \frac{bd}{2} R_m \left(\frac{\pi A}{\omega\mu_0 a}\right)^2 \end{aligned}$$

The total power loss is

$$\begin{aligned} P_{lc} = P_{lc1} + P_{lc2} + P_{lc3} &= \left(\frac{\pi A}{\omega\mu_0}\right)^2 R_m \left(\frac{ab}{2d^2} + \frac{d}{4a} + \frac{a}{4d} + \frac{bd}{2a^2}\right) \\ &= R_m \left(\frac{\pi A}{2\omega\mu_0 ad}\right)^2 (2a^3b + ad^3 + a^3d + 2bd^3) \qquad (10.140) \end{aligned}$$

The total time-average energy stored in the cavity is, from (10.133) and (10.134),

$$W = 2W_e = \frac{abd}{8} \epsilon_0 A^2$$

Hence the Q of the cavity, for the TE_{101} mode, is given by

$$Q = \frac{\omega}{2\alpha} = \omega \frac{W}{P_{lc}} = \frac{\omega abd\epsilon_0}{8\pi^2 R_m} \frac{(2\omega\mu_0 ad)^2}{2a^3b + a^3d + ad^3 + 2bd^3}$$
$$= \frac{(k_0 ad)^3 bZ_0}{2\pi^2 R_m(2a^3b + 2d^3b + a^3d + d^3a)} \quad (10.141)$$

where $k_0^2 = (\pi/a)^2 + (\pi/d)^2$.

As a typical example, consider a copper cavity ($\sigma = 5.8 \times 10^7$ mhos per meter) with dimensions $a = b = c = 3$ centimeters. The resonant frequency is found from (10.132) to be 7,070 megacycles per second. The surface resistance R_m is 0.022 ohm. The Q of the cavity is found to be 12,700, and α equals 1.74×10^6 nepers per second. A rather startling property of a cavity is its extremely high Q as compared with the Q of LC circuits at low frequencies, which is usually of the order of a few hundred only.

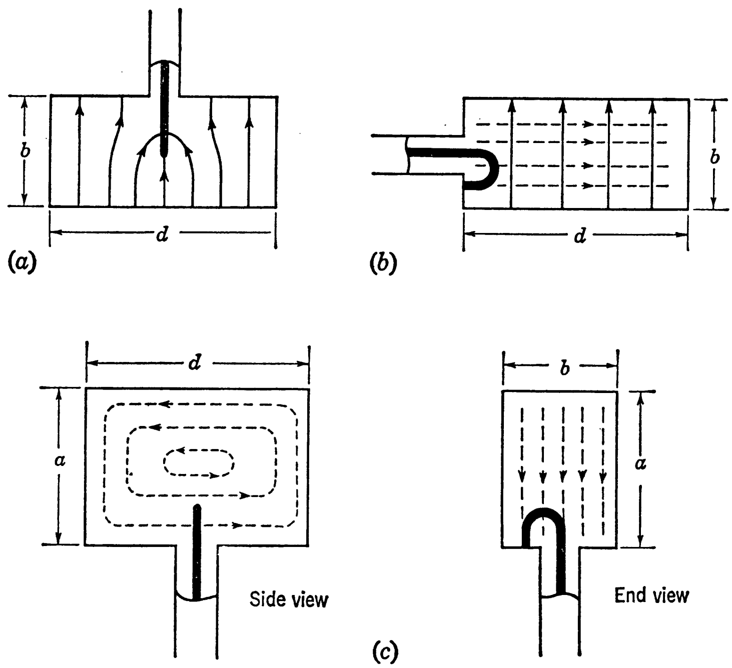

FIG. 10.19. Methods of exciting the TE_{101} mode from a coaxial line. (a) Probe coupling with E_y; (b) loop coupling with H_x; (c) loop coupling with H_z.

The oscillations in a cavity may be excited from a coaxial line by means of a small probe or loop antenna, as illustrated in Fig. 10.19. The probe couples to the electric field of the mode and is hence located in the center of the broad wall where E_y is a maximum. The loop antenna must be located at a point where the magnetic flux of the mode, through the loop, will be large. Similar probe and loop antennas may be used to excite the fields in a waveguide. The field configuration for the TE_{101} cavity mode is also illustrated in Fig. 10.19. Of course, the frequency of the incident wave in the coaxial line must be equal to the resonant frequency of the cavity if the mode is to be excited.

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Chapter 10

10.1. A parallel-polarized TEM wave is incident on a dielectric interface at an angle θ_i. Find the total reflected and transmitted fields and verify Eqs. (10.46). With reference to Fig. 10.3, let the incident electric field be

$$\mathbf{E} = A_1(\mathbf{a}_x \cos \theta_i - \mathbf{a}_z \sin \theta_i)e^{-jk_0(x \sin \theta_i + z \cos \theta_i)}$$

10.2. The region $z > 0$ is occupied by a dielectric medium with dielectric constant κ_2. In front of this medium a slab of material of thickness d and dielectric constant κ_1 is placed. For a TEM wave incident normally on this structure from the left show that no reflection occurs if $\kappa_1 = \kappa_2^{1/2}$ and $d = \frac{1}{4}\lambda_0\kappa_1^{-1/2}$, where λ_0 is the free-space wavelength. The intermediate dielectric slab forms a quarter-wave transformer which matches the dielectric medium for $z > 0$ to free space. This technique is used in optics to reduce reflections from lenses and is known as lens blooming.

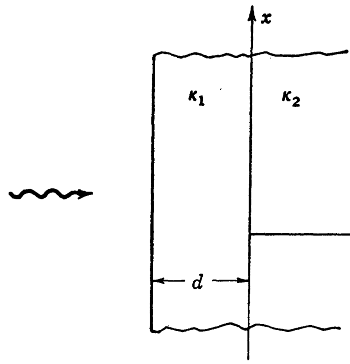

FIG. P 10.2

10.3. When a perpendicular-polarized TEM wave is incident on the structure of Fig. P 10.2, at an angle θ_i with respect to the interface normal, show that the parameters of the quarter-wave matching layer are given by

$$\kappa_1 - \sin^2\theta_i = (\kappa_2 - \sin^2\theta_i)^{1/2}\cos\theta_i$$
$$d = \frac{\lambda_0}{4}(\kappa_1 - \sin^2\theta_i)^{-1/2}$$

Note that the requirement for a match (no reflection) is that the wave impedance of the intermediate layer be the geometric mean of the wave impedances of free space and the medium to be matched and that the thickness be equal to one-fourth of the effective wavelength in the z direction in the matching layer.

For a parallel-polarized wave show that the equation for d is the same as for perpendicular polarization but that κ_1 is given by

$$\frac{\kappa_1 - \sin^2\theta_i}{\kappa_1^2} = \frac{(\kappa_2 - \sin^2\theta_i)^{1/2}\cos\theta_i}{\kappa_2}$$

Can both polarizations be matched simultaneously?

10.4. A perpendicular-polarized wave is incident on a magnetic material at an angle θ_i, as in Fig. 10.3. The electrical parameters of the medium for $z > 0$ are $\epsilon = \epsilon_0$, $\mu = \kappa_m\mu_0$. Show that a Brewster angle exists such that no reflection takes place. Show also that in the present case a similar phenomenon does not occur for the parallel-polarized wave.

10.5. A perpendicular-polarized wave is incident on a dielectric-air interface at an angle θ (relative to the interface normal) from the dielectric side. Show that a critical angle θ_c exists such that the emerging ray on the air side just grazes the surface. For angles of incidence greater than θ_c show that the angle of propagation on the air side must be complex and that the field is exponentially damped in a direction normal and away from the interface on the air side. Note that no energy is transmitted past the interface since the modulus of the reflection coefficient is unity. However, the fields on the air side are not zero. Contrast this rigorous solution with that based on ray optics.

10.6. An infinite dielectric slab of thickness d is placed above a perfectly conducting plane, as illustrated. Along such a structure a field known as a "surface wave" may propagate. This wave consists essentially of a TEM wave propagating along a zigzag

10.10. A 50-ohm transmission line is terminated in a load $Z_L = 20 + j10$. Find the reflection coefficient and the standing-wave ratio. At what distance l from the load is the input impedance equal to $Z_{\text{in}} = 50 + jX$? At this point a reactance $-jX$ may be added in series with the line in order to match the load to the line.

10.11. For a rectangular guide with inner dimensions $a = 0.9$ inch, $b = 0.4$ inch, evaluate the parameters λ_0, λ_g, v_p, v_g, α when the frequency is 10^{10} cycles per second. Assume a copper guide with $\sigma = 5.8 \times 10^7$ mhos per meter.

10.12. A rectangular guide of dimensions a, b is filled with a dielectric material with $\epsilon = \kappa\epsilon_0$. Obtain the solution for a TE_{10} mode. Show that the guide wavelength is given by

$$\lambda_g = \frac{\lambda_0}{[\kappa - (\lambda_0/2a)^2]^{1/2}}$$

10.13. A rectangular guide is filled with a dielectric medium ($\epsilon = \kappa\epsilon_0$) for $z \geq 0$. A TE_{10} mode is incident from the region $z < 0$. Find the reflected and transmitted fields.

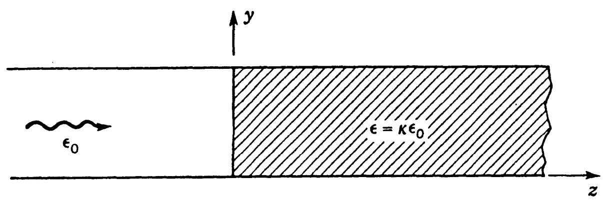

FIG. P 10.13

10.14. For Prob. 10.13 find the thickness and dielectric constant of a quarter-wave dielectric matching layer that will match the empty guide to the dielectric-filled guide.

10.15. A rectangular pulse of width 1 microsecond (frequency components up to 1 megacycle) is used to modulate a carrier of frequency 10^{10} cycles per second. This signal is transmitted through a rectangular guide with $a = 2b = 2.5$ centimeters as a TE_{10} mode. What length of guide is required to produce a signal delay of 2 microseconds?

10.16. Find the solutions for a TE_{111} mode in a rectangular cavity of sides a, b, d. Obtain expressions for the resonant frequency Q and the decay constant α. Evaluate the Q for the case when $a = b = c = 3$ centimeters and $\sigma = 5.8 \times 10^7$ mhos per meter.

10.17. Obtain the solutions for axially symmetric TE_{n0m} modes in a cylindrical cavity of height d and radius a.

10.18. Prove that k_c^2 is always real for TE and TM modes in an arbitrary perfectly conducting waveguide.

HINT: Start with the two-dimensional divergence theorem taken over the waveguide cross section:

$$\int_{cs} (\nabla_t \cdot \mathbf{A})\, dS = \oint \mathbf{A} \cdot \mathbf{n}\, dl$$

and let

$$\mathbf{A} = E_z^* \nabla_t E_z \; (= H_z^* \nabla_t H_z)$$