CHAPTER 1

VECTOR ANALYSIS

This chapter develops the mathematics of vector analysis that will be needed in the succeeding chapters of the book. Based on the work of this chapter, it is possible to considerably simplify the formulation of the physical laws of electromagnetic theory. Furthermore, manipulation of the equations with the goal of solving physical problems is also greatly facilitated. One of the purposes of this chapter is to lay the necessary groundwork in the use of vector algebra and vector calculus.

Another purpose is also sought in this chapter. For along with the mathematical simplifications in the use of vector analysis, there go certain concomitant philosophical concepts. This chapter, consequently, contains a discussion of fields, the flux representation of vector fields, and some general remarks concerning sources of fields. The definition of the divergence and curl of a field can then be understood as measures of the strength of the sources and vortices of a field. When in the succeeding chapters the specific nature of the electric and magnetic fields is considered, the student will have an appropriate framework into which to fit them.

Although much effort has been directed to the development of a physical basis for the mathematical definitions of this chapter, they may still seem somewhat artificial. The full justification of their utility, and a deepening of their meaning, will become apparent when the physical laws of electromagnetics are considered in the later chapters.

1.1. Scalars and Vectors

In this book we deal with physical quantities that can be measured. The measurement tells how many times a given unit is contained in the quantity measured. The simplest physical quantities are those that are completely specified by a single number, along with a known unit. Such quantities are called scalars. Volume, density, and mass are examples of scalars.

Another group of physical quantities are called vectors. We may see how the vector arises if we consider as an example a linear displacement of a point from a given initial position. It is true that the final position of the point could be described in terms of three scalars, e.g., the cartesian coordinates of the final point with respect to axes chosen through the initial point. But this obscures the fact that the concept of displacement is a single idea and does not depend on a coordinate system. Consequently, we introduce displacements as quantities of a new type and establish a system of rules for their use. All physical quantities which can be represented by such displacements and which obey their respective rules are called vectors.

The vector can be represented graphically by a straight line drawn in the direction of the vector, the sense being indicated by an arrowhead and its length made proportional to the magnitude of the vector. Examples of vector quantities include displacement, acceleration, and force. In this book all vector quantities are designated by boldface type, while their magnitudes only are indicated through the use of italics.

1.2. Addition and Subtraction of Vectors

From the definition of a vector, just given, it is possible to deduce the rule for addition of vectors. Thus, consider two vectors **A** and **B** as illustrated in Fig. 1.1. Vector **A** represents the displacement of a movable point from point 1 to point 2. Vector **B** represents a displacement from point 2 to point 3. The result is equivalent to a total displacement of a point from 1 to 3. This linear displacement from 1 to 3 is called the resultant, or geometric sum of the two displacements (1,2) and (2,3). It is represented by the vector **C**, which we call the sum of the vectors **A** and **B**:

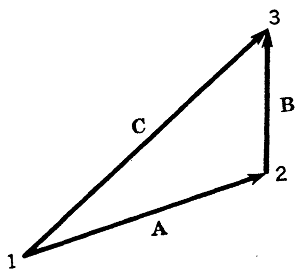

FIG. 1.1. Vector addition.

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.1}$$

Note that vectors **A** and **B** are of the same dimensions and type and that the geometric construction of Fig. 1.1 requires that the origin of one be placed at the head of the other. We may inquire whether the order of addition is of significance.

Consider that the displacement **B** is made first and then the displacement **A**. In this case the movable point describes the path 143 as in Fig. 1.2 and consequently produces the same resultant. Vector addition thus obeys the commutative law; i.e., the geometric sum of two vectors is independent of the order of addition so that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.2}$$

The path 143 and 123 together make up a parallelogram whose diagonal is the resultant of the displacement represented by the two adjacent sides.

Accordingly, the law of vector addition is often referred to as the parallelogram law. This law of addition is characteristic of the quantities called vectors. Thus it is proved in statics that forces acting on a rigid body follow the parallelogram law of addition; consequently, such forces are vectors.

It is easy to show that vectors satisfy the associative law of addition, which states that the order of adding any number of vectors is immaterial. Thus the sum of three vectors **A**, **B**, **C** can be expressed as

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1.3}$$

The proof can be established by considering Fig. 1.3, in which the same resultant (1,4) is arrived at by carrying out the summation indicated by

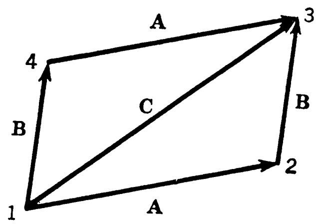

FIG. 1.2. Illustration of parallelogram of vector addition.

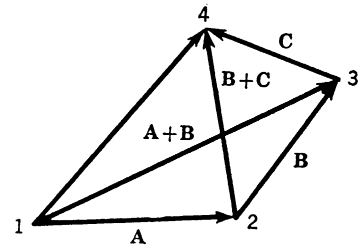

FIG. 1.3. Illustration of associative law of vector addition.

either the left- or right-hand side of Eq. (1.3). The former path is (1,3,4); the latter is (1,2,4).

To obtain the difference of two vectors $\mathbf{A} - \mathbf{B}$, it becomes necessary to define the negative of a vector. This is taken to mean a vector of the same magnitude but of opposite direction to the original vector. Thus

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) \tag{1.4}$$

We may therefore define vector subtraction as follows: A vector **B** is subtracted from a vector **A** by adding to **A** a vector of the same magnitude as **B** but in the opposite direction. In the parallelogram of Fig. 1.2 a diagonal from 4 to 2 would represent the geometric difference $\mathbf{A} - \mathbf{B}$.

1.3. Unit Vectors and Vector Components

The result of multiplying a vector **A** by a positive scalar m is to produce a new vector in the same direction as **A** but whose magnitude is that of **A** times m. The resultant **P** is thus related to **A** and m by the following

$$\mathbf{P} = m\mathbf{A} \tag{1.5}$$

$$|\mathbf{P}| = m|\mathbf{A}| \qquad \text{or} \qquad P = mA \tag{1.6}$$

A unit vector is one whose magnitude is unity. It is often convenient to express a vector as the product of its magnitude and a unit vector having the same direction. Thus if **a** is a unit vector having the direction of **A**, then $\mathbf{A} = A\mathbf{a}$. The result expressed by (1.5) follows immediately, since $m\mathbf{A} = mA\mathbf{a}$. The three unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ parallel to the right-hand rectangular axes x, y, z, respectively, are of particular importance.

The components of a vector are any vectors whose sum is the given vector. We shall often find it convenient to choose as components the three rectangular components of cartesian coordinates. Thus, if A_x, A_y, A_z are the magnitude of the projections of vector **A** on the x, y, z axes, its rectangular components are $\mathbf{a}_xA_x$, $\mathbf{a}_yA_y$, $\mathbf{a}_zA_z$. The vector **A** is completely determined by its components since the magnitude is given by

$$A^2 = A_x^2 + A_y^2 + A_z^2 \tag{1.7}$$

and the direction cosines l, m, n are given by

$$l = \frac{A_x}{A} \qquad m = \frac{A_y}{A} \qquad n = \frac{A_z}{A} \tag{1.8}$$

For brevity, we shall usually designate A_x, A_y, A_z, without the associated unit vectors, as the components of **A**.

Equal vectors have the same magnitude and direction; consequently, their respective rectangular components are equal. Therefore a vector equation can always be reduced, in general, to three scalar equations. For example, $\mathbf{A} + \mathbf{B} = \mathbf{C}$ can be expressed as

$$\mathbf{a}_x(A_x + B_x) + \mathbf{a}_y(A_y + B_y) + \mathbf{a}_z(A_z + B_z) = \mathbf{a}_xC_x + \mathbf{a}_yC_y + \mathbf{a}_zC_z \tag{1.9}$$

i.e., addition is commutative and associative. Since the vector represented by the left-hand side of (1.9) equals that of the right-hand side, we are led to the result

$$A_x + B_x = C_x \qquad A_y + B_y = C_y \qquad A_z + B_z = C_z \tag{1.10}$$

1.4. Vector Representation of Surfaces

Figure 1.4 illustrates a plane surface of arbitrary shape. We may represent this surface by a vector **S** whose length corresponds to the magnitude of the surface area and whose direction is specified by the normal to the surface. To avoid ambiguity, however, some convention must be adopted which establishes the positive sense of the normal.

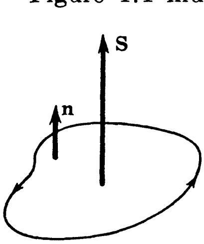

FIG. 1.4. Vector surface area. **n** is a unit surface normal.

When the surface forms part of a closed surface, the positive normal is usually taken as directed outward. For an open surface the positive normal can be associated with the positive sense of describing the periphery. This relationship is defined by taking the positive normal in the direction that a right-hand screw would advance when turned so as to describe the positive periphery. This definition actually arises out of a mathematical description of certain physical phenomena which will be discussed in later chapters. One can choose either positive periphery or positive normal arbitrarily.

If the surface is not plane, it is subdivided into elements which are sufficiently small so that they may be considered plane. The vector representing the surface is then found by vector addition of these components. This means that an infinite number of surfaces correspond to a given surface vector. The unit surface normal is designated by **n**.

1.5. The Vector Product of Two Vectors

Certain rules have been set up governing multiplication of vectors. The vector or cross product **A** × **B** of two vectors **A** and **B** is, by definition,

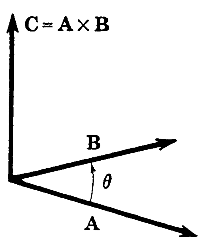

FIG. 1.5. Vector cross product.

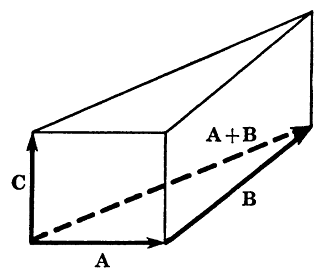

FIG. 1.6. Illustration for the distributive law of vector multiplication.

a vector of magnitude $AB \sin \theta$ in the direction of the normal to the plane determined by **A** and **B**. Its sense is that of advance of a right-hand screw rotated from the first vector to the second through the angle θ between them, as in Fig. 1.5. Since the direction reverses if the order of multiplication is interchanged, the commutative law of multiplication does not hold. Actually, we have

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.11}$$

This definition of vector product was chosen because it corresponds to a class of physically related quantities. Geometrically, the magnitude $|\mathbf{A} \times \mathbf{B}|$ is the area of a parallelogram formed by **A** and **B** as the sides. If we think of the periphery of the parallelogram as described from the origin to head of **A** followed by the origin to head of **B**, then, in accordance with the definitions in the last section, **A** × **B** represents the vector area of the parallelogram.

The preceding geometric interpretation is the basis for a proof that vector multiplication follows the distributive law. Thus, consider the prism described in Fig. 1.6, whose sides are **A**, **B**, **A** + **B**, and **C**. Since the total surface is closed, the vector representing the total surface of the prism is zero (see Prob. 1.6). Consequently, taking the positive normal as directed outward, the sum of the component surface areas may be set equal to zero, giving

$$\tfrac{1}{2}(\mathbf{A} \times \mathbf{B}) + \tfrac{1}{2}(\mathbf{B} \times \mathbf{A}) + \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times (\mathbf{A} + \mathbf{B}) = 0 \qquad (1.12)$$

from which we obtain

$$\mathbf{C} \times (\mathbf{A} + \mathbf{B}) = \mathbf{C} \times \mathbf{A} + \mathbf{C} \times \mathbf{B} \qquad (1.13)$$

Equation (1.13) expresses the distributive law of multiplication.

The vector product of two vectors can be expressed in terms of the rectangular components of each vector. Since the distributive law holds,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) \times (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z) \\ &= \mathbf{a}_x \times \mathbf{a}_x A_x B_x + \mathbf{a}_x \times \mathbf{a}_y A_x B_y + \mathbf{a}_x \times \mathbf{a}_z A_x B_z \\ &\quad + \mathbf{a}_y \times \mathbf{a}_x A_y B_x + \mathbf{a}_y \times \mathbf{a}_y A_y B_y + \mathbf{a}_y \times \mathbf{a}_z A_y B_z \\ &\quad + \mathbf{a}_z \times \mathbf{a}_x A_z B_x + \mathbf{a}_z \times \mathbf{a}_y A_z B_y + \mathbf{a}_z \times \mathbf{a}_z A_z B_z \end{aligned} \qquad (1.14)$$

The sine of the angle between two vectors is zero when they are in the same or opposite directions and is ± 1 when they are orthogonal. It is thus easy to verify that

$$\begin{gathered}\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \qquad \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \qquad \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \\ \mathbf{a}_x \times \mathbf{a}_x = \mathbf{a}_y \times \mathbf{a}_y = \mathbf{a}_z \times \mathbf{a}_z = 0 \end{gathered} \qquad (1.15)$$

so that (1.14) simplifies to

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x) \qquad (1.16)$$

A convenient way of remembering the formula given by (1.16) is to note that it is obtained from the formal expansion of the following determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (1.17)$$

Once one term of the expansion is found, the remaining can be obtained by cyclical permutation; that is, replace x by y, y by z, and z by x. For example, the first term in (1.17) is $\mathbf{a}_x(A_y B_z - A_z B_y)$, from which the second term is found to be $\mathbf{a}_y(A_z B_x - A_x B_z)$ by replacing x, y, z by y, z, x, respectively.

1.6. The Scalar Product of Two Vectors

As mentioned, vector multiplication is useful in mathematically describing the relationship between vectors that arise out of a class of physical problems. In handling another class of physically related quantities, it will be desirable to define a scalar product of two vectors.

The scalar, or dot, product of two vectors **A** and **B**, written $\mathbf{A} \cdot \mathbf{B}$, is a scalar and has the magnitude $AB \cos \theta$, where θ is the angle between the two vectors, as illustrated in Fig. 1.5. From the definition it is clear that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1.18}$$

and consequently the commutative law of multiplication applies.

The scalar product of **A** and **B** may be interpreted as the algebraic product of the magnitude of one of them (say **A**) by the component of the other (**B**) in the direction of the first. Referring to Fig. 1.7, we apply this concept to establish a basic property of the scalar product. We have

$$\mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{D} = OP|\mathbf{D}| + PQ|\mathbf{D}| = OQ|\mathbf{D}| \tag{1.19}$$

But $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{D} = OQ|\mathbf{D}|$; consequently,

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{D} \tag{1.20}$$

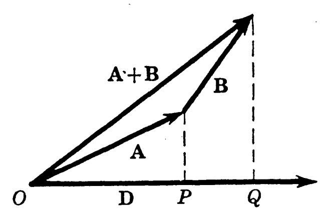

FIG. 1.7. Illustration of distributive law of scalar multiplication.

In other words, the distributive law applies to scalar multiplication.

The cosine of the angle between the directions of two vectors is $+1$ when the directions are the same, -1 when they are opposite, and 0 when they are at right angles. Consequently, the following vector relationships involving the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ must be true:

$$\begin{aligned} \mathbf{a}_x \cdot \mathbf{a}_x &= \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \\ \mathbf{a}_x \cdot \mathbf{a}_y &= \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0 \end{aligned} \tag{1.21}$$

Since the commutative and distributive laws hold, it follows that scalar multiplication of vectors is carried out by the rules of ordinary algebra. In particular, we may expand $\mathbf{A} \cdot \mathbf{B}$ in terms of rectangular components.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) \cdot (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z) \\ &= \mathbf{a}_x \cdot \mathbf{a}_x A_x B_x + \mathbf{a}_x \cdot \mathbf{a}_y A_x B_y + \mathbf{a}_x \cdot \mathbf{a}_z A_x B_z + \mathbf{a}_y \cdot \mathbf{a}_x A_y B_x \\ &\quad + \mathbf{a}_y \cdot \mathbf{a}_y A_y B_y + \mathbf{a}_y \cdot \mathbf{a}_z A_y B_z + \mathbf{a}_z \cdot \mathbf{a}_x A_z B_x + \mathbf{a}_z \cdot \mathbf{a}_y A_z B_y \\ &\quad + \mathbf{a}_z \cdot \mathbf{a}_z A_z B_z \end{aligned} \tag{1.22}$$

With the aid of (1.21) this simplifies to

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.23}$$

A special case is the dot product of **A** with itself, which gives

$$\mathbf{A} \cdot \mathbf{A} = A^2 = A_x A_x + A_y A_y + A_z A_z \tag{1.24}$$

1.7. Product of Three Vectors

Three vectors can be multiplied in three different ways. As an example of one such possibility we consider $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$. This is nothing more than the product of a scalar $(\mathbf{B} \cdot \mathbf{C})$ times a vector $\mathbf{A}$, and the resultant may be evaluated by the rules given in Sec. 1.3.

A second arrangement is known as the triple scalar product. An example is $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$. The vector product is necessarily formed before taking the scalar product, and the resultant is a scalar. This product has a simple geometrical interpretation that is evident in Fig. 1.8. For $\mathbf{B} \times \mathbf{C}$ is clearly the vector area of the base, whereupon $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ is the volume of the parallelepiped. But this volume is calculated equally well by the expressions $\mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$ and $\mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$. Consequently,

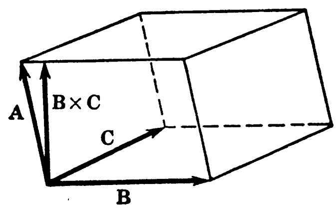

FIG. 1.8. Geometric interpretation of triple scalar product.

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$$

This result is often expressed by the statement that in the triple scalar product the dot and cross may be interchanged and/or the order of the vectors altered by a cyclic rearrangement.

The third arrangement involving the product of three vectors is the triple vector product, exemplified by $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The parentheses indicate which product is taken first, since the result depends on the order of forming the product; that is, the associative law does not hold and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is not the same as $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. By inspection we note that the resultant of $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ lies in the plane of $\mathbf{B}$ and $\mathbf{C}$ and is orthogonal to $\mathbf{A}$. If each vector is expressed in terms of its rectangular components and the indicated operations carried out, it can be verified that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1.25}$$

The details are left as an exercise for the student (see Prob. 1.7).

1.8. Scalar and Vector Fields

If a particle is in motion, then at any instant its velocity can be designated by a vector; i.e., the velocity possesses the properties characteristic of a vector. But if we examine the state of motion of a fluid filling space, then the velocities of the different particles will not be the same, in general. In this case every point has its own velocity vector and the moving continuous fluid can be represented by what is called a vector field.

In mathematical physics the field of a physical quantity refers to nothing more than the dependence of that quantity on position in a region of space. It is assumed that the variation is, ordinarily, a continuous one. The field may be a vector field, as illustrated above, or a scalar field.

The scalar field is simply a scalar function of position in space; that is, there is associated with each point in space a definite scalar magnitude. For example, the barometric pressure at each point on the earth's surface constitutes a scalar field. The field is a scalar field because pressure is a scalar quantity.

Since a field is a function of x, y, z, say, it can also be expressed as a function of a new set of coordinates x', y', z' by an appropriate transformation. Ordinarily, such a transformation brings in the direction cosines of the new axes measured relative to the old axes. But the presence of direction cosines would make any physical law involving the scalar field depend on the choice of axes, which is contrary to the character of the laws of nature. Consequently, the only scalar functions of the coordinates which can enter physical laws are those in which the direction cosines do not appear in an arbitrary transformation of axes. Such a function is called a proper scalar function of the coordinates.

To each point on the surface of the earth a temperature can be measured and a temperature field established. It is convenient to organize this information graphically (or conceptually) by connecting points that are at the same temperature, choosing certain specific values of temperature. In this way the isotherms of a weather map give a rough idea of the temperature field. Scalar fields are sometimes called potential fields, and lines or surfaces over which the field has a constant magnitude are referred to as equipotentials. For example, points on the same contour line of a topological map correspond to points on the earth with the same potential energy, that is, equipotentials.

A vector function of position associates a definite vector with each point of a special region, the aggregate of these vectors constituting a vector field. A simple example is the position vector $\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$, which is a function of position of the point (x,y,z) relative to the origin of the axes chosen. There is thus associated with every point a vector having the magnitude and direction of the line drawn from the origin to the point in question. The field strength at a few points of the field is illustrated in Fig. 1.9.

A vector field may be described in terms of its components at every point in space. In this way one can form three scalar fields from the given vector field. If the component fields are proper scalar functions, then the vector function is a proper vector function.

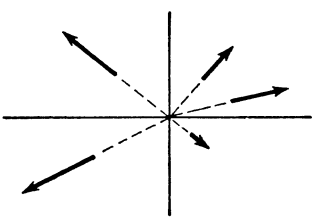

FIG. 1.9. The vector field $\mathbf{r}$.

Among the vector fields that arise in physics are those involving the flow of some quantity. In fluid flow, if $\mathbf{v}$ is the fluid velocity and ρ is the density at (x,y,z), then $\rho\mathbf{v}$ is the vector representing the flow of mass per unit area at each point (x,y,z). Similarly, the electric current density $\mathbf{J}(x,y,z)$ represents the flow of charge per unit area in a current flow field.

As we have already said, a vector field is defined by specifying a vector at each point in space, for example, $\mathbf{J}(x,y,z)$. In most cases of interest this vector is a continuous function of (x,y,z), except possibly at isolated

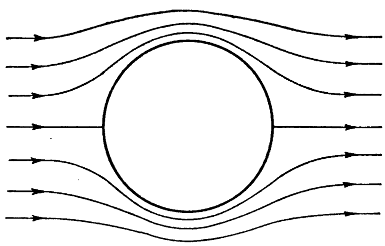

FIG. 1.10. Illustration of flow lines around a sphere.

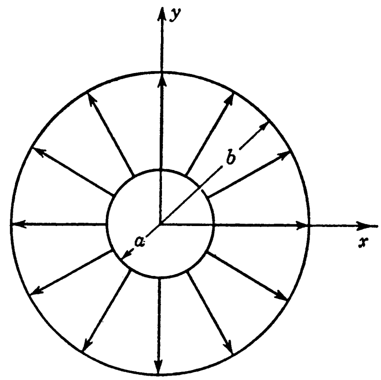

FIG. 1.11. Flow lines between two coaxial cylinders.

points or singularities or along isolated lines or singular lines. Where the vector field is continuous we can define lines of flow of the field, which are lines at every point tangent to the vector at that point. The differential equation for the line can be found by forming a proportionality between the components of displacement along the line dx, dy, dz and the corresponding components of the vector field at the same point:

$$\frac{dx}{J_x} = \frac{dy}{J_y} = \frac{dz}{J_z} \tag{1.26}$$

Just as equipotential surfaces proved convenient in visualizing a scalar field, flow lines are useful in "mapping" vector fields. Obviously, only a few of the infinite number of lines would be drawn in such a map. As a matter of fact, in a two-dimensional field we can represent the field intensity by taking equal quantity of flow between adjacent lines. In this way the density of lines transverse to the direction of flow is proportional to the magnitude of the flow vector. Figure 1.10 illustrates fluid flow through a figure of revolution; adjacent lines actually symbolize concentric tubes, each of which carries the same quantity of fluid.

As a simple example of the evaluation of flow lines, consider the radial flow of current between two coaxial cylinders as illustrated in Fig. 1.11. The current density is given by

$$\mathbf{J} = \frac{I}{2\pi}\frac{\mathbf{a}_r}{r} = \frac{I}{2\pi}\left(\frac{x\mathbf{a}_x}{r^2} + \frac{y\mathbf{a}_y}{r^2}\right)$$

and hence the differential equation for the flow lines is

$$\frac{dx}{J_x} = \frac{dy}{J_y} \qquad \text{or} \qquad \frac{dx}{x} = \frac{dy}{y}$$

Integration gives the result

$$\ln y = \ln x + \ln c = \ln cx$$

or

$$y = cx$$

where c is a constant of integration. Every different value of the constant c gives a new flow line. A complete flow map as illustrated in Fig. 1.11 is obtained by considering a range of values of c. Note that with the values selected, the relative density of the lines graphically corresponds to the relative current density. Figure 1.11 is therefore a particularly appropriate representation of the vector field **J**. A more complete discussion of flux plotting is given in Chap. 5.

The representation of a vector field by means of flow lines seems fairly obvious where the field represents the flow of some quantity. But the technique for construction of flow lines as given in the previous paragraph contains no restrictions on such fields. It may, indeed, be used to represent any vector field. We shall see that the representation of vector fields through the concept of flow or flux (which means the same thing) will prove very useful in our future work.

1.9. Gradient

Let us suppose that $\Phi(x,y,z)$ represents a scalar field and that Φ is a single-valued, continuous, and differentiable function of position. These properties will always be true of the physical fields that we shall encounter. An equipotential surface, then, has the equation

$$\Phi(x,y,z) = C \tag{1.27}$$

where C is a constant. By assigning to C a succession of values, a family of equivalued surfaces is obtained. No two such surfaces will intersect since we have taken Φ to be single-valued.

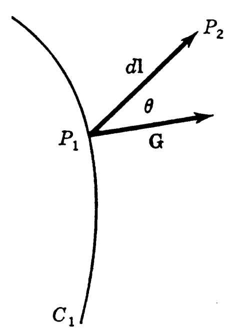

FIG. 1.12

Consider two closely spaced points P_1 and P_2, where P_1 lies on the equipotential surface C_1, and P_2 may or may not be on this surface, as in Fig. 1.12. Let the coordinates of P_1 be (x,y,z); then P_2 is located at $(x + dx,\ y + dy,\ z + dz)$. The displacement $d\mathbf{l}$ from P_1 to P_2 can be expressed in terms of its rectangular components:

$$d\mathbf{l} = \mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz \tag{1.28}$$

We desire to evaluate the difference in potential $d\Phi$ between P_1 and P_2. From the rules of calculus this can be written

$$d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \tag{1.29}$$

It will prove very useful to define the following vector $\mathbf{G}$:

$$\mathbf{G} = \mathbf{a}_x \frac{\partial \Phi}{\partial x} + \mathbf{a}_y \frac{\partial \Phi}{\partial y} + \mathbf{a}_z \frac{\partial \Phi}{\partial z} \tag{1.30}$$

for we may now combine (1.28) and (1.30) to express $d\Phi$ as

$$d\Phi = \mathbf{G} \cdot d\mathbf{l} \tag{1.31}$$

Let us deduce some of the properties of the vector $\mathbf{G}$, as defined above. Suppose that P_2 lies on surface C_1 (see Fig. 1.12). Since P_2 is a differential distance from P_1, $d\mathbf{l}$ will be tangent to the surface C_1 at P_1. Furthermore, $d\Phi$ will, of course, be zero, since Φ is constant on C_1. In order for (1.31) to be satisfied, it is clear that $\mathbf{G}$ is normal to the equipotential surface at P_1.

To determine the magnitude of $\mathbf{G}$, let P_2 be chosen in such a way that $d\mathbf{l}$ makes an angle θ with the normal to C, as illustrated in Fig. 1.12. Since $\mathbf{G}$ is in the direction of the normal, then

$$d\Phi = \mathbf{G} \cdot d\mathbf{l} = G \cos \theta \, dl \tag{1.32}$$

and consequently

$$\frac{d\Phi}{dl} = G \cos \theta \tag{1.33}$$

In words, the component of $\mathbf{G}$ in the direction $d\mathbf{l}$ is the rate of increase of Φ in that direction. The latter is also termed the directional derivative of Φ in the direction $d\mathbf{l}$. If $\theta = 0$, $d\mathbf{l}$ becomes an element normal to the equipotential surface, written dn, and the directional derivative is maximum and equal to the magnitude G; that is,

$$G = \frac{\partial \Phi}{\partial n} \tag{1.34}$$

and is the maximum rate of increase of Φ. Because $\mathbf{G}$ thus coincides with the maximum space rate of increase of Φ, in both direction and magnitude, it is called the gradient of Φ.

The vector $\mathbf{G}$, for the reason just expressed, is often written grad Φ. Another more useful notation involves the ∇ (read del) operator, which is defined as

$$\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \tag{1.35}$$

We shall have much more to say about this operator, but for the present it is sufficient to note that $\mathbf{G} = \text{grad } \Phi = \nabla \Phi$.

The gradient operation may be viewed formally as converting a scalar field into a vector field. We shall discover later that certain vector fields can be specified as the gradient of some fictitious scalar field. The scalar field may then play a simplifying role in the ensuing mathematical analysis.

1.10. The Divergence

We have noted that it may be convenient to think of any vector field as representing the flow of a fluid. It is desirable to specify that the fluid is incompressible. But this restriction would ordinarily prevent us from representing any arbitrary vector field. For example, a consequence of incompressibility is the requirement that as much fluid enter as leave a fixed region. Only special vector fields could be represented by fluid motion of this type (solenoidal fields).

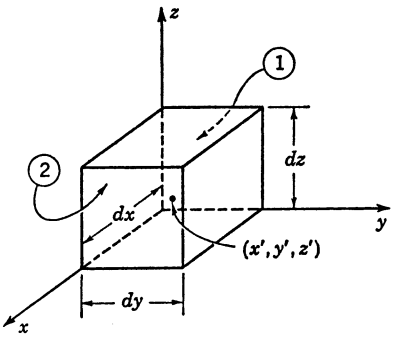

FIG. 1.13. Illustration for evaluation of divergence.

To avoid this limitation we also suppose that fluid at certain points may be created or destroyed. Points of the first sort will be called sources; points of the second kind, sinks, or negative sources. By setting up a suitable source system (including vortices to be discussed later), we may represent any arbitrary vector field by a steady motion of an incompressible fluid. In general, the sources will be found to be continuously distributed in space. We shall now describe how the source strength can be calculated.

To find the source magnitude within a volume V we may measure the volume of fluid leaving in a unit time. Since the fluid is incompressible and in steady motion, this volume of fluid represents the algebraic sum of all the sources contained within V.

Now this same evaluation can be performed by calculating the fluid flow through the bounding surface S of V. If we actually desire the source strength at a point (x',y',z'), we take V to be a differential volume element $dx\,dy\,dz$ enclosing this point as in Fig. 1.13. We assume the existence of a vector field $\mathbf{F}$. A Taylor expansion of F_x (the x component of $\mathbf{F}$) in the vicinity of the point (x',y',z') can be made. The leading terms are

$$F_x = F_x(x',y',z') + \frac{\partial F_x}{\partial x}(x - x') + \frac{\partial F_x}{\partial y}(y - y') + \frac{\partial F_x}{\partial z}(z - z') \quad (1.36)$$

where the partial derivatives are evaluated at (x',y',z'). If (x',y',z') is taken at the center of the volume element, then the outflow through surface 2, shown in Fig. 1.13, is

$$\int_2 F_x\,dS = dy\,dz\left[F_x(x',y',z') + \frac{1}{2}\frac{\partial F_x}{\partial x}\,dx\right] + \text{higher-order terms} \tag{1.37}$$

Notice that the vector is in the positive x direction and hence represents an outflow from the volume V. The outflow through surface 1 involves, in this case, an x component $-F_x$; consequently,

$$-\int_1 F_x\,dS = -dy\,dz\left[F_x(x',y',z') - \frac{1}{2}\frac{\partial F_x}{\partial x}\,dx\right] + \text{higher-order terms} \tag{1.38}$$

The total outflow through surfaces 1 and 2, neglecting higher-order terms, is $(\partial F_x/\partial x)\,dx\,dy\,dz$. In a similar way the contributions from the remaining two pairs of surfaces will be found to be $(\partial F_y/\partial y)\,dx\,dy\,dz$ and $(\partial F_z/\partial z)\,dx\,dy\,dz$. The net outflow is therefore

$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz \tag{1.39}$$

Here $\mathbf{F}\cdot d\mathbf{S}$ is the rate of flow of material through the surface element $d\mathbf{S}$. For example, $\int_\sigma \mathbf{F}\cdot d\mathbf{S}$ evaluates the total flux through a surface σ. In (1.39) the notation $\oint$ signifies that the integral is over a closed surface. It therefore correctly expresses, in vector notation, the net flow of flux through the surface S which bounds V. Integrals of this type are known as surface integrals.

If we divide (1.39) by $dx\,dy\,dz$, the left-hand side becomes the net outflow per unit volume at (x',y',z'). This is a scalar quantity and is called the divergence of the vector $\mathbf{F}$ at the point (x',y',z') and is written

$$\operatorname{div}\mathbf{F} = \lim_{V\to 0}\frac{\oint \mathbf{F}\cdot d\mathbf{S}}{V} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.40}$$

The divergence may be thought of as equal to the rate of increase of lines of flow per unit volume. The div $\mathbf{F}$ is a scalar field which at each point is a measure of the strength of the source of the vector field $\mathbf{F}$ at that point.

Since the operator ∇ defined earlier has the formal properties of a vector, we may form its product with any vector according to the usual rules. Thus if we write the scalar product

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.41}$$

we discover a convenient representation for div $\mathbf{F}$ which leads to the correct expansion in cartesian coordinates. Accordingly, we adopt the notation

$$\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F} \tag{1.42}$$

1.11. Line Integral

There are several properties of vector fields, exhibited through their representation by flux lines, that will concern us. One has to do with the "spreading out," or "outflow," of lines of flow in a given region. As we have already noted, this is a measure of the net strength of the sources within that region. The net outflow is most readily evaluated by means of a surface integral $\oint_S \mathbf{F} \cdot d\mathbf{S}$ over the bounding surface of the region. The magnitude of the resultant is also sometimes referred to as "the number of flux lines crossing S."

The outflow per unit volume at a point measures the divergence of the field at that point and consequently the strength of the source located there. An evaluation of the divergence of a vector field in rectangular coordinates is given in (1.40).

Now, instead of integrating the normal component of a vector over a surface, we can integrate its component along a line. Thus if $d\mathbf{l}$ is a vector element of length along an arbitrary path C, the integral $\int_1^2 \mathbf{F} \cdot d\mathbf{l}$ taken over the extent of the path C is called the line integral. The value of the integral, in general, depends on the path, although for certain vector functions it depends only on the end points. A simple problem that leads to a line integral is the calculation of the work done in moving a particle over the path C, where $\mathbf{F}$ represents the force applied at each point along the contour.

When the path is a closed one, the line integral is denoted by $\oint_C \mathbf{F} \cdot d\mathbf{l}$ and the quantity is called the "net circulation integral" for $\mathbf{F}$ around the chosen path. It is a measure of another vector field property, namely, the curling up of the field lines. For example, if the flux lines are closed loops, then the circulation integral over any such loop will obviously yield a nonzero result. For the fluid-flow analog, $\oint_C \mathbf{F} \cdot d\mathbf{l}$ measures the circulation of fluid around the path C; hence the expression "circulation integral."

For a special class of vector fields, $\mathbf{F}$ can be derived from the gradient of a scalar Φ. The line integral over a path C joining points 1 and 2 can then be written

$$\int_1^2 \mathbf{F} \cdot d\mathbf{l} = \int_1^2 \nabla\Phi \cdot d\mathbf{l} = \int_1^2 \frac{d\Phi}{dl}\, dl \tag{1.43}$$

The last expression arises out of the definition of the directional derivative. Consequently,

$$\int_1^2 \mathbf{F} \cdot d\mathbf{l} = \Phi_2 - \Phi_1 \tag{1.44}$$

where Φ_2 is the value of the scalar potential at point 2, while Φ_1 is the value at point 1. What is most important about this result is that it depends only on the location of the end points and is independent of the path C. Consequently, if the path is a closed one, the line integral vanishes; i.e., we may state that the line integral of a gradient over any closed path vanishes. The absence of circulation in such functions is expressed by designating them irrotational.

We have shown that any vector function that can be derived as the gradient of a scalar potential function is irrotational. It follows, conversely, that if the line integral of a vector function around any closed loop is zero, then the vector function is derivable as the gradient of a scalar function.

1.12. The Curl

We have already defined the divergence operator and noted how it evaluates a certain property of a vector field, namely, its rate of increase of lines of flow. Another very important property of a vector field, just noted, is the amount of "curl" in the flow lines. The latter is related to the magnitude of the circulation. By analogy with fluid flow, the region that "produces" the circulation is a vortex region; hence the "curling" of flow lines may be thought of as related to the "vorticity" of the field. We should like to specify a differential operator that measures the "vorticity." Following our experience with the divergence operator, one might suggest that the vorticity of a field at some point P be found by computing the net circulation integral around an element of area at P and dividing by the area of the element, taking the limit as the area approaches zero.

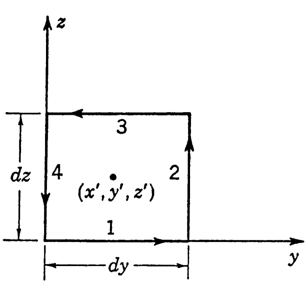

FIG. 1.14. Contour for evaluation of x component of the curl of a vector.

The above definition is not quite satisfactory because the orientation of the differential area is not specified, and the result may be expected to depend on that choice. The process of evaluating the vorticity is thus more complicated than that for finding the divergence. Let us, however, carry out the suggested operation for a differential area in the yz plane as illustrated in Fig. 1.14. If the path shown by the arrows (counterclockwise) is followed, then the net circulation integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_1 F_y\, dy + \int_2 F_z\, dz - \int_3 F_y\, dy - \int_4 F_z\, dz \qquad (1.45)$$

where the total line integral is expressed in terms of its component parts along paths 1, 2, 3, 4. If we let the point (x',y',z') be at the center of the path chosen and expand the field components in a Taylor series expansion about that point [as in (1.36)], then (1.45) can be written

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{l} &= F_y(x',y',z')\, dy - \frac{\partial F_y}{\partial z}\frac{dz}{2}\, dy + F_z(x',y',z')\, dz + \frac{\partial F_z}{\partial y}\frac{dy}{2}\, dz \\ &\quad - F_y(x',y',z')\, dy - \frac{\partial F_y}{\partial z}\frac{dz}{2}\, dy - F_z(x',y',z')\, dz + \frac{\partial F_z}{\partial y}\frac{dy}{2}\, dz \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) dy\, dz \qquad (1.46)\end{aligned}$$

where the partial derivatives are evaluated at the point (x',y',z'). If we divide by the area $dy\, dz$, we obtain $\partial F_z/\partial y - \partial F_y/\partial z$, which is a measure of the circulation per unit area in the yz plane at the point (x',y',z'). This quantity is defined as the x component of the curl of $\mathbf{F}$. The direction of the curl is thus that associated with the direction of advance of a right-hand screw when rotated in the direction in which the fluid circulates.

A similar derivation may be applied to elementary contours in the xy and xz planes to obtain the z and y components of the curl of $\mathbf{F}$. We may obtain the other components of the curl of $\mathbf{F}$ by cyclic permutation of the variables x, y, z as well. The final result is

$$\text{curl } \mathbf{F} = \mathbf{a}_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \qquad (1.47)$$

The curl of $\mathbf{F}$ is a vector quantity. Its component along an arbitrary direction equals the circulation per unit area in the plane normal to that direction. This is clearly true for the direction of the coordinate axes, and will be true for any direction once it is shown that the curl $\mathbf{F}$ is a proper vector function. Proof of the latter is left to the student (see Prob. 1.10).

The vector curl $\mathbf{F}$ clearly is a measure of what we have called the vorticity of the field. It corresponds to the maximum circulation per unit area at a point, the maximum being obtained when the area $d\mathbf{S}$ is so oriented that curl $\mathbf{F}$ is normal to it. When $\mathbf{F}$ represents a fluid velocity, the direction of curl $\mathbf{F}$ at a point P is along the axis of rotation of the fluid close to P, using the right-hand rule.

If we take the vector product $\nabla \times \mathbf{F}$, treating ∇ as a vector according to the rules of vector multiplication, then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \mathbf{a}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad (1.48)$$

But this result is nothing more than curl $\mathbf{F}$ in rectangular coordinates. Because of this, the usual notation for designating the curl of $\mathbf{F}$ is $\nabla \times \mathbf{F}$. The importance of the curl (and divergence) property will be clear later, when we show that an arbitrary vector field is specified by giving its divergence and curl.

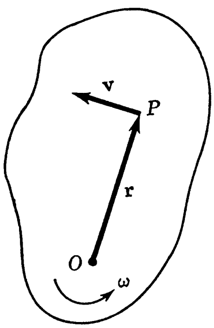

FIG. 1.15. A rotating rigid body.

Example 1.1. We wish to calculate the curl of the vector field $\mathbf{v}$, where $\mathbf{v}$ is found in the following way. Consider an arbitrary rigid body, as illustrated in Fig. 1.15, rotating about an axis through 0 with an angular velocity ω. Then, as we know, the linear velocity at any point P in the body is the product of the angular velocity by the moment arm of the point about the axis. The latter is the perpendicular distance from P to the axis. In vector notation the relation between the linear velocity $\mathbf{v}$, the moment arm vector $\mathbf{r}$, and the angular velocity $\boldsymbol{\omega}$ (the vector angular velocity $\boldsymbol{\omega}$ is defined to be directed along the axis of rotation with a sense established by the advance of a right-hand screw) is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad (1.49)$$

The velocity $\mathbf{v}$ as given above defines a vector field whose curl we desire.

If (1.49) is reduced to rectangular coordinates, the curl operation may be carried out. Thus

$$(\nabla \times \mathbf{v})_x = \frac{\partial(\omega_x y - x\omega_y)}{\partial y} - \frac{\partial(\omega_z x - \omega_x z)}{\partial z} \quad (1.50)$$

Note that $\boldsymbol{\omega}$ is a fixed vector and consequently a constant in the differentiation. Thus

$$(\nabla \times \mathbf{v})_x = 2\omega_x \quad (1.51)$$

Similarly, $(\nabla \times \mathbf{v})_y = 2\omega_y$, $(\nabla \times \mathbf{v})_z = 2\omega_z$, and hence

$$\nabla \times \mathbf{v} = 2\boldsymbol{\omega} \quad (1.52)$$

This result verifies the concept of the curl operation as a measure of the vorticity. Intuitively, $\boldsymbol{\omega}$ measures the vorticity of the $\mathbf{v}$ field. Note that the result is independent of position within the rotating body.

1.13. Successive Application of ∇

It is possible to form scalar and vector products in which the operator ∇ appears more than once. For example, since the gradient is a vector, it is possible to take the divergence of the gradient. If this is expanded in rectangular coordinates, there results

$$\nabla \cdot \nabla\Phi = \nabla \cdot \left(\mathbf{a}_x \frac{\partial \Phi}{\partial x} + \mathbf{a}_y \frac{\partial \Phi}{\partial y} + \mathbf{a}_z \frac{\partial \Phi}{\partial z}\right) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{1.53}$$

The same result is obtained if we think of $\nabla \cdot \nabla$ as a new operator ∇^2 with properties

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.54}$$

The operator ∇^2 is called the Laplacian and is a scalar. It may also be applied to a vector, with the result

$$\nabla^2 \mathbf{F} = \frac{\partial^2 \mathbf{F}}{\partial x^2} + \frac{\partial^2 \mathbf{F}}{\partial y^2} + \frac{\partial^2 \mathbf{F}}{\partial z^2} \tag{1.55}$$

This result is interpreted as three scalar equations; e.g., the x component is

$$\nabla^2 F_x = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2} \tag{1.56}$$

In addition to taking the divergence of the gradient, it is also possible to form the curl of the gradient. The reader should verify that

$$\nabla \times \nabla\Phi = \nabla \times \left(\mathbf{a}_x \frac{\partial \Phi}{\partial x} + \mathbf{a}_y \frac{\partial \Phi}{\partial y} + \mathbf{a}_z \frac{\partial \Phi}{\partial z}\right) = 0 \tag{1.57}$$

by expanding the curl of $\nabla\Phi$ by the determinant rule. The result is not unexpected, since it has already been noted that the gradient is irrotational. Consequently, its curl must vanish everywhere, as is verified by (1.57).

The divergence of the curl of a vector is also identically zero. This can be verified by expansion in rectangular components. Thus

$$\nabla \cdot \nabla \times \mathbf{F} = 0 \tag{1.58}$$

Any vector field that has zero divergence is called solenoidal. This describes the fact that flux lines are closed on themselves since there are no sources or sinks in the field for the lines of flux to terminate on. Equation (1.58) specifies that the curl of any vector field is solenoidal.

An important identity for work involving the wave equation is

$$\nabla \times (\nabla \times \mathbf{F}) \equiv \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \tag{1.59}$$

This can be readily verified by expansion in rectangular coordinates. Equation (1.59) can also be considered as a defining equation for $\nabla^2\mathbf{F}$, when the vectors are expanded in other than rectangular coordinates.

A number of additional vector identities involving composite vectors are summarized at the end of this chapter. They can be verified by expansion in rectangular coordinates, carrying out the indicated operation. As an example, let us consider $\nabla \cdot (\mathbf{A} \times \mathbf{B})$.

Example 1.2. Consider

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \nabla \cdot \Sigma \mathbf{a}_x(A_yB_z - B_yA_z) \tag{1.60}$$

The summation, here, represents cyclical permutation; that is, the second term is obtained by replacing x by y, y by z, z by x. The third term is obtained from the second in a similar fashion. This notation avoids the need to write what is essentially repetitious material. Expanding (1.60) gives

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \sum \frac{\partial}{\partial x}(A_yB_z - B_yA_z) \\ &= \sum \left(A_y \frac{\partial B_z}{\partial x} + B_z \frac{\partial A_y}{\partial x} - B_y \frac{\partial A_z}{\partial x} - A_z \frac{\partial B_y}{\partial x} \right) \\ &= \sum A_y \frac{\partial B_z}{\partial x} + \sum B_z \frac{\partial A_y}{\partial x} - \sum B_y \frac{\partial A_z}{\partial x} - \sum A_z \frac{\partial B_y}{\partial x}\end{aligned} \tag{1.61}$$

Since it does not matter which of the three terms indicated by the summation is chosen to represent the summation, we can write

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \sum A_z \frac{\partial B_x}{\partial y} + \sum B_z \frac{\partial A_y}{\partial x} - \sum B_z \frac{\partial A_x}{\partial y} - \sum A_z \frac{\partial B_y}{\partial x} \\ &= \sum B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \sum A_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)\end{aligned} \tag{1.62}$$

But $\partial B_y/\partial x - \partial B_x/\partial y$ is the z component of the curl of $\mathbf{B}$, and $\partial A_y/\partial x - \partial A_x/\partial y$ is the z component of the curl of $\mathbf{A}$. Accordingly,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \Sigma B_z(\nabla \times A)_z - \Sigma A_z(\nabla \times B)_z \tag{1.63}$$

Expanding the indicated summation yields three terms which are clearly the three terms of a scalar product. Accordingly, the right-hand side of (1.63) can be written in vector notation, and the following identity results:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{1.64}$$

1.14. Gauss' Law

In Sec. 1.9 we noted that the net outflow of "fluid" from a given volume can be found by integrating the scalar product of the vector function (considered as a flow function) and an element of area over the boundary surface. This same quantity can also be found by integrating the divergence throughout the volume since the divergence evaluates the net outflow per unit volume. Expressed mathematically,

$$\int_V \nabla \cdot \mathbf{F}\, dV = \oint_S \mathbf{F} \cdot d\mathbf{S} \tag{1.65}$$

In this equation S is the boundary surface for the volume V, and $d\mathbf{S}$ a surface element with positive normal drawn outward. Equation (1.65) is known as Gauss' law, or sometimes as the divergence theorem.

It is worthwhile pointing out that (1.65) applies even when the bounding surface is not simply connected. For example, in Fig. 1.16 the volume is bounded by the closed surfaces S_1, S_2, and S_3, where S_3 contains S_1 and S_2. In applying Gauss' theorem the volume integral of $\nabla \cdot \mathbf{F}$ is taken throughout the designated volume, while the surface integral of $\mathbf{F} \cdot \mathbf{n}\, dS$ is taken over the bounding surface, in this case the separate component surfaces S_1, S_2, S_3. We conclude that Gauss' theorem is applicable to a volume enclosed by a multiply connected surface; the surface integral in (1.65) then designates an integral over each separate surface involved. It is important to remember that positive surface area is outward from the volume; this accounts for the direction of $\mathbf{n}$ as shown in Fig. 1.16.

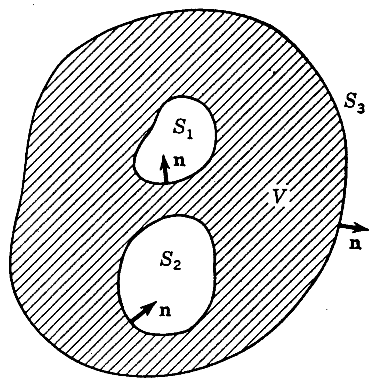

FIG. 1.16. Volume bounded by multiply connected surfaces.

A useful corollary of Gauss' law is known as Green's theorem. To derive this we let $\mathbf{F}$ be the product of a scalar Φ and a vector $\nabla\psi$. Then, using the vector identity (1.118) given at the end of the chapter,

$$\nabla \cdot \mathbf{F} = \nabla \cdot \Phi\, \nabla\psi = \Phi\, \nabla^2\psi + \nabla\Phi \cdot \nabla\psi \tag{1.66}$$

Integrating (1.66) over an arbitrary volume and making use of Gauss' law leads to Green's first theorem:

$$\int_V \nabla \cdot \mathbf{F}\, dV = \oint_S \Phi\, \nabla\psi \cdot d\mathbf{S} = \int_V (\Phi\, \nabla^2\psi + \nabla\Phi \cdot \nabla\psi)\, dV \tag{1.67}$$

The result expressed by (1.67) holds for any two scalar functions Φ and ψ which are finite and continuous and can be differentiated twice in the region V. If, from (1.67), we subtract the equation obtained by interchanging ψ and Φ, we obtain

$$\oint_S \left(\Phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \Phi}{\partial n}\right) dS = \int_V (\Phi \nabla^2 \psi - \psi \nabla^2 \Phi)\, dV \qquad (1.68)$$

Note that $\nabla\psi \cdot d\mathbf{S}$ is replaced by $(\partial\psi/\partial n)\, dS$ in (1.68), where $\mathbf{n}$ is the outward unit normal to S. Equation (1.68) is known as Green's second theorem and is very important in the solution of boundary-value problems in electromagnetic theory.

1.15. Stokes' Theorem

Stokes' theorem may be stated in the form

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \qquad (1.69)$$

where S is an arbitrary surface (not necessarily plane) bounded by the contour C. The positive direction of $d\mathbf{S}$ is related to the positive sense of describing C according to the right-hand rule, as discussed in Sec. 1.4.

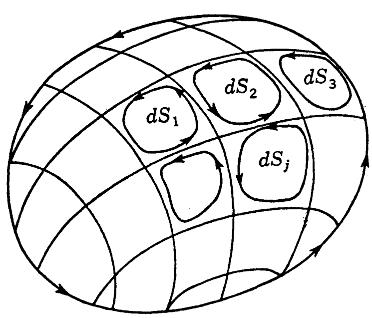

FIG. 1.17. Illustration for proof of Stokes' theorem.

To establish Stokes' theorem, consider the arbitrary surface S illustrated in Fig. 1.17. Let us divide the surface into differential elements of area $d\mathbf{S}_1$, $d\mathbf{S}_2$, $d\mathbf{S}_3$, etc. For each such area element we form $\oint \mathbf{F} \cdot d\mathbf{l}$, taking the contour direction to correspond with the positive sense of S. If now all such integrals are summed, the contributions arising from the common boundary of any two elements (e.g., $d\mathbf{S}_1$ and $d\mathbf{S}_2$) exactly cancel each other, since they are described in opposite directions in the adjoining differential areas. Thus after addition there is left only the integral over the original bounding contour. Consequently,

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \oint_{dS_1} \mathbf{F} \cdot d\mathbf{l} + \oint_{dS_2} \mathbf{F} \cdot d\mathbf{l} + \cdots \qquad (1.70)$$

Now for each integral on the right-hand side of (1.70) the definition of $\nabla \times \mathbf{F}$ can be applied. Thus we have

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = (\nabla \times \mathbf{F}) \cdot d\mathbf{S}_1 + (\nabla \times \mathbf{F}) \cdot d\mathbf{S}_2 + \cdots \qquad (1.71)$$

Note that in view of the choice of describing the contour around each element, each differential area element $d\mathbf{S}_1$, $d\mathbf{S}_2$, . . . corresponds in the same way to the positive contour C. Consequently, the summation indicated on the right-hand side of (1.71) can be expressed as a surface integral. We are thus led to Stokes' theorem as postulated earlier; i.e.,

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{1.72}$$

If we consider two surfaces S_1 and S_2 which have the same contour C, then from Stokes' theorem,

$$\int_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S}_1 = \int_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}_2 \tag{1.73}$$

By reversing the direction of the normal to one of the surfaces, say surface 1, then the direction of positive area is outward (or inward) to $S_1 + S_2$ considered as a closed surface, and we have

$$-\int_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S}_1 + \int_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}_2 = \oint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0 \tag{1.74}$$

In other words, the vector function $\mathbf{A} = \nabla \times \mathbf{F}$ has no net outflow from an arbitrary region. It is therefore solenoidal, a result that we have already noted in Sec. 1.13.

1.16. Orthogonal Curvilinear Coordinates

It should be noted that the fundamental definitions of gradient, divergence, and curl do not involve a particular coordinate system. That we have expressed them, so far, in rectangular coordinates reflects merely that it is easiest to do so. But in a wide variety of problems other coordinate systems will be more appropriate. Accordingly, it is desirable to develop expansions for the preceding differential operations in other such systems. The easiest way of doing this is to work out general formulas in orthogonal curvilinear coordinates. Then the expansions for a specific system (e.g., spherical, cylindrical) can be obtained by substitution of appropriate parameters.

A generalized coordinate system consists of three families of surfaces whose equations in terms of rectangular coordinates are

$$u_1(x,y,z) = \text{constant} \qquad u_2(x,y,z) = \text{constant} \qquad u_3(x,y,z) = \text{constant} \tag{1.75}$$

We are interested only in the case where these three families of surfaces are orthogonal to each other (problems requiring nonorthogonal coordinates practically never can be solved exactly, and approximate techniques usually involve use of orthogonal coordinate systems). Equation (1.75) can be inverted so that (x,y,z) are expressed in terms of (u_1,u_2,u_3).

The lines of intersection of the coordinate surfaces constitute three families of lines, in general curved. At the point (x,y,z) or (u_1,u_2,u_3) we assign three unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ tangent to the corresponding coordinate line at the point. The vector field $\mathbf{F}$ may be expressed in terms of components along these unit vectors, as we have been doing for rectangular coordinates. For the assumed orthogonal system the unit vectors are mutually perpendicular at any point.

Consider the infinitesimal parallelepiped, illustrated in Fig. 1.18, whose faces coincide with the surfaces u_1 or u_2 or u_3 = constant. Since the coordinates need not express a distance directly (e.g., the angles of spherical coordinates), the differential elements of length must be expressed as $dl_1 = h_1\,du_1$, $dl_2 = h_2\,du_2$, $dl_3 = h_3\,du_3$, where h_1, h_2, h_3 are suitable scale factors and may be functions of u_1, u_2, u_3. As an illustration, in cylindrical coordinates (r,ϕ,z), $h_1 = 1$, $h_2 = r$, $h_3 = 1$, since the elements of length along the coordinate curves r, ϕ, z are dr, $r\,d\phi$, and dz. The square of the diagonal dl of the parallelepiped may be written

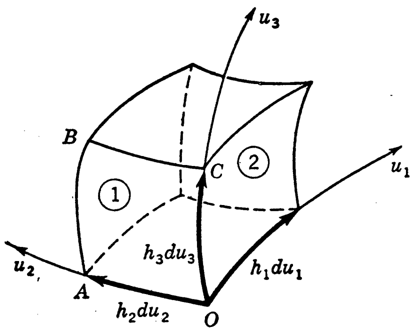

FIG. 1.18. Orthogonal curvilinear coordinates.

$$dl^2 = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2 \tag{1.76}$$

and its volume is $h_1h_2h_3\,du_1\,du_2\,du_3$.

Gradient

Let $\Phi(u_1,u_2,u_3)$ be a scalar function. Then, according to the properties of $\nabla\Phi$, it is a vector whose component in any direction is given by the directional derivative of Φ in that direction. Thus, for the u_1 component, we have

$$(\nabla\Phi)_1 = \frac{\partial\Phi}{\partial l_1} = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} \tag{1.77}$$

and similarly for directions 2 and 3. The resultant vector expansion is

$$\nabla\Phi = \frac{\mathbf{a}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\mathbf{a}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\mathbf{a}_3}{h_3}\frac{\partial\Phi}{\partial u_3} \tag{1.78}$$

Divergence

To calculate the divergence of a vector $\mathbf{A}$ it is necessary to evaluate the net outflow per unit volume in the limit as the volume approaches zero.

If we refer to the differential volume of Fig. 1.18, it is possible to proceed as we did in rectangular coordinates. Let the components of **A** be $(\mathbf{a}_1 A_1, \mathbf{a}_2 A_2, \mathbf{a}_3 A_3)$. Then the flux through surface 1 ($OABC$) taking the outward normal is

$$\text{Flux}_{S_1} = -A_1 h_2 h_3 \, du_2 \, du_3 + \frac{1}{2} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \, du_1 \, du_2 \, du_3$$

while the flux through surface 2 is

$$\text{Flux}_{S_2} = A_1 h_2 h_3 \, du_2 \, du_3 + \frac{1}{2} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \, du_1 \, du_2 \, du_3$$

If we add the outflow for the remaining two surface pairs, the net flux will be found to be

$$\left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] du_1 \, du_2 \, du_3$$

The second and third terms can also be written down by cyclical permutation of the first. From the definition of divergence we can now write

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] \quad (1.79)$$

Curl

The component 1 of the curl can be found by calculating the circulation around contour $OABC$ and dividing by the enclosed surface area. Thus

$$\int_0^A A_2 \, dl_2 + \int_B^C A_2 \, dl_2 = -\frac{\partial}{\partial u_3} (A_2 h_2) \, du_2 \, du_3$$

and

$$\int_A^B A_3 \, dl_3 + \int_C^0 A_3 \, dl_3 = \frac{\partial}{\partial u_2} (A_3 h_3) \, du_2 \, du_3$$

The details are analogous to those in Sec. 1.12. In vector notation the above result, by definition of the curl, leads to

$$(\nabla \times \mathbf{A})_1 = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right] \quad (1.80)$$

By cyclic changes in the indices the remaining components are obtained. Consequently,

$$\begin{aligned}(\nabla \times \mathbf{A}) = \frac{\mathbf{a}_1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right] \\ + \frac{\mathbf{a}_2}{h_3 h_1} \left[\frac{\partial}{\partial u_3} (A_1 h_1) - \frac{\partial}{\partial u_1} (A_3 h_3) \right] \\ + \frac{\mathbf{a}_3}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (A_2 h_2) - \frac{\partial}{\partial u_2} (A_1 h_1) \right] \quad (1.81)\end{aligned}$$

which may be written in determinant form as

$$\nabla \times \mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{a}_1 & h_2\mathbf{a}_2 & h_3\mathbf{a}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \tag{1.82}$$

Laplacian

The Laplacian of a scalar is defined as the divergence of the gradient of the scalar and may be formed by combining (1.79) and (1.78). The result is

$$\nabla^2\Phi = \nabla \cdot \nabla\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right] \tag{1.83}$$

The results of this section are used in Sec. 1.19 to evaluate the aforementioned vector operations in rectangular, cylindrical, and spherical coordinates.

1.17. Point Sources

In physical problems vector fields arise from source distributions which are continuous in space. Nevertheless, it is convenient, mostly from a

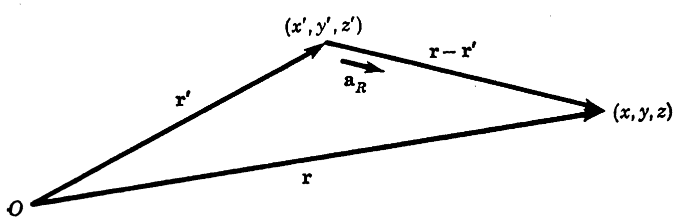

FIG. 1.19. Illustration of notation for source and field points.

mathematical standpoint, to assume the source distribution discontinuous. We shall consider here the characteristics of fields set up by point sources. It should be noted that by properly superposing such point sources, an arbitrary distribution can be represented.

In field theory it is necessary to clearly distinguish between the coordinates that determine the location of the source and the coordinates that designate the point at which the field is being evaluated. In this book primed coordinates x', y', z' will be used to designate the source point while unprimed coordinates x, y, z will be used to designate the field point. The vector $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$ is a vector from the origin to the source point, while $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is a vector from the origin to the field point, as in Fig. 1.19. The vector from the source point to the field point is

$$\mathbf{r} - \mathbf{r}' = (x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z \tag{1.84}$$

with the magnitude of the distance given by

$$|\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \tag{1.85}$$

We shall also use an abbreviated notation, described below, when there is no danger of confusion. In this notation, the magnitude $|\mathbf{r} - \mathbf{r}'|$ will be designated by R and a unit vector directed from the source point to the field point by $\mathbf{a}_R$; thus

$$|\mathbf{r} - \mathbf{r}'| = R \tag{1.86a}$$

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \mathbf{a}_R \tag{1.86b}$$

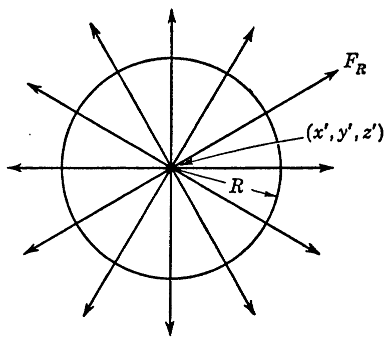

FIG. 1.20. Flow lines from a point source.

The capital letter R is used in order to avoid confusion with the usual notation for spherical coordinates.

For a single point source located at (x',y',z') symmetry requires that the flow lines be radial and diverge uniformly. If we choose any spherical surface whose center is at the point source, as illustrated in Fig. 1.20, the total flux crossing the surface will be independent of the radius. In particular, the total flux computed is a measure of the total outflow from the source—hence is a measure of the strength of the source. If we call this quantity Q and let $\mathbf{F}$ be the vector field, then

$$Q = k \oint_S \mathbf{F} \cdot d\mathbf{S} = k4\pi R^2 F_R \tag{1.87}$$

The surface integral is over a spherical surface of radius R, and it can be evaluated because $\mathbf{F}$ is everywhere radial and of the same magnitude on S. In (1.87) k is a constant of proportionality to be determined on the basis of our definition of source strength. We shall choose $k = 1$, so that

$$Q = 4\pi R^2 F_R \tag{1.88}$$

and hence we must have

$$\mathbf{F} = \frac{Q}{4\pi R^2}\mathbf{a}_R \tag{1.89}$$

where $\mathbf{a}_R$ is a unit vector in the radial direction.

The vector field is irrotational, a fact readily established by demonstrating that $\mathbf{F}$ can be derived as the gradient of a scalar Φ. By inspection it is clear that if $\Phi = Q/4\pi R$, then

$$\mathbf{F} = -\nabla\Phi = -\frac{Q}{4\pi}\mathbf{a}_R\frac{\partial R^{-1}}{\partial R} = \frac{Q}{4\pi R^2}\mathbf{a}_R \tag{1.90}$$

One advantage of representing the vector field $\mathbf{F}$ in terms of a scalar potential Φ becomes apparent when the field due to a series of point sources is required. We could superpose the vector fields $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , $\mathbf{F}_n$ due to Q_1, Q_2, Q_3, . . . , Q_n vectorially, but it is much easier to add algebraically Φ_1, Φ_2, Φ_3, . . . , Φ_n. Then, if $\mathbf{F}_S$ is the resultant field, it can be expressed as

$$\mathbf{F}_S = \sum_{i=1}^{n} \mathbf{F}_i = -\nabla\Phi \qquad \Phi = \sum_{i=1}^{n} \frac{Q_i}{4\pi R_i} \tag{1.91}$$

where R_i is the distance from the source at (x'_i,y'_i,z'_i) to the field point at (x,y,z). This equation shows how any irrotational field can be calculated when its sources are given.

If x'_i, y'_i, z'_i are the coordinates of the ith source, then the potential $\Phi(x,y,z)$ can be expressed by the following, according to (1.91):

$$\Phi(x,y,z) = \sum_{i=1}^{n} \frac{Q_i}{4\pi[(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2]^{1/2}} \tag{1.92}$$

In any physical problem the sources will be confined to some finite region. It is often of interest to compute the potential set up at distances which are very large compared with the extent of the source region. To find this potential, take the origin of coordinates in the neighborhood of the source system and expand (1.92) in powers of x'_i, y'_i, z'_i which are small compared with x, y, z. Then by Taylor's theorem,

$$\Phi = \Phi_0 + \sum_{i=1}^{n} \left(\left.\frac{\partial\Phi}{\partial x'_i}\right|_0 x'_i + \left.\frac{\partial\Phi}{\partial y'_i}\right|_0 y'_i + \left.\frac{\partial\Phi}{\partial z'_i}\right|_0 z'_i + \cdots \right) \tag{1.93}$$

where the index 0 means that the quantities in the parentheses are evaluated for $x'_i = y'_i = z'_i = 0$. Thus

$$\Phi_0 = \sum_{i=1}^{n} \frac{Q_i}{4\pi(x^2 + y^2 + z^2)^{1/2}} = \sum_{i=1}^{n} \frac{Q_i}{4\pi r} \qquad r = (x^2 + y^2 + z^2)^{1/2}$$

Carrying out the remaining operations indicated in (1.93) with respect to (1.92) leads to

$$\Phi(x,y,z) = \frac{1}{4\pi r} \sum_{i=1}^{n} Q_i + \frac{1}{4\pi r^2} \left(\frac{x}{r} \sum_{i=1}^{n} Q_i x'_i + \frac{y}{r} \sum_{i=1}^{n} Q_i y'_i + \frac{z}{r} \sum_{i=1}^{n} Q_i z'_i + \cdots \right) \tag{1.94}$$

The total source strength is obviously

$$Q = \sum_{i=1}^{n} Q_i$$

We define the moment of the source system as

$$\mathbf{m} = \sum_{i=1}^{n} Q_i \mathbf{r}'_i$$

where the vector $\mathbf{r}'_i = \mathbf{a}_x x'_i + \mathbf{a}_y y'_i + \mathbf{a}_z z'_i$. The first two terms may now be written as

$$\Phi = \frac{Q}{4\pi r} + \frac{\mathbf{m} \cdot \mathbf{r}}{4\pi r^3} \tag{1.95}$$

Note that to a first approximation the system of sources acts at great distances like a point source of strength

$$Q = \sum_{i=1}^{n} Q_i$$

The second term is a dipole term, about which more will be said in a later chapter.

1.18. Helmholtz's Theorem

All vector fields will be found to be made up of one or both of two fundamental types: solenoidal fields that have identically zero divergence everywhere and irrotational fields that have zero curl everywhere. The most general vector field will have both a nonzero divergence and a nonzero curl. We shall show that this field can always be considered as the sum of a solenoidal and an irrotational field. This statement is essentially the content of Helmholtz's theorem. Another way of stating the Helmholtz theorem is that a vector field is completely specified by its divergence and curl. The latter constitute the source and vortex source of the field, respectively. Before proceeding to the general case we shall treat the two special cases mentioned above first. Many of the properties of a vector field, whether it be an electric, magnetic, velocity, etc., field, stem directly from its solenoidal or irrotational characteristic. When we consider the electric and magnetic fields in later chapters, it will be seen that they fit into the general framework presented in this section.

Case 1. *Irrotational Field*

A vector field **F** that has zero curl or rotation everywhere is called an irrotational field. Thus $\nabla \times \mathbf{F} = 0$, but at the same time the divergence of **F** cannot be identically zero or else the field **F** would vanish everywhere.† Hence let

$$\nabla \cdot \mathbf{F} = \rho(x,y,z) \tag{1.96}$$

where ρ is now interpreted as the source function for the field **F**.

The gradient of any scalar function Φ has zero curl, as was noted in Sec. 1.13, and hence the condition $\nabla \times \mathbf{F} = 0$ is satisfied if we take

$$\mathbf{F} = -\nabla\Phi \tag{1.97}$$

since $\nabla \times \nabla\Phi = 0$. The minus sign is chosen arbitrarily so that these results compare directly with later work in the book; a positive sign would be an equally correct choice. Substituting into (1.96) we get

$$-\nabla \cdot \mathbf{F} = \nabla^2\Phi = -\rho \tag{1.98}$$

Thus the scalar function Φ, which is called the scalar potential, is a solution of (1.98), a partial differential equation known as Poisson's equation. Once a solution for Φ has been found, we may obtain our vector field **F** at once from (1.97).

Case 2. Solenoidal Fields

A vector field for which $\nabla \cdot \mathbf{F} = 0$ is called a solenoidal field. In a field of this type all the flow lines are continuous and close upon themselves. If $\nabla \cdot \mathbf{F} = 0$, we cannot have an identically vanishing curl or again our field **F** would vanish.† Thus let

$$\nabla \times \mathbf{F} = \mathbf{J}(x,y,z) \tag{1.99}$$

The vector function **J** is the vortex source for the field **F**. It must be a vector source function, since $\nabla \times \mathbf{F}$ is a vector.

A mathematical identity that has been established is $\nabla \cdot \nabla \times \mathbf{A} = 0$, where **A** is any vector function. Thus $\nabla \times \mathbf{A}$ is a solenoidal field, and hence we may take

$$\mathbf{F} = \nabla \times \mathbf{A} \tag{1.100}$$

The vector **A** is called the vector potential since it plays a role similar to that of the scalar potential Φ. Whether **A** has any significant physical properties is usually of little importance since the use of a vector potential is mainly to facilitate the integration of (1.99).

If we substitute (1.100) into (1.99), we obtain

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} = \mathbf{J} \tag{1.101}$$

after expanding the curl-curl operation. If $\nabla \cdot \mathbf{A}$ could be taken as zero,

† Although this result is plausible in that $\nabla \times \mathbf{F} = \nabla \cdot \mathbf{F} = 0$ signifies that there are no sources or vortex sources, we have yet to demonstrate the conclusion. It is a consequence of the Helmholtz theorem

(1.101) would simplify to

$$\nabla^2 \mathbf{A} = -\mathbf{J} \tag{1.102}$$

and **A** would be a solution of the vector Poisson equation; i.e., each component of **A** satisfies the scalar Poisson equation. For example,

$$\nabla^2 A_x = -J_x$$

On the basis of the Helmholtz theorem the divergence of **A** is at our disposal since only its curl has been specified thus far. Consequently, we can always choose **A** so that $\nabla \cdot \mathbf{A} = 0$. This can also be demonstrated in the following way. In place of the potential **A** we could use a potential

$$\mathbf{A}' = \mathbf{A} + \nabla\psi$$

where ψ is an arbitrary scalar function. This will not change the value of **F** obtained from (1.100) since

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \nabla\psi = \nabla \times \mathbf{A}$$

If **A** does not have a zero divergence, then we use the potential **A′** and choose ψ so that $\nabla \cdot \mathbf{A}' = 0$, that is, so that $\nabla \cdot \mathbf{A} + \nabla^2\psi = 0$. Since a function ψ can always be found that satisfies this (Poisson's) equation, a function **A′** with zero divergence and with curl equal to **F** can always be obtained.

Case 3. *General Vector Field*

Helmholtz's theorem states that the most general vector field will have both a nonzero divergence and a nonzero curl and, furthermore, can be derived from the negative gradient of a scalar potential Φ and the curl of a vector potential **A**. In view of our discussion above, this statement is fairly obvious, since a general field would be simply a superposition of the two types of fields discussed separately. It will, nevertheless, be instructive to examine the mathematical statement of Helmholtz's theorem. A proof of the theorem is to be found in the following section.

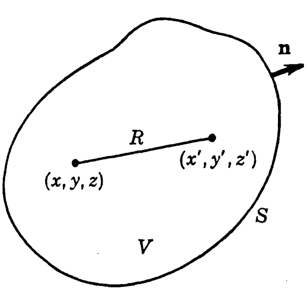

FIG. 1.21. Illustration of Helmholtz's theorem.

Consider a volume V bounded by a closed surface S, as in Fig. 1.21. A mathematical identity (proved later) states that the vector field **F** at the point (x,y,z) is given by

$$\mathbf{F}(x,y,z) = -\nabla\left[\int_V \frac{\nabla' \cdot \mathbf{F}(x',y',z')}{4\pi R}\,dV' - \oint_S \frac{\mathbf{F}(x',y',z') \cdot \mathbf{n}}{4\pi R}\,dS'\right] + \nabla \times \left[\int_V \frac{\nabla' \times \mathbf{F}(x',y',z')}{4\pi R}\,dV' + \oint_S \frac{\mathbf{F}(x',y',z') \times \mathbf{n}}{4\pi R}\,dS'\right] \tag{1.103}$$

where $\nabla' \equiv \mathbf{a}_x(\partial/\partial x') + \mathbf{a}_y(\partial/\partial y') + \mathbf{a}_z(\partial/\partial z')$ operates on the source coordinates, the integration is over the source coordinates (x',y',z'), and $\mathbf{n}$ is a unit normal directed out of the volume V. This is the mathematical statement of Helmholtz's theorem.

The term $\nabla' \cdot \mathbf{F}(x',y',z')$ gives the source function $\rho(x',y',z')$, while the term $\nabla' \times \mathbf{F}(x',y',z')$ determines the (vortex)† source function $\mathbf{J}(x',y',z')$.

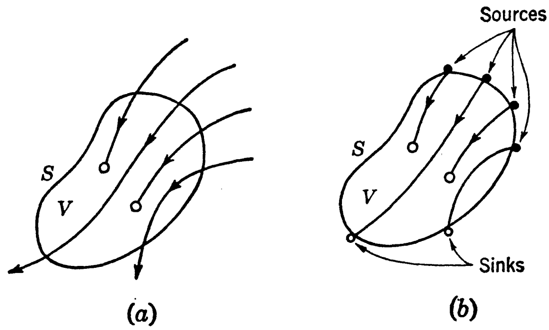

FIG. 1.22. Illustration of need for surface sources in Helmholtz's theorem.

The surface integrals represent integration over the surface sources on S. If S recedes to infinity, the field $\mathbf{F}$ will generally vanish at infinity, and hence the surface sources will vanish also. If V is finite, however, sources will occur on the surface S in general.

The physical significance of the surface sources may be seen as follows. Consider the situation where the flow lines of $\mathbf{F}$ extend into the volume V from outside the surface S, as in Fig. 1.22*a*. If we sever or cut off these flow lines at the surface S, then the field inside V can be maintained at its original value only if we place an equivalent source on the surface S to produce the same flow into the volume V as was produced by the original sources outside V. This situation is illustrated in Fig. 1.22*b*. The strength of the surface source must be equal to the original flow per unit area across S and hence equal to $-\mathbf{F} \cdot \mathbf{n}$. The minus sign arises since $\mathbf{F} \cdot \mathbf{n}$ is a measure of the outward flow, whereas the source strength must equal the inward flow. The other surface source $\mathbf{F} \times \mathbf{n}$ arises for similar reasons and is the equivalent vortex source that must be placed on S in order to maintain the proper circulation for the field $\mathbf{F}$ in V.

We now let $-\mathbf{F} \cdot \mathbf{n} = \sigma$ and $\mathbf{F} \times \mathbf{n} = \mathbf{K}$, where σ and $\mathbf{K}$ are the equivalent surface sources. The scalar and vector potentials are next

† In the future little effort will be made to distinguish between the two types of sources; both will be referred to as sources, and the context will clarify whether it is a source or vortex source.

defined to be

$$\Phi(x,y,z) = \int_V \frac{\rho(x',y',z')}{4\pi R}\, dV' + \oint_S \frac{\sigma(x',y',z')}{4\pi R}\, dS' \qquad (1.104a)$$

$$\mathbf{A}(x,y,z) = \int_V \frac{\mathbf{J}(x',y',z')}{4\pi R}\, dV' + \oint_S \frac{\mathbf{K}(x',y',z')}{4\pi R}\, dS' \qquad (1.104b)$$

Thus in place of (1.103) we have

$$\mathbf{F}(x,y,z) = -\nabla\Phi(x,y,z) + \nabla \times \mathbf{A}(x,y,z) \qquad (1.105)$$

which is the mathematical statement of the second part of Helmholtz's theorem.

In summary, we can thus state:

1. If the curl of **F** is identically zero, then **F** is an irrotational field and can be obtained from the gradient of a scalar potential function.
2. If the divergence of **F** is identically zero, then **F** is a solenoidal field and may be derived from the curl of a vector potential function.
3. A general vector field can be derived from the negative gradient of a scalar potential and the curl of a vector potential.
4. The potentials are determined by the volume and surface source functions ρ, **J** and σ, **K**.

Integration of Poisson's Equation

Let us return to a consideration of the integrals stated in (1.104) and show that the potentials are, indeed, solutions of the Poisson equation. From our discussion on point sources at the beginning of this section we obtained the result that for a point source Q

$$\Phi = \frac{Q}{4\pi R}$$

If instead of a point source Q we have a distribution of point sources with a volume density $\rho(x',y',z')$, it follows, by superposition, that the potential is given by

$$\Phi(x,y,z) = \int_V \frac{\rho(x',y',z')}{4\pi R}\, dV' \qquad (1.106)$$

Consequently, from (1.90) and (1.98), Φ defined by (1.106) satisfies Poisson's equation. Although this proof is probably satisfactory from an intuitive point of view, it is worthwhile to show mathematically that Φ as given by (1.106) is a solution of the Poisson equation

$$\nabla^2\Phi(x,y,z) = -\rho(x,y,z) \qquad (1.107)$$

The mathematical details involved are themselves of great importance.

The Laplacian of (1.106) is

$$\nabla^2\Phi(x,y,z) = \int_V \frac{\rho(x',y',z')}{4\pi}\, \nabla^2\left(\frac{1}{R}\right) dV'$$

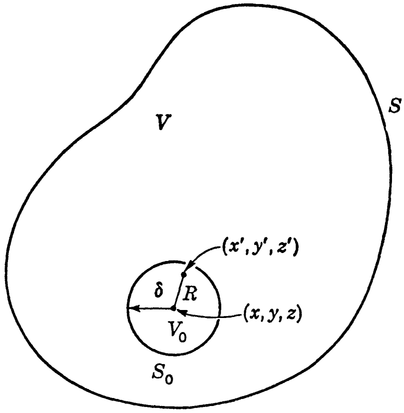

FIG. 1.23. Sphere surrounding singularity point (x,y,z).

We may bring the ∇^2 operator inside the integral because ∇^2 affects the x, y, z variables only while the integration is over the x', y', z' variables. Next we note that $\nabla^2(1/R)$ equals zero at all points except at the singularity point $R = 0$ (see Prob. 1.11). Thus the volume integral is zero except, possibly, for a contribution from the singular point $R = 0$. As x', y', z' approach x, y, z, R tends toward zero. Our procedure is to surround the singular point (x,y,z) by a small sphere of radius δ, surface S_0, and volume V_0, as in Fig. 1.23. Since $\rho(x',y',z')$ is a continuous function, we may choose δ so small that for all values of x', y', z' inside the sphere, ρ is essentially equal to its value $\rho(x,y,z)$ at the singular point.

Our integral now becomes

$$\int_V \frac{\rho(x',y',z')}{4\pi} \nabla^2 \left(\frac{1}{R}\right) dV' = \frac{\rho(x,y,z)}{4\pi} \int_{V_0} \nabla^2 \left(\frac{1}{R}\right) dV'$$

We designate by ∇' the del operator, which has been defined as

$$\nabla' \equiv \mathbf{a}_x \frac{\partial}{\partial x'} + \mathbf{a}_y \frac{\partial}{\partial y'} + \mathbf{a}_z \frac{\partial}{\partial z'}$$

so that ∇' operates on the source coordinates. Similarly,

$$\nabla'^2 \equiv \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$$

Since $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$, then we can confirm by direct expansion that $\nabla(1/R) = -\nabla'(1/R)$, and $\nabla^2(1/R) = \nabla'^2(1/R)$. Using the latter identity and the divergence theorem in the above volume integral, we obtain

$$\frac{\rho}{4\pi} \int_{V_0} \nabla'^2 \left(\frac{1}{R}\right) dV' = \frac{\rho}{4\pi} \oint_{S_0} \nabla' \left(\frac{1}{R}\right) \cdot d\mathbf{S}_0'$$

Now $\nabla'(1/R) = -\mathbf{a}_R'/R^2$ and $d\mathbf{S}_0' = \mathbf{a}_R' R^2 \, d\Omega$, where $\mathbf{a}_R'$ is a unit vector directed outward from the point (x,y,z) and $d\Omega$ is an element of solid angle. Substitution of these relations now shows finally that

$$\nabla^2\Phi(x,y,z) = -\frac{\rho(x,y,z)}{4\pi} \oint_{S_0} d\Omega = -\rho(x,y,z)$$

and hence verifies that Φ as given by (1.106) is a solution of Poisson's equation.

For the vector potential **A** each component is a solution of the scalar Poisson equation; so it follows by vector addition that the solution to $\nabla^2\mathbf{A} = -\mathbf{J}$ is given by

$$\mathbf{A} = \int_V \frac{\mathbf{J}(x',y',z')}{4\pi R}\,dV'$$

For surface sources the solutions are the same, with the exception that the integration is now over a surface instead of throughout a volume.

Proof of Helmholtz's Theorem

In view of the properties of the function $\nabla^2(1/R)$ as discussed in connection with the integration of Poisson's equation, it is clear that the vector function $\mathbf{F}(x,y,z)$ can be represented as

$$\begin{aligned}\mathbf{F}(x,y,z) &= -\int_V \frac{\mathbf{F}(x',y',z')}{4\pi}\nabla^2\left(\frac{1}{R}\right)dV' \\ &= -\nabla^2\int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV'\end{aligned}$$

Using the vector identity $\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2$, we may rewrite the above as

$$\mathbf{F}(x,y,z) = \nabla \times \nabla \times \int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' - \nabla\nabla \cdot \int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' \tag{1.108}$$

Consider the divergence term first. We have

$$\nabla \cdot \int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' = \frac{1}{4\pi}\int_V \mathbf{F}(x',y',z') \cdot \nabla\left(\frac{1}{R}\right)dV'$$

since ∇ does not operate on the primed variables. Next we note that

$$\begin{aligned}\mathbf{F}(x',y',z') \cdot \nabla\left(\frac{1}{R}\right) &= -\mathbf{F}(x',y',z') \cdot \nabla'\left(\frac{1}{R}\right) \\ &= -\nabla' \cdot \frac{\mathbf{F}(x',y',z')}{R} + \frac{\nabla' \cdot \mathbf{F}(x',y',z')}{R}\end{aligned}$$

Hence

$$\begin{aligned}\nabla \cdot \int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' &= -\int_V \nabla' \cdot \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' + \int_V \frac{\nabla' \cdot \mathbf{F}(x',y',z')}{4\pi R}\,dV' \\ &= -\oint_S \frac{\mathbf{F}(x',y',z') \cdot \mathbf{n}}{4\pi R}\,dS' + \int_V \frac{\nabla' \cdot \mathbf{F}(x',y',z')}{4\pi R}\,dV' = \Phi\end{aligned} \tag{1.109}$$

which is the desired form for the scalar potential Φ.

We now return to the curl term in (1.108) and note that

$$\begin{aligned}\nabla \times \int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' &= -\frac{1}{4\pi}\int_V \mathbf{F}(x',y',z') \times \nabla\left(\frac{1}{R}\right)dV' \\ &= \frac{1}{4\pi}\int_V \mathbf{F}(x',y',z') \times \nabla'\left(\frac{1}{R}\right)dV'\end{aligned}$$

We next use the relation

$$\nabla' \times \frac{\mathbf{F}(x',y',z')}{R} = -\mathbf{F}(x',y',z') \times \nabla'\left(\frac{1}{R}\right) + \frac{\nabla' \times \mathbf{F}(x',y',z')}{R}$$

to get

$$\nabla \times \int_V \frac{\mathbf{F}(x',y',z')\,dV'}{4\pi R} = \int_V \frac{\nabla' \times \mathbf{F}(x',y',z')}{4\pi R}\,dV' - \frac{1}{4\pi}\int_V \nabla' \times \frac{\mathbf{F}(x',y',z')}{R}\,dV' \tag{1.110}$$

The first integral on the right-hand side is in the desired form. The remaining step is to show that

$$-\frac{1}{4\pi}\int_V \nabla' \times \frac{\mathbf{F}(x',y',z')}{R}\,dV' = \oint_S \frac{\mathbf{F}(x',y',z') \times \mathbf{n}}{4\pi R}\,dS' \tag{1.111}$$

To prove this result let $\mathbf{C}$ be a constant vector and apply the divergence theorem to the quantity $\nabla' \cdot \mathbf{C} \times \mathbf{F}/R$ to obtain

$$\begin{aligned}\int_V \nabla' \cdot \mathbf{C} \times \frac{\mathbf{F}}{R}\,dV' &= -\int_V \mathbf{C} \cdot \nabla' \times \frac{\mathbf{F}}{R}\,dV' \\ &= \oint_S \mathbf{C} \times \frac{\mathbf{F}}{R} \cdot \mathbf{n}\,dS'\end{aligned} \tag{1.112}$$

In the surface integral we have

$$\mathbf{C} \times \frac{\mathbf{F}}{R} \cdot \mathbf{n} = \mathbf{C} \cdot \frac{\mathbf{F} \times \mathbf{n}}{R}$$

and (1.112) becomes

$$-\mathbf{C} \cdot \int_V \nabla' \times \frac{\mathbf{F}}{R}\,dV' = \mathbf{C} \cdot \oint_S \frac{\mathbf{F} \times \mathbf{n}}{R}\,dS'$$

Since $\mathbf{C}$ is an arbitrary vector, the two integrals are equal and the relation (1.111) is verified. Thus we have

$$\nabla \times \int_V \frac{\mathbf{F}(x',y',z')}{4\pi R}\,dV' = \int_V \frac{\nabla' \times \mathbf{F}(x',y',z')}{4\pi R}\,dV' + \oint_S \frac{\mathbf{F}(x',y',z') \times \mathbf{n}}{4\pi R}\,dS' = \mathbf{A} \tag{1.113}$$

Consequently, it now follows that

$$\mathbf{F} = -\nabla\Phi + \nabla \times \mathbf{A}$$

when (1.109) and (1.113) are used in (1.108). This completes the proof of Helmholtz's theorem.

1.19. Vector Summary

From the general equations in orthogonal curvilinear coordinates the vector operations in the three most common systems are found to be as follows.

Rectangular Coordinates

$$u_1 = x \qquad u_2 = y \qquad u_3 = z$$
$$h_1 = 1 \qquad h_2 = 1 \qquad h_3 = 1$$

$$\nabla\Phi = \mathbf{a}_x \frac{\partial \Phi}{\partial x} + \mathbf{a}_y \frac{\partial \Phi}{\partial y} + \mathbf{a}_z \frac{\partial \Phi}{\partial z}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \mathbf{F} = \mathbf{a}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{a}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{a}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\nabla^2\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Cylindrical Coordinates

$$u_1 = r \qquad u_2 = \phi \qquad u_3 = z$$
$$h_1 = 1 \qquad h_2 = r \qquad h_3 = 1$$

$$\nabla\Phi = \mathbf{a}_r \frac{\partial \Phi}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial \Phi}{\partial \phi} + \mathbf{a}_z \frac{\partial \Phi}{\partial z}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \mathbf{F} = \mathbf{a}_r \left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) + \mathbf{a}_\phi \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) + \mathbf{a}_z \left[\frac{1}{r}\frac{\partial (rF_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Spherical Coordinates

$$u_1 = r \qquad u_2 = \theta \qquad u_3 = \phi$$
$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$$

$$\nabla\Phi = \mathbf{a}_r \frac{\partial \Phi}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{\mathbf{a}_\phi}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{F} = \frac{\mathbf{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right] + \frac{\mathbf{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right] + \frac{\mathbf{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}$$

Vector Identities

$$\nabla(\Phi + \psi) = \nabla\Phi + \nabla\psi \tag{1.114}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{1.115}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{1.116}$$

$$\nabla(\Phi\psi) = \Phi\, \nabla\psi + \psi\, \nabla\Phi \tag{1.117}$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A} \tag{1.118}$$

$$\nabla \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{1.119}$$

$$\nabla \times (\Phi\mathbf{A}) = \nabla\Phi \times \mathbf{A} + \Phi\nabla \times \mathbf{A} \tag{1.120}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{1.121}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{1.122}$$

$$\nabla \cdot \nabla\Phi = \nabla^2\Phi \tag{1.123}$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \tag{1.124}$$

$$\nabla \times \nabla\Phi = 0 \tag{1.125}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{1.126*}$$

$$\int_V \nabla\Phi\, dV = \oint_S \Phi\, d\mathbf{S} \tag{1.127}$$

$$\int_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{1.128}$$

$$\int_V \nabla \times \mathbf{A}\, dV = \oint_S \mathbf{n} \times \mathbf{A}\, dS \tag{1.129}$$

$$\int_S \mathbf{n} \times \nabla\Phi\, dS = \oint_C \Phi\, d\mathbf{l} \tag{1.130}$$

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l} \tag{1.131}$$

* In rectangular coordinates $\nabla^2\mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z$, but in curvilinear coordinates (1.126) defines $\nabla^2\mathbf{A}$, i.e., $\nabla^2\mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A}$. The simple expansion in rectangular coordinates is possible only because the orientations of the unit vectors are independent of position.

PROBLEMS

Chapter 1

1.1. (*a*) Find the sum and difference of the following two vectors:

$$\mathbf{A} = 4\mathbf{a}_x + 2\mathbf{a}_y - 2\mathbf{a}_z \qquad \mathbf{B} = 2\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$$

(*b*) Show that the two vectors in part *a* are orthogonal.

1.2. Derive the law of cosines by squaring both sides of the equation $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

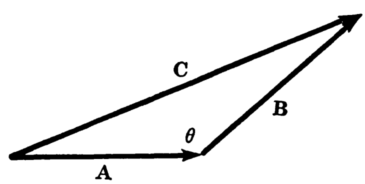

FIG. P 1.2

1.3. (*a*) Show that the direction cosines between each of the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ and the unit vectors $(\mathbf{a}_r, \mathbf{a}_\phi, \mathbf{a}_z)$ in a cylindrical coordinate system are $(\cos\phi, -\sin\phi, 0)$, $(\sin\phi, \cos\phi, 0)$, and $(0,0,1)$, respectively.

HINT: Note that the direction cosine between $\mathbf{a}_x$ and $\mathbf{a}_r$ is given by $\mathbf{a}_x \cdot \mathbf{a}_r$, etc.

(*b*) Show that the direction cosines between each of the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ and the unit vectors $(\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi)$ in a spherical coordinate system are $(\sin\theta\cos\phi, \cos\theta\cos\phi, -\sin\phi)$, $(\sin\theta\sin\phi, \cos\theta\sin\phi, \cos\phi)$, and $(\cos\theta, -\sin\theta, 0)$, respectively, where θ is the polar angle measured from the z axis.

HINT: Find the projection of $\mathbf{a}_r$ and $\mathbf{a}_\theta$ on the xy plane first.

1.4. Consider a force $\mathbf{F}$ acting at a point which is specified by the position vector $\mathbf{r}$. Show that the torque about an axis defined by the unit vector $\mathbf{a}$ is given by $T = (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{a}$.

1.5. Find the components of the following vector along the coordinate directions in a cylindrical and spherical coordinate system, $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y - 3\mathbf{a}_z$.

HINT: The component A_ϕ along the unit vector $\mathbf{a}_\phi$ in a cylindrical coordinate system is the projection of A on $\mathbf{a}_\phi$; that is, $A_\phi = \mathbf{A} \cdot \mathbf{a}_\phi$, etc. To evaluate the dot products use the results of Prob. 1.3.

1.6. Show that the total vector surface of a closed surface is zero.

HINT: Consider a small plane area, and first show that its projection on any coordinate plane is the component of the vector surface on the axis perpendicular to the coordinate plane. Then by superposition for an arbitrary curved surface, the net component of its vector surface along any coordinate direction is the projection of the surface on the coordinate plane normal to that direction.

1.7. Express **A**, **B**, and **C** in rectangular components, and verify that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

1.8. A five-sided prism as illustrated has its corners at (0,0,0), (2,0,0), (0,2,0), (0,2,3), (0,0,3), and (2,0,3). Evaluate the vector area of each side, and show that the total vector area is zero.

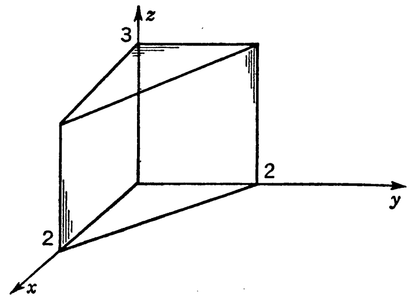

FIG. P 1.8

1.9. Let **A**, **B**, **C** represent position vectors from the origin to three arbitrary points A, B, and C. Prove that the vector

$$\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}$$

is orthogonal to the plane determined by A, B, C.

HINT: Note that the vector $\mathbf{A} - \mathbf{B}$, $\mathbf{B} - \mathbf{C}$, or $\mathbf{A} - \mathbf{C}$ lies in the plane determined by the points A, B, C.

1.10. The curl of **F** will be a proper vector function if its form is independent of the choice of axes. Establish this fact by showing that

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{x'} & \mathbf{a}_{y'} & \mathbf{a}_{z'} \\ \dfrac{\partial}{\partial x'} & \dfrac{\partial}{\partial y'} & \dfrac{\partial}{\partial z'} \\ F_{x'} & F_{y'} & F_{z'} \end{vmatrix}$$

where the primed and unprimed rectangular coordinate systems are arbitrarily oriented.

HINT: **F** itself is tacitly assumed to be a proper vector function, so that

$$\mathbf{F} = \mathbf{a}_x F_x + \mathbf{a}_y F_y + \mathbf{a}_z F_z = \mathbf{a}_{x'} F_{x'} + \mathbf{a}_{y'} F_{y'} + \mathbf{a}_{z'} F_{z'}$$

It is then sufficient to show that

$$\mathbf{a}_{x'} \frac{\partial \Phi}{\partial x'} + \mathbf{a}_{y'} \frac{\partial \Phi}{\partial y'} + \mathbf{a}_{z'} \frac{\partial \Phi}{\partial z'} = \mathbf{a}_x \frac{\partial \Phi}{\partial x} + \mathbf{a}_y \frac{\partial \Phi}{\partial y} + \mathbf{a}_z \frac{\partial \Phi}{\partial z}$$

since then

$$\mathbf{a}_{x'} \frac{\partial}{\partial x'} + \mathbf{a}_{y'} \frac{\partial}{\partial y'} + \mathbf{a}_{z'} \frac{\partial}{\partial z'} = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

This will be facilitated by noting the transformation of a point from one system to the other; i.e.,

$$\begin{aligned} x' &= l_{11}x + l_{12}y + l_{13}z & \qquad x &= l_{11}x' + l_{21}y' + l_{31}z' \\ y' &= l_{21}x + l_{22}y + l_{23}z & \qquad y &= l_{12}x' + l_{22}y' + l_{32}z' \\ z' &= l_{31}x + l_{32}y + l_{33}z & \qquad z &= l_{13}x' + l_{23}y' + l_{33}z' \end{aligned}$$

where l_{11}, l_{12}, l_{13} are the direction cosines of x' relative to x, y, z, etc.

1.11. By direct differentiation show that $\nabla^2(1/R) = 0$ at all points $R \neq 0$, where

$$R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$$

1.12. Find the gradient of the function $\Psi = x^2yz$ and also the directional derivative of Ψ in the direction specified by the unit vector $(3/\sqrt{50})\mathbf{a}_x + (4/\sqrt{50})\mathbf{a}_y + (5/\sqrt{50})\mathbf{a}_z$, at the point $x = 2$, $y = 3$, $z = 1$.

1.13. By direct differentiation show that $\nabla(1/r) = -\nabla'(1/r)$, where $r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$ and ∇' indicates differentiation with respect to x', y', z'. Also show that, for any function $f(r)$, $\nabla f(r) = -\nabla' f(r)$.

HINT: Note that

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial x'} = \frac{df}{dr}\frac{\partial r}{\partial x'} \qquad \text{etc.}$$

1.14. Find the divergence of the vector function $\mathbf{A} = x^2\mathbf{a}_x + (xy)^2\mathbf{a}_y + 24x^2y^2z^3\mathbf{a}_z$. Evaluate the volume integral of $\nabla \cdot \mathbf{A}$ throughout the volume of a unit cube centered at the origin. Also evaluate the total outward flux of $\mathbf{A}$ over the surface of the cube and thus verify Gauss' law for this particular example.

1.15. Show that the following functions satisfy Laplace's equation in their respective coordinate systems: $\sin kx \sin ly\ e^{-hz}$, where $h^2 = l^2 + k^2$; $r^n(\cos n\phi + A \sin n\phi)$, $r^{-n} \cos n\phi$ (cylindrical coordinates); $r \cos \theta$, $r^{-2} \cos \theta$ (spherical coordinates, no azimuth variation).

1.16. Evaluate the line integral of the vector function $\mathbf{F} = x\mathbf{a}_x + x^2y\mathbf{a}_y + y^2x\mathbf{a}_z$ around the rectangular contour C in the xy plane as illustrated. Also integrate the $\nabla \times \mathbf{F}$ over the surface bounded by C and thus verify that Stokes' law holds for this example.

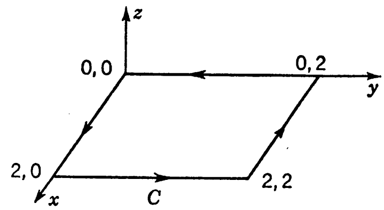

FIG. P 1.16

1.17. Prove the following vector identities: $\nabla \times \nabla\psi = 0$, $\nabla \cdot \nabla \times \mathbf{F} = 0$, $\nabla \times \psi\mathbf{F} = (\nabla\psi) \times \mathbf{F} + \psi\nabla \times \mathbf{F}$, $\nabla \cdot \psi\mathbf{F} = \mathbf{F} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{F}$, where ψ is an arbitrary scalar function and $\mathbf{F}$ is an arbitrary vector function.

1.18. Prove that $\int_V \psi\nabla \cdot \mathbf{F}\, dV = \oint_S \psi\mathbf{F} \cdot \mathbf{n}\, dS - \int_V \mathbf{F} \cdot \nabla\psi\, dV$. This is the vector equivalent of integration by parts where $\mathbf{n}$ is a unit normal to S.

1.19. Evaluate the line integral of the vector function $\mathbf{F} = x^2\mathbf{a}_x + xy^2\mathbf{a}_y$ around the circle $x^2 + y^2 = a^2$. Repeat, making use of Stokes' theorem.

1.20. Prove the following:

$$\nabla \cdot \mathbf{r} = 3 \qquad \nabla \times \mathbf{r} = 0 \qquad \nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$$

where $\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$, and $\mathbf{A}$ is a constant vector.

1.21. Show that $(1/F)(\mathbf{F} \cdot \nabla)(\mathbf{F}/F)$ gives the curvature of the flux lines of the vector field $\mathbf{F}$.

HINT: Note that $(\mathbf{F}/F)$ represents a unit tangent to the lines of flux of $\mathbf{F}$.

1.22. Consider a compressible fluid of density ρ and having a velocity $\mathbf{v}(x,y,z)$. Prove the continuity equation $\nabla \cdot \mathbf{v}\rho = -(\partial\rho/\partial t)$.

HINT: The total mass of fluid flowing out through a closed surface S is given by $\oint \rho\mathbf{v} \cdot d\mathbf{S}$ and must equal the rate at which the enclosed mass of fluid is decreasing, i.e., must equal $-(\partial/\partial t)\int_V \rho\, dV$. Use the divergence theorem (Gauss' law) to convert the surface integral to a volume integral. The results hold for any arbitrary volume, and hence the integrands may be equated.

1.23. Water flowing along a channel with sides along $x = 0$, a has a velocity distribution $\mathbf{v}(x,y) = \mathbf{a}_y(x - a/2)^2z^2$. A small freely rotating paddle wheel with its axis parallel to the z axis is inserted into the fluid as illustrated. Will the paddle wheel rotate? What are the relative rates of rotation at the points $x = a/4$, $z = 1$; $x = a/2$, $z = 1$; $x = 3a/4$, $z = 1$? Will the paddle wheel rotate if its axis is parallel to the x axis or y axis?

HINT: The paddle wheel will rotate provided the fluid is curling or rotating at the point in question. The rate of rotation will be proportional to the z component of the curl of the fluid velocity. The small paddle wheel could form the basis of a curl meter to measure the curl of the fluid velocity.

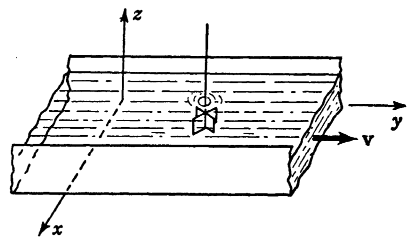

FIG. P 1.23

1.24. Prove that for an arbitrary vector function $\mathbf{A}(x,y,z)$ that is continuous at the point (x',y',z'),

$$\int_V \mathbf{A}(x,y,z)\nabla^2\left(\frac{1}{r}\right) dx\, dy\, dz = \begin{cases} -4\pi\mathbf{A}(x',y',z') & (x',y',z') \text{ inside } V \\ 0 & (x',y',z') \text{ outside volume } V \end{cases}$$

HINT: See Sec. 1.18 (integration of Poisson's equation).

1.25. (*a*) Consider the following vector fields **A**, **B**, **C**, and state which may be completely derived from the gradient of a scalar function and which from the curl of a vector function.

(*b*) Describe a possible source distribution that could set up the field.

$$\mathbf{A} = \sin\theta\cos\phi\mathbf{a}_r + \cos\phi\cos\theta\mathbf{a}_\theta - \sin\phi\mathbf{a}_\phi$$
$$\mathbf{B} = z^2\sin\phi\mathbf{a}_r + z^2\cos\phi\mathbf{a}_\phi + 2rz\sin\phi\mathbf{a}_z$$
$$\mathbf{C} = (3y^2 - 2x)\mathbf{a}_x + x^2\mathbf{a}_y + 2z\mathbf{a}_z$$