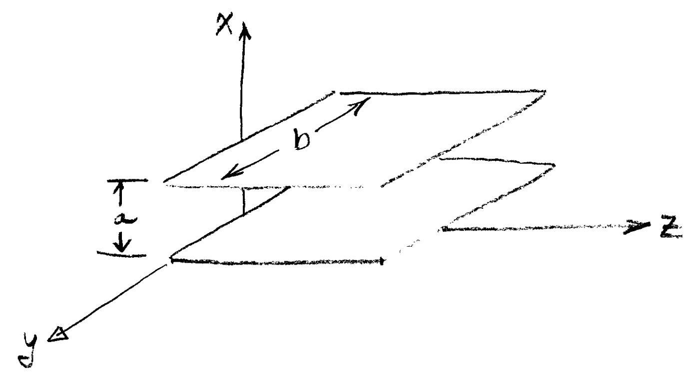
4.1 Waves between parallel metal plates



typically $b \gg a$
infinite in z

Assume propagation of a wave in the $+z$ direction

We define $\bar{\gamma} = \bar{\alpha} + j\bar{\beta}$ and let the wave propagate as $e^{-\bar{\gamma} z}$

Note $\bar{\alpha}, \bar{\beta}$ are NOT α, β for infinite extent waves and depend upon the dimensions of the guiding structure as well as $\omega, \sigma, \epsilon, \mu$.

If $\bar{\beta} = 0$ $\quad Re\{e^{-\bar{\gamma} z} e^{j\omega t}\} \rightarrow e^{-\bar{\gamma} z} \cos(\omega t)$ evanescent wave

If $\bar{\alpha} = 0$ $\quad Re\{e^{-\bar{\gamma} z} e^{j\omega t}\} \rightarrow \cos(\omega t - \beta z)$ propagating wave

In general $\nabla^2 \underline{E} = -\omega^2 \mu\epsilon \underline{E}$ and $\nabla^2 \underline{H} = -\omega^2 \mu\epsilon \underline{H}$

$$\frac{\partial^2 \underline{E}}{\partial x^2} + \frac{\partial^2 \underline{E}}{\partial y^2} + \frac{\partial^2 \underline{E}}{\partial z^2} = -\omega^2 \mu\epsilon \underline{E}$$

$$\underbrace{\frac{\partial^2 \underline{H}}{\partial x^2}}_{\text{imposes boundary conditions}} + \underbrace{\frac{\partial^2 \underline{H}}{\partial y^2}}_{\text{assume no variation}} + \underbrace{\frac{\partial^2 \underline{H}}{\partial z^2}}_{\text{assume } e^{-\bar{\gamma} z} \text{ dependence}} = -\omega^2 \mu\epsilon \underline{H}$$

i.e. $H_y(x,z) = H_y^0(x) e^{-\bar{\gamma} z}$

$$\therefore \frac{\partial}{\partial z} H_y(x,z) \rightarrow -\bar{\gamma} H_y^0(x) e^{-\bar{\gamma} z}$$

With this approach the wave equations become

$$\frac{\partial^2 \underline{E}}{\partial x^2} + \bar{\gamma}^2 \underline{E} = -\omega^2 \mu\epsilon \underline{E}$$

$$\frac{\partial^2 \underline{H}}{\partial x^2} + \bar{\gamma}^2 \underline{H} = -\omega^2 \mu\epsilon \underline{H}$$

Maxwell's equations also apply in the region between the plates

$$\nabla \times \underline{H} = j\omega\epsilon \underline{E}$$

$$\nabla \times \underline{E} = -j\omega\mu \underline{H}$$

where we assumed that $\sigma = 0$, $\epsilon'' = 0$. Examing the components

$$\frac{\partial H_z}{\partial y}\Big(\to 0\Big) - \frac{\partial H_y}{\partial z}\Big(\to -\bar{\gamma} H_y\Big) = j\omega\epsilon E_x \qquad \frac{\partial E_z}{\partial y}\Big(\to 0\Big) - \frac{\partial E_y}{\partial z}\Big(\to -\bar{\gamma} E_y\Big) = -j\omega\mu H_x$$

$$\frac{\partial H_x}{\partial z}\Big(\to -\bar{\gamma} H_x\Big) - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \qquad \frac{\partial E_x}{\partial z}\Big(\to -\bar{\gamma} E_x\Big) - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\Big(\to 0\Big) = j\omega\epsilon E_z \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\Big(\to 0\Big) = -j\omega\mu H_z$$

and eliminating all partial derivatives wrt y and replacing partial derivatives wrt z by $-\bar{\gamma}$ we get

$$+\bar{\gamma} H_y = j\omega\epsilon E_x \qquad \bar{\gamma} E_y = -j\omega\mu H_x$$

$$-\bar{\gamma} H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \qquad -\bar{\gamma} E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial H_y}{\partial x} = j\omega\epsilon E_z \qquad \frac{\partial E_y}{\partial x} = -j\omega\mu H_z$$

define $h^2 = \bar{\gamma}^2 + \omega^2\mu\epsilon$ and rewrite these eqns

$$-\bar{\gamma} H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y = j\omega\epsilon\left(-\frac{j\omega\mu H_x}{\bar{\gamma}}\right)$$

$$-\bar{\gamma}^2 H_x - \bar{\gamma}\frac{\partial H_z}{\partial x} = +\omega^2\mu\epsilon H_x$$

$$(-\bar{\gamma}^2 - \omega^2\mu\epsilon) H_x = +\bar{\gamma}\frac{\partial H_z}{\partial x}$$

$$H_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x}$$

$$\bar{\gamma} H_y = j\omega\epsilon E_x$$

$$\bar{\gamma} H_y = j\omega\epsilon \left(\frac{\frac{\partial E_z}{\partial x} - j\omega\mu H_y}{-\bar{\gamma}} \right)$$

$$-\bar{\gamma}^2 H_y = j\omega\epsilon \frac{\partial E_z}{\partial x} + \omega^2 \mu\epsilon H_y$$

$$(-\bar{\gamma}^2 - \omega^2\mu\epsilon) H_y = j\omega\epsilon \frac{\partial E_z}{\partial x}$$

$$H_y = -\frac{j\omega\epsilon}{h^2} \frac{\partial E_z}{\partial x}$$

$$-\bar{\gamma} E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y = -j\omega\mu \left(\frac{j\omega\epsilon E_x}{\bar{\gamma}} \right)$$

$$-\bar{\gamma}^2 E_x - \bar{\gamma} \frac{\partial E_z}{\partial x} = +\omega^2\mu\epsilon E_x$$

$$\left[-\bar{\gamma}^2 - \omega^2\mu\epsilon \right] E_x = \bar{\gamma} \frac{\partial E_z}{\partial x}$$

$$E_x = -\frac{\bar{\gamma}}{h^2} \frac{\partial E_z}{\partial x}$$

$$\bar{\gamma} E_y = -j\omega\mu H_x$$

$$\bar{\gamma} E_y = -j\omega\mu \left(\frac{+\frac{\partial H_z}{\partial x} + j\omega\epsilon E_y}{-\bar{\gamma}} \right)$$

$$-\bar{\gamma}^2 E_y = -j\omega\mu \frac{\partial H_z}{\partial x} + \omega^2\mu\epsilon E_y$$

$$-\bar{\gamma}^2 E_y - \omega^2\mu\epsilon E_y = -j\omega\mu \frac{\partial H_z}{\partial x}$$

$$E_y = +\frac{j\omega\mu}{h^2} \frac{\partial H_z}{\partial x}$$

4.1.1 Field solutions for TE & TM waves

Three categories of guided-wave solutions

$E_z = 0$, $H_z \neq 0$ } transverse electric (TE) waves

$E_z \neq 0$, $H_z = 0$ } transverse magnetic (TM) waves

$E_z = H_z = 0$ } transverse electromagnetic (TEM) waves.

TE Waves.

There is always and everywhere an electric field vector which is transverse to the direction of propagation & $E_z = 0$.

Use the wave equation to find E_y

$$\frac{\partial^2 E_y}{\partial x^2} + \bar{\gamma}^2 E_y = -\omega^2 \mu \epsilon E_y \qquad \text{since } \frac{\partial E_y}{\partial y} = 0$$

$$\frac{\partial^2 E_y}{\partial x^2} = -(\omega^2 \mu\epsilon + \bar{\gamma}^2) E_y = -h^2 E_y$$

Now write E_y as a product of functions

$$E_y(x,z) = E_y^o(x)\, e^{-\bar{\gamma} z}$$

$$\frac{d^2 E_y^o}{dx^2} e^{-\bar{\gamma} z} = -h^2 E_y^o(x)\, e^{-\bar{\gamma} z}$$

$$\frac{d^2 E_y^o}{dx^2} = -h^2 E_y^o(x)$$

Solutions are $E_y^o(x) = C_1 \sin hx + C_2 \cos hx$

The boundary conditions are

$E_y = 0$ at $x = 0$

$E_y = 0$ at $x = a$

at $x=0$ $E_y=0$ requires $C_2=0$

$$\therefore \quad E_y(x,z) = C_1 \sin hx\, e^{-\bar{\gamma} z}$$

For $E_y=0$ at $x=a$ $\quad ha = m\pi$

or $h = \frac{m\pi}{a}$, $m=1,2,3,\ldots$

h is a "characteristic value" or eigenvalue.

$$\therefore \quad E_y(x,z) = C_1 \sin\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

To find the H fields

$$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z$$

$$H_z = -\frac{1}{j\omega\mu}\frac{\partial E_y}{\partial x} = -\frac{m\pi}{j\omega\mu a}\cos\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

$$\bar{\gamma} E_y = -j\omega\mu H_x$$

$$H_x = \frac{-\bar{\gamma}}{j\omega\mu} E_y = -\frac{\bar{\gamma}}{j\omega\mu} C_1 \sin\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

Transverse magnetic (TM) fields.

There is always and everywhere a magnetic field vector transverse to the direction of propagation. $\quad H_z = 0$

Use the wave equation to find H_y

$$\frac{\partial^2 H_y}{\partial x^2} + \bar{\gamma}^2 H_y = -\omega^2 \mu\epsilon H_y \qquad \text{since} \quad \frac{\partial H_y}{\partial y} = 0$$

$$\frac{\partial^2 H_y}{\partial x^2} = -(\omega^2\mu\epsilon + \bar{\gamma}^2) H_y = -h^2 H_y$$

Write H_y as a product of functions $\quad H_y = H_y^0(x)\, e^{-\bar{\gamma} z}$

$$\frac{d^2 H_y^0}{dx^2} e^{-\bar{\gamma} z} = -h^2 E_y^0(x)\, e^{-\bar{\gamma} z}$$

$$\frac{d^2 H_y^0}{dx^2} = -h^2 H_y^0(x)$$

Solutions are $H_y^0 = C_3 \sin hx + C_4 \cos hx$

The boundary conditions do not directly apply to H_y but can be applied to E_z by using Maxwell's Equations

$$E_z = \frac{1}{j\omega\epsilon}\frac{\partial H_y}{\partial x} \qquad (p.\ 2)$$

$$= \frac{1}{j\omega\epsilon}\frac{\partial}{\partial x}\left[(C_3 \sin hx + C_4 \cos hx)e^{-\bar{\gamma} z}\right]$$

$$= \frac{1}{j\omega\epsilon}(h\, C_3 \cos hx - h\, C_4 \sin hx)\, e^{-\bar{\gamma} z}$$

$$E_z = \frac{h}{j\omega\epsilon}(C_3 \cos hx - C_4 \sin hx)\, e^{-\bar{\gamma} z}$$

We now apply B.C. that $E_z = 0$ at $x = 0, a$

$E_z = 0$ at $x = 0$ requires $C_3 = 0$

$E_z = 0$ at $x = a$ requires $h = \frac{m\pi}{a}$, $m = 0, \pm 1, \ldots$

Transverse electromagnetic (TEM) waves.

This is similar to the previous solutions EXCEPT there are <u>no</u> z fields $H_z = E_z = 0$

The TE modes vanish since $H_z = 0$

All but the $m=0$ TM mode vanishes

Since E_z vanishes when $h = 0$ we still have a TM_0 mode, which is the TEM mode.

$$H_y = C_4 e^{-\bar{\gamma} z}$$

$$E_x = \frac{\bar{\gamma}}{j\omega\epsilon} C_4 e^{-\bar{\gamma} z}$$

$$E_z = 0$$

$H_y = C_4 \underbrace{\cos\left(\frac{m\pi}{a}x\right)}_{\text{goes to "1" if } m=0} e^{-\bar{\gamma} z}$

so we have an E_x, H_y but $E_z = \frac{j m\pi}{\omega\epsilon a} C_4 \sin\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z} \rightarrow 0$

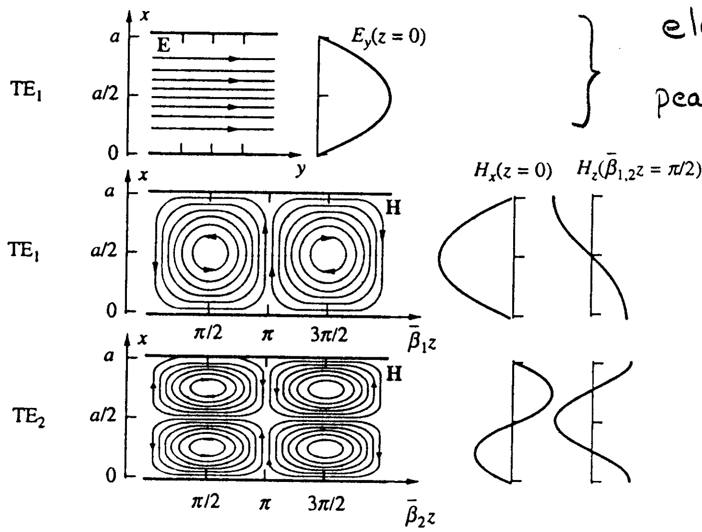
Parallel-plate TE_m, $m = 0, \pm1, \pm2, \ldots$

$$\boxed{\begin{aligned} E_y &= C_1 \sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \\ H_z &= -\frac{1}{j\omega\mu}\frac{\partial E_y}{\partial x} = -\frac{m\pi}{j\omega\mu a}C_1\cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \\ H_x &= -\frac{\bar{\gamma}}{j\omega\mu}E_y = -\frac{\bar{\gamma}}{j\omega\mu}C_1\sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \end{aligned}} \quad [4.12]$$

TE_m modes have $H_z \neq 0$, $E_z = 0$

the H fields wrap around the E fields

This is now a higher order mode.



electric field in transverse plane peak at center

magnetic fields in longitudinal plane

FIGURE 4.2. **TE_1 and TE_2 modes.** The electric and magnetic field distributions for the TE_1 and the magnetic field distribution for the TE_2 modes in a parallel-plate waveguide. Careful examination of the field structure for the TE_1 mode indicates that the magnetic field lines encircle the electric field lines (i.e., displacement current) in accordance with [4.1*a*] and the right-hand rule. The same is true for the TE_2 mode, although the electric field distribution for this mode is not shown.

Parallel-plate TM_m, $m = 0, \pm 1, \pm 2, \ldots$

$$H_y = C_4 \cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}$$

$$E_x = \frac{\bar{\gamma}}{j\omega\epsilon}H_y = \frac{\bar{\gamma}}{j\omega\epsilon}C_4\cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \quad [4.13]$$

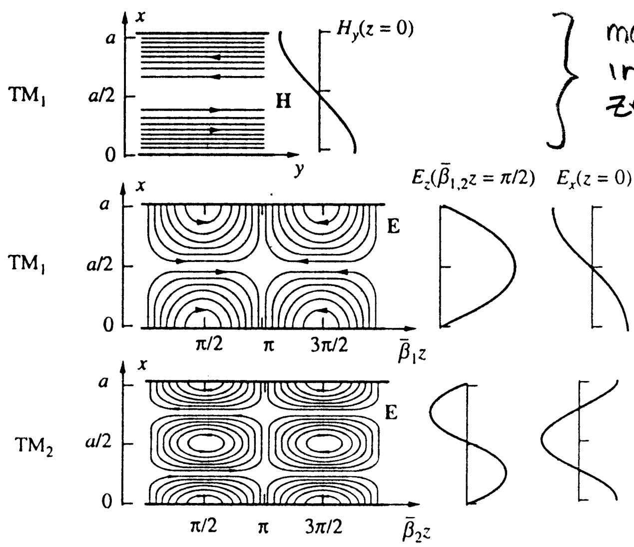
$$E_z = \frac{jm\pi}{\omega\epsilon a}C_4\sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}$$

$$E_z = \frac{1}{j\omega\epsilon}\frac{\partial H_y}{\partial x}$$

TM_m modes have $E_z \neq 0$, $H_z = 0$

The magnetic field lines increase near the plates.

The electric field lines wrap around the magnetic fields. Note that E_z vanishes on the plates. This is the important BC.



magnetic field in transverse plane zero in center

electric fields in longitudinal plane. $E_z = 0$ on plates

FIGURE 4.3. TM_1 and TM_2 modes. The electric and magnetic field distributions for the TM_1 and TM_2 modes in a parallel-plate waveguide. Note that for TM_1 the electric field lines encircle the magnetic field lines (Faraday's law) in accordance with [4.1c]. The same is true for TM_2, although the magnetic field distribution for TM_2 is not shown.

TE

$$H_z = -\frac{1}{j\omega\epsilon}\frac{\partial E_y}{\partial x}$$

$$H_z = -\frac{m\pi}{j\omega\mu a} C_1 \cos\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

$$ha = m\pi$$

At $m=0$ $\cos() \rightarrow 1$
$\Rightarrow m=0$ doesn't exist since H_z must be zero

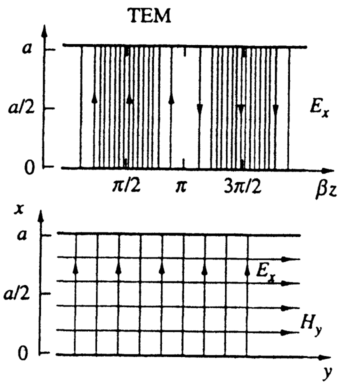
TM

$$E_z = \frac{1}{j\omega\epsilon}\frac{\partial H_y}{\partial x}$$

$$E_z = \frac{jm\pi}{\omega\epsilon a} C_4 \sin\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

$$ha = m\pi$$

At $m=0$ $\sin() \rightarrow 0$
$\Rightarrow m=0$ mode exists



} This is the longitudinal plane.

This is the transverse plane }

Note that the E & H fields are perpendicular, with $\underline{E} \times \underline{H}$ in the $+z$ direction.

FIGURE 4.4. **TEM mode.** Electric and magnetic fields between parallel planes for the TEM (TM_0) mode. Only electric field lines are shown in the top panel; the magnetic field lines are out of (or into) the page.

Transverse Electromagnetic (TEM) Waves Note that contrary to the TE case, the TM solutions do not all vanish for $m = 0$. Since E_z is zero for $m = 0$, the TM_0 mode is actually a transverse electromagnetic (or TEM) wave. In this case, we have

Parallel-plate TEM

$$H_y = C_4 e^{-\bar{\gamma} z}$$
$$E_x = \frac{\bar{\gamma}}{j\omega\epsilon} C_4 e^{-\bar{\gamma} z} \qquad [4.14]$$
$$E_z = 0$$

This is actually a TM_0 mode since there is a E_x but $E_z = 0$.

For TM_m:

$$H_y = C_4 \cos\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$
$$E_x = \frac{\bar{\gamma}}{j\omega\epsilon} C_4 \cos\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

Which for $m=0$ becomes

$$H_y = C_4 e^{-\bar{\gamma} z}$$
$$E_x = \frac{\bar{\gamma}}{j\omega\epsilon} C_4 e^{-\bar{\gamma} z}$$

4.1.2. Cutoff Frequency, Phase Velocity, Wavelength.

TE and TM modes have similar characteristics

(i) $\underline{E}$ & $\underline{H}$ have sinusoidal standing wave distributions in the x direction

(ii) xy planes are equiphase planes, i.e. surfaces of constant phase

(iii) the equiphase surfaces propagate along the waveguide with velocity $v_p = \frac{\omega}{\beta}$

Consider E_y for TE waves

$$E_y = C_1 \sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}$$

Assume $\bar{\alpha}=0$ so $\bar{\gamma} \to j\bar{\beta}$. Then write the real wave as

$$\mathcal{E}_y(x,z,t) = \underbrace{C_1 \sin\left(\frac{m\pi}{a}x\right)}_{\text{transverse standing wave}} \cos(\omega t - \bar{\beta}z)$$

By definition $h^2 = \bar{\gamma}^2 + \omega^2\mu\epsilon$. But $h^2 = \frac{m^2\pi^2}{a^2}$ from B.C.'s

Solving for $\bar{\gamma}$ $\qquad \bar{\gamma} = \sqrt{h^2 - \omega^2\mu\epsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\epsilon}$

There is a cutoff frequency f_{c_m} for which $\bar{\gamma} = 0$ or

$$\frac{m^2\pi^2}{a^2} = 4\pi^2 f_{c_m}^2 \mu\epsilon$$

Solving for f
$$f_{c_m} = \frac{m}{2a\sqrt{\mu\epsilon}} = \frac{m v_p}{2a}$$

For $f > f_{cm}$ $\quad \left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\epsilon < 0$ ← and wave propagates

Using the expression for f_{cm}
$$\bar{\gamma} = j\bar{\beta}_m = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2} = j\sqrt{\omega^2\mu\epsilon - \left(\frac{2 f_{cm}\pi}{v_p}\right)^2}$$

$$\bar{\gamma} = j\sqrt{\omega^2\mu\epsilon - \left(2\pi f_{cm}/v_p\right)^2}$$

$$\bar{\gamma} = j\sqrt{\beta^2 - \left(2\pi f_{cm}/v_p\right)^2}$$

$$\bar{\gamma} = j\beta\sqrt{1 - \frac{\left(2\pi f_{cm}/v_p\right)^2}{\beta}}$$

$$\bar{\gamma} = j\beta\sqrt{1 - \frac{\left(2\pi f_{cm}/v_p\right)^2}{\left(2\pi f/v_p\right)^2}}$$

$$\bar{\gamma} = j\beta\sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2} \quad \text{where } f > f_{cm}$$

$$\bar{\gamma} = j\bar{\beta}_m\sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2}, \quad f > f_{cm}$$

β_m emphasizes that this is for mode m.

Note that for $f < f_{cm}$ $\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\epsilon > 0$ and

$$\bar{\gamma} = \underbrace{\bar{\alpha}_m \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\epsilon}} = \beta\sqrt{\left(\frac{f_{cm}}{f}\right)^2 - 1} \quad f < f_{cm}$$

this is simply recognizing that the square root is negative so this becomes α_m.

This is known as an evanescent wave.
Attenuation NOT due to energy loss but from boundary conditions.

$$\bar{\lambda}_m = \frac{2\pi}{\bar{\beta}_m} = \frac{2\pi}{\bar{\beta}_m\sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2}} = \frac{2\pi/\beta}{\sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2}}$$

$$\bar{\lambda}_m = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2}}$$

similarly

$$\bar{v}_{pm} = \frac{\omega}{\bar{\beta}_m} = \frac{v_p}{\sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2}}$$

We see that $\bar{\lambda}_m$ and $\bar{v}_{pm}$ vary as a function of the mode frequency

4.1.3 TE & TM modes as superpositions of TEM waves

$$E_y(x,z) = C_1 \sin\left(\frac{m\pi}{a}x\right) e^{-j\bar{\beta}z}$$

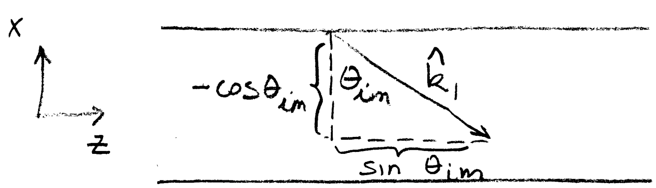
Rewrite the sin as $\dfrac{e^{j\frac{m\pi x}{a}} - e^{-j\frac{m\pi x}{a}}}{2j}$

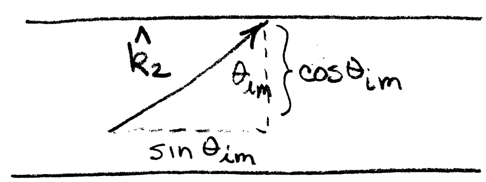
$$E_y(x,z) = C_1 \frac{1}{2j}\left[e^{j\frac{m\pi x}{a}} - e^{-j\frac{m\pi x}{a}}\right] e^{-j\bar{\beta}z}$$

$$= \frac{C_1}{2} e^{-j\frac{\pi}{2}}\left[e^{+j\frac{m\pi x}{a}} - e^{-j\frac{m\pi x}{a}}\right] e^{-j\bar{\beta}z}$$

$$E_y(x,z) = \frac{C_1}{2}\left[e^{j\left(\frac{m\pi x}{a} - \bar{\beta}z - \frac{\pi}{2}\right)} - e^{-j\left(\frac{m\pi x}{a} + \bar{\beta}z + \frac{\pi}{2}\right)}\right]$$

Since E_y: first term — $\perp$ polarized wrt xz plane TEM wave propagating in the $\hat{k}_1 = -\hat{x}\cos\theta_{im} + \hat{z}\sin\theta_i$ direction; second term — $\perp$ polarized wrt xz plane TEM wave propagates in the $\hat{k}_2 = \hat{x}\cos\theta_{im} + \hat{z}\sin\theta_i$ direction





$$\hat{k}_1 = \frac{-\beta_x\hat{x} + \beta_z\hat{z}}{\beta} \qquad \hat{k}_2 = \frac{\beta_x\hat{x} + \beta_z\hat{z}}{\beta}$$

Identify $\beta_x = \beta\cos\theta_{im} = \dfrac{m\pi}{a}$ (1)

$$\beta_z = \beta\sin\theta_{im} = \bar{\beta} = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2} = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{\omega_{cm}}{\omega_c}\right)^2}$$

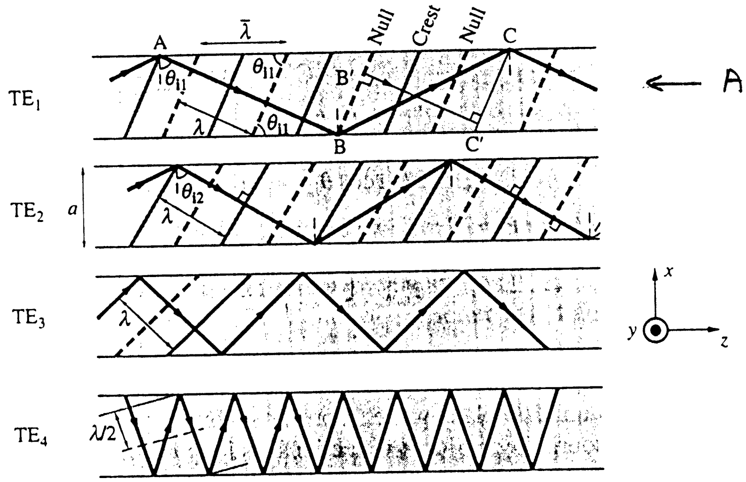
from 4.16

$$\text{From (1)} \quad \cos\theta_{im} = \frac{\frac{m\pi}{a}}{\beta} = \frac{\frac{m\pi}{a}}{\frac{2\pi}{\lambda}} = \frac{\frac{m}{2a}}{\frac{1}{\lambda}} = \frac{f_{cm}\sqrt{\mu\epsilon}}{\frac{1}{\lambda}} = \frac{f_{cm}}{\frac{c}{\lambda}} = \frac{f_{cm}}{f}$$

where we used $f_{cm} = \dfrac{m}{2a\sqrt{\mu\epsilon}}$

$$\theta_{im} = \cos^{-1}\left[\frac{m\lambda}{2a}\right] \qquad m = 0, 1, 2, \ldots$$

Waves only propagate for discrete θ_{im}



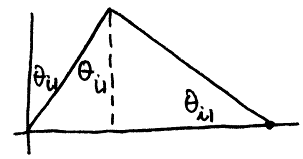


FIGURE 4.9. **Representation of propagating waveguide modes as "rays" reflecting between the parallel plates.** The dashed lines represent the phase fronts, and the solid lines represent the ray paths.

The different θ_{im} look like waves propagating at different θ_{im}

We can do a geometric wave analysis.

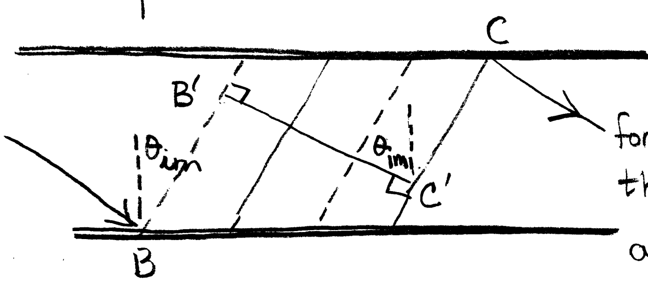
BB' represents a constant phase front for a wave from A to B

For the wave to propagate

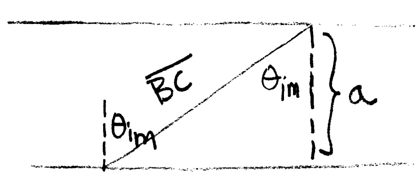
CC' must also be a constant phase front.

$\Rightarrow$ phase change along B–C and B'–C' must be multiple of 2π

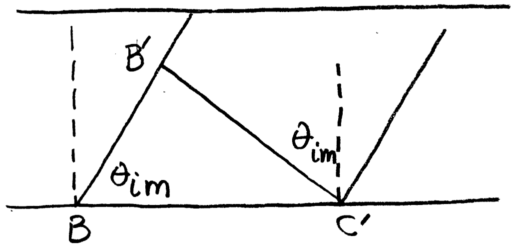
$$\beta(\overline{BC} - \overline{B'C'}) = m\,2\pi$$



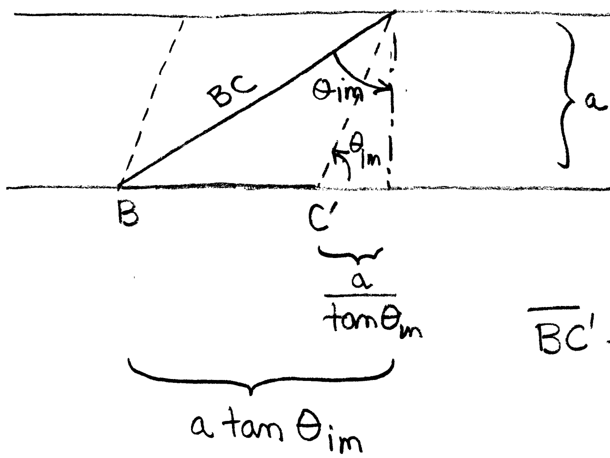
for wavefronts to pick up here the phase shift along $\overline{BC}$ and the phase shift along $\overline{B'C'}$ can only differ by a multiple of 2π



$$\overline{BC} = \frac{a}{\cos\theta_{im}}.$$



$$\sin\theta_{im} = \frac{\overline{B'C'}}{\overline{BC'}}$$



Note: all wavefronts are at θ_{im}

$$\overline{BC'} = a\tan\theta_{im} - \frac{a}{\tan\theta_{im}}$$

Combining

$$\beta(\overbrace{\overline{BC} - \overline{B'C'}}^{\Delta z}) = m2\pi$$

$$\beta\left(\frac{a}{\cos\theta_{im}} - \overline{BC'}\sin\theta_{im}\right) = m2\pi$$

$$\beta\left(\frac{a}{\cos\theta_{im}} - \left(a\tan\theta_{im} - \frac{a}{\tan\theta_{im}}\right)\sin\theta_{im}\right) = m2\pi$$

$$\beta\left(\frac{a}{\cos\theta_{im}} - a\frac{\sin\theta_{im}}{\cos\theta_{im}}\sin\theta_{im} + a\frac{\sin\theta_{im}}{\sin\theta_{im}}\cos\theta_{im}\right) = m2\pi$$

$$\beta a\left(\frac{1-\sin^2\theta_{im}}{\cos\theta_{im}} + \cos\theta_{im}\right) = m2\pi$$

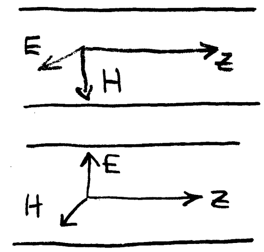
$$\beta a\left(\frac{\cos^2\theta_{im}}{\cos\theta_{im}} + \cos\theta_{im}\right) = m2\pi$$

$$2\beta a\cos\theta_{im} = m2\pi$$

$$2\left(\frac{2\pi}{\lambda}\right)a\cos\theta_{im} = m2\pi$$

$$\cos\theta_{im} = \frac{m\lambda}{2a} \qquad m = 0, 1, 2, \ldots$$

A simple physical model is to break the



TE mode into $\perp$ polarized TEM waves

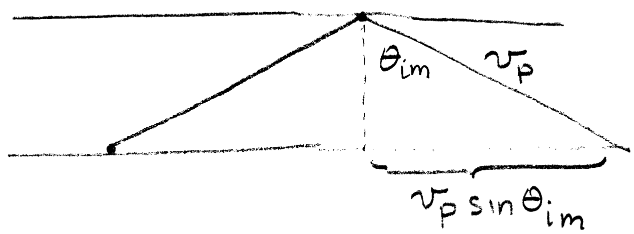
TM mode into $\parallel$ polarized TEM waves

for both $\lambda_{c,m} = \frac{2a}{m}$ cutoff wavelength

at $\lambda_{c,m}$ $\theta_{im} = 0$

Because of this zig-zag path of the wave the energy propagates at a velocity less than that in free space, called the group velocity

$$v_g = v_p \sin\theta_{im}$$



4.1.4. Attenuation in Parallel-Plate Waveguides

Practical waveguides made of copper or brass usually coated with silver

Assume losses are very small so that they have a negligible effect on the field distribution, i.e. small perturbation approach

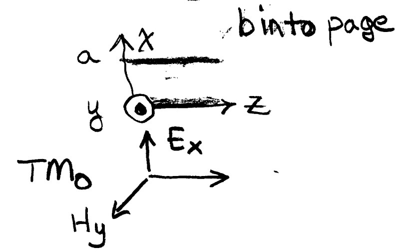
Assume fields vary as $e^{-\alpha_c z}$ so power goes as $e^{-2\alpha_c z}$

$$-\frac{\partial P_{AV}}{\partial z} = +2\alpha_c P_{AV}$$

$$\frac{\text{Power lost per unit length}}{\text{Power transmitted}} = \frac{2\alpha_c P_{AV}}{P_{AV}} = 2\alpha_c$$

$$\alpha_c = \frac{\text{Power lost per unit length}}{2 \times \text{Power Transmitted}} = \frac{P_{LOSS}}{P_{AV}}$$

Consider losses due to conduction currents for TEM waves.



For TEM waves

$$H_y = C_4 e^{-j\bar{\beta} z}$$

$$E_x = \frac{\bar{\beta}}{\omega \epsilon} C_4 e^{-j\bar{\beta} z}$$

surface current density on each plate is

$$\underline{J}_s = \hat{n} \times \underline{H} = \hat{x} \times C_4 e^{-j\bar{\beta} z} \hat{y} = \hat{z}\, C_4 e^{-j\bar{\beta} z}$$

$$|J_s| = C_4$$

Total loss for a length of 1 meter is

$$P_{LOSS} = 2 \int_0^1 \int_0^b \frac{1}{2} |J_s|^2 R_s \, dy \, dz = C_4^2 R_s b$$

(2: upper & lower plates; 1: unit length; b: width; R_s: surface resistance)

$$R_s = \frac{1}{\delta \sigma_2} \quad [3.41]$$

Total time average power transmitted

$$\underline{S}_{AV} = \frac{1}{2} \mathrm{Re} \{ \underline{E} \times \underline{H}^* \}$$

$$|S_{AV}| = \frac{1}{2} \frac{|E_x|^2}{\eta} = \frac{1}{2} \eta |H_y|^2 = \frac{1}{2} \eta C_4^2$$

← write in terms of H_y since simpler

this is a power density

$$P_T = |S_{AV}|\, ab = \frac{1}{2} \eta C_4^2 ab$$

this is a power

(ab: cross-sectional area of guide)

Using definition of α_c

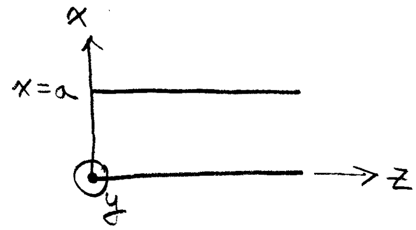
$$\therefore \alpha_{c_{TEM}} = \frac{C_4^2 R_s b}{2\left(\frac{1}{2} \eta C_4^2 ab\right)} = \frac{R_s}{\eta a} = \frac{1}{\eta a} \sqrt{\frac{\omega \mu_0}{2\sigma}}$$

Now for
parallel plate TE waves.

$$E_y = C_1 \sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}$$

$$H_z = -\frac{m\pi}{j\omega\mu a} C_1 \cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}$$

$$H_x = -\frac{\bar{\gamma}}{j\omega\mu} C_1 \sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}$$



Just as for TEM waves there is a surface current due to the tangential (H_z) field.

$$\therefore \quad |J_{sy}| = |H_z| = \frac{m\pi C_1}{j\omega\mu a} \qquad \text{this is a } y \text{ directed current}$$

Power loss is ⊥ to direction of propagation

$$P_{LOSS} = 2\int_0^1\int_0^b \frac{1}{2}|J_{sy}|^2 R_s\, dy\, dz = 2(1)(b)\underbrace{\frac{1}{2}\frac{m^2\pi^2 C_1^2}{j^2\omega^2\mu^2 a^2}\sqrt{\frac{\omega\mu_0}{2\sigma}}}_{\text{substitute expressions for } |J_{sy}| \text{ and } R_s}$$

Now compute

$$|S_{AV}| = \frac{1}{2}\text{Re}\{\underset{\hat{y}}{\underline{E}} \times \underset{\hat{x}}{\underline{H}^*}\}\cdot\hat{z} = \frac{1}{2}E_y H_x^* = \frac{1}{2}\left[C_1\sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z}\right]\left[\frac{-\bar{\gamma}}{-j\omega\mu}C_1\sin(\;)e^{+\bar{\gamma}z}\right]$$

$$\hat{y}\times\hat{x} = -\hat{z}$$

$$|S_{AV}| = +\frac{1}{2}C_1^2\sin^2\left(\frac{m\pi}{a}x\right)\frac{\bar{\gamma}}{j\omega\mu} \qquad \bar{\gamma} \to \bar{\alpha}+j\bar{\beta} \cong j\bar{\beta} \text{ assume losses are small}$$

$$|S_{AV}| = +\frac{\bar{\beta}C_1^2}{2\omega\mu}\sin^2\left(\frac{m\pi}{a}x\right) \qquad \text{this is a power density}$$

$$P_{AV} = \int_0^b\int_0^a \frac{\bar{\beta}C_1^2}{2\omega\mu}\sin^2\left(\frac{m\pi}{a}x\right)dx\,dy = \frac{\bar{\beta}C_1^2}{2\omega\mu}\int_0^b\left[\frac{1}{2}x - \frac{1}{4\frac{m\pi}{a}}\sin\frac{2m\pi}{a}x\right]_0^a dy$$

$$P_{AV} = \frac{\bar{\beta}C_1^2}{2\omega\mu}\frac{1}{2}ba$$

by integrating over the surface of the plates we get power loss

$$\alpha_{c,TE_m} = \frac{P_{LOSS}}{2P_{AV}} = \frac{b\frac{m^2\pi^2C_1^2}{\omega^2\mu^2a^2}\sqrt{\frac{\omega\mu_0}{2\sigma}}}{2\frac{\bar{\beta}C_1^2}{4\omega\mu}ba} = \frac{2m^2\pi^2}{\bar{\beta}\omega\mu a^3}\sqrt{\frac{\omega\mu_0}{2\sigma}}$$

use definition
and our expressions
for TE waves to determine losses

We can re-write this in a more usable form

$$\alpha_{c,TE_m} = \frac{2m^2\pi^2}{\bar{\beta}\omega\mu a^3}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{2m^2\pi^2 R_s}{\bar{\beta}\omega\mu a^3}$$

$$= \frac{2\epsilon\pi^2 \cancel{R_s} 4}{\bar{\beta}\omega a} \cdot \underbrace{\frac{m^2}{4a^2\mu\epsilon}}_{f_{cm}^2} R_s$$

$$= \frac{8\epsilon\pi^2 \cancel{R_s} 2\pi f}{\bar{\beta} a} \frac{f_{cm}^2}{(2\pi)f(2\pi)f} R_s = \frac{16\pi^3\epsilon f}{4\pi^2\bar{\beta} a} \left(\frac{f_{cm}}{f}\right)^2 R_s$$

$$= \frac{4\pi\epsilon f}{a\beta\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}} \left(\frac{f_{cm}}{f}\right)^2 R_s = \frac{4\pi\epsilon\frac{\omega}{2\pi}}{\beta a} \frac{\left(\frac{f_{cm}}{f}\right)^2}{\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}} R_s$$

$$= \frac{2\epsilon\cancel{\omega}}{\cancel{\omega}\sqrt{\mu\epsilon}\, a} \frac{\left(\frac{f_{cm}}{f}\right)^2}{\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}} R_s$$

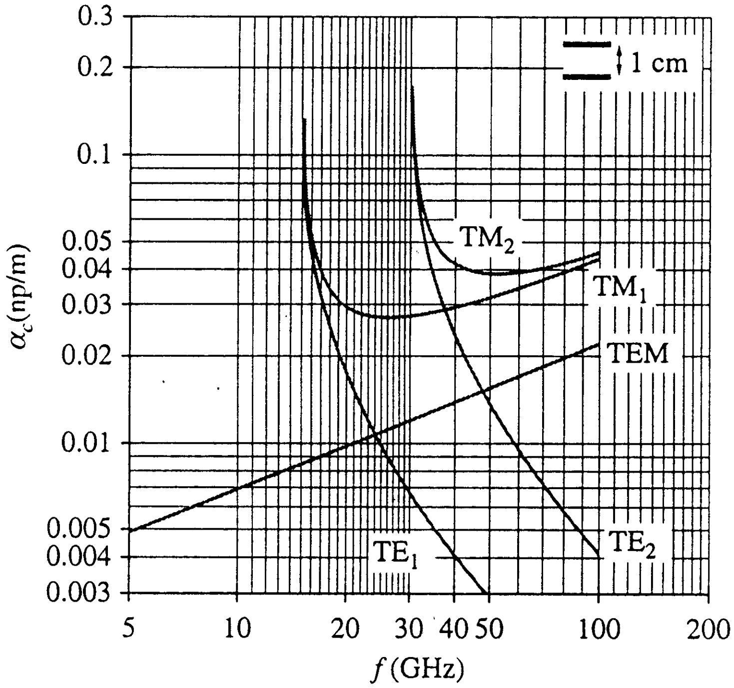
$$= \frac{2}{a}\sqrt{\frac{\epsilon}{\mu}} \frac{\left(\frac{f_{cm}}{f}\right)^2}{\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}} R_s = \frac{2}{\eta a} \frac{\left(\frac{f_{cm}}{f}\right)^2}{\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}} R_s$$

Similar calculations give for TM modes

$$\alpha_{c,TM_m} = \frac{2\cancel{R_s}}{\eta a\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}} R_s$$

This is a graphical presentation of the mode losses.

cutoff frequencies for different modes



TEM waves exist for all frequencies

FIGURE 4.11. Attenuation versus frequency for parallel-plate waveguide. Attenuation-versus-frequency characteristics of waves guided by parallel plates.

This figure shows the attenuation as a function of frequency for a few modes. Higher order modes have higher losses.

TM modes have higher losses than TE modes since TM modes have a tangential $\underline{J}$ due to tangential H_y

$$\text{i.e. } H_y = C_4 \cos\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

TE modes have lower losses at higher frequencies

$$H_z = -\frac{m\pi}{j\omega\mu a} C_1 \cos\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z}$$

since as ω increases surface currents decrease
very strong frequency dependence

Attenuation due to dielectric losses.

Just like for TEM waves we have losses associated with the dielectric (i.e. polarization currents)

$$\epsilon_c = \epsilon' - j\epsilon''$$

For TEM modes just compute α just like for a uniform plane wave.

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{\frac{1}{2}}$$

use $\tan\delta_c = \frac{\sigma}{\omega\epsilon} \rightarrow \frac{\omega\epsilon''}{\omega\epsilon'}$ for a dielectric

Just as we did for conduction losses we have to assume the losses are not too large so the field configurations don't change, i.e., a perturbation analysis.
However, the propagation constant will change.
For $f > f_{cm}$ we know

$$\bar{\gamma} = j\bar{\beta} = j\left[\mu\epsilon(\omega^2 - \omega_{cm}^2)\right]^{\frac{1}{2}} = j\left[\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}} \qquad [4.16]$$

If we substitute $\epsilon \rightarrow \epsilon' - j\epsilon''$

$$\bar{\gamma} = j\left[\omega^2\mu(\epsilon' - j\epsilon'') - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}} =$$

$$= j\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}}\left[\frac{\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2 - j\omega^2\mu\epsilon''}{\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]}\right]^{\frac{1}{2}}$$

Add these terms

$$= j\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}}\left[1 - \frac{j\omega^2\mu\epsilon''}{\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2}\right]^{\frac{1}{2}}$$

$$= j\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}}\left[1 - \frac{j\omega^2\mu\epsilon''}{2\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]} + \ldots\right]$$

binomial expansion

$$\bar{\gamma} \cong \frac{\omega^2\mu\epsilon}{2\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]} + j\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}}$$

keeping only the first term since ϵ'' is small

Now, if we recall that $\omega_{c,m} = \dfrac{2\pi m}{2a\sqrt{\mu\epsilon}} = \dfrac{\pi m}{a\sqrt{\mu\epsilon}}$

we define $\omega_{c,m}$ for a lossy dielectric as

$$\omega_{c,m} = \frac{\pi m}{a\sqrt{\mu\epsilon'}}$$

Rewriting $\bar{\gamma}$

$$\bar{\gamma} = \frac{\omega^2\mu\epsilon''}{2\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}}} + j\left[\omega^2\mu\epsilon' - \left(\frac{m\pi}{a}\right)^2\right]^{\frac{1}{2}}$$

$$\bar{\gamma} = \frac{\omega^2\mu\epsilon''}{2\sqrt{\omega^2\mu\epsilon'}\left[1 - \left(\frac{m\pi}{a\omega\sqrt{\mu\epsilon'}}\right)^2\right]^{\frac{1}{2}}} + j\sqrt{\mu\epsilon'\left[\omega^2 - \left(\frac{m\pi}{a\sqrt{\mu\epsilon'}}\right)^2\right]}$$

$$\bar{\gamma} = \frac{\omega\sqrt{\frac{\mu}{\epsilon'}}\,\epsilon''}{2\sqrt{1 - \left(\frac{\omega_{cm}}{\omega}\right)^2}} + j\underbrace{\sqrt{\mu\epsilon'(\omega^2 - \omega_{cm}^2)}}$$

exactly the same as for a lossless dielectric

$$\bar{\gamma} = \alpha_d + j\beta_m$$

↑ attenuation constant due to dielectric losses (α_d)

$$\alpha_d = \frac{\omega\sqrt{\mu\epsilon'}\,\frac{\epsilon''}{\epsilon'}}{2\sqrt{1 - \left(\frac{\omega_{cm}}{\omega}\right)^2}}$$

you get TEM mode by substituting $\omega_{cm} = 0$

4.1.5. Voltage, Current & Impedance

TEM mode corresponds to voltage & current waves of transmission line analysis.

The voltage between the plates for a TEM wave is

$$V_{TEM}(z) = -\int_{x=0}^{a} \underline{E} \cdot \underline{dl} = \int_{a}^{0} E_x(z)dx = \frac{\beta}{\omega\epsilon} C_4 e^{-j\beta z} \int_{a}^{0} dx$$

$$V_{TEM}(z) = -\frac{\beta a C_4}{\omega\epsilon} e^{-j\beta z} = V^+ e^{-j\beta z}$$

$$I_{TEM}(z) = |J_s|(1m) = |\hat{n} \times H| = \left| \pm\hat{x} \times \hat{y} [H_x]_{x=0} \right| = |\hat{x} \times \hat{y} C_4 e^{-j\beta z}|$$

($1m$: 1 meter wide; $\pm\hat{x}$: top or bottom plate; $[H_x]_{x=0}$: tangential field on lower plate)

$$I_{TEM}(z) = C_4 e^{-j\beta z}$$

$$I_{TEM}(z) = \frac{V^+}{Z_{TEM}} e^{-j\beta z}$$

$$V^+ \equiv -\frac{\beta C_4 a}{\omega\epsilon} = -\eta a C_4$$

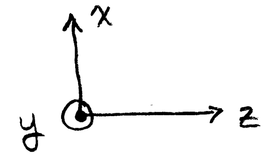
is the peak voltage

$$Z_{TEM} \stackrel{\cdot}{=} \frac{E_x}{H_y} = \frac{\frac{\beta}{\omega\epsilon} C_4 e^{-j\beta z}}{C_4 e^{-j\beta z}} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}}$$

$$Z_{TEM} = \eta$$

Voltage and current definitions are ambiguous for TE & TM modes.

Consider a TE_m mode



$$E_y = C_1 \sin\left(\frac{m\pi}{a}x\right) e^{-j\bar{\beta}z}$$

$$H_z = -\frac{m\pi}{j\omega\mu a} C_1 \cos\left(\frac{m\pi}{a}x\right) e^{-j\bar{\beta}z}$$

$$H_x = -\frac{\bar{\beta}}{\omega\mu} C_1 \sin\left(\frac{m\pi}{a}x\right) e^{-j\bar{\beta}z} \qquad \bar{\beta} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2}$$

Electric field is in y direction so you would normally define voltage in y direction. But

$$-\int_0^a \underline{E}\cdot \underline{dl} = -\int_0^a \hat{y}E_y \cdot \hat{x} = 0$$

Also, the current flows in the y-direction

$$\underline{J_s} = \hat{n} \times \hat{z}\underbrace{\left[H_z\right]_{x=0}}_{\text{lower plate}} = \hat{x}\times\hat{z}\left[\frac{-m\pi C_1}{j\omega\mu a}\right]e^{-j\bar{\beta}z} = \hat{y}\left[\frac{m\pi C_1}{j\omega\mu a}\right]e^{-j\bar{\beta}z}$$

Practice is to define V and I such that

(i) line voltage $\propto$ transverse electric field component

(ii) line current $\propto$ transverse H field component

(iii) $P_{AV} = \frac{1}{2}\mathrm{Re}\{VI^*\}$

$$Z_{TM\text{ or }TE} \equiv \underset{+\hat{z}}{\frac{E_x}{H_y}} = \underset{-(-\hat{z})}{-\frac{E_y}{H_x}}$$

$$Z_{TE_m} = -\frac{E_y}{H_x} = -\frac{C_1\sin\left(\frac{m\pi}{a}x\right)e^{-j\bar{\beta}z}}{-\frac{\bar{\beta}}{\omega\mu}C_1\sin\left(\frac{m\pi}{a}x\right)e^{-j\bar{\beta}z}} = \frac{\omega\mu}{\bar{\beta}} = \frac{\omega\mu}{\beta\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}}$$

$$Z_{TE_m} = \frac{\eta}{\sqrt{1-\left(\frac{f_{cm}}{f}\right)^2}}$$

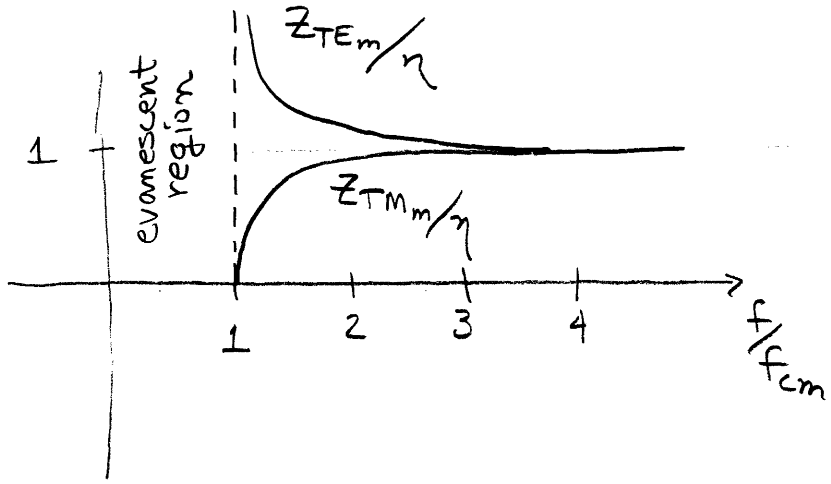
resistive

larger than η

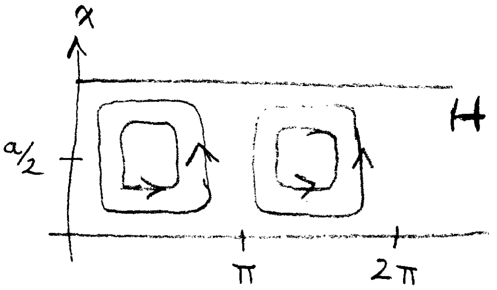
For TM modes

$$Z_{TM_m} = \frac{E_x}{H_y} = \frac{\frac{\bar{\beta}}{\omega\epsilon} C_4 \cos\left(\frac{m\pi}{a}x\right) e^{-j\bar{\beta}z}}{C_4 \cos\left(\frac{m\pi}{a}x\right) e^{-j\bar{\beta}z}} = \frac{\bar{\beta}}{\omega\epsilon}$$

$$Z_{TM_m} = \eta \sqrt{1 - \left(\frac{f_{cm}}{f}\right)^2}$$



4.1.6 E&M field distributions



How do you plot field lines?
Start with time domain expressions.

$$\mathcal{E}_y(x,z,t) = \mathrm{Re}\left\{E_y^0(x)e^{-j\beta z}\right\} = C_1 \sin\left(\frac{m\pi}{a}x\right)\cos(\omega t - \bar{\beta}z)$$

$$\mathcal{H}_x(x,z,t) = -\frac{\bar{\beta}}{\omega\mu} C_1 \sin\left(\frac{m\pi}{a}x\right)\cos(\omega t - \bar{\beta}z)$$

$$\mathcal{H}_z(x,z,t) = -\frac{m\pi}{\omega\mu a} C_1 \cos\left(\frac{m\pi}{a}x\right)\sin(\omega t - \bar{\beta}z)$$

The magnetic field must vary as $\frac{dx}{dz}$ where

$$\frac{dx}{dz} = \frac{\mathcal{H}_x(x,z,t)}{\mathcal{H}_z(x,z,t)} = \left(\frac{\bar{\beta}a}{m\pi}\right)\frac{\sin\left(\frac{m\pi}{a}x\right)\cos(\omega t - \bar{\beta}z)}{\cos\left(\frac{m\pi}{a}x\right)\sin(\omega t - \bar{\beta}z)}$$

rearranging

$$\frac{\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)dx}{\sin\left(\frac{m\pi}{a}x\right)} = \frac{\bar{\beta}\cos(\omega t - \bar{\beta}z)\,dz}{\sin(\omega t - \bar{\beta}z)}$$

Recognizing

$$\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)dx = d\left[\sin\left(\frac{m\pi}{a}x\right)\right]$$

$$\bar{\beta}\cos(\omega t - \bar{\beta}z)\,dz = -d\left[\sin(\omega t - \bar{\beta}z)\right]$$

and substituting

$$\frac{d\left[\sin\left(\frac{m\pi}{a}x\right)\right]}{\sin\left(\frac{m\pi}{a}x\right)} = \frac{-d\left[\sin(\omega t - \bar{\beta}z)\right]}{\sin(\omega t - \bar{\beta}z)}$$

Integrating

$$\ln\left[\sin\left(\frac{m\pi}{a}x\right)\right] = -\ln\left[\sin(\omega t - \bar{\beta}z)\right] + \text{const.}$$

$$\ln\left[\sin\left(\frac{m\pi}{a}x\right)\right] + \ln\left[\sin(\omega t - \bar{\beta}z)\right] = \text{const}$$

$$\ln\left[\sin\left(\frac{m\pi}{a}x\right)\sin(\omega t - \bar{\beta}z)\right] = \text{const.}$$

$$\Rightarrow \sin\left(\frac{m\pi}{a}x\right)\sin(\omega t - \bar{\beta}z) = \text{const.}$$

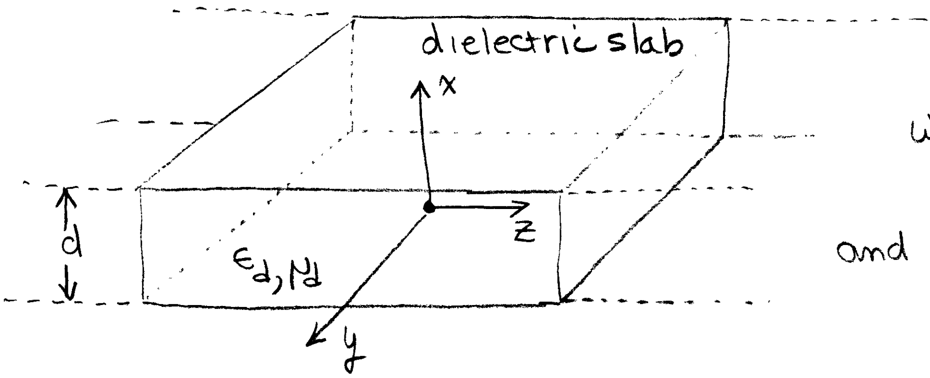
which can be used to plot H

4.2 dielectric waveguides

especially appropriate for optical wavelengths
1.3μm (minimum distortion) and 1.5μm (minimum loss)

Mode Theory.

want solutions with $e^{-\bar{\gamma} z}$ z variation
consider TM solutions written in terms of $E_z(x,z)$



where $\frac{\partial}{\partial y} \to 0$
and $E_z(x,z) = E_z^o(x) e^{-\bar{\gamma} z}$

We write a wave equation in the slab.

$$\frac{\partial^2 \underline{E}}{\partial x^2} + \bar{\gamma}^2 \underline{E} = -\omega^2 \mu \epsilon \underline{E}$$

Rewriting for E_z component

$$\frac{\partial^2 E_z}{\partial x^2} + \bar{\gamma}^2 E_z = -\omega^2 \mu \epsilon E_z$$

use $E_z^o(x) e^{-\bar{\gamma} z}$

$$\frac{\partial^2 E_z}{\partial x^2} + h^2 E_z = 0$$

giving

$$\frac{d^2 E_z^o(x)}{dx^2} + h^2 E_z^o(x) = 0$$

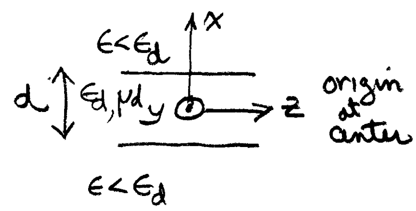
Must solve in both the slab and outside
define $h_d^2 = \omega^2 \mu_d \epsilon_d - \bar{\beta}^2$

The general solution to this equation inside the slab is

$$E_z^o(x) = C_o \sin(\beta_x x) + C_e \cos(\beta_x x) \qquad |x| \le \frac{d}{2}$$

where $\beta_x^2 = \omega^2 \mu_d \epsilon_d - \bar{\beta}^2 = h_d^2$ just like before but just in the dielectric

dielectric assumed lossless



The coefficients C_o and C_e refer to odd and even respectively.

For the waves to be guided by the slab the fields outside the slab should exponentially decay.

$$E_z^o(x) = \begin{cases} C_a e^{-\alpha_x (x - \frac{d}{2})} & x \ge \frac{d}{2} \quad \text{above the slab} \\ C_b e^{\alpha_x (x + \frac{d}{2})} & x \le \frac{d}{2} \quad \text{below the slab} \end{cases}$$

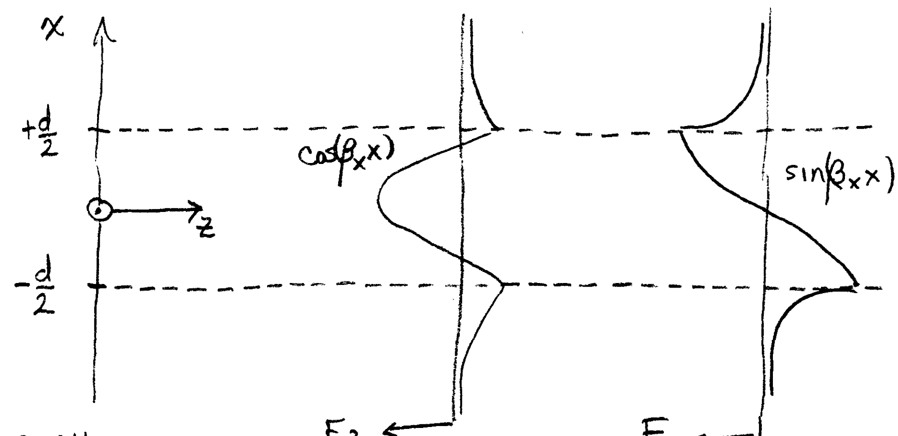
$$\alpha_x^2 = \bar{\beta}^2 - \omega^2 \mu_o \epsilon_o = -h_o^2$$

Note that the sign reverses since it decays

$$\alpha_x = \sqrt{\bar{\beta}^2 - \omega^2 \mu_o \epsilon_o}$$

here it is assumed that $\epsilon = \epsilon_o$ outside the slab

We require continuity of the field at the boundary



$f = 30\text{GHz}$
$d = 0.75\text{cm}$
$\epsilon_d = 2\epsilon_0$

We use the even/odd property of the transverse E fields to classify the solutions. Other classification schemes are possible.

The other field components can be calculated from

$$E_z(x, z) = E_z^o(x)\, e^{-j\bar{\beta} z}$$

We assumed that $H_z = 0$ (this is a TM mode)

We can use the previously written wave equations

$$H_y = -\frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x}$$

The complete "odd" solutions are

$x \geq \frac{d}{2}$ above slab

$$E_z^o(x) = \left[C_o \sin\left(\frac{\beta_x d}{2}\right)\right] e^{-\alpha_x (x - d/2)}$$

$$E_x^o(x) = -\frac{j\bar{\beta}}{\alpha_x}\left[C_o \sin\left(\frac{\beta_x d}{d}\right)\right] e^{-\alpha_x (x - d/2)}$$

$$H_y^o(x) = \frac{j\omega\epsilon_o}{\alpha_x}\left[C_o \sin\left(\frac{\beta_x d}{2}\right)\right] e^{-\alpha_x (x - d/2)}$$

dielectric slab $|x| \leq \frac{d}{2}$

$$E_z^o(x) = C_o \sin(\beta_x x)$$ ← this is what defines odd

$$E_x^o(x) = -\frac{j\bar{\beta}}{\beta_x} C_o \cos(\beta_x x)$$

$$H_y^o(x) = \frac{j\omega\epsilon_d}{\beta_x} C_o \cos(\beta_x x)$$

$x \leq -\frac{d}{2}$ below slab

$$E_z^o(x) = \left[C_o \sin\left(\frac{\beta_x d}{2}\right)\right] e^{-\alpha_x (x + \frac{d}{2})}$$

$$E_x^o(x) = -\frac{j\bar{\beta}}{\alpha_x}\left[C_o \sin\left(\frac{\beta_x d}{2}\right)\right] e^{-\alpha_x (x + \frac{d}{2})}$$

$$H_y^o(x) = \frac{j\omega\epsilon_o}{\alpha_x}\left[C_o \sin\left(\frac{\beta_x d}{2}\right)\right] e^{-\alpha_x (x + \frac{d}{2})}$$

There are several observations

The transverse E_x and H_y are in phase so power flow is in $+z$ direction

E_z & H_y are out of phase so no average power flows in x direction

α_x and β_x come from continuity of H_y at $x = \pm d/2$ (Continuity of E_z was already used)

$$H_y^o(x=+\tfrac{d}{2}) = \frac{j\omega\epsilon_o}{\alpha_x} C_o \sin\left(\frac{\beta_x d}{2}\right) \quad \text{free space above slab}$$

$$H_y^o(x=+\tfrac{d}{2}) = \frac{j\omega\epsilon_d}{\beta_x} C_o \cos\left(\frac{\beta_x d}{2}\right) \quad \text{within slab}$$

$$\frac{j\omega\epsilon_o}{\alpha_x} C_o \sin\left(\frac{\beta_x d}{2}\right) = \frac{j\omega\epsilon_d}{\beta_x} C_o \cos\left(\frac{\beta_x d}{2}\right)$$

$$\frac{\epsilon_o}{\epsilon_d}\,\frac{\sin\left(\frac{\beta_x d}{2}\right)}{\cos\left(\frac{\beta_x d}{2}\right)} = \frac{\alpha_x}{\beta_x}$$

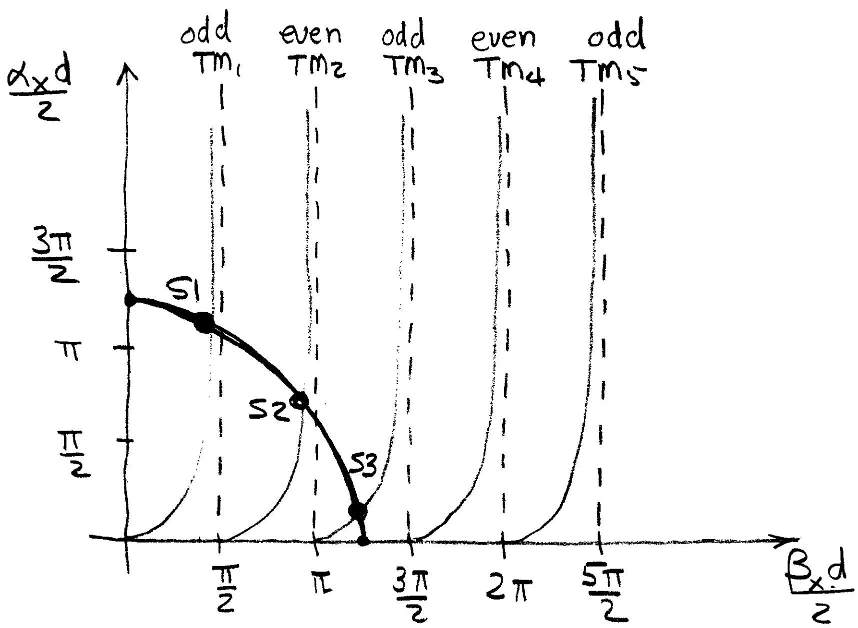
We also know that

$$\beta_x^2 = \omega^2 \mu_d \epsilon_d - \bar{\beta}^2$$

$$\alpha_x^2 = \bar{\beta}^2 - \omega^2 \mu_o \epsilon_o$$

So that $\quad \alpha_x^2 + \beta_x^2 = \omega^2(\mu_d\epsilon_d - \mu_o\epsilon_o)$

Good way to find α_x and β_x is graphically.



for $\epsilon_d = 2\epsilon_0$
$d = 1.25\lambda_0$

You can plot $\alpha_x^2 + \beta_x^2 = \omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0)$ as the circle shown above.

$$\alpha_x^2 \frac{d^2}{4} + \beta_x^2 \frac{d^2}{4} = \omega^2(2\mu_0\epsilon_0 - \mu_0\epsilon_0)\frac{d^2}{4}$$

$$\left(\frac{\alpha_x d}{2}\right)^2 + \left(\frac{\beta_x d}{2}\right)^2 = \frac{\omega^2 d^2}{4} = \frac{\left(\frac{2\pi}{\lambda}\right)^2\left(\frac{5}{4}\lambda\right)^2}{4} = \frac{\frac{4\pi^2}{\lambda^2}\cdot\frac{25\lambda^2}{16}}{4}$$

$$\left(\frac{\alpha_x d}{2}\right)^2 + \left(\frac{\beta_x d}{2}\right)^2 = \left(\frac{5\pi}{4}\right)^2$$

S1, S2 and S3 are the only possible solutions for these parameters.

The even modes are VERY similar to the odd mode solutions

Except

$$E_z^o(x) = C_e \cos(\beta_x x) \qquad |x| \le \frac{d}{2}$$

and the mode expression slightly changes to

$$\frac{\alpha_x}{\beta_x} = -\frac{\epsilon_o}{\epsilon_d}\cot\left(\frac{\beta_x d}{2}\right)$$

Cutoff frequencies

$$\omega\sqrt{\mu_o\epsilon_o} < \bar{\beta} < \omega\sqrt{\mu_d\epsilon_d}$$

as $\bar{\beta} \to \omega\sqrt{\mu_o\epsilon_o}$
$\alpha_x \to 0$
so the slab does no confining.
This represents cutoff since no bound modes exist.

set $\alpha_x = 0$ for cutoff

$$\beta_x = \omega_c\sqrt{\mu_d\epsilon_d - \mu_o\epsilon_o}$$

cutoff occurs when $\alpha_x = 0$

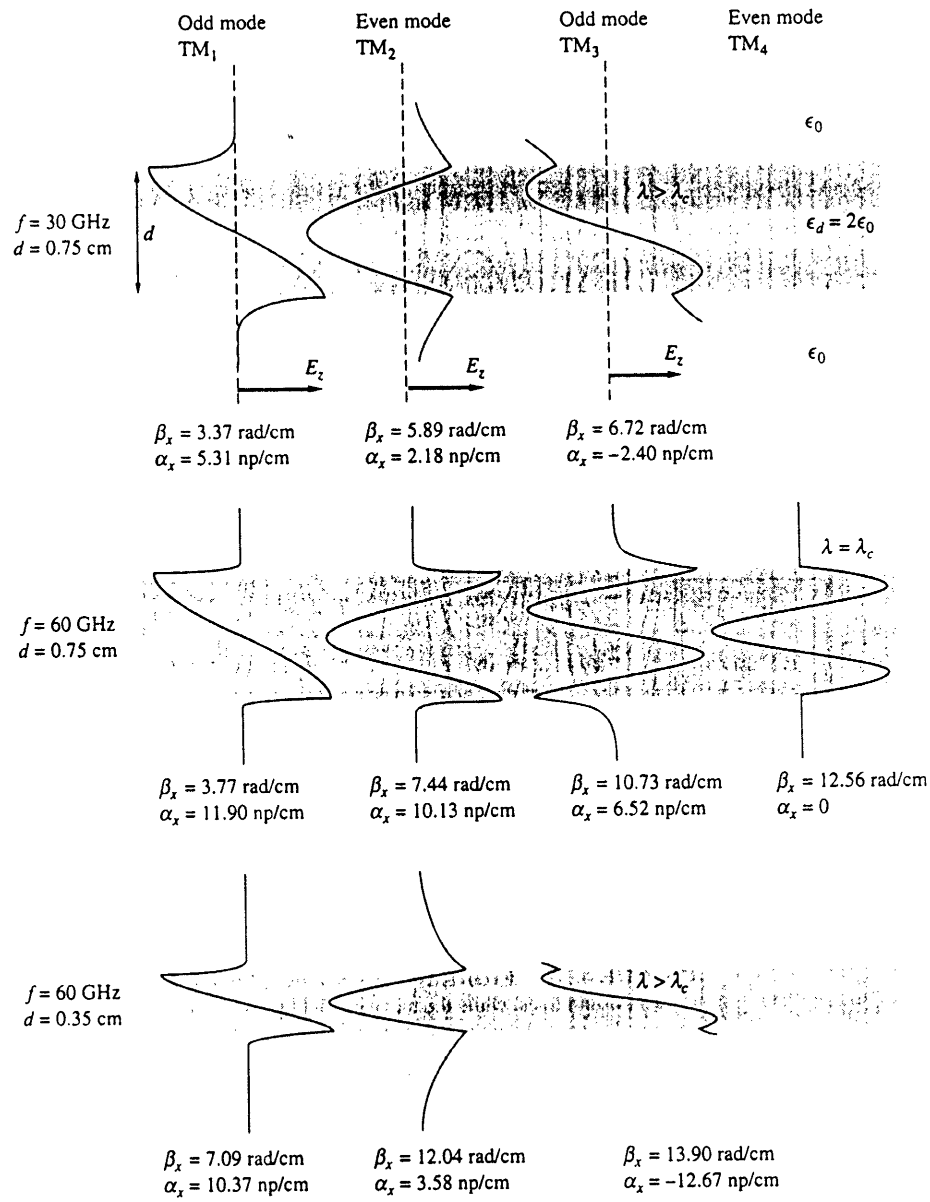
This requires that $\frac{\epsilon_o}{\epsilon_d}\tan\left(\frac{\beta_x d}{2}\right) = \frac{\alpha_x}{\beta_x} = 0$

Then $\frac{\beta_x d}{2} = (m-1)\frac{\pi}{2}$ where $m = 1, 3, 5,$ etc. "odd" modes

$$\frac{\omega_c\sqrt{\mu_d\epsilon_d - \mu_o\epsilon_o}\, d}{2} = (m-1)\frac{\pi}{2}$$

$$\frac{2\pi f_c\sqrt{\mu_d\epsilon_d - \mu_o\epsilon_o}\, d}{2} = (m-1)\frac{\pi}{2}$$

$$f_{c,TM_m} = \frac{m-1}{2\sqrt{\mu_d\epsilon_d - \mu_o\epsilon_o}}$$



frequency goes up, modes go up.

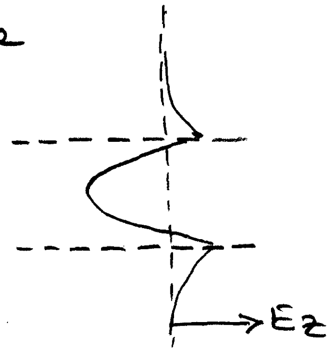
Thinner slab modes go down

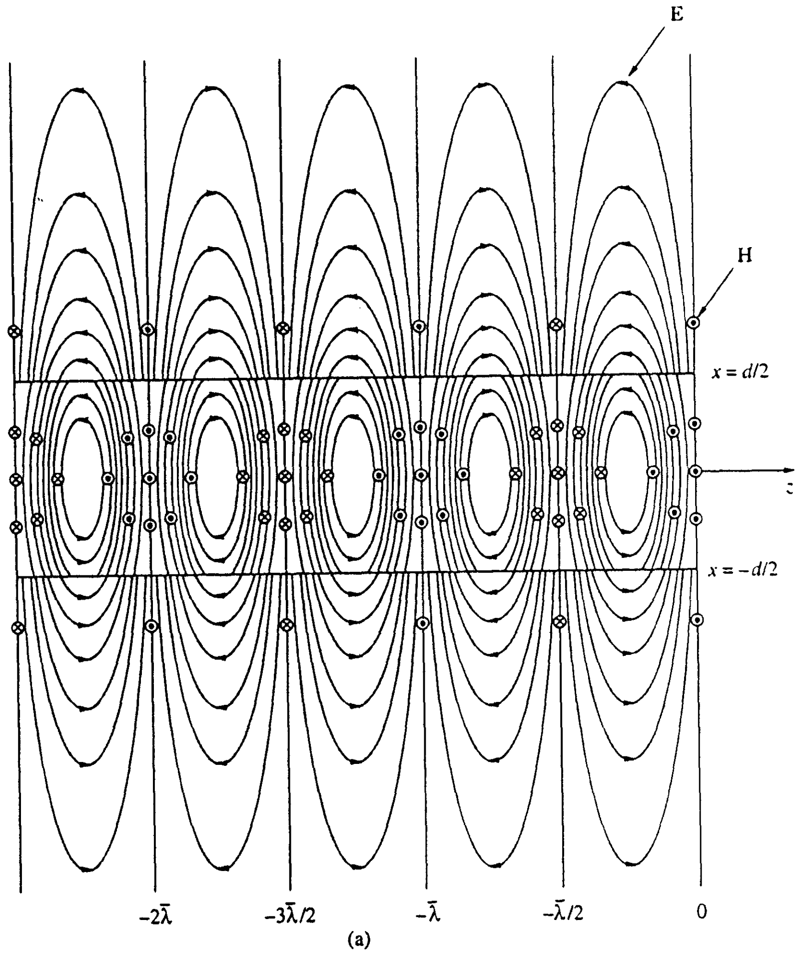
FIGURE 4.17. Variation of field distributions for various modes with respect to thickness and frequency. Dependence of the dielectric waveguide mode structure on frequency and slab thickness. The plots show the variation of the axial field component E_z over the vertical cross section of the dielectric slab. For all cases, $\epsilon_d = 2\epsilon_0$. Note that the numerical values are the same as those for Example 4-6.

Observations

1. increasing the operating frequency increases the number of propagating modes in the guide
2. increasing the slab thickness also increases the number of propagating modes

Basic TM_2 "even" mode





E field intensity decreases as one gets farther away from the slab

FIGURE 4.18. Even TM_2 mode field distribution. The electric field lines are shown as solid lines, while the magnetic field lines are orthogonal to the page and are indicated alternately with circles or crosses. Figure taken (with permission) from H. A. Haus, ***Waves and Fields in Optoelectronics***, Prentice Hall, Englewood Cliffs, New Jersey, 1984.

TE Modes Look very similar to TM modes

Fundamental solution inside slab is

$$H_z^o(x) = C_o \sin(\beta_x x) + C_e \cos(\beta_x x)$$

The exact form is usually not necessary

We apply the boundary conditions at $x = \pm d/2$ to get

$$\frac{\alpha_x}{\beta_x} = \frac{\mu_o}{\mu_d} \tan\left(\frac{\beta_x d}{2}\right) \quad \text{odd TE modes}$$

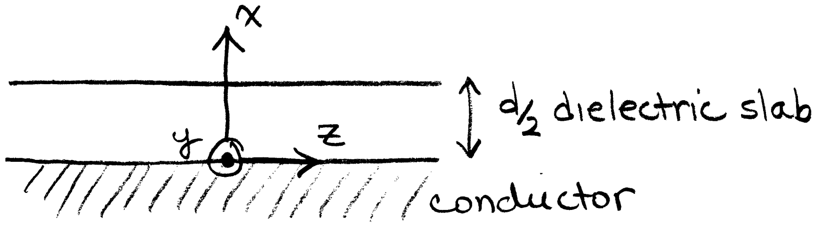
$$\frac{\alpha_x}{\beta_x} = -\frac{\mu_o}{\mu_d} \cot\left(\frac{\beta_x d}{2}\right) \quad \text{Even TE modes}$$

Cutoff occurs when $\alpha_x = 0$, i.e. no attenuation with distance

The cutoff frequencies for the TE modes are the same as for the TM modes

$$f_{c,TE_m} = \frac{(m-1)}{2d\sqrt{\mu_d \epsilon_d - \mu_o \epsilon_o}} \qquad \begin{array}{l} m = 1, 3, 5 \text{ odd modes} \\ m = 2, 4, 6 \text{ even modes} \end{array}$$

Dielectric Covered ground plane



this is a realistic structure for miniature microwave circuits

However, we have boundary conditions for a conductor at $x=0$, i.e., $E_y=0$ and $E_z=0$

Let's look at the E fields of the TM modes

Even TM $0 \le x \le d/2$

$$\left\{\begin{array}{l} E_y = 0 \\ E_x = \frac{j\bar{\beta}}{\beta_x} C_e \sin(\beta_x x) e^{-j\bar{\beta} z} \\ E_z = C_e \cos(\beta_x x) e^{-j\bar{\beta} z} \end{array}\right.$$

Not allowed since $E_z(z=0)$ must be zero.

Odd TM $0 \le x \le d/2$

$$\left\{\begin{array}{l} E_y = 0 \\ E_x = -\frac{j\bar{\beta}}{\beta_x} C_o \cos(\beta_x x) e^{-j\bar{\beta} z} \\ E_z = C_o \sin(\beta_x x) e^{-j\bar{\beta} z} \end{array}\right.$$

Even TE $0 \le x \le d/2$

$$\left\{\begin{array}{l} E_y = -\frac{j\omega\mu}{\beta_x} C_e \sin(\beta_x x) e^{-j\bar{\beta} z} \\ E_x = 0 \\ E_z = 0 \end{array}\right.$$

Odd TE $0 \le x \le \frac{d}{2}$

$$\left\{\begin{array}{l} E_y = \frac{j\omega\mu}{\beta_x} C_o \cos(\beta_x x) e^{-j\bar{\beta} z} \\ E_x = 0 \\ E_z = 0 \end{array}\right.$$

Not allowed since $E_y(z=0)$ must be zero.

Only Even TM and odd TE modes are allowed.

$$\beta_x^2 = \omega^2 \mu_d \epsilon_d - \bar{\beta}^2 \cong \omega^2 \mu_d \epsilon_d - \omega^2 \mu_o \epsilon_o$$

since for thin guides $\bar{\beta} \to \omega^2 \mu_o \epsilon_o$

$$\alpha_x = \frac{\epsilon_o}{\epsilon_d} \frac{\beta_x^2 d}{2} = \frac{\epsilon_o}{\epsilon_d} \frac{d}{2} \left[\omega^2 \mu_d \epsilon_d - \omega^2 \mu_o \epsilon_o\right]$$

$$\alpha_x = \frac{d}{2} \frac{\epsilon_o}{\epsilon_d} \left[\omega^2 \mu_d \epsilon_d - \omega^2 \mu_o \epsilon_o\right] = \frac{d}{2} \frac{\epsilon_o}{\epsilon_d} \omega^2 \mu_o \epsilon_o \left[\frac{\mu_d \epsilon_d}{\mu_o \epsilon_o} - 1\right]$$

$$\alpha_x = \frac{d}{2} \beta^2 \frac{\epsilon_o}{\epsilon_d} \left[\frac{\mu_d}{\mu_o} \frac{\epsilon_d}{\epsilon_o} - 1\right] = \frac{d}{2} \beta^2 \left[\frac{\mu_d}{\mu_o} - \frac{\epsilon_o}{\epsilon_d}\right]$$

$$\alpha_x = \frac{d}{2} \frac{2\pi}{\lambda} \beta \left[\frac{\mu_d}{\mu_o} - \frac{\epsilon_o}{\epsilon_d}\right]$$

$$\alpha_x = 2\pi\beta \left[\frac{\mu_d}{\mu_o} - \frac{\epsilon_o}{\epsilon_d}\right] \frac{d}{2\lambda}$$

For these "surface wave" modes continuity of fields requires

$$\frac{\alpha_x}{\beta_x} = \frac{\epsilon_0}{\epsilon_d}\tan\left(\frac{\beta_x d}{2}\right) \quad \text{odd TM}$$

$$\frac{\alpha_x}{\beta_x} = -\frac{\mu_0}{\mu_d}\cot\left(\frac{\beta_x d}{2}\right) \quad \text{Even TE}$$

For all modes

$$\alpha_x^2 + \beta_x^2 = \omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0)$$

cutoff frequencies

$$f_{c\,TM\,or\,TE} = \frac{(m-1)}{2d\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}} \qquad \begin{array}{l} m = 1,3,5 \text{ odd TM} \\ m = 2,4,6 \text{ Even TE} \end{array}$$

If you make the dielectric thick then $\bar{\beta} \to \beta_d = \omega^2\mu_d\epsilon_d$

In this case

$$\alpha_x = \sqrt{\bar{\beta}^2 - \omega^2\mu_0\epsilon_0} \cong \sqrt{\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0)}$$

or $\alpha_x \cong \beta\sqrt{\frac{\mu_d\epsilon_d}{\mu_0\epsilon_0} - 1}$ which is large and

which indicates the transverse fields attenuate rapidly.

If you make the dielectric thin then $\bar{\beta} \to \beta = \omega\sqrt{\mu_0\epsilon_0}$

$$\beta_x^2 = \omega^2\mu_d\epsilon_d - \bar{\beta}^2 \cong \omega^2\mu_d\epsilon_d - \omega^2\mu_0\epsilon_0$$

In this thin limit $\frac{d}{\lambda_x} \to 0$ and

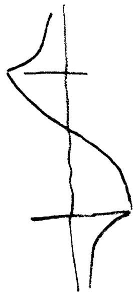
$$\frac{\alpha_x}{\beta_x} = \frac{\epsilon_0}{\epsilon_d}\tan\left(\frac{\beta_x d}{2}\right) \cong \frac{\epsilon_0}{\epsilon_d}\left(\frac{\beta_x d}{2}\right) + \cdots$$

Recall (p.27)
$\beta_x^2 = \omega^2\mu_d\epsilon_d - \beta^2 = \beta_d^2$

$$\alpha_x = \frac{\epsilon_0}{\epsilon_d}\frac{\beta_x^2 d}{2} = \frac{d}{2}\frac{\epsilon_0}{\epsilon_d}\omega^2\mu_0\epsilon_0\left(\frac{\epsilon_d\mu_d}{\epsilon_0\mu_0} - 1\right) = 2\pi\beta\left[\frac{\mu_d}{\mu_0} - \frac{\epsilon_d}{\epsilon_0}\right]\frac{d}{2\lambda}$$

so α_x is small.

Something very unusual occurs in dielectric slab waveguides



$f_c = 0$ for the TM_1 mode! for both the dielectric slab and for the dielectric above a conducting ground plane

$f_c = 0$ for the TE_1 mode for a dielectric slab in free space.

$$f_{c,\,TM\text{ or }TE} = \frac{(m-1)}{2d\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}$$

However, this is not really true that you can go down to DC. As f decreases α_x also decreases

$$\alpha_x = \sqrt{\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - \beta_x^2}$$

$$\rightarrow \omega\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0} \text{ as } \omega \text{ gets large.}$$

As $\omega \rightarrow \infty$ $\alpha_x \rightarrow \infty$

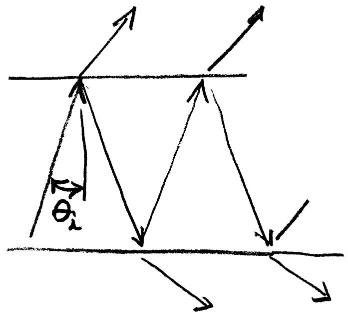
and $\frac{1}{\alpha_x} \rightarrow 0$ This is the depth of penetration <u>outside</u> the slab so the field is essentially confined to the slab as $\omega \rightarrow \infty$.

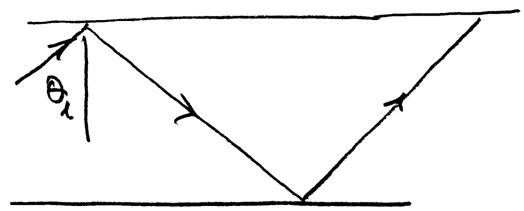
However, at $\omega = \omega_c$ we already know that $\alpha_x = 0$ since at cutoff the fields do not decay.

This means that $\frac{1}{\alpha_x} \rightarrow \infty$ as $\omega \rightarrow \omega_c$

The physical consequence is that the field spreads out to infinity at $\omega = \omega_c$.

4.2.3 Dielectric slab waveguides: ray theory

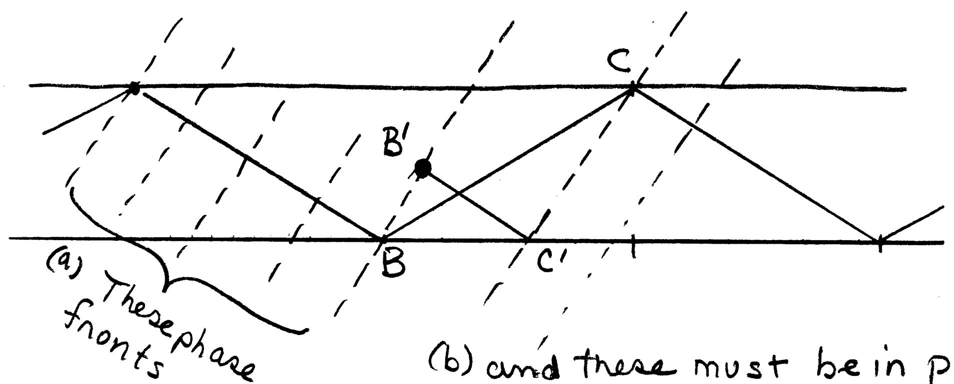




unguided wave since $\theta_i < \theta_c$ and wave refracts out of guide

guided wave since $\theta_i > \theta_c$ gives total internal reflection

However, not any angle can propagate



(b) and these must be in phase

so

(c) This distance must be $n \cdot 2\pi$
$B'C'$

reflections at B and C

$$\therefore \quad \beta_d(\overline{BC} - \overline{B'C'}) - 2\phi_r = (m-1)\,2\pi \qquad m = 1, 2, 3$$

$$\beta_d = \frac{2\pi}{\lambda_d} = \frac{2\pi}{\lambda}\sqrt{\frac{\epsilon_d}{\epsilon_0}}$$

ϕ_r is the phase shift from TIR at either B or C

Geometry gives

$$2\beta_d d \cos\theta_i - 2\phi_r = (m-1)2\pi$$

Note: $\tan\theta_i = \dfrac{\bar{\beta}}{\beta_x}$ $\qquad \beta_d^2 = \beta_x^2 + \bar{\beta}^2 = \omega^2 \mu_d \epsilon_d.$

Assume $\perp$ polarization ($\underline{E}$ out of plane)

$$\Gamma_\perp = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}} = 1e^{j\phi_\perp}$$

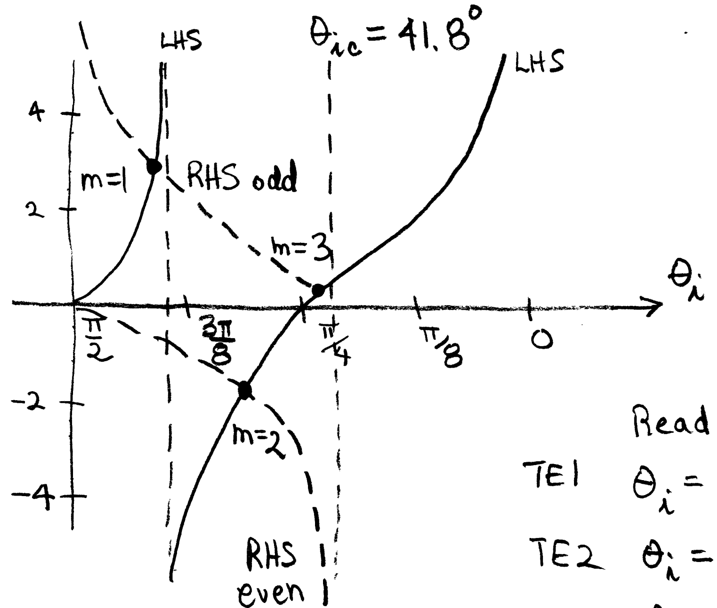
where $\phi_r = \phi_\perp = 2\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}}{\cos\theta_i}\right)$

$$\therefore\quad 2\beta_d d\cos\theta_i - 4\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}}{\cos\theta_i}\right) = (m-1)\,2\pi$$

$$\mathrm{Tan}\left(\frac{\beta_d d\cos\theta_i}{2} - \frac{(m-1)\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - \epsilon_0/\epsilon_d}}{\cos\theta_i}$$

Do graphically. Plot LHS (solid) and RHS (dashed)

Example 4.9



even $m = 2, 4, 6, \ldots$

odd $m = 1, 3, 5, \ldots$

Note: axis goes from $\frac{\pi}{2}$ to 0 which is the physical limit.

Read θ_i's from graph as

TE1 $\theta_i = 75.03^\circ$

TE2 $\theta_i = 59.47^\circ$

TE3 $\theta_i = 43.86^\circ$

Parameters: $f = 30$ GHz, $d = 1$ cm, $\epsilon_d = 2.25\epsilon_0$ (glass) surrounded by air tan can be ± depending on quadrant

$$\tan\left(\frac{\beta_d d\cos\theta_i}{2} - \frac{(m-1)\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}}{\cos\theta_i} \qquad m=1,2,3,\ldots$$

$$\tan\left(\frac{2\pi\frac{d}{\lambda_d}\cos\theta_i}{2}\right) = \frac{\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}}{\cos\theta_i} \qquad m=1,3,5,\ldots$$

positive quadrants

$$\tan\left(2\pi\frac{d}{\lambda_d}\cos\theta_i\right) = -\frac{\cos\theta_i}{\sqrt{\sin^2\theta_i - \frac{\epsilon_0}{\epsilon_d}}} \qquad m=2,4,6,\ldots$$

negative quadrants

4.3 Wave velocities and waveguide dispersion

for a parallel plate waveguide

$$\bar{\beta} = \omega\sqrt{\mu\epsilon'}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

or $$\bar{v}_p = \frac{\omega}{\bar{\beta}} = \frac{1}{\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Note that $\bar{v}_p$ inside the waveguide is always greater than

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$$ in the unbounded dielectric.

Note further that $\bar{v}_p \to \infty$ as $f \to f_c$

The phase velocity of uniform plane waves in lossless <u>unbounded</u> (unguided) media does not change.

$$\beta = \omega\sqrt{\mu\epsilon}$$

and $$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$$

4.3.1 Group velocity

The envelope (modulation) of a wave travels at

$$v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\bar{\beta}} \to \frac{d\omega}{d\bar{\beta}}$$

For lossless media

$$v_g = \frac{d\omega}{d\bar{\beta}} = \frac{1}{\frac{d\bar{\beta}}{d\omega}} = \frac{1}{\sqrt{\mu\epsilon}} = v_p.$$

wave packets of closely related frequencies travel at v_g

Consider two waves

$$\mathcal{E}_{1x}(z,t) = C\cos(\omega t - \bar{\beta} z)$$

$$\mathcal{E}_{2x}(z,t) = C\cos\left((\omega+\Delta\omega)t - (\bar{\beta}+\Delta\bar{\beta})z\right)$$

using trig. identities

$$\mathcal{E}_x(z,t) = \mathcal{E}_{1x} + \mathcal{E}_{2x} = 2C\cos\left[\tfrac{1}{2}(\Delta\omega t - \Delta\bar{\beta} z)\right]\cos\left[\left(\omega + \tfrac{\Delta\omega}{2}\right)t - \left(\bar{\beta} + \tfrac{\Delta\bar{\beta}}{2}\right)z\right]$$

(first cosine:) this is the so-called "amplitude envelope" It contains only Δ terms

(second cosine:) high frequency

$\Delta\omega t - \Delta\bar{\beta} z$ = constant for a propagating plane wave

differentiating

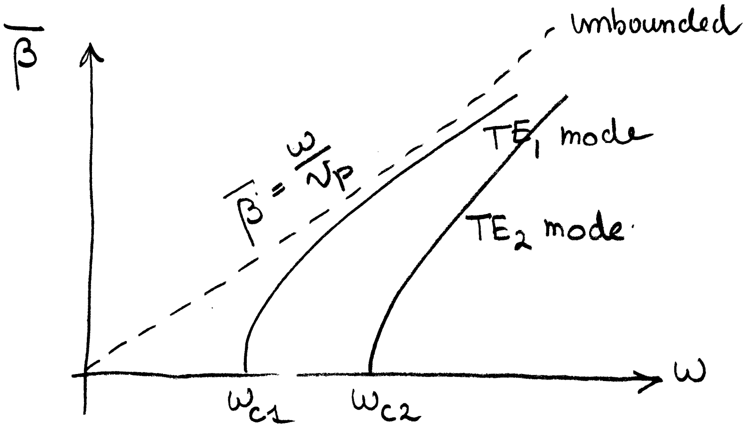
$$\Delta\omega\, dt - \Delta\bar{\beta}\, dz = 0$$

$$\frac{dz}{dt} = \frac{\Delta\omega}{\Delta\bar{\beta}}$$

Group velocity $v_g \triangleq \dfrac{dz}{dt} = \dfrac{\Delta\omega}{\Delta\bar{\beta}}$

4.3.2 Dispersion ($\bar{\beta}$–ω) diagrams

Consider the plot of $\bar{\beta}$ versus ω for TE_1/TM_1 modes in a parallel plate waveguide.



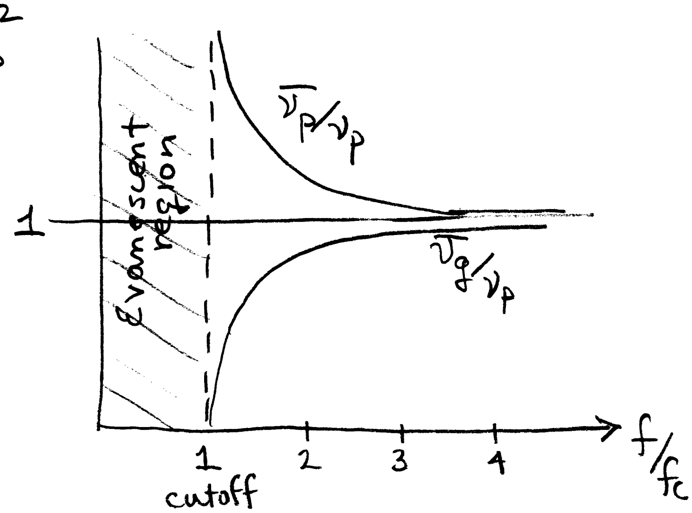
$$\bar{\beta} = \omega\sqrt{\mu\epsilon}\,\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$ as a function of ω

$$\bar{v}_p = \bar{v}_p(\omega) = \frac{\omega}{\bar{\beta}} = \frac{v_p}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

$v_p = \frac{1}{\sqrt{\mu\epsilon}}$ as a function of ω

$$\bar{v}_g = \frac{1}{\frac{d\bar{\beta}}{d\omega}} = v_p\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$ group velocity for modes in a parallel plate waveguide

Note: $\bar{v}_p \bar{v}_g = v_p^2$



See Sect. 4.3.3. of text

The velocity of energy flow is the group velocity

$$v_E = v_g = v_p\sqrt{1-\left(\frac{m\lambda}{2a}\right)^2}$$

which is also the component of each mode's velocity in the z-direction.

$$\bar{\beta} = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

$$\frac{d\bar{\beta}}{d\omega} = \sqrt{\mu\epsilon}\left[1-\left(\frac{\omega_c}{\omega}\right)^2\right]^{\frac{1}{2}} + \omega\sqrt{\mu\epsilon}\;\frac{\omega_c^2}{\omega^3}\left[1-\left(\frac{\omega_c}{\omega}\right)^2\right]^{-\frac{1}{2}}$$

$$\frac{d\bar{\beta}}{d\omega} = \frac{\sqrt{\mu\epsilon}\left[1-\left(\frac{\omega_c}{\omega}\right)^2\right] + \frac{\omega_c}{\omega^2}\sqrt{\mu\epsilon}}{\left[1-\left(\frac{\omega_c}{\omega}\right)^2\right]^{\frac{1}{2}}}$$

$$\frac{d\bar{\beta}}{d\omega} = \frac{\sqrt{\mu\epsilon}}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

$$v_g = \frac{1}{\frac{d\bar{\beta}}{d\omega}} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = v_p\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$

From Ramo, Whinnery, Van Duzer, 3/e Ch. 8

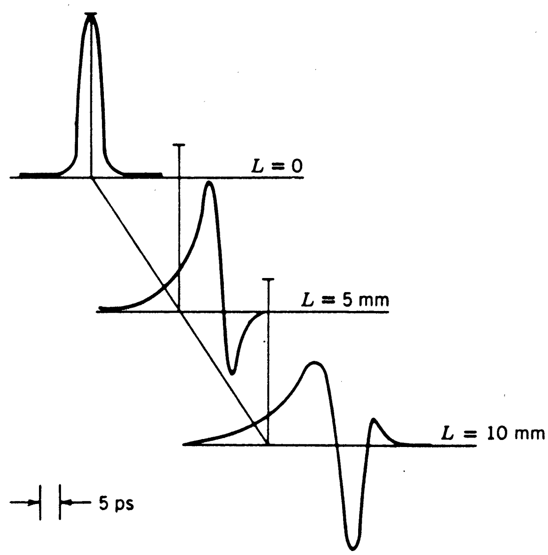


FIG. 8.16a Propagation of a 5-ps gaussian pulse along a microstrip line. Strip width = 0.32 mm, dielectric thickness = 0.4 mm, and ε_r = 6.9. Reproduced by permission from K. K. Li, G. Arjavalingam, A. Dienes, and J. R. Whinnery, *IEEE Trans.* **MTT-30**, 1270 (1982). © 1982 IEEE.

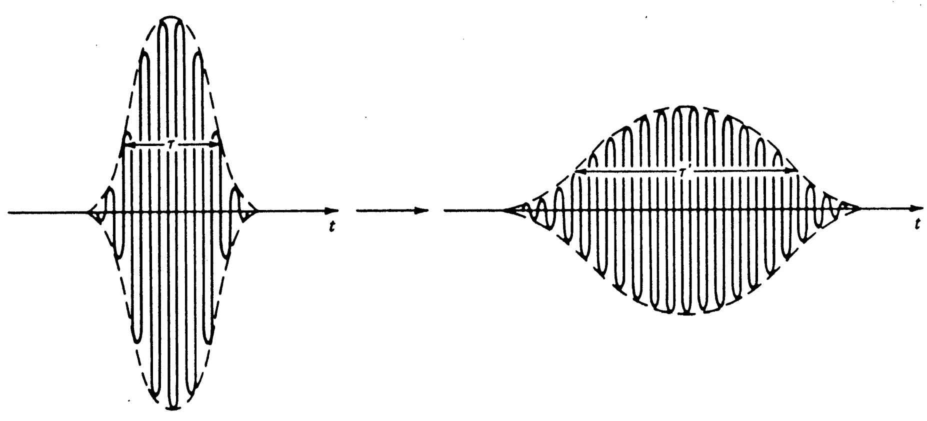


FIG. 8.16b Illustration of the spread of the modulated envelope of a pulse as it travels down a system with group dispersion.